EQUIDISTRIBUTIONS OF MESH PATTERNS OF LENGTH TWO AND KITAEV AND ZHANG'S CONJECTURES

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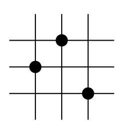
ABSTRACT. A systematic study of *avoidance* of mesh patterns of length 2 was conducted by Hilmarsson *et al.* in 2015. In a recent paper Kitaev and Zhang examined the distribution of the aforementioned patterns. The aim of this paper is to prove more equidistributions of mesh pattern and confirm Kitaev and Zhang's four conjectures by constructing two involutions on permutations.

1. Introduction

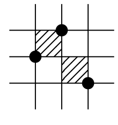
Patterns in permutations and words have implicitly appeared in the mathematics literature for over a century, but interest in them has blown up in the past four decades (see [3, 6, 8, 12, 15] and references therein), and the research of this area continues to increase gradually.

Let S_n be the set of all permutations of length n. A (classical permutation) pattern is a permutation $\tau \in S_n$. We could draw the pattern $231 \in S_3$ as follows, where the horizontal lines represent the values and the vertical lines denote the positions in the pattern.



To study the explicit expansions for certain permutation statistics as, possibly infinite, linear combinations of (classical) permutation patterns, Brändén and Claesson [3] first introduced the notion of a *mesh pattern*, which generalize several classes of patterns.

A pair (τ, R), where τ is a permutation of length k and R is a subset of $[\![0,k]\!] \times [\![0,k]\!]$, where $[\![0,k]\!]$ denotes the interval of the integers from 0 to k, is a *mesh pattern* of length k. Let (i,j) denote the box whose corners have coordinates $(i,j),(i,j+1),(i+1,j+1)$, and $(i+1,j)$. Mesh patterns can be depicted by shading the boxes in R. A mesh pattern with $\tau = 231$ and $R = \{(1,2),(2,1)\}$ is drawn as follows.



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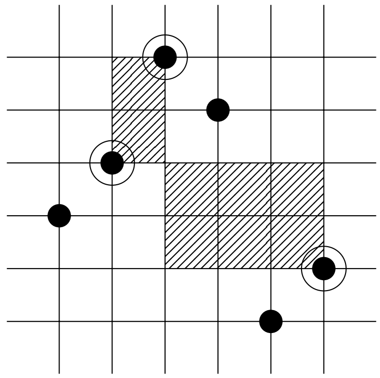
For example, the permutation 346512 depicted in the following picture contains the mesh pattern $(231, \{(1,2),(2,1)\})$ since the subsequence 462 forms the classical pattern 231 and there are no points in the shaded areas.

$346512 =$ 

The mesh patterns and their generalizations were studied in many papers; e.g. see [1,2,6,7,9–12,16,17]. In the first systemically study of the mesh patterns avoidance, Hilmarsson et al. [6] solved 25 out of 65 non-equivalent *avoidance* cases of patterns of length 2. In a recent paper [12], Kitaev and Zhang further studied the distributions of mesh patterns considered in [6] by giving 27 distribution results see [12, Table 1]. Moreover, for the unsolved case, they gave an equidistribution result and conjectured 6 more equidistributions (see Table 1). In this paper, we prove 3 conjectured equidistributions and 2 more equidistribuions (see Table 2) by constructing two involutions.

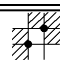
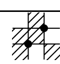
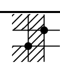
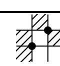
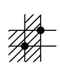
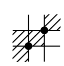
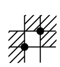
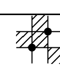
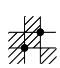
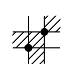
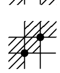
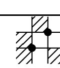
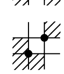
	Nr.	Repr. p	Ref.	Nr.	Repr. p	Ref.
proved equidistributions	48		[12, Theorem 5.1]			
	49					
conjectured equidistributions	23		Theorem 1.9	53		Theorem 1.6
	24			54		
	48		Theorem 1.6 and [12, Theorem 5.1]	57		N/A
	49			58		
	50			61		N/A
				62		

TABLE 1. Equidistributions for which enumeration is unknown. Pattern's numbers are adopted from [6,12]

	Nr.	Repr. p	Ref.	Nr.	Repr. p	Ref.
proved	1*		Theorem 1.9	3*		Theorem 1.9
equidistributions	2*			4*		

TABLE 2. More proved equidistributions. Pattern's numbers are not considered in [6, 12]

For a pattern p and a permutation π, we let $p(\pi)$ denote the number of occurrences of p in π. Kitaev and Zhang [12, Conjecture 6.1] conjectured a Stieltjes continued fraction formula for the distribution of pattern Nr. 3 = (see [13, A200545]), which is equivalent to the following identity.

Conjecture 1.1. [12, Conjecture 6.1] We have

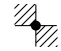
$$\sum_{n\geq 0} t^n \sum_{\pi\in S_n} y^{\,p(\pi)} = \cfrac{1}{1-\cfrac{\alpha_1 t}{1-\cfrac{\alpha_2 t}{1-\cdots}}} \tag{1.1a}$$

with coefficients

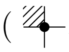
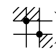
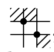
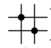
$$\alpha_{2k-1} = k, \quad \alpha_{2k} = y + k - 1. \tag{1.1b}$$

Presenting their conjecture in this way, we notice that the S-continued fraction (1.1a) appears in a recent paper of Sokal and Zeng [14]. Let us reformulate the relevant permutation statistics in [14] in terms of mesh patterns. Given a permutation $\pi \in S_n$, an index $i \in [n]$ (or a value $\pi(i) \in [n]$) is called

- an *excedance* if $\pi(i) > i$;
- an *inversion* if $\pi(j) > \pi(i)$ for $1 \leq j < i$; in other words, an inversion of π is one occurrence of pattern of π;
- a *record* (rec) (or *left-to-right maximum*) if $\pi(j) < \pi(i)$ for all $j < i$ [note in particular that the index 1 is always a record and that the value n is always a record]; in other words, a record of π is one occurrence of pattern of π;
- an *antirecord* (arec) (or *right-to-left minimum*) if $\pi(j) > \pi(i)$ for all $j > i$ [note in particular that the index n is always an antirecord and that the value 1 is always an antirecord]; in other words, an antirecord of π is one occurrence of pattern of π;
- an *exclusive record* (erec) if it is a record and not also an antirecord; in other words, an exclusive record of π is one occurrence of pattern of π, see (1.12);
- an *exclusive antirecord* (earec) if it is an antirecord and not also a record; in other words, an exclusive antirecord of π is one occurrence of pattern of π, see (1.11);

- a *record-antirecord* (rar) (or *pivot*) if it is both a record and an antirecord; in other words, a record-antirecord of π is one occurrence of pattern  of π.

We denote the number of excedances, records, antirecords, exclusive records, exclusive antirecords and record-antirecords in π by $\mathrm{exc}(\pi)$, $\mathrm{rec}(\pi)$, $\mathrm{arec}(\pi)$, $\mathrm{erec}(\pi)$, $\mathrm{earec}(\pi)$ and $\mathrm{rar}(\pi)$, respectively.

Dumont and Kreweras [5] gave the joint distribution of (, ), Zeng [18] gave the joint distribution of (, , ). Recently Sokal and Zeng [14] proved much more general results. For example, define the generating function of the generalized Eulerian polynomials

$$F(x,y,z,v,q;t) = \sum_{n=0}^{\infty} t^n \sum_{\sigma\in\mathfrak{S}_n} x^{\mathrm{arec}(\sigma)} y^{\mathrm{erec}(\sigma)} z^{\mathrm{rar}(\sigma)} v^{\mathrm{exc}(\sigma)} q^{\mathrm{inv}(\sigma)}. \tag{1.2}$$

From [14, Theorems 2.7 and 2.8] we derive the following result.

Theorem 1.2. *We have*

$$F(x,y,z,v,q;t) = \frac{F(x,y,1,v,q;t)}{1+x(1-z)tF(x,y,1,v,q;t)}, \tag{1.3a}$$

where

$$F(x,y,1,v,q;t) = \cfrac{1}{1-\cfrac{\alpha_1 t}{1-\cfrac{\alpha_2 t}{1-\cdots}}} \tag{1.3b}$$

with coefficients

$$\alpha_{2k-1} = q^{k-1}(x+q+q^2+\cdots+q^{k-1}) \tag{1.3c}$$

$$\alpha_{2k} = q^k v(y+q+q^2+\cdots+q^{k-1}). \tag{1.3d}$$

Proof. This follows from [14, Theorems 2.7 and 2.8] by specializing the parameters. We just indicate the appropriate specialisation and refer the reader to [14] for further details. In the specialization (2.57) of [14, Theorem 2.7], if we choose $w_0 = xz$ (instead of $w_0 = x$ in [14]) and

$$y = qv, \quad u = 1, \quad v_1 = v_2 = qv, \quad p_+ = p_- = q, \quad q_+ = q_- = q^2,$$

then equation (2.52) of [14] reduces to

$$F(x,y,z,v,q;t) = \cfrac{1}{1-\gamma_0 t-\cfrac{\beta_1 t}{1-\gamma_1 t-\cfrac{\beta_2 t}{1-\cdots}}} \tag{1.4}$$

with

$$\gamma_0 = xz, \quad \gamma_n = \alpha_{2n}+\alpha_{2n+1}, \quad \beta_n = \alpha_{2n-1}\alpha_{2n}. \tag{1.5}$$

Therefore, the J-fraction formula can be written as (by contracting the S-fraction starting from the second line),

$$F(x, y, z, v, q; t) = \cfrac{1}{1 + x(1-z)t - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{1 - \cdots}}}, \tag{1.6}$$

which is equivalent to (1.3a). □

Remark 1.3.
- We can also prove (1.3a) by following the same steps in the special case as in [12] and then derive (1.3b) directly from [14, Theorem 2.8].
- The case $x = y = v = q = 1$ of Theorem 1.2 is Theorem 1.1 in [12].
- Since $(\mathrm{arec}, \mathrm{inv})\pi = (\mathrm{rec}, \mathrm{inv})\pi^{-1}$ we derive from [18] that

$$F(x, 1, 1, 1, q; t) = \sum_{n=0}^{\infty} x(x+q)\ldots(x+q+\cdots q^{n-1})t^n. \tag{1.7}$$

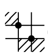
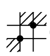
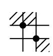
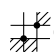
For $\pi = \pi(1)\ldots\pi(n) \in S_n$ we define the following three associated permutations:

$$\pi^{-1} := \pi^{-1}(1)\pi^{-1}(2)\cdots\pi^{-1}(n) \tag{1.8}$$

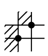
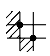
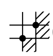
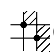
$$\pi^{\mathrm{r}} := \pi(n)\cdots\pi(2)\pi(1) \tag{1.9}$$

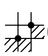
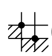
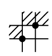
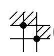
$$\pi^{c} := (n+1-\pi(1))(n+1-\pi(2))\cdots(n+1-\pi(n)) \tag{1.10}$$

Obviously we have

$(\pi) =$ $(\pi^c) =$ $(\pi^{roc}) =$ (π^r)

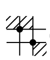
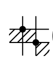
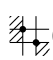
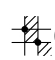
and

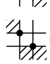
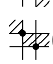
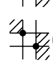
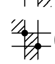
$(\pi) =$ $(\pi^c) =$ $(\pi^{roc}) =$ (π^r)

$=$ $(\pi^{-1}) =$ $(\tau^r) =$ $(\tau^{roc}) =$ (τ^c)

with $\tau = \pi^{-1}$.

Lemma 1.4. *For $\pi \in S_n$, we have*

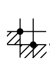
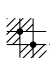
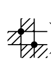
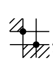
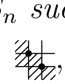
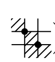
$\mathrm{earec}(\pi) =$ $(\pi) =$ $(\pi) =$ $(\pi) =$ (π), (1.11)

$\mathrm{erec}(\pi) =$ $(\pi) =$ $(\pi) =$ $(\pi) =$ (π). (1.12)

Proof. We just prove (1.11) as the proof of (1.12) is similar. In the rook placement representation of a permutation $\pi \in S_n$ the rook $y = (i, \pi(i))$ is an exclusive antirecord iff there is a another rook $x = (j, \pi(j))$ at left of y, i.e., $j < i$ and higher than x, i.e., $\pi(j) > \pi(i)$. Hence there are four unique choices for such a rook x: the *highest, lowest, farthest* and *nearest*. This corresponds to the four mesh patterns in (1.11), respectively. □

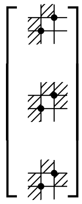
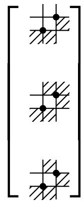
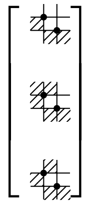
Remark 1.5. As $\mathrm{earec}(\pi) = \mathrm{erec}(\pi^{\mathrm{roc}})$ for $\pi \in S_n$, we can also derive (1.12) from (1.11).

Theorem 1.6. *There exists an involution Φ on S_n such that for $\pi \in S_n$,*

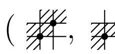
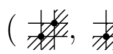
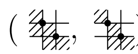
(, , )$\pi =$ (, , )$\Phi(\pi)$.

Corollary 1.7. *The triple pattern* (Nr.3, Nr.48, Nr.53) *is equidistributed with* (erec, Nr.50, Nr.54) *on* S_n.

Proof. For any $\pi \in S_n$ we have

$\begin{bmatrix} \text{Nr.3} \\ \text{Nr.48} \\ \text{Nr.53} \end{bmatrix} \pi =$  $\pi =$  $\pi^{-1} =$  $(\pi^{-1})^{\mathrm{r}}$ (1.13)

and

$(\text{Nr.50}, \text{Nr.54})\pi =$  $\pi =$  $\pi^{-1} =$  $(\pi^{-1})^{\mathrm{r}}$. (1.14)

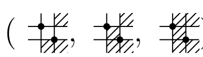
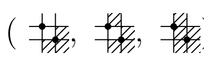
By Theorem 1.6 the result follows from (1.12), (1.13) and (1.14). □

Corollary 1.8. *Conjecture 1.1 holds true.*

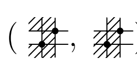
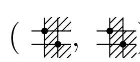
Proof. By Corollary 1.7 this follows from (1.3b) with $x = v = q = 1$. □

As the equidistribution of Nr.48 and Nr.49 is known [12, Theorem 5.1], Corollary 1.7 confirms two conjectured equidistributions in Table 1.

Theorem 1.9. *There exist an involution* Ψ *on* S_n *such that for* $\pi \in S_n$,

 $(\pi) =$  $\Psi(\pi))$.

For the patterns Nr.23 and Nr.24, we have

$(\text{Nr.23}, \text{Nr.24})\pi =$  $\pi =$  π^r.

By Theorem 1.9, we confirm another conjecture in Table 1, i.e., the patterns Nr.23 and Nr.24 are equidistributed.

We shall prove Theorem 1.6 and Theorem 1.9 in Section 2 and Section 3, respectively, and make a connection between pattern Nr. 14 and the statistic *succession* in permutations in Section 4.

2. Proof of Theorem 1.6

For $\pi \in S_n$ let $\text{AREC}(\pi) = (i_1, i_2, \dots, i_l)$ be the sequence of antirecord positions of π from left to right. So $\pi(i_1) = 1$, $i_1 < \cdots < i_l$ and $i_l = n$. For each antirecord position i_k define two mappings

$$\varphi_1^{(i_k)} : \pi \mapsto \pi' \tag{2.1a}$$

$$\varphi_2^{(i_k)} : \pi \mapsto \pi'' \tag{2.1b}$$

as follows:

- let $w = w_1 \dots w_r$ be the subword of π consisting of letters greater than $\pi(i_k)$ on the left of $\pi(i_k)$ (resp. $\pi(i_{k-1})$);

- let $w' = w'_1 \dots w'_r$ be the word obtained by substituting the jth largest letter with the jth smallest letter in w for $j = 1, \dots, r$;
- let π' (resp. π'') be the word obtained by replacing w_j with w'_j for $j = 1, \dots, r$ in π.

Remark 2.1. By convention, we define $\varphi_2^{(i_1)}$ to be the identity mapping. Clearly the two operations keep the sequence of antirecords for both values and positions, that is,

$$\mathrm{AREC}(\pi) = \mathrm{AREC}(\pi') = \mathrm{AREC}(\pi'') \tag{2.2a}$$

$$\pi'(i_k) = \pi''(i_k) = \pi(i_k) \quad \text{for} \quad k = 1, \dots l. \tag{2.2b}$$

Let $P = \{p_1 < \cdots < p_r\}$ and $Q = \{q_1 < \cdots < q_r\}$ be two ordered sets and $\pi = p_1 \dots p_r$ and $\tau = q_1 \dots q_r$ are permutations of P and Q, respectively. We say that π and τ are *order isomorphic* and write $\pi \sim \tau$ if for any two indices r and s we have the equivalence $p_r < p_s$ if and only if $q_r < q_s$. In other words, τ is the permutation obtained from π by substituting p_i by q_i for $i = 1, \dots, r$.

Let $w = w_1 \dots w_n$ be a permutation of $a_1 < a_2 < \cdots < a_n$. We define the *complement* of w by w^c[1], which is the word obtained by substituting a_i by a_{n+1-i} in w for $i = 1, \dots, n$. If x is a subset of the letters in w, we write $[w]_x$ as the subword of w consisting of the letters $a \in x$.

Lemma 2.2. *(1) If $w = w_1 w_2$ and $w^c = w'_1 w'_2$, then $(w'_1)^c \sim w_1$.*[2]

(2) Let $w = w_1 w_2 w_3$ and $v = v_1 v_2 v_3$ with $|w_1| = |v_1|$. If $w_1 w_2 \sim v_1 v_2$ with $(w_1 w_2)^c = w'_1 w'_2$ and $(v_1 v_2)^c = v'_1 v'_2$, then $w_1 \sim v_1$, $w_2 \sim v_2$, $w'_1 \sim v'_1$ and $w'_2 \sim v'_2$. Moreover, we have $(w'_1)^c \sim (v'_1)^c$ and $(w'_2)^c \sim (v'_2)^c$.

(3) If $w \sim v$ and $[w]_x = [v]_x$ for some set x of some common letters in w and v, then
- *$w^c \sim v^c$ and $[w^c]_x = [v^c]_x$.*
- *$[w]_y \sim [v]_z$, where y (resp. z) is the complementary of x in the alphabet of w (resp. v).*

Proof. The verification is easy and left to the reader. □

For example, if $w = 3\,5\,9\,1\,4\,7\,2\,8\,6$, then $w^c = 7\,5\,1\,9\,6\,3\,8\,2\,4$. Let $w = w_1 w_2$ with $w_1 = 3\,5\,9\,1\,4\,7$ and $w_2 = 2\,8\,6$, then $w'_1 = 7\,5\,1\,9\,6\,3$ and $(w'_1)^c = 3\,6\,9\,1\,5\,7$. Clearly $(w'_1)^c \sim w_1$ and $[(w'_1)^c]_x = [w_1]_x$ with $x = \{1, 3, 9\}$. We see that $w_1^c = 7\,4\,1\,9\,5\,3$ and $[w'_1]_x = [w_1^c]_x = 1\,9\,3$.

Lemma 2.3. *For any antirecord position i of $\pi \in S_n$ the mappings $\varphi_1^{(i)}$ and $\varphi_2^{(i)}$ are involutions and commute, namely,*

$$\varphi_1^{(i)} \circ \varphi_1^{(i)}(\pi) = \varphi_2^{(i)} \circ \varphi_2^{(i)}(\pi) = \pi \tag{2.3}$$

and

$$\varphi_2^{(i)} \circ \varphi_1^{(i)}(\pi) = \varphi_1^{(i)} \circ \varphi_2^{(i)}(\pi). \tag{2.4}$$

[1] When $a_i = i$, w^c reduces to π^c, see (1.10).

[2] The word w'_1 is the complement of w_1 in the alphabet of w, while $(w'_1)^c$ is the complement of w'_1 in the alphabet of w'_1.

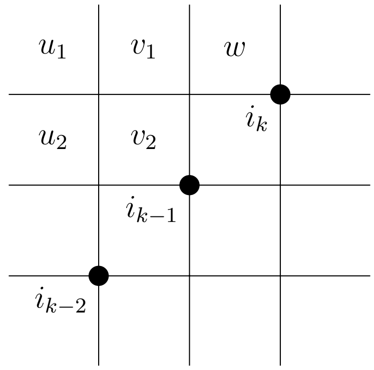


FIGURE 1. The decomposition of three consecutive anti-records of π

Proof. From the definitions of $\varphi_1^{(i)}$ and $\varphi_2^{(i)}$ in Eq. (2.1), it is easy to check Eq. (2.3) holds and

$$\varphi_2^{(i)} \circ \varphi_1^{(i)}(\pi) \sim \varphi_1^{(i)} \circ \varphi_2^{(i)}(\pi).$$

Since the set of letters greater than $\pi(i)$ on the left of $\pi(i)$ are invariant under the operation $\varphi_1^{(i)}$ and $\varphi_2^{(i)}$ on π, we obtain Eq.(2.4) immediately. □

Let $\pi \in S_n$ with sequence of antirecord positions $\mathrm{AREC}(\pi) = (i_1, i_2, \ldots, i_l)$. We define the operation Φ on π by

$$\Phi(\pi) = \varphi^{(i_1)} \circ \varphi^{(i_2)} \circ \cdots \circ \varphi^{(i_l)}(\pi) \tag{2.5}$$

with $\varphi^{(i_k)} = \varphi_2^{(i_k)} \circ \varphi_1^{(i_k)}$ for $k = 1, \ldots, l$.

Lemma 2.4. *For $\pi \in S_n$ with $\mathrm{AREC}(\pi) = \{i_1, \ldots, i_l\}$. The mappings $g := \varphi^{(i_{k-1})}$ and $f := \varphi^{(i_k)}$ commute, i.e.,*

$$g \circ f(\pi) = f \circ g(\pi).$$

Proof. We write the permutation $\pi = \pi(1)\ldots\pi(n)$ as $\pi = u\pi(i_{k-2})v\pi(i_{k-1})w\pi(i_k)x$ for $3 \le k \le l$, and

(i) u_1 (resp. v_1, w) as the subword consisting of letters greater than $\pi(i_k)$ in u (resp. v, w);

(ii) u_2 (resp. v_2) as the subword consisting of letters between $\pi(i_{k-1})$ and $\pi(i_k)$ in u (resp. v);

see Figure 1. For convenience, we introduce the following notations:

$$f(\pi) := u'\pi(i_{k-2})v'\pi(i_{k-1})w'\pi(i_k)x \tag{2.6a}$$

$$g \circ f(\pi) := \tilde{u}\pi(i_{k-2})\tilde{v}\pi(i_{k-1})w'\pi(i_k)x, \tag{2.6b}$$

$$g(\pi) := u''\pi(i_{k-2})v''\pi(i_{k-1})w\pi(i_k)x, \tag{2.6c}$$

$$f \circ g(\pi) := \hat{u}\pi(i_{k-2})\hat{v}\pi(i_{k-1})w'\pi(i_k)x. \tag{2.6d}$$

We will use the similar notations $u_i', \hat{u}_i, \tilde{u}_i, u_i'', v_i', \hat{v}_i, \tilde{v}_i, v_i''$ as in (i) and (ii) for $i = 1, 2$.

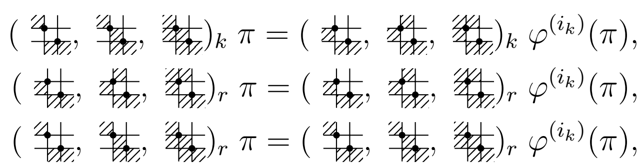
By the definition of f and g and Lemma 2.2, we have the following facts,

(1) Under the operation f (cf. (2.6a)), as $u_1v_1 \sim u_1'v_1'$, $u_2'v_2' = u_2v_2$, we have $u'v' \sim uv$;
(2) Applying g to $f(\pi)$ and π (cf. (2.6b) and (2.6c)), respectively, we see that $\tilde{u}\tilde{v} \sim u''v''$ by (1) and Lemma 2.2, hence $\tilde{u}_2\tilde{v}_2 = u_2''v_2''$;
(3) Applying f to $g(\pi)$ (cf. (2.6c) and (2.6d)), we have $u_2''v_2'' = \hat{u}_2\hat{v}_2$, combining with (2) yields $\hat{u}_2\hat{v}_2 = \tilde{u}_2\tilde{v}_2$, that is $\hat{u}_2 = \tilde{u}_2$ and $\hat{v}_2 = \tilde{v}_2$;
(4) Applying f to $g(\pi)$ (cf. (2.6c) and (2.6d)), using Lemma 2.2 we have $\hat{u}\hat{v} \sim u''v''$, from $u''v'' \sim \tilde{u}\tilde{v}$ (cf. (2)) we derive $\hat{u}\hat{v} \sim \tilde{u}\tilde{v}$. Combining with Lemma 2.2 and (3) yields $\hat{u}_1\hat{v}_1 \sim \tilde{u}_1\tilde{v}_1$.
(5) Let $\{w\}$ denote the set of letters in w.
   - Applying f to π we have $\{u_1'v_1'w'\} = \{u_1v_1w\}$,
   - applying g to $f(\pi)$ we have $\{u_1'v_1'w'\} = \{\tilde{u}_1\tilde{v}_1w'\}$,
   - applying g to π we have $\{u_1''v_1''w\} = \{u_1v_1w\}$,
   - applying f to $g(\pi)$ we have $\{u_1''v_1''w\} = \{\hat{u}_1\hat{v}_1w'\}$.

   Thus $\{\tilde{u}_1\tilde{v}_1\} = \{\hat{u}_1\hat{v}_1\}$. It follows from (4) that $\hat{u}_1 = \tilde{u}_1$ (resp. $\hat{v}_1 = \tilde{v}_1$).

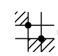
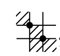
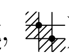
Summarizing the above facts we have proved $f \circ g = g \circ f$. □

Lemma 2.5. *The mapping $\varphi^{(i_k)}$ is an involution such that for $\pi \in S_n$ and $r \neq k$,*



(2.7a)
(2.7b)
(2.7c)

where $(pattern)_k$ *means the number of the patterns between* $\pi(i_{k-1})$ *and* $\pi(i_k)$.

Proof. If the pair $(\pi(j), \pi(i_k))$ with $j < i_k$ contributes the pattern (resp. , ), then $j > i_{k-1}$ because $\pi(i_{k-1}) < \pi(i_k)$ and $\pi(i) > \pi(i_k)$ for $j \leq i < i_k$. Also, for $i < j$, we have the equivalence

$$\pi(i) < \pi(j) \iff \pi(i_k) < \varphi_1^{(i_k)}(\pi(i)) > \varphi_1^{(i_k)}(\pi(j)),$$

as $\varphi_2^{(i_k)}$ will affect only the letters at the left of $\pi(i_{k-1})$. Thus we have proved (2.7a).

Next, recall that the operation $\varphi^{(i_k)}$ keeps the sequence of antirecords for both positions and values. The two identities (2.7b) and (2.7c) are clear if $r > k$. Assume that $r < k$ and $[\pi]_r = \pi(1)\ldots\pi(i_{k-1})$. By Lemma 2.2, after two operations $\varphi_1^{(i_k)}$ and $\varphi_2^{(i_k)}$ the permutation $\varphi^{(i_k)}([\pi]_r)$ is isomorphic with $[\pi]_r$. This proves (2.7b) and (2.7c).

□

Proof of Theorem 1.6. By (2.5) the reverse of the mapping Φ is given by

$$\Phi^{-1}(\pi) = \varphi^{(i_l)} \circ \cdots \circ \varphi^{(i_2)} \circ \varphi^{(i_1)}(\pi). \tag{2.8}$$

Theorem 1.6 follows from Lemma 2.3, Lemma 2.4 and Lemma 2.5. □

Example 2.6. We show the process of the involution Φ in Figure 2, For $\pi = 2\,5\,7\,1\,8\,9\,4\,6\,3$, we have $\mathrm{AREC}(\pi) = (4, 9)$. We proceed from right to left.

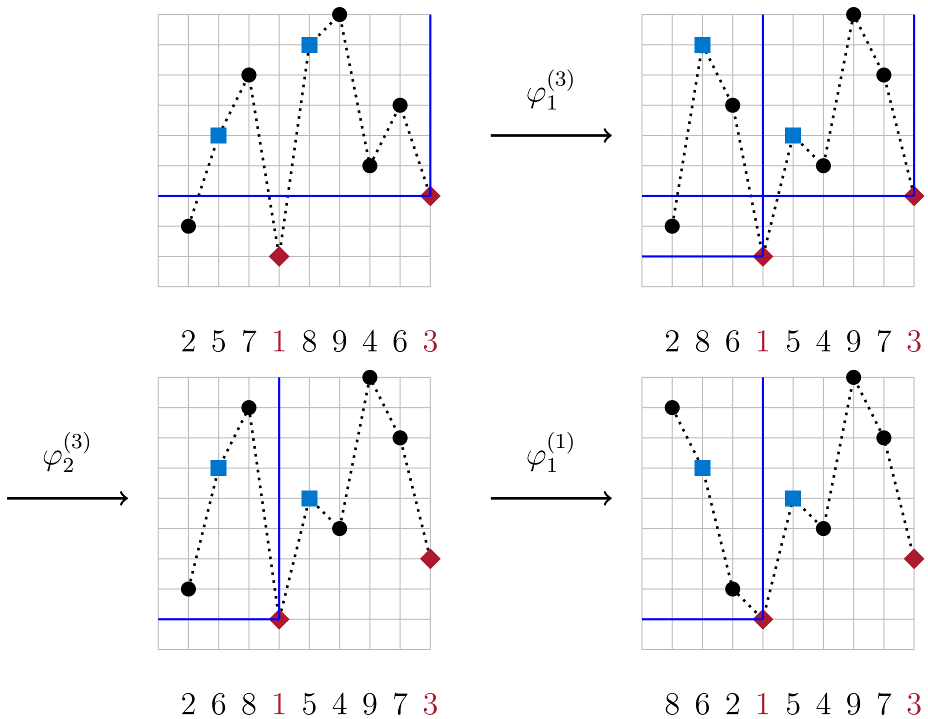


FIGURE 2. The involution Φ on the permutation 257189463

(1) For position 9 with value 3, w have $w = 578946$ and $w' = 865497$. Thus $\varphi_1^{(9)} : \pi \mapsto \pi' = 286154973$. Next, we have $w = 86$ and $w' = 68$. Thus $\varphi_2^{(9)} : \pi' \mapsto \pi'' = 268154973$.
(2) For position 4 with value 1 we have $w = 268$ and $w' = 862$. Finally we obtain $\Phi(\pi) = 862154973$.

Now, we check the mesh patterns.

- First, $\varphi_1^{(9)} : \pi = 257189463 \mapsto \pi' = 286154973$, the pair $(8,3)$ of π contributes the pattern without the patterns ,), the pair $(5,3)$ of π' contributes the pattern without the patterns , , the operations $\varphi_2^{(9)}, \varphi_1^{(1)}$ do not change the corresponding mesh pattens at position 9 of π'.
- Second, $\varphi_2^{(9)} \circ \varphi_1^{(9)} : \pi = 257189463 \mapsto \pi'' = 268154973$ it is easy to see that $257 \sim 268$. The pair $(5,1)$ of π contributes the patterns , without the pattern , the pair $(6,1)$ of π'' also contributes the pattern , without the pattern , $\varphi_1^{(1)} : \pi'' = 268154973 \mapsto \pi''' = 862154973$, the pair $(6,1)$ of π''' contributes the pattern , without the pattern .

3. Proof of Theorem 1.9

First we introduce two mappings different from Section 2. For $\pi \in S_n$, recall that $\mathrm{AREC}(\pi) = (i_1, i_2, \dots, i_l)$ be the sequence of antirecord positions of π from left to right. For any antirecord position i_k we define two mappings

$$\psi_1^{(i_k)} : \pi \mapsto \pi' \tag{3.1a}$$

$$\psi_2^{(i_k)} : \pi \mapsto \pi'' \tag{3.1b}$$

as follows:

- let $w = w_1 \dots w_r$ is the subword of π consisting of letters greater than $\pi(i_k)$ on the right side of $\pi(i_{k-1})$ (resp. $\pi(i_k)$) with $\pi(i_0) = 0$;
- let $w' = w'_1 \dots w'_r$ be the word obtained by substituting the jth largest letter with the jth smallest letter in w for $j = 1, \dots, r$;
- let π' (resp. π'') is defined to be the word obtained by replacing w_j with w'_j in π.

Note that $\pi'(i_k) = \pi(i_k)$.

Lemma 3.1. *For any antirecord positions i_{k-1} and i_k of $\pi \in S_n$ the mappings $\psi_1^{(i_k)}$ and $\psi_2^{(i_k)}$ are involutions and commutate, namely,*

$$\psi_1^{(i_k)} \circ \psi_1^{(i_k)}(\pi) = \psi_2^{(i_k)} \circ \psi_2^{(i_k)}(\pi) = \pi \tag{3.2}$$

and

$$\psi_2^{(i_k)} \circ \psi_1^{(i_k)}(\pi) = \psi_1^{(i_k)} \circ \psi_2^{(i_k)}(\pi). \tag{3.3}$$

Let $\psi^{(i_k)} = \psi_2^{(i_k)} \circ \psi_1^{(i_k)}$. Then $\psi^{(i_k)}(\pi)$ and π have the same sequence of antirecord positions.

Proof. From the definitions of $\psi_1^{(i_k)}$ and $\psi_2^{(i_k)}$ in Eq. (3.1), it is easy to check Eq. (3.2) holds and

$$\psi_2^{(i_k)} \circ \psi_1^{(i_k)}(\pi) \sim \psi_1^{(i_k)} \circ \psi_2^{(i_k)}(\pi).$$

Since the set of letters greater than $\pi(i_k)$ on the right of $\pi(i_{k-1})$ are invariant under the operation $\psi_1^{(i_k)}$ and $\psi_2^{(i_k)}$ on π, we obtain Eq. (3.3) immediately. □

Lemma 3.2. *For $\pi \in S_n$ with $\mathrm{AREC}(\pi) = \{i_1, \dots, i_l\}$. For $k = 2, \dots, l$ the mappings $\psi^{(i_{k-1})}$ and $\psi^{(i_k)}$ commute, i.e.,*

$$\psi^{(i_k)} \circ \psi^{(i_{k-1})}(\pi) = \psi^{(i_{k-1})} \circ \psi^{(i_k)}(\pi).$$

Proof. For the permutation $\pi = \pi(1) \dots \pi(n)$ we write $\pi = u\pi(i_{k-1})v\pi(i_k)w$ and

$$\psi^{(i_{k-1})}(\pi) := \pi' = u'\pi(i_{k-1})v'\pi'(i_k)w' \tag{3.4a}$$

$$\psi^{(i_k)} \circ \psi^{(i_{k-1})}(\pi) := \tilde{\pi} = u'\pi(i_{k-1})\tilde{v}\pi'(i_k)\tilde{w}, \tag{3.4b}$$

$$\psi^{(i_k)}(\pi) := \pi'' = u\pi(i_{k-1})v''\pi(i_k)w'', \tag{3.4c}$$

$$\psi^{(i_{k-1})} \circ \psi^{(i_k)}(\pi) := \hat{\pi} = u'\pi(i_{k-1})\hat{v}\hat{\pi}(i_k)\hat{w}. \tag{3.4d}$$

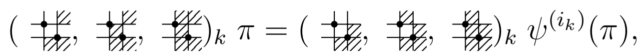
By the definition of $\psi^{(i_k)}$, we have the following facts.

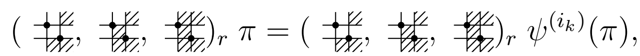
(1) Applying $\psi^{(i_{k-1})}$ to π (cf. (3.4a)) we have $v'\pi'(i_k)w' \sim v\pi(i_k)w$;
(2) Applying $\psi^{(i_k)}$ to $\psi^{(i_{k-1})}(\pi)$ and π (cf.(3.4b) and (3.4c)), respectively, we take complement of v' and v once, while twice for w' and w. By Lemma 2.2 and (1) we get $\tilde{v}\pi'(i_k)\tilde{w} \sim v''\pi(i_k)w''$;
(3) Applying $\psi^{(i_{k-1})}$ to $\psi^{(i_k)}(\pi)$ (cf. (3.4c) and (3.4d)), by (1) we have $\hat{v}\hat{\pi}(i_k)\hat{w} \sim v''\pi(i_k)w''$, combining with by (2) yields $\hat{v}\hat{\pi}(i_k)\hat{w} \sim \tilde{v}\pi'(i_k)\tilde{w}$.
(4) Let $\{w\}$ denote the set of letters in w.
   - Applying $\psi^{(i_k)}$ to $\psi^{(i_{k-1})}(\pi)$ we have $\{u'v'\pi'(i_k)w'\} = \{u'\tilde{v}\pi'(i_k)\tilde{w}\}$,
   - applying $\psi^{(i_{k-1})}$ to π we have $\{u'v'\pi'(i_k)w'\} = \{uv\pi(i_k)w\}$,
   - applying $\psi^{(i_k)}$ to π we have $\{uv''\pi(i_k)w''\} = \{uv\pi(i_k)w\}$,
   - applying $\psi^{(i_{k-1})}$ to $\psi^{(i_k)}(\pi)$ we have $\{u'\hat{v}\hat{\pi}(i_k)\hat{w}\} = \{uv''\pi(i_k)w''\}$,
   - it follows that $\{u'\hat{v}\hat{\pi}(i_k)\hat{w}\} = \{u'\tilde{v}\pi'(i_k)\tilde{w}\}$ and $\{\hat{v}\hat{\pi}(i_k)\hat{w}\} = \{\tilde{v}\pi'(i_k)\tilde{w}\}$.

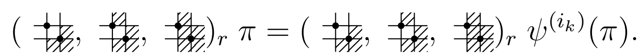
(5) It follows from (3) and (4) that $\hat{v}\hat{\pi}(i_k)\hat{w} = \tilde{v}\pi'(i_k)\tilde{w}$, that is, Thus $\hat{v} = \tilde{v}$, $\hat{\pi}(i_k) = \pi'(i_k)$ and $\hat{w} = \tilde{w}$.

□

Lemma 3.3. *The mapping $\psi^{(i)}$ is an involution such that for $\pi \in S_n$ and $r \neq k$*


(3.5a)


(3.5b)


(3.5c)

where (pattern)$_k$ means the number of the patterns between $\pi(i_{k-1})$ and $\pi(i_k)$.

Proof. If the pair $(\pi(j), \pi(i_k))$ with $j < i_k$ contributes the pattern , then $j > i_{k-1}$ because $\pi(i_{k-1}) < \pi(i_k)$ and $\pi(i) > \pi(i_k)$ for $j \leq i < i_k$. Also, for $j < i_k < i$, we have the equivalence

$$\pi(i_k) < \pi(i) < \pi(j) \Longleftrightarrow \pi(i_k) < \psi_1^{(i_k)}(\pi(j)) < \psi_1^{(i_k)}(\pi(i)),$$

as $\psi_2^{(i_k)}$ will affect only the letters at the right of $\pi(i_k)$. Thus we have proved (3.5a).

Next, recall that the operation $\psi^{(i_k)}$ keeps the sequence of antirecord positions. The two identities (3.5b) and (3.5c) are clear if $r < k$. Assume that $r > k$ and $[\pi]_{>r} = \pi(i_k + 1) \ldots \pi(n)$. By Lemma 2.2 after two operations $\psi_1^{(i_k)}$ and $\psi_2^{(i_k)}$ the permutation $\psi^{(i_k)}([\pi]_{>r})$ is isomorphic with $[\pi]_{>r}$. This proves (3.5b) and (3.5c).

□

Proof of Theorem 1.9. For $\pi \in S_n$ and $\text{AREC}(\pi) = (i_1, i_2, \ldots, i_l)$, we define the operation Ψ on π by

$$\Psi(\pi) = \psi^{(i_l)} \circ \cdots \circ \psi^{(i_2)} \circ \psi^{(i_1)}(\pi). \tag{3.6}$$

By (3.6) the mapping Ψ is reversible with reverse

$$\Psi^{-1}(\pi) = \psi^{(i_1)} \circ \psi^{(i_2)} \circ \ldots \circ \psi^{(i_l)}(\pi).$$

Theorem 1.9 follows from Lemma 3.1, Lemma 3.2 and Lemma 3.3. □

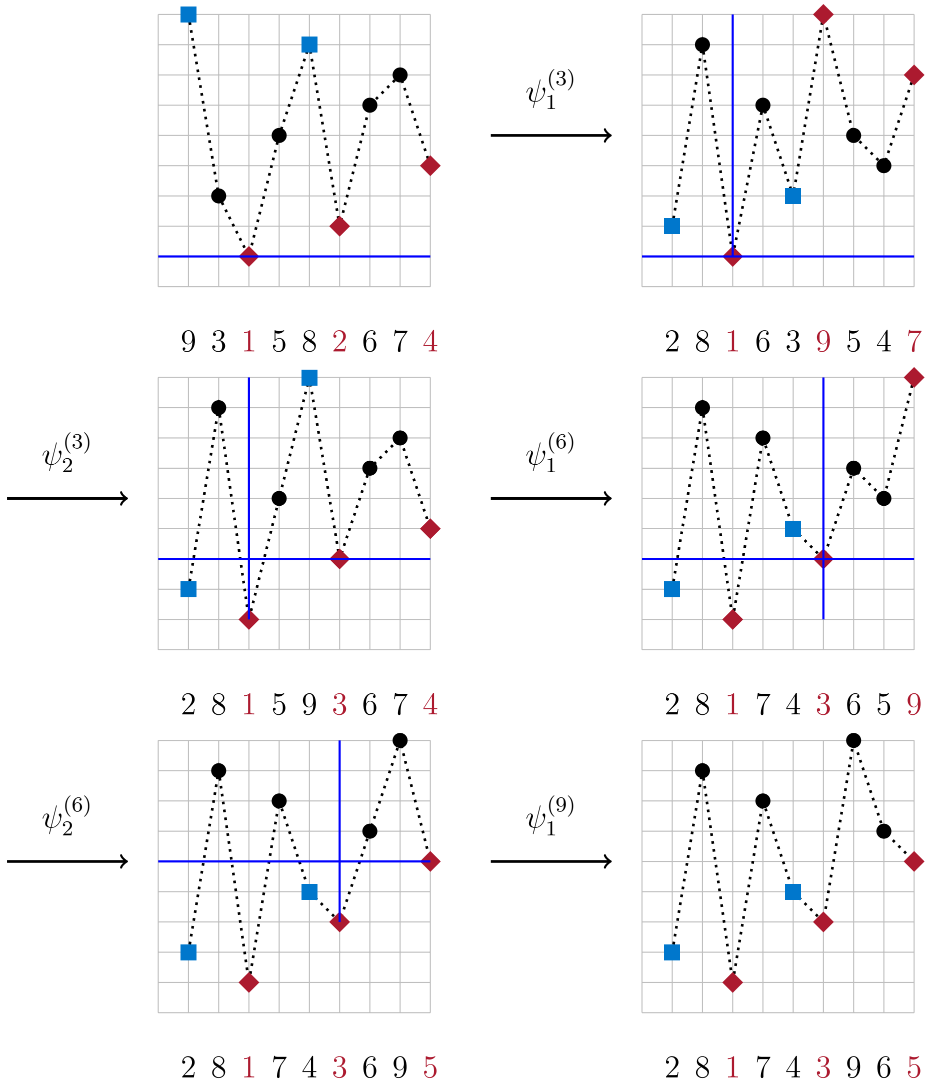


FIGURE 3. The involution Ψ on the permutation 931582674

Example 3.4. Figure 3. For $\pi = 931582674$, we have $\mathrm{AREC}(\pi) = (3, 6, 9)$. We procced from left to right.

(1) For position 3 with value 1, we have $w = 93582674$ and $w^c = 28639547$ Thus $\psi_1^{(3)} : \pi \mapsto \pi' = 281639547$. Next, we have $w = 639547$ and $w^c = 593674$ Thus $\psi_2^{(3)} : \pi' \mapsto \pi'' = 281593674$.

(2) For position 6 with value 3, we have $w = 59674$ and $w^c = 74659$. So $\psi_1^{(6)}(\pi'') = 281743659$. Next, we have $w = 659$ and $w^c = 695$. Thus we have $\psi^{(6)}(\pi'') = 281743695$.

(3) For position 9 with value 5 we have $w = 69$ and $w^c = 96$. Finally we obtain $\Psi(\pi) = 281743965$.

Now, we check the mesh patterns.

- First, $\psi_1^{(3)} : \pi = 9\,3\,1\,5\,8\,2\,6\,7\,4 \mapsto \pi' = 2\,8\,1\,6\,3\,9\,5\,4\,7$, the pair $(9,1)$ contributes the pattern without the patterns , , then the pair $(2,1)$ of π' contributes the pattern without the patterns , , the operations $\psi_2^{(i)}(i = 3,6), \psi_1^{(j)}(j = 6,9)$ do not change the corresponding mesh pattens at $(2,1)$ of π'.
- Second, $\psi_2^{(3)} \circ \psi_1^{(3)} : \pi = 9\,3\,1\,5\,8\,2\,6\,7\,4 \mapsto \pi'' = 2\,8\,1\,5\,9\,3\,6\,7\,4$, it is easy to see $5\,8\,2\,6\,7\,4 \sim 5\,9\,3\,6\,7\,4$. The pair $(8,2)$ of π contributes the pattern (resp. ,), then the pair $(9,3)$ of π'' also contributes the pattern (resp. ,), $\psi_1^{(6)} : \pi'' = 2\,8\,1\,5\,9\,3\,6\,7\,4 \mapsto \pi''' = 2\,8\,1\,7\,4\,3\,6\,5\,9$, the pair $(4,3)$ contributes the pattern (resp. ,), the operations $\psi_2^{(6)}, \psi_1^{(9)}$ do not change the corresponding mesh pattens at $(4,3)$ of π'''.

4. A remark on pattern Nr. 14

Recall that an index i (with $1 < i \le n$) is a *succession* of $\sigma \in S_n$ if $\sigma(i) = \sigma(i-1) + 1$, see [4, Section 5]. Thus an occurrence of the pattern Nr. 14 = corresponds to a succession and we can translate the results on successions in [4, Section 5] to this pattern. For example, letting

$$S_n(x) = \sum_{\pi \in S_n} x^{(\pi)}$$

and differtentiating the generating function [4, (5.6)]

$$\sum_{n\ge 0} S_n(x)\frac{t^n}{n!} = \frac{e^{(x-1)t}}{1-t} + (1-x)\int_0^t \frac{e^{(x-1)z}}{1-z}dz \tag{4.1}$$

yields

$$\sum_{n\ge 0} S_{n+1}(x)\frac{t^n}{n!} = \frac{e^{(x-1)t}}{(1-t)^2}. \tag{4.2}$$

This is the exponential generating function given in [13, A123513]. We note that the ordinary generating function (cf. [4, (5.8)]) reads

$$\sum_{n\ge 0} S_n(x)t^n = \sum_{n\ge 0} \frac{n!t^n}{[1-(x-1)t]^n}. \tag{4.3}$$

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References

[1] S. Avgustinovich, S. Kitaev, A. Valyuzhenich, Avoidance of boxed mesh patterns on permutations, Discrete Appl. Math. 161 (2013), 43–41. 2

[2] N. Borie, On the combinatorics of quadrant marked mesh patterns in 132-avoiding permutations, Australas. J. Comb. 64(1) (2016), 140–153. 2

[3] P. Brändén, A. Claesson, Mesh patterns and the expansion of permutation statistics as sums of permutation patterns, Electron. J. Combin. 18(2) (2011), #P5. 1

[4] R. J. Clarke, G. N. Han, J. Zeng, A combinatorial interpretation of the Seidel generation of q-derangement numbers, Ann. Comb. 1 (1997), no. 4, 313–327. 14

[5] D. Dumont, G. Kreweras, Sur le développement d'une fraction continue liée à la série hypergéométrique et son interprétation en termes de records et anti-records dans les permutations, European J. Combin. 9, 27–32 (1988). 4

[6] I. Hilmarsson, I. Jónsdóttir, S. Sigurdardóttir, L. Vidarsdóttir, H. Ulfarsson, Wilf-classification of mesh patterns of short length, Electron. J. Combin. 22(4) (2015), #P4.13. 1, 2, 3

[7] M. Jones, S. Kitaev, J. Remmel, Frame patterns in n-cycles, Discrete Math. 338 (2015), 1197–1215. 2

[8] S. Kitaev, Patterns in Permutations and Words, Springer, 2011. 1

[9] S. Kitaev, J. Liese, Harmonic numbers, Catalan's triangle and mesh patterns, Discrete Math. 313 (2013), 1515–1531. 2

[10] S. Kitaev, J. Remmel, Quadrant marked mesh patterns in alternating permutations, Sém. Lothar. Combin, B68a (2012). 2

[11] S. Kitaev, J. Remmel, M. Tiefenbruck, Quadrant marked mesh patterns in 132-avoiding permutations, Pure Math. and Appl. (Pu.M.A.), 23(3) (2012), 219–256. 2

[12] S. Kiatev, P. B. Zhang, Distributions of mesh patterns of short lengths. Adv. in Appl. Math. 110 (2019), 1–32. 1, 2, 3, 5, 6

[13] N. J. A. Sloane, The On−Line Encyclopedia of Integer Sequences, published electronically at https://oeis.org. 3, 14

[14] A. D. Sokal, J. Zeng, Some multivariate master polynomials for permutations, set partitions, and perfect matchings, and their continued fractions, arXiv preprint (2020). (arXiv:2003.08192) 3, 4, 5

[15] E. Steingrímsson, Some open problems on permutation patterns. Surveys in combinatorics 2013, 239–263, London Math. Soc. Lecture Note Ser. 409, Cambridge Univ. Press, Cambridge, 2013. 1

[16] B. E. Tenner, Coincidental pattern avoidance, J. Comb. 4 (2013), 311–326. 2

[17] B. E. Tenner, Mesh patterns with superfluous mesh, Adv. in Appl. Math. 51 (2013), 606–618. 2

[18] J. Zeng, Records, antirecords et permutations discordantes, European J. Combin. 10, 103–109 (1989). 4, 5

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