Polynomial interpolation of modular forms for Hecke groups

Barry Brent

4 October 2020

Abstract

For $m = 3, 4, ...$, let $\lambda_m = 2\cos\pi/m$ and let $G(\lambda_m)$ be a Hecke group. Let $J_m(m = 3, 4, ...)$ be a triangle function for $G(\lambda_m)$ such that, when normalized appropriately, J_3 becomes Klein's j-invariant $j(z) = 1/e^{2\pi iz} + 744 + ...$, and $J_m(z)$ has a Fourier expansion $J_m(z) = \sum_{n=-1}^{\infty} a_n(m)q_m^n$ with $q_m(z) = \exp 2\pi iz/\lambda_m$. Raleigh [12] conjectures that, for each non-negative integer n, there is a rational function $F_n(x)$ with coefficients in $\mathbb{Q}$ such that $a_n(m) = F_n(m)/a_{-1}(m)^n$. We offer a similar conjecture (involving polynomials instead of rational functions) for normalizations of the J_m and extend it to normalizations of other modular forms on the $G(\lambda_m)$. We discuss a modular form for $m = 6$ that appears to detect the integers represented by the quadratic form $x^2 + xy + y^2$, and we study cusp forms Δ_m^* such that $\Delta_3^*(\tau) = \Delta(z) = \sum_{n=1}^{\infty} \tau(n)e^{2\pi inz}$ where τ is Ramanujan's function. The article is concerned with numerical experiments. The only theorems are quotations from the literature.

1 Introduction

Let us write $\mathbb{H}^*$ for the extension of the upper half plane $\mathbb{H}$ by adjoining cusps at the points on the real axis with rational abscissa and at $i\infty$, regarded as a metric space with the Poincaré metric. Figures T made by three geodesics of $\mathbb{H}^*$ are called hyperbolic or circular-arc triangles. Schwarz [15], Lehner [10] and others studied Schwarz triangle functions, which map hyperbolic triangles T in the extended upper half z-plane onto the extended upper half w-plane. Let $G(\lambda_m)$ be the Hecke group $G(2\cos\pi/m)$. For certain $T = T_m$, a triangle function $\phi_{\lambda_m} : T \to \mathbb{H}^*$ extends to a map $J_m : \mathbb{H}^* \to \mathbb{H}^*$ invariant under modular transformations from $G(\lambda_m)$. Suitably normalized, the J_m become analogues j_m of the normalized Klein's modular invariant

$$j(z) = -1/q + 744 + 196884q + ...$$

where $q = q(z) = \exp(2\pi iz)$ and $j_3(z) = j(z)$. With $\lambda_m = 2\cos\pi/m$ and $q_m(z) = \exp(2\pi iz/\lambda_m)$, the original J_m have Fourier series $J_m(z) = \sum_{n\geqslant -1} a_n(m)q_m(z)^n$. Raleigh [12] conjectured that the $a_n(m)$ are interpolated by polynomials $R_n(x)$ in $\mathbb{Q}[x]$ in the sense that $a_n(m) = m^{-2n-2}a_{-1}(m)^{-n}R_n(m)$. Under a certain normalization, the Fourier coefficients of j_m appear to interpolate polynomials in $\mathbb{Q}[x]$ (not rational functions as in Raleigh's original conjecture); the a_{-1} factors are not seen either.

The plan of the article is as follows. We sketch the theory of Schwartz triangles; then, the construction of their triangle functions; from the triangle functions, the modular functions for the Hecke groups; and, from them, entire modular forms for these groups. To this point, the article is just a summary of background material. By methods familiar from the classical case, we then construct modular forms with rational Fourier expansions and describe experiments on them, especially on cusp forms, using code adapted from the results of several authors, in particular, Hecke's theory.

J. Leo [11], anticipated the author in calculating Fourier expansions of triangle functions with computers. His code was based on Lehner's construction. We based ours on a variant in [12].

2 A glossary

1. The digamma function $\psi(z) := \Gamma'(z)/\Gamma(z)$.
2. The Schwarzian derivative ([6], p. 130, equation 370.8)
$$\{w, z\} = \frac{2w'w''' - 3w''^2}{2w'^2} \tag{1}$$
for $w = w(z)$.
3. The Pochhammer symbol
$$(a)^0 := 1 \text{ and, for } n \geqslant 1, (a)^n := a(a+1)...(a+n-1) = \Gamma(a+n)/\Gamma(a).$$
4. The function
$$c_\nu = c_\nu(\alpha, \beta, \gamma) := \frac{(\alpha)^n(\beta)^\nu}{\nu!(\gamma)^\nu}, \nu \geqslant 0.$$
To facilitate comparison with Raleigh's equation (9^1) [12], we remark that
$$c_\nu = \frac{\Gamma(\alpha+\nu)}{\Gamma(\alpha)} \cdot \frac{\Gamma(\beta+\nu)}{\Gamma(\beta)} \cdot \frac{\Gamma(1)}{\Gamma(1+\nu)} \cdot \frac{\Gamma(\gamma)}{\Gamma(\gamma+\nu)}. \tag{2}$$
In the terms of this article's Theorem 1 below, Raleigh is treating the case $\lambda = 0$, for which (equation (6) below) $\gamma = 1$ and the expression on the right side of (2) becomes, as in Raleigh,
$$\frac{\Gamma(\alpha+\nu)\Gamma(\beta+\nu)}{\Gamma(\alpha)\Gamma(\beta)(\nu!)^2}.$$

5. The function e_ν given by ([12], equation 9^1)

$$e_\nu = e_\nu(\alpha, \beta) := \sum_{p=0}^{\nu-1} \left(\frac{1}{\alpha+p} + \frac{1}{\beta+p} - \frac{2}{1+p} \right).$$

   Here, we are dealing with the same ambiguity present in the definition of c_ν: this is a specialization to the case $\gamma = 1$ of the e_ν for $\nu \geqslant 1$ given by ([6], p. 153, equation 387.5)

$$e_\nu = e_\nu(\alpha, \beta, \gamma) := \sum_{p=0}^{n-1} \left(\frac{1}{\alpha+p} + \frac{1}{\beta+p} - \frac{2}{\gamma+p} \right).$$

   Unless it is explicitly indicated to be otherwise, we intend the first (Raleigh's) definition.

6. Gauss's hypergeometric series

$$F(\alpha, \beta, \gamma; \tau) := \sum_{\nu=0}^{\infty} c_\nu(\alpha, \beta, \gamma)\tau^\nu.$$

   (F is occasionally written in Carathéodory as ϕ_1.)

7. With $F = F(\alpha, \beta, \gamma; \tau)$, a special function

$$F^*(\alpha, \beta, \gamma; \tau) := \frac{\partial F}{\partial \alpha} + \frac{\partial F}{\partial \beta} + 2\frac{\partial F}{\partial \gamma};$$

   by [6], equation (387.4) on p. 153, F^* may be written

$$F^*(\alpha, \beta, \gamma; \tau) = \sum_{\nu=1}^{\infty} c_\nu(\alpha, \beta, \gamma) e_\nu(\alpha, \beta, \gamma)\tau^\nu.$$

8. A special function $\phi_2^*(\tau)$ is defined as a certain limit ([6], p. 152, equation 386.2) but is immediately (equation 386.3) reduced to

$$\phi_2^*(\tau) = F(\alpha, \beta, 1; \tau) \log \tau + F^*(\alpha, \beta, 1; \tau).$$

9. The set $\mathscr{Q} = \{2, 5, 6, 8, 10, 11, 14, 15, 17, 18, 20, 22, 23, ...\}$ of positive integers not represented by the quadratic form $x^2 + xy + y^2$. (It is conjectured by Benoit Cloitre [17] that $\mathscr{Q}$ is also the set of non-square natural numbers with divisor sums divisible by three.)

3 Schwarz triangles, triangle functions and Hecke groups

For our purposes, Schwarz triangles T are hyperbolic triangles in $\mathbb{H}^*$ with certain restrictions on the angles at the vertices. From a Euclidean point of view, their sides are vertical rays, segments of vertical rays, semicircles orthogonal to the real axis and meeting it at points $(r, 0)$ with r rational, or arcs of such semicircles. We choose λ, μ and ν, all non-negative, such that $\lambda + \mu + \nu < 1$; then the angles of T are $\lambda\pi, \mu\pi$, and $\nu\pi$. By reflecting T across one of its edges, we get another Schwarz triangle. The reflection between two triangles in $\mathbb{H}^*$ is effected by a Möbius transformation, so the orbit of T under repeated reflections is associated to a collection of Möbius transformations. The group generated by these transformations is a triangle group M, say. By the Riemann Mapping Theorem there is a conformal, onto map $\phi : T \mapsto \mathbb{H}^*$ called a triangle function.

Hecke groups [8] are triangle groups M that act properly discontinuously on $\mathbb{H}$. This means that for compact $K \subset \mathbb{H}$, the set $\{\mu \in M \text{ s.t.} K \cap \mu(K) \neq \emptyset\}$ is finite. Recall that $G(\lambda_m)$ is the Hecke group generated by the maps $\tau \mapsto -1/\tau$ and $\tau \mapsto \tau + \lambda_m$. Apparently it was Hecke who, also in [8], established that $G(\lambda_m)$ has the structure of a free product of cyclic groups $C_2 * C_m$, generalizing the relation [16, 4] $SL(2, \mathbb{Z}) = C_2 * C_3$.

Let $\rho = -\exp(-\pi i/m) = -\cos(\pi/m) + i\sin(\pi/m)$, and let $T_m \subset \mathbb{H}^*$ denote the hyperbolic triangle with vertices ρ, i, and $i\infty$. The corresponding angles are $\pi/m, \pi/2$ and 0 respectively. Let ϕ_{λ_m} be a triangle function for T_m. The function ϕ_{λ_m} has a pole at $i\infty$ and period λ_m. For $P, Q \in \mathbb{H}^*$, let us us write $P \equiv_M Q$ when $\mu \in M$ and $Q = \mu(P)$. Then ϕ_{λ_m} extends to a function $J_m : \mathbb{H}^* \to \mathbb{H}^*$ by declaring that $J_m(P) = J_m(Q)$ if and only if $P \equiv_M Q$. J_m is a modular function for $G(\lambda_m)$.

4 Calculation of Schwarz's inverse triangle function

Schwarz proved

Theorem 1. *([6], §374)*

1. *Let the half-plane $\Im z > 0$ be mapped conformally onto an arbitrary circular-arc triangle whose angles at its vertices A, B, and C are $\pi\lambda, \pi\mu$, and $\pi\nu$, and let the vertices A, B, C be the images of the points $z = 0, 1, \infty$, respectively. Then the mapping function $w(z)$ must be a solution of the third-order differential equation*

$$\{w, z\} = \frac{1-\lambda^2}{2z^2} + \frac{1-\mu^2}{2(1-z^2)} + \frac{1-\lambda^2-\mu^2+\nu^2}{2z(1-z)}. \tag{3}$$

2. *If $w_0(z)$ is any solution of equation (3) that satisfies $w_0'(z) \neq 0$ at all interior points of the half-plane, then the function*

$$w(z) = \frac{aw_0(z)+b}{cw_0(z)+d} \qquad (ad - bc \neq 0)$$

is likewise a solution of equation 3.

3. *Also, every solution of equation (3) that is regular and non-constant in the half-plane $\Im z > 0$ represents a mapping of this half-plane onto a circular-arc triangle with angles $\pi\lambda, \pi\mu$, and $\pi\nu$.*

In Carathéodory's lexicon ([5] p. 124), a regular function is one that is differentiable on an open connected set. Carathéodory writes the left side of (3) as "$\{w, z\} = \frac{w' w''' - 3w''^2}{w'^2} = ...$", but this is not the case. We infer that $\{w, z\}$ is intended from the automorphy property of clause 2.

Let us write

$$\alpha = \frac{1}{2}(1 - \lambda - \mu + \nu), \tag{4}$$

$$\beta = \frac{1}{2}(1 - \lambda - \mu - \nu), \tag{5}$$

and

$$\gamma = 1 - \lambda. \tag{6}$$

The solutions w of (3) are inverse to triangle functions; they are quotients of arbitrary solutions of

$$u'' + p(z)u' + q(z)u = 0 \tag{7}$$

when ([6], p. 136, equation (376.4))

$$p = \frac{1 - \lambda}{z} - \frac{1 - \mu}{1 - z}$$

and

$$q = -\frac{\alpha\beta}{z(1 - z)}.$$

Equation (7) reduces ([6], p. 137, equations 376.5-7) to the hypergeometric differential equation

$$z(1 - z)u'' + (\gamma - (\alpha + \beta + 1)z)u' - \alpha\beta u = 0. \tag{8}$$

As long as γ is not a non-positive integer, $u = F(\alpha, \beta, \gamma; z)$ is a solution of (8); it is the only solution regular at $z = 0$, and it satisfies $F(\alpha, \beta, \gamma; 0) = 1$ (final paragraph of [6], §377, p. 138.)

In [6], §§386-388 (pp. 151-155), we find that when $\gamma = 1$ and $\lambda = 0$, another, linearly independent, solution of equation (7) is $\phi_2^*(z)$. [6], Section 394, pp. 165 - 167 is devoted to the case $\lambda = 0$. The mapping function w of Theorem 1 satisfies ([6], p. 166, equation 394.4)

$$w = \frac{1}{\pi i}\left[\frac{\phi_2^*}{\phi_1} - (2\psi(1) - \psi(1 - \alpha) - \psi(1 - \beta))\right] + i\frac{\sin \pi\mu}{\cos \pi\mu + \cos \pi\nu}. \tag{9}$$

5 Inversion of Schwarz's inverse triangle function

Following Lehner and Raleigh, we consider the Schwarz triangle T_m with vertices at $\rho = -\exp(-\pi i/m), i$, and $i\infty$. In terms of Theorem 1, T_m has $\lambda = 0$ (an angle 0 at the vertex $i\infty$), $\mu = 1/2$ (an angle $\pi/2$ at i), and $\nu = 1/m$ (an angle π/m at ρ.) In this situation, $\gamma = 1$.

Let J_m be automorphic for $G(\lambda_m)$ with $J_m(\rho) = 0, J_m(i) = 1$, and $J_m(i\infty) = \infty$. In terms of Theorem 1, w and J_m are inverse functions. We are going to write down the Fourier expansion of J_m.

By clause 2 of Theorem 1, if w satisfies equations (3) and (9), so does $\tau = \tau(z) = \lambda_m w(z)/2$, and therefore

$$2\pi i\tau/\lambda_m = \frac{\phi_2^*}{\phi_1} - (2\psi(1) - \psi(1-\alpha) - \psi(1-\beta)) - \pi\sec(\pi/m).$$

Let us write $\log A_m = -2\psi(1) + \psi(1-\alpha) + \psi(1-\beta) - \pi\sec(\pi/m)$. Recalling the definitions of ϕ_1 and ϕ_2^* from our glossary items 6 and 8, we find (abbreviating $J_m(\tau)$ as J_m) that

$$2\pi i\tau/\lambda_m = -\log J_m + \frac{F^*(\alpha,\beta,1;1/J_m)}{F(\alpha,\beta,1;1/J_m)} + \log A_m. \tag{10}$$

Equation (10) is equation (6) of [12], but Raleigh suppresses the subscripts. He also writes $\exp 2\pi i\tau/\lambda_m$ as x_m, so that (in our earlier notation) $x_m = q_m(\tau)$.

In Raleigh's notation, after taking exponentials,

$$x_m/A_m = \frac{1}{J_m}\exp\frac{F^*(\alpha,\beta,1;1/J_m)}{F(\alpha,\beta,1;1/J_m)}, \tag{11}$$

the right side of which has a power series in J_m with rational coefficients. Writing $X_m = x_m/A_m$ we can regard $X_m = X_m(J_m)$ as a power series in J_m. Following [10] and [12], we inverted this power series to obtain one for the modular function J_m. Let $\mathscr{I}$ be a formal operation taking a power series $\sigma(v)$ to its inverse; that is, if $u = \sigma(v)$ then $v = \mathscr{I}(\sigma)(u)$. Let $Y_m(J)$ be a power series such that

$$Y_m(J_m) = J_m \exp\frac{F^*(\alpha,\beta,1;J_m)}{F(\alpha,\beta,1;J_m)} = X_m\left(1/J_m\right)$$

and hence

$$Y_m(1/J_m) = \frac{1}{J_m}\exp_m\frac{F^*(\alpha,\beta,1;1/J_m)}{F(\alpha,\beta,1;1/J_m)} = X_m(J_m),$$

so that $\mathscr{I}(Y_m)(X_m(J)) = 1/J_m$ and, therefore, $J_m = 1/\mathscr{I}(Y_m)(X_m)$.

6 Modular forms from modular functions

When the w-image of $\mathbb{H}^*$ is T_m, the inverse of w is ϕ_{λ_m}. J_m, the extension by modularity of ϕ_{λ_m} to $\mathbb{H}^*$, is periodic with period λ_m and maps ρ to 0, i to 1, and $i\infty$ to ∞ ([10], equation (2).) These mapping properties allow us, following Berndt's exposition of Hecke, to construct positive weight modular forms for $G(\lambda_m)$ from J_m. This section describes results of Hecke that are perhaps most easily accessible for the classical case in Schoeneberg [14], and, for the general case, in Berndt [2].

6.1 $m = 3$.

By keeping track of the weights, zeros and poles of the constituent factors in the numerator and denominator of the fraction defining

$$f_{a,b,c} = \frac{J'^a}{J^b(J-1)^c},$$

Schoeneberg [14] (Theorem 16, p.45) demonstrates that $f_{a,b,c}$ is an entire modular form of weight $2a$ for $SL(2,\mathbb{Z})$ if $a \geqslant 2, 3c \leqslant a, 3b \leqslant 2a$, $b + c \geqslant a$ and a, b, c are integers. (Schoeneberg speaks of "dimension $-2a$.") Thus he is able to write down a weight 4 entire modular form $E_4^* = f_{2,1,1}$ for $SL(2,\mathbb{Z})$ with a zero of order $\frac{1}{3}$ at $\rho = e^{2\pi i/3}$ and a weight 6 entire modular form $E_6^* = f_{3,2,1}$ for $SL(2,\mathbb{Z})$ with a zero of order $\frac{1}{2}$ at i. (Schoeneberg writes G_4^*, G_6^*.) It is well known that the (vector space) dimension of the spaces of weight 4 and 6 entire modular forms for $SL(2,\mathbb{Z})$ is equal to one, so E_4^* and E_6^* may be identified with the usual weight 4 and weight 6 Eisenstein series, up to a normalization. Finally, Schoeneberg defines the weight 12 cusp form $\Delta^* = E_4^{*3} - E_6^{*2}$ with a zero of order 1 at $i\infty$. It is a multiple of Δ.

6.2 $m \geqslant 3$.

We quote statements from Berndt [2], which is an exposition of Hecke's [7] and other writings. We depart occasionally from Berndt's choices of variable to avoid clashes with our earlier notation.

Definition 1. *([2], Definition 2.2) We say that f belongs to the space $M(\lambda, k, \gamma)$ if*

1. $$f(\tau) = \sum_{n=0}^{\infty} a_n e^{2\pi i n\tau/\lambda},$$

   where $\lambda > 0$ and $\tau \in \mathbb{H}$, and

2. $f(-1/\tau) = \gamma(\tau/i)^k f(\tau)$, *where $k > 0$ and $\gamma = \pm 1$.*

We say that f belongs to the space $M_0(\lambda, k, \gamma)$ if f satisfies conditions (i), (ii), and if $a_n = O(n^c)$ for some real number c, as n tends to ∞.

After defining the notion of a fundamental region in the usual way and defining as $G(\lambda)$ the group of linear fractional transformations generated by $\tau \mapsto -1/\tau$ and $\tau \mapsto \tau + \lambda$, Berndt states

Theorem 2. *([2], Theorem 3.1) Let* $B(\lambda) = \{\tau \in \mathbb{H} : x < \lambda/2, |\tau| > 1\}$. *Then if* $\lambda \geqslant 2$ *or if* $\lambda = 2\cos(\pi/m)$, *where* $m \geqslant 3$ *is an integer,* $B(\lambda)$ *is a fundamental region for* $G(\lambda)$.

Definition 2. *([2], Definition 3.4) Let* $T_A = \{\lambda : \lambda = 2\cos(\pi/m), m \geqslant 3, m \in \mathbb{Z}\}$.

Berndt states in his Theorem 5.4 that $G(\lambda)$ is discrete if and only if λ belongs to T_A. This discreteness is the premise of the theory of automorphic functions generally. He embeds within the proof of his Lemma 3.1 (which we omit), the

Definition 3.

1. *Let* τ_λ *denote the intersection in* $\mathbb{H}$ *of the line* $x = -\lambda/2$ *and the unit circle* $|\tau| = 1$. *(Berndt also remarks on page 35 that* τ_λ *is the lower left corner of* $B(\lambda)$.*)*
2. *"Letting* $\pi\theta = \pi - \arg(\tau_\lambda)$ *(so that* $\lambda = 2\cos\theta$*) ...."*

To characterize Eisenstein series, we need to keep track of some analytical properties. The next definition summarizes the second paragraph of Berndt's Chapter 5. (Throughout his Chapter 5, $\lambda < 2$.)

Definition 4. *Let* $f \in M(\lambda, k, \gamma), f$ *not identically zero.*

1. $N = N_f$ *counts the zeros of* f *on* $\overline{B(\lambda)}$ *with multiplicities.*
2. N_f *does not count zeros at* $\tau_\lambda, \tau_\lambda + \lambda, i$ *or* $i\infty$.
3. *If* $\tau_0 \in \overline{B(\lambda)}, f(\tau_0) = 0$ *and* $\Re(\tau_0) = -\lambda/2$, *then* $f(\tau_0 + \lambda) = 0$ *and* N_f *counts only one of the two zeros.*
4. *If* $\tau_0 \in \overline{B(\lambda)}, f(\tau_0) = 0$, *and* $|\tau_0| = 1$, *then,* $f(-1/\tau_0) = 0$, *and* N_f *counts only one of these two zeros.*
5. *The numbers* n_λ, n_i, *and* n_∞ *are the orders of the zeros of* f *at* τ_λ, i *and* $i\infty$, *repectively. The order* n_∞ *is measured in terms of* $\exp(2\pi i\tau/\lambda)$.

The multiplier γ is given by

Theorem 3. *([2], Corollary 5.2) Let* $f \in M(\lambda, k, \gamma)$ *and let* n_i *be the order of the zero of* f *at* $\tau = i$. *Then*

$$\gamma = (-1)^{n_i}.$$

The next two results tell us that the only nontrivial case in this theory is the one that we are interested in.

Theorem 4. *([2], Lemma 5.1) If* $\dim M(\lambda, k, \gamma) \neq 0$,

$$N_f + n_\infty + \frac{1}{2}n_i + \frac{n_\lambda}{m} = \frac{1}{2}k\left(\frac{1}{2} - \theta\right).$$

By Berndt's equation (5.16), if $m \geqslant 3$ then the right side can be written as $k(m-2)/4m$.

Theorem 5. *([2], Theorem 5.2) If* $\dim M(\lambda, k, \gamma) \neq 0$*, then* $\theta = 1/m$ *where* $m \geqslant 3$ *and* $m \in \mathbb{Z}$*.*

We are concerned with $\lambda \in T_A$. This makes $\lambda < 2$ as in all the results of Berndt's Chapter 5.

One estimate for $\dim M(\lambda, k, \gamma)$ is

Theorem 6. *([2], Theorem 5.6) If* $\lambda \notin T_A$*, then* $\dim M(\lambda, k, \gamma) = 0$*. If* $\lambda = 2\cos(\pi/m) \in T_A$*, then for nontrivial* $f \in M(\lambda, k, \gamma)$*, the weight* k *has the form*

$$k = \frac{4h}{m-2} + 1 - \gamma,$$

where $h \geqslant 1$ *is an integer. Furthermore,*

$$\dim M(\lambda, k, \gamma) = 1 + \left\lfloor \frac{h + (\gamma - 1)/2}{m} \right\rfloor.$$

Eliminating h, we find that

$$\dim M(\lambda, k, \gamma) = 1 + \left\lfloor k\left(\frac{1}{4} - \frac{1}{2m}\right) + \frac{\gamma}{4} - \frac{1}{4} \right\rfloor. \tag{12}$$

Berndt ([2], Remark 5.3) proves that the dimension formula above holds also when $h = 0$.

The existence of certain modular forms is provided by

Theorem 7. *([2], Theorem 5.5) Let* $\lambda \in T_A$*. Then there exist functions* f_λ, f_i*, and* $f_\infty \in M(\lambda, k, \gamma)$ *such that each has a simple zero at* τ_λ, i*, and* $i\infty$*, respectively, and no other zeros. Here,* γ *is given by [this article's Theorem 3], and* k *is determined in each case from [Theorem 4 of this article]. Thus,* $f_\lambda \in M(\lambda, 4/(m-2), 1), f_i \in M(\lambda, 2m/(m-2), -1)$*, and* $f_\infty \in M(\lambda, 4m/(m-2), 1)$*.*

Remark 1. *([2], pages 47-48) By the Riemann mapping theorem there exists a function* $g(\tau)$ *that maps the simply connected region* $B(\lambda)$ *one-to-one and conformally onto* $\mathbb{H}$*. If we require that* $g(\tau_\lambda) = 0, g(i) = 1$*, and* $g(i\infty) = \infty$*, then* g *is determined uniquely.*

Now we can write down f_λ, f_i, and f_∞ explicitly. The next theorem is extracted from the proof of Theorem 7. f_λ and f_i correspond to Eisenstein series and f_∞ to a cusp form. In our code, we take g to be a normalized form of J_m.

Theorem 8. *([2], page 50)*

$$f_\lambda(\tau) = \left\{ \frac{g'(\tau)^2}{g(\tau)(g(\tau)-1)} \right\}^{1/(m-2)},$$

$$f_i(\tau) = \left\{ \frac{g'(\tau)^m}{g(\tau)^{m-1}(g(\tau)-1)} \right\}^{1/(m-2)},$$

and

$$f_\infty(\tau) = \left\{ \frac{g'(\tau)^{2m}}{g(\tau)^{2m-2}(g(\tau)-1)^m} \right\}^{1/(m-2)}.$$

In our applications to Lehmer's problem, we will be interested in the dimensions of the weight 12 cusp spaces for $\lambda = \lambda_m = 2\cos \pi/m$.

Definition 5. *([2], Definition 5.2) If $f \in M(\lambda, k, \gamma)$ and $f(i\infty) = 0$, then we call f a cusp form of weight k and multiplier γ with respect to $G(\lambda)$. We denote by $C(\lambda, k, \gamma)$ the vector space of all cusp forms of this kind.*

Remark 2. *([2], equation (5.25))*

$$\dim C(\lambda, k, \gamma) \geqslant \dim M(\lambda, k, \gamma) - 1.$$

Remark 3. *In view of (i) Theorem 6, (ii) equation (12), (iii) Remark 2, and (iv) the fact that $\gamma = \pm 1$, we see that $\dim C(\lambda_m, 12, \gamma) > 1$ when m is greater than or equal to 12.*

7 Normalizations

In this section we write down normalizations of the functions from Theorem 8 such that, at $m = 3$, the normalizations reduce to corresponding functions in the classical theory, namely (in Serre's notation [16]), j, E_2, E_3, and Δ— Klein's invariant, the weight 4 and weight 6 Eisenstein series, and the normalized discriminant Δ. Also, the coefficients of their Fourier expansions should be rational numbers, but this we have verified only experimentally. For simplicity, we will abuse our notation and write $f(q_m)$ for functions $f(z)$ with Fourier expansions $f(z) = \sum c_n q_m(z)^n$. Let us also write $W_m(X_m)$ for the series $1/\mathscr{I}(Y_m)(X_m)$ in the last equation of section 5.

The section 5 parameter $A_3 = 1/1728 = 1/(2^6 3^3)$ and $J_3(\tau) = W_3(X_3) = W_3(x_3/A_3) = W_3(1728x_3) = W_3(2^6 3^3 q_3(\tau))$. The Fourier expansion of J_3 and that of the Klein j-invariant agree, but if $m \neq 3, 4$, or 6, the Fourier expansion of J_m has irrational coefficients because the residue a_{-1} of $J_m(q_m)$ is transcendental if $m \neq 3, 4$ or 6, ([18], according to [11]) and

$$a_{-1} = A_m. \tag{13}$$

This equation can be justified by reference to Raleigh, but that author does, it seems, commit a sign error in his equation (10), which it is necessary to compare to his equation (I) to conclude that our equation (13) is true. The same comparison indicates the sign error, because in (I) the signs of $\pi \sec(\pi/m)$ and $2\psi(1)$ disagree, whereas they agree in Raleigh's equation (10).

Therefore, following [11], we set

$$j_m(x) := W_m(2^6 m^3 x_m)/B_m,$$

where B_m is the coefficient of $1/x_m$ in the series $W_m(2^6 m^3 x_m)$; by construction, $B_m = 2^{-6} m^{-3}$. Corresponding to f_λ, we set

$$H_{4,m}(\tau) := \left\{ \frac{j_m'(\tau)^2}{j_m(\tau)(j_m(\tau) - 2^6 m^3)} \right\}^{1/(m-2)}.$$

Let

$$K_{6,m}(\tau) := \left\{ \frac{j_m'(\tau)^m}{j_m(\tau)^{m-1}(j_m(\tau) - 2^6 m^3)} \right\}^{1/(m-2)}.$$

We set $H_{6,m} := K_{6,m}/\epsilon$ where $\epsilon = e^{i\pi/(m-2)}$ or 1, depending on whether m is odd or even, respectively.

Corresponding to f_∞, we set

$$\Delta^\star(\tau) = \left\{ \frac{j_m'(\tau)^{2m}}{j_m(\tau)^{2m-2}(j_m(\tau) - 2^6 m^3)^m} \right\}^{1/(m-2)}.$$

Finally, we set $\Delta_m^\dagger = H_{4,m}^3 - H_{6,m}^2$ and $\Delta_m^\diamond = H_{4,m}^3/j_m$. Because J_3' is a weight-2 modular function ([14], p.44), and the weight-4 and weight-6 spaces and the weight-12 cusp space are one-dimensional when $m = 3$, the desired property follows after checking the first few terms of the Fourier expansions at issue.

8 Interpolation by polynomials of the Fourier coefficients of normalized modular forms and functions for Hecke groups

Let

$$J_m(z) = \sum_{n=-1}^{\infty} a_n(m) q_m(z)^n.$$

For integers $m \geqslant 3$ Raleigh [12] showed that

$$a_{-1}(m) = \exp(\pi \sec(\pi/m) - 2\psi(1) + \psi(1/4 + 1/(2m)) + \psi(1/4 - 1/(2m))$$

and that, for $n = 0, 1, 2, 3, a_n(m) = m^{-2n-2} a_{-1}(m)^{-n} R_n(m)$ where $R_n(x)$ is a polynomial with rational coefficients and degree $2n + 2$. He conjectured that similar relations exist among the a_n for all positive n.

We made numerical studies to explore how this conjecture might extend to the j_m. We computed the Fourier expansions of $j_m = 1/q_m + \sum_{n\geqslant 0} c_n(m) q_m^n$ to order 23 and used *Mathematica* functions to generate polynomials r_n (not merely rational functions) with rational coefficients which, we conjecture, interpolate the sequences $\{c_n(3), c_n(4), ...\}$). This procedure has obvious drawbacks. On the other hand, the polynomials we will mention exhibit regularities which may improve the credibility of the following conjectures. *Mathematica* notebooks and associated data files are here [3].

Conjecture 1. *For each integer n greater than -2, there exists a polynomial $r_n(x) \in \mathbb{Q}[x]$ that satisfies the relation $c_n(m) = r_n(m)$ for $m = 3, 4, ...,$ with $r_{-1}(x) \equiv 1, r_0(x) = 8x(3x^2 + 4)$, and $r_1(x) = 4x^2(69x^4 - 8x^2 - 48)$. For n greater than one, $r_n(x) = (x-2)(x+2)x^{n+1}p_n(x)$ where $p_n(x)$ is an irreducible polynomial over $\mathbb{Q}$ of degree $2n$.*

Conjecture 1 implies that for all integers m greater than or equal to three, $r_n(m)$ is nonzero, and therefore that

Conjecture 2. *For each fixed n greater than or equal to -1 and all integers m greater than or equal to three, $c_n(m)$ is nonzero.*

Conjecture 3. *(i) If $n > 1, p_n(\rho) = 0$ and $\rho \neq \pm 2$, then*

$$|\rho| \leqslant n/\log(n). \tag{14}$$

(ii) Consequently (even supposing that conjecture 2 is false), if $n \geqslant 0$ and $m > \max(n/\log(n), 2)$ then $c_n(m) \neq 0$.
(iii) For each n, there is a closed curve P_n symmetric about both axes in the complex plane, such that the roots of p_n lie on P_n or on one of the axes. Exactly two non-zero roots of p_n are imaginary. P_n has exactly one self-intersection, at zero.

We used the argument principle to count the zeros in the disks of radius $n/\log(n)$ to test clause (i). For small enough n, we plotted the zeros to discern the curve P_n, which resembles a lemniscate. For larger n this was infeasible, and we used a different plotting method to see P_n: points in the complex plane that the function $z \mapsto p_n(z)$ send to a given quadrant were assigned one of four colors, and so zeros were visible as points where four colors meet. The plots are included in [3], in the documents "conjecture 3 no2.nb" and "conjecture 3 no3.nb" (readable with *Mathematica* software.) Remark: it is already known that, for all integers $n \geqslant -1$, $c_n(3)$ is positive. (See, for example, page 199 in [13].)

We identified the set $\mathscr{Q}$ mentioned in the next conjecture and defined in our glossary with the assistance of Sloane's article [17].

Conjecture 4. *The function $H_{4,m}$ has a Fourier expansion*

$$H_{4,m}(z) = \sum_{n=0}^{\infty} \beta_n(m) q_m(z)^n.$$

For each n there is a polynomial $B_n(x)$ with rational coefficients such that (i) $\beta_n(m) = B_n(m)$ for $m = 3, 4, ...$, (ii) $B_0(x) \equiv 1$ identically, (iii) If n is positive, *then the degree of* $B_n(x)$ *is* $3n-1$. *(iii)* $B_1(x) = 16x(x+2)$, *(iv)* $B_2(x) = -16x(x-2)(x+2)(x+6)$ *(v) If* n *is larger than one and* $n \notin \mathcal{Q}$, *then* $B_n(x) = (x^2-4)x^n b_n(x)$ *where* $b_n(x)$ *is an irreducible polynomial. (vi) If* n *is larger than two and* $n \in \mathcal{Q}$, *then* $B_n(x) = (x^2-4)(x-6)x^n b_n(x)$, *where for* $b_n(x)$ *is an irreducible polynomial.*

Thus, in the range of our observations ($3 \leqslant m \leqslant 302, 0 \leqslant n \leqslant 100$), the only integer value of m such that $H_{4,m}$ has any vanishing coefficients is six, and $\beta_n(6)$ is zero just if n is in $\mathcal{Q}$.

Conjecture 5. *The function* $H_{6,m}$ *has a Fourier expansion*

$$H_{6,m}(z) = \sum_{n=0}^{\infty} \gamma_n(m) q_m(z)^n.$$

For each n *there is a polynomial* $C_n(x)$ *such that (i)* $\gamma_n(m) = C_n(m)$ *for* $m = 3, 4, ...$, *(ii)* $C_0(x) \equiv 1$ *identically, (iii)* $C_1(x) = -8x^2(3x-2)$, *(iv) and, for* n *larger than one,* $C_n(x) = (x-2)(3x-2)x^{n+1}d_n(x)$ *where* $d_n(x)$ *is a polynomial, irreducible over* $\mathbb{Q}$, *of degree* $2n-3$.

Let Δ be usual normalized discriminant, a weight 12 cusp form for $SL(2,\mathbb{Z}) = G(\lambda_3)$ with integer coefficients. Its Fourier expansion is written

$$\Delta(z) = \sum_{n=1}^{\infty} \tau(n) q^n$$

where $q = e^{2\pi i z}$ and $\tau(n)$ is Ramanujan's function. Whether or not the equation $\tau(n) = 0$ has any solutions is an open question [9]. (Recently, Balakrishnan, Craig, and Ono [1] ruled out $\pm 1, \pm 3, \pm 5, \pm 7$, and ± 691 from membership in the image of Ramanujan's function.)

Conjecture 6. *1. Let* $\Delta_m = \Delta_m^\star, \Delta_m^\dagger$ *or* $\Delta_m^\diamond$ *and let the Fourier expansion of* $\Delta_m(z)$ *be*

$$\Delta_m(z) = \sum_{n=1}^{\infty} \tau_m(n) q_m^n.$$

where $\tau_m = \tau_m^*, \tau_m^\dagger$ *or* $\tau_m^\diamond$ *respectively. Then there is a set of polynomials* $T_n = T_n^\star, T_n^\dagger$, *or* $T_n^\diamond$, *respectively, with coefficients in* $\mathbb{Q}$ *such that* $\tau_m(n) = T_n(m)$ *for each* $m = 3, 4, ...$

2. $T_1^\star(x) \equiv 1$ *identically, and, if* $n > 1$, *then* $T_n^\star(x) = (x-2)^2 x^{n-1} t_n^\star(x)$, *where* $t_n^\star(x)$ *is an irreducible polynomial over* $\mathbb{Q}$ *of degree* $2n-4$.

3. (i) $T_1^\dagger(x) = 16x(3x^2+x+6)$.
(ii) $T_2^\dagger(x) = -16x^2(39x^4 - 95x^3 + 66x^2 - 260x - 120)$.
(iii) $T_3^\dagger(x) =$

$$64x^3(189x^6 - 3021x^5 + 9574x^4 - 12520x^3 + 19136x^2 - 2960x - 2208)/9.$$

(iv) If $n > 3$, then $T_n^\dagger(x) = (x-2)x^n t_n^\dagger(x)$, where $t_n^\dagger(x)$ is an irreducible polynomial over $\mathbb{Q}$ of degree $2n-1$.

4. (i) $T_1^\diamond(x), T_2^\diamond(x)$, and $T_3^\diamond(x)$ are irreducible polynomials over $\mathbb{Q}$ of degrees $3, 6$, and 9, respectively.
(ii) If n is greater than 2, $T_1^\diamond(x) = (x-2)x^{n-1}t_n^\diamond(x)$, where $t_n^\diamond(x)$ is an irreducible polynomial over $\mathbb{Q}$ of degree $2n-3$.

Conjecture 7. *None of the $T_n(x)$ takes an integer greater than two to zero; consequently, none of the τ_m vanish for $m = 3, 4, \ldots$.*

Obviously, one basis of conjecture 7 is conjecture 6, but it is also a consequence of the following

Conjecture 8. *For $T_n = T_n^\diamond(x), T_n^\diamond(x)$, or $T_n^\diamond(x)$, for each positive integer n, and for each integer m greater than two, let the minimum distance from m of any root of T_n be denoted as $d(m,n)$. For fixed n, $d(m,n)$ is never zero. For fixed m, $d(m,n)$ decays exponentially as n increases. If m is greater than three, then $d(m,n) > d(3,n)$. (We note, however, that for $T = T^\star$ and $n = 1$, the interpolating polynomial is identically equal to one and the corresponding root set is therefore empty.)*

References

[1] Jennifer S Balakrishnan, William Craig, and Ken Ono. "Variations of Lehmer's Conjecture for Ramanujan's tau-function". In: *Journal of Number Theory* (2020).

[2] Bruce C Berndt and Marvin Isadore Knopp. *Hecke's theory of modular forms and Dirichlet series*. Vol. 5. World Scientific, 2008.

[3] Barry Brent. *Supporting documents*. `https://drive.google.com/drive/folders/1Dl4r-ElSBRTGz5UB` 2020.

[4] I. N. Cangül. "The group structure of Hecke groups H(λ_q)". In: *Turkish Journal of Mathematics* 20.2 (1996), pp. 203–207.

[5] Constantin Carathéodory. *Theory of functions of a complex variable, Second English Edition*. Vol. 1. Translated by F. Steinhardt. Chelsea Publishing Company., 1958.

[6] Constantin Carathéodory. *Theory of functions of a complex variable, Second English Edition*. Vol. 2. Translated by F. Steinhardt. Chelsea Publishing Company., 1981.

[7] Erich Hecke. *Lectures on Dirichlet series, modular functions, and quadratic forms*. Vandenhoeck & Ruprecht, 1983.

[8] Erich Hecke. "Über die bestimmung dirichletscher reihen durch ihre funktionalgleichung". In: *Mathematische Annalen* 112.1 (1936), pp. 664–699.

[9] D. H. Lehmer. “The vanishing of Ramanujan’s function $\tau(n)$”. In: *Duke Mathematical Journal* 14.2 (1947), pp. 429–433.

[10] Joseph Lehner. “Note on the Schwarz triangle functions.” In: *Pacific Journal of Mathematics* 4.2 (1954), pp. 243–249.

[11] John Garrett Leo. *Fourier coefficients of triangle functions, Ph.D. thesis*. http://halfaya.org/ucla/research/thesis.pdf. 2008.

[12] J Raleigh. “On the Fourier coefficients of triangle functions”. In: *Acta Arithmetica* 8 (1962), pp. 107–111.

[13] R.A. Rankin. *Modular Forms and Functions*. Cambridge University Press, Cambridge, 1977.

[14] Bruno Schoeneberg. *Elliptic modular functions*. Springer, 1974, p. 45.

[15] Hermann Amandus Schwarz. “Ueber diejenigen Fälle, in welchen die Gaussische hypergeometrische Reihe eine algebraische Function ihres vierten Elementes darstellt.” In: *Journal für die reine und angewandte Mathematik* 75 (1873), pp. 292–335.

[16] Jean-Pierre Serre. *A course in arithmetic*. Springer-Verlag, 1970.

[17] Neil J. A. Sloane. *The On-Line Encyclopedia of Integer Sequences, A034020*. https://oeis.org/A034020.

[18] Jurgen Wolfart. “Transzendente Zahlen als Fourierkoeffizienten von Heckes Modulformen”. In: *Acta Arithmetica* 2.39 (1981), pp. 193–205.