

Counting the Nontrivial Equivalence Classes of S_n Under $\{1234, 3412\}$ -Pattern-Replacement

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Abstract

We study the $\{1234, 3412\}$ pattern-replacement equivalence relation on the set S_n of permutations of length n , which is conceptually similar to the Knuth relation. In particular, we enumerate and characterize the nontrivial equivalence classes, or equivalence classes with size greater than 1, in S_n for $n \geq 7$ under the $\{1234, 3412\}$ -equivalence. This proves a conjecture by Ma, who found three equivalence relations of interest in studying the number of nontrivial equivalence classes of S_n under pattern-replacement equivalence relations with patterns of length 4, enumerated the nontrivial classes under two of these relations, and left the aforementioned conjecture regarding enumeration under the third as an open problem.

1 Introduction

A permutation $\pi \in S_n$ is said to *contain a pattern* $\sigma \in S_c$, $c \leq n$, if there is a c -letter subsequence $\pi_{i_1}, \pi_{i_2}, \dots, \pi_{i_c}$ of π that is *order-isomorphic* with σ (i.e., for indices $j, k \in [c]$, $\pi_{i_j} < \pi_{i_k}$ if and only if $\sigma_j < \sigma_k$).

In the past thirty years, the topic of permutation patterns has risen to the forefront of combinatorics (see Kitaev [7] for a survey) and has even spawned its own annual conference [6].

The focus of this paper is *permutation pattern-replacement equivalences*. Given a set of patterns $P \subseteq S_c$ and a permutation $\pi \in S_n$, one can perform a *P-pattern-replacement* on π by taking a subsequence $\pi_{i_1}, \dots, \pi_{i_c}$ of π that forms a pattern $\sigma \in P$ and rearranging the relative order of the characters $\pi_{i_1}, \dots, \pi_{i_c}$ so that they form a different pattern in P . We say that two permutations $\alpha, \beta \in S_n$ are *P-replacement equivalent* if α can be reached from β by a series of *P*-pattern-replacements. This defines an equivalence relation on S_n , which is known as the *P*-replacement equivalence [9, 10, 19, 12, 16, 8, 15, 13, 5].

The study of permutation pattern-replacement equivalences is closely related to the study of permutation pattern avoidance [7, 17, 4, 11, 14, 1, 2, 3], which seeks to count the number of singleton equivalence classes (i.e., the number of permutations containing no patterns in P). The dual problem of counting the number of non-singleton (i.e., *nontrivial*) equivalence classes has recently received a large amount of attention in the literature [9, 10, 19, 12, 16, 8, 15, 13, 5].

Most of the research so far on permutation pattern-replacement equivalences has worked to systematically understand the equivalence classes for all pattern-replacement equivalence relations involving patterns of length three [9, 10, 19, 12, 16, 8, 15]. Many interesting number sequences have arisen (e.g., the Catalan numbers, Motzkin numbers, tribonacci numbers, central binomial coefficients, and many more complicated number sequences).

Recently, Ma [13] initiated the systematic study of pattern-replacement equivalences with patterns of length four. Because the number of pattern-replacement sets $P \subseteq S_4$ is very large, Ma took a computational approach to identifying which of the pattern-replacement sets were most interesting to study. In particular, Ma computed the number of nontrivial equivalence classes under *P*-equivalence for all sets $P \in S_4 \times S_4$, and then matched the resulting number sequences with the Online Encyclopedia of Integer Sequences (OEIS) [18] in order to identify pattern-replacement sets P for which the number of nontrivial equivalence classes has a natural formula. Ma identified three formulas of particular interest and was able to enumerate the nontrivial equivalence classes for the equivalence relations corresponding to two of them. Enumerating the equivalence classes for the third equivalence relation, namely the $\{1234, 3412\}$ -equivalence, has remained an open question.

In this paper, we resolve the aforementioned open problem by proving the following theorem:

Theorem 1 (Conjectured by Ma [13]). *For $n \geq 7$, the number of nontrivial equivalence classes of S_n under the $\{1234, 3412\}$ -equivalence is $\frac{n^3+6n^2-55n+54}{6}$ (given by sequence [A330395](#)).*

Recall that we define two permutations $\alpha, \beta \in S_n$ to be equivalent under the $\{1234, 3412\}$ -equivalence if α can be reached from β by performing a series of $1234 \rightarrow 3412$ and $3412 \rightarrow 1234$ pattern-replacements. A $1234 \rightarrow 3412$ pattern-replacement in a permutation π simply takes an increasing 4-letter subsequence $\pi_{i_1}, \pi_{i_2}, \pi_{i_3}, \pi_{i_4}$ (where $i_1 < i_2 < i_3 < i_4$), and places each of $\pi_{i_3}, \pi_{i_4}, \pi_{i_1}, \pi_{i_2}$ in positions i_1, i_2, i_3, i_4 of the permutation. For example, in the permutation $\pi = 7162435$, the subsequence 1, 2, 3, 5 forms a 1234-pattern, and we can perform a $1234 \rightarrow 3412$ pattern-replacement to obtain the new permutation $\pi' = 7365412$. Similarly, a $3412 \rightarrow 1234$ pattern-replacement takes any four-letter subsequence that forms a 3412 pattern, and rearranges the letters in that subsequence to instead be in increasing order, thereby forming a 1234 pattern.

Theorem 1 counts the number of *nontrivial* equivalence classes in S_n under the $\{1234, 3412\}$ -equivalence. These are the equivalence classes of size greater than one, or alternatively, the equivalence classes consisting of non- $\{1234, 3412\}$ -avoiding permutations. In the remainder of the paper, we prove Theorem 1.

1.1 Formal definitions

Definition 1 (Standardization of a Permutation). Given a word ρ of length n consisting of n distinct letters in \mathbb{N} , the *standardization* of ρ is the permutation in S_n obtained by replacing the i -th smallest letter in ρ with i for each $i \in \{1, 2, \dots, n\}$.

Definition 2 (Sub-standardization of a Permutation). Given a permutation $\pi \in S_n$, and a subword ρ of π , the standardization of ρ is known as a *sub-standardization* of π . If removing the letter a from π gives ρ , then the standardization of ρ is also called the sub-standardization of π formed by removing a .

Definition 3 (Pattern Formation). Given a pattern $p \in S_c$ and a permutation $\pi \in S_n$, we say that π contains pattern p if p is a sub-standardization of π . If a subword ρ of π has standardization p , then we say that ρ *forms* pattern p .

Definition 4 (Pattern-Replacement). Given two patterns $p, q \in S_c$ and a permutation $\pi \in S_n$, a $p \rightarrow q$ *pattern-replacement* can be performed by taking any subword ρ of π that forms pattern p , and rearranging the letters in ρ to instead form pattern q .

Definition 5 (Pattern-Replacement Equivalences). Given a set of patterns $P \subseteq S_c$, we say that two permutations $p, q \in S_n$ are *equivalent* under P -equivalence if p can be reached from q by a sequence of pattern-replacements using patterns in P . Maximal collections of P -equivalent permutations are known as *equivalence classes*, and an equivalence class is said to be *nontrivial* if it contains more than one permutation.

1.2 Paper outline

To prove Theorem 1, we will use three lemmas in order to characterize the equivalence classes under the $\{1234, 3412\}$ -equivalence. To begin, we use the principle of inclusion-exclusion to establish a recurrence relation that allows us to express the equivalence classes in S_n in terms of the equivalence classes in S_{n-1} and S_{n-2} (Section 2). This reduces the proof of Theorem 1 to proving that the number of equivalence classes of a certain form is $n + 1$. We then characterize $n - 1$ of these classes that have a natural combinatorial structure, consisting of all permutations one transposition away from one of $n - 1$ different representative permutations (Section 3). Finally, we show that the remaining permutations fall into exactly two classes depending on the parity of the number of inversions they contain (Section 4); here, we use a proof structure that exploits the *stooge-sort technique* of Kuszmaul [9]. Combining these steps, we prove Theorem 1.

2 Reducing to permutations neither beginning with n nor ending with 1

In this section, we reduce the proof of Theorem 1 to counting the number of nontrivial equivalence classes that contain no permutations with n in the first position or 1 in the last position.

Lemma 1. *Theorem 1 can be reduced to proving that there are exactly $n + 1$ nontrivial equivalence classes of S_n where no permutation begins with n or ends with 1.*

Proof. Let A_n be the total number of nontrivial equivalence classes of S_n under the $\{1234, 3412\}$ -equivalence. Let B_n be the number of nontrivial equivalence classes of S_n under the $\{1234, 3412\}$ -equivalence that do not contain any permutation beginning with n or ending with 1. Then we have the following recurrence relation.

Claim 1. *For $n \geq 3$,*

$$A_n - B_n = 2A_{n-1} - A_{n-2}. \tag{1}$$

Proof. Since the largest element is never first and the smallest element is never last in the patterns 1234 and 3412, no pattern-replacements under the $\{1234, 3412\}$ -equivalence can move a leading n or an ending 1. Therefore, appending n to the front (resp., 1 to the end) of all permutations of an equivalence class in S_{n-1} results in an equivalence class in S_n . Call this the *lifting observation*.

We can count the number of equivalence classes where all permutations have either n in the first position or 1 in the last position (or both) in two different ways. The left side of (1) represents this number with complementary counting, where all equivalence classes not satisfying the requisite are subtracted from the total number of equivalence classes. In

the right side of (1), we count the same number of equivalence classes using the principle of inclusion-exclusion. By the lifting observation, we can append an n to the front of all permutations in S_{n-1} or append a 1 to the back of all permutations in S_{n-1} to form all possible equivalence classes where all permutations have either n in the first position or 1 in the last position (or both). (This contributes $2A_{n-1}$ to (1).) The repeats, which are formed by appending both an n to the front and a 1 to the back of all permutations in S_{n-2} , are then subtracted off. (This removes A_{n-2} .)

Since the two sides of (1) count the same quantity, we must have equality. \square

Using the recursion given in the preceding claim, we now analyze the value that B_n must take in order for A_n to have the formula stated in Theorem 1.

Claim 2. *Showing that $B_n = n + 1$ suffices to prove that $A_n = \frac{n^3+6n^2-55n+54}{6}$ for all $n \geq 7$.*

Proof. Suppose $B_n = n + 1$ for all $n \geq 7$. We use this to prove that $A_n = \frac{n^3+6n^2-55n+54}{6}$ for all $n \geq 7$.

The base cases of $n = 7, 8$ for $A_n = \frac{n^3+6n^2-55n+54}{6}$ have been checked by computer [13].

Assume by induction that our formula for A_n holds for $n = k - 1$ and $n = k - 2$ for some integer $k \geq 9$. Then by Claim 1,

$$\begin{aligned} A_k &= B_k + 2A_{k-1} - A_{k-2} \\ &= k + 1 + 2 \cdot \frac{(k-1)^3 + 6(k-1)^2 - 55(k-1) + 54}{6} \\ &\quad - \frac{(k-2)^3 + 6(k-2)^2 - 55(k-2) + 54}{6} \\ &= k + 1 + 2 \cdot \frac{k^3 + 3k^2 - 64k + 114}{6} - \frac{k^3 - 67k + 180}{6} \\ &= \frac{k^3 + 6k^2 - 55k + 54}{6}, \end{aligned}$$

which is our formula for A_k .

By induction, $A_n = \frac{n^3+6n^2-55n+54}{6}$ holds for all $n \geq 7$. \square

This completes the proof of the lemma. \square

3 Characterizing the $n - 1$ small classes

In this section we characterize $n - 1$ of the nontrivial equivalence classes of S_n that contain no permutation beginning with n or ending with 1. Each of these classes is associated with a “leader permutation” that is one transposition away from each permutation in the class.

Below we define the notion of a leader permutation and what we mean when we call two permutations “adjacent.” The equivalence classes then correspond to the sets of permutations that are adjacent to each leader permutation.

Definition 6 (Leader Permutation). Define a *leader permutation* of length n to be a permutation of the form $a_1 a_2 \cdots a_n$ such that for some integer $k \in [2, n]$, $a_i = k - i$ for all $1 \leq i < k$ and $a_i = n + k - i$ for all $k \leq i \leq n$.

Example 1. The 7 leader permutations of length 8 are 18765432, 21876543, 32187654, 43218765, 54321876, 65432187, and 76543218.

Definition 7 (Adjacent Permutations). We define two permutations $\pi, \rho \in S_n$ to be *adjacent* if neither π nor ρ begins with n or end with 1 and if π is a transposition of ρ such that the positive difference between the two letters in the transposition is neither 1 nor $n - 1$.

We now provide our characterization of the equivalence classes associated with the leader permutations in S_n .

Proposition 1. For any leader permutation π , the set of permutations adjacent to π is an equivalence class. Moreover, there are $n - 1$ of these classes.

To prove Proposition 1, we begin by proving that the equivalence classes are disjoint.

Lemma 2. For distinct leader permutations π, ψ , the set of permutations adjacent to π is disjoint from the set of permutations adjacent to ψ .

Proof. Assume for the sake of contradiction that the two sets share a common element. This would imply that it is possible to reach π from ψ in two transpositions. Since any leader permutation is a derangement of any other leader permutation, the permutations π and ψ differ in all $n \geq 7$ positions. However, since each transposition can only switch the positions of two letters, it is impossible for two transpositions to get π from ψ , a contradiction. \square

Next we prove that

Lemma 3. If we have a permutation adjacent to some leader permutation π and we apply a pattern-replacement to it, the resulting permutation is also adjacent to π .

Proof. Note that π consists of a decreasing set of consecutive letters followed by another decreasing set of consecutive letters that are all larger than those of the first set; let A and B denote these two sets, respectively.

Consider a permutation $a_1 a_2 \cdots a_n$ such that if a transposition is applied to two letters a_i and a_j , $i \neq j$, the resulting permutation is equal to a leader permutation π . We claim that any pattern-replacement applied to $a_1 a_2 \cdots a_n$ must use both a_i and a_j and result in a permutation adjacent to π . We show this using casework based on which type of pattern in $a_1 a_2 \cdots a_n$ is replaced.

Case 1: A 1234 pattern is replaced. Since A and B are decreasing, the longest increasing subsequence in π has length 2, so any 1234 pattern in $a_1 a_2 \cdots a_n$ must use both letters a_i and a_j involved in the transposition. Moreover, the two swapped letters a_i and a_j would need to be in the same set (i.e., both in A or both in B), because swapping a letter in A

with a letter in B would not place the two letters in increasing order. In order to have an increasing sequence of length 4, the pattern must additionally contain one letter between a_i and a_j , and one letter from the set A or B that does not contain a_i and a_j . In particular, if the two letters besides a_i and a_j are both from A or are both from B then they are in decreasing order (a contradiction). If either of the two letters is from the same set (i.e., out of A and B) as a_i, a_j are in but is not between a_i and a_j then it does not form an increasing sequence with a_i and a_j (a contradiction). Since a $\{1234, 3412\}$ -pattern replacement swaps the first and third letters and swaps the second and fourth letters, a_i and a_j are swapped back to their original order in π , and a new transposition is applied. The new permutation is adjacent to π .

Case 2: A 3412 pattern is replaced. Consider the four three-term subpatterns of the 3412 pattern, 341, 342, 312, and 412; we show that π cannot contain any of these. Assume for the sake of contradiction that some three letters $b_x, b_y, b_z, x < y < z$, of π follow one of these patterns. Then, in any of the patterns, $b_x > b_z$. In addition, note that b_y is either the largest or the smallest letter. If b_y is the largest letter, then $b_x < b_y$, so b_x must be in A and b_y must be in B , implying that b_z must be in B as well. Similarly, if b_y is the smallest letter, then $b_y < b_z$, so b_y must be in A and b_z must be in B , and thus b_x must be in A . In either case, b_x is in A and b_z is in B , which is a contradiction because $b_x > b_z$. Thus π cannot contain any length-3 subpattern of 3412.

As a consequence, any 3412 pattern in $a_1 a_2 \cdots a_n$ would have to contain both a_i and a_j . Furthermore, a_i and a_j would need to be in different sets, and the 3412 pattern would need to contain exactly one letter between a_i and a_j . To see this, note that in π , any four term subsequence corresponds to one of the patterns 4321, 3214, 2143, or 1432 depending on how many letters are from each of A and B . The only transpositions of two terms that could result in 3412 are swapping the 2 and 4 in 3214 and swapping the 1 and 3 in 1432. In either case, the smaller term is from A and the larger term is from B , and there is exactly one term between these two terms.

Since a $\{1234, 3412\}$ -pattern-replacement swaps the first and third letters and swaps the second and fourth letters, a_i and a_j are swapped back to their original order in π , and a new transposition is applied. The new permutation is adjacent to π .

This completes the proof of the claim. □

Finally, we show that any two permutations adjacent to the same leader permutation are always equivalent.

Lemma 4. *For any permutations τ and σ that are both adjacent to a leader permutation π , τ can be reached from σ by a sequence of $\{1234, 3412\}$ -pattern-replacements.*

Proof. We prove this using induction on n . Our base cases of $n = 7$ and 8 can be checked by computer. Assume as an inductive hypothesis that the result holds for $n - 1 \geq 8$. Consider two permutations τ and σ of length n that are adjacent to some leader permutation π . Pick one of their common letters a that is (1) not 1 or n ; (2) not part of either transposition from the leader permutation; (3) not adjacent in position to both letters in the transposition

from π to τ ; and (4) not adjacent in position to both letters in the transposition from π to σ . Such an a is guaranteed to exist because $n \geq 9$. Now, consider the sub-standardizations τ' , σ' , and π' respectively formed by removing a from τ , σ , and π . Note that π' is a leader permutation of length $n - 1$ (because $a \neq 1$ and $a \neq n$). In addition, τ' and σ' are both adjacent to π' . Thus, by our inductive hypothesis, π' can be reached from τ' by a sequence of pattern-replacements. By looking at the letters in τ corresponding to those of τ' , we see that π can be reached from τ by a sequence of pattern-replacements. This completes the proof by induction. \square

Combined, Lemmas 2, 3, and 4 complete the proof of Proposition 1.

4 Showing that there are two remaining classes

We now proceed to show that there are only two nontrivial classes remaining to be analyzed. To do this, we will find two representative permutations, one from each class. First, we will show that these representative permutations cannot be in the same equivalence class because they differ in parity and, hence, that there are at least two remaining equivalence classes. Then, we will show that every permutation that we have yet to analyze is equivalent to one of these two representative permutations, hence showing that there are at most, and thus exactly, two remaining nontrivial equivalence classes. To show the second part, we will use induction on the size of the permutation. We will begin by removing one letter of the permutation and applying the inductive hypothesis to the remaining letters to transform them into a representative permutation (of length $n - 1$). We then select a different letter in the new permutation (specifically, we pick a letter that is already in the “final” position that we want it to be in), we remove that letter from the permutation, and we apply the inductive hypothesis on the remaining letters in order to transform them into a representative permutation (of length $n - 1$). If done correctly, this construction causes the entire final permutation to form a representative permutation of length n , completing the proof.

We start out with some preliminary definitions.

Definition 8 (Primary Permutation). We call a permutation in S_n *primary* if it is not adjacent to a leader permutation and does not start with n or end with 1.

Definition 9 (Primary Class). We define a *primary class* to be a nontrivial equivalence class of primary permutations in S_n .

Lemma 5. *All of the permutations in each equivalence class have the same parity.*

Proof. Each $1234 \leftrightarrow 3412$ pattern-replacement consists of two transpositions. Since parity is invariant under both transpositions, the lemma follows. \square

In the remainder of the section, we will attempt to start by applying induction. To do this, we will look at a sub-standardization of an arbitrary primary permutation formed by excluding one letter. The goal is to find conditions on the letter removed such that the sub-standardization is also primary, hence allowing us to apply the inductive hypothesis to the sub-standardization. There are two possible problems that we would run into: either the sub-standardization is adjacent to a leader permutation, or the sub-standardization could start with its largest letter or end in 1. The latter is easily dealt with and we will put off to Case 2 of Lemma 7. The former, however, can be dealt with by an intuitive claim that is proven in Lemma 6. We show in Lemma 6 that, if we remove some letter and end up with a permutation adjacent to a leader permutation, we can instead choose a different letter with some constraints to yield a permutation that is not adjacent to a leader permutation.

Lemma 6 (Creating Primary Permutations). *Let $\rho \in S_k$ with $k \geq 5$ be a primary permutation. Suppose that there is a letter a in ρ such that removing a results in a sub-standardization $\rho_{-(a)}$ that is leader-permutation-adjacent. Let τ_1 be the transposition taking $\rho_{-(a)}$ to a leader permutation, and suppose b is some letter in both ρ and $\rho_{-(a)}$ that is not operated on by τ_1 and such that $a \not\equiv b \pm 1 \pmod{k}$. Also, suppose that if τ_1 operates on two letters with only one letter of $\rho_{-(a)}$ between them, then b is not that letter. Then the sub-standardization $\rho_{-(b)}$ obtained by removing b from ρ is not adjacent to any leader permutation.*

Proof. Suppose, for the sake of contradiction, that $\tau_2 \cdot \rho_{-(b)}$ is a leader permutation for some transposition τ_2 . Let $\rho_{-(a,b)}$ be the sub-standardization of ρ that contains all letters of ρ other than a and b (here we say that a transposition of a sub-standardization swaps the same physical letters as it did in the original permutation).

Define a *semi-leader permutation* to be a permutation that is either a leader permutation or the decreasing permutation. Note that for any semi-leader permutation, if we remove one letter and take the standardization, the result is always a semi-leader permutation.

We claim that $\tau_1 \cdot \rho_{-(a,b)}$ is a semi-leader permutation (note that τ_1 does not involve a or b). In particular, $\tau_1 \cdot \rho_{-(a)}$ is a leader permutation, and thus removing b to get $\tau_1 \cdot \rho_{-(a,b)}$ gives a semi-leader permutation. Moreover, $\tau_1 \cdot \rho_{-(a,b)}$ is the unique semi-leader permutation adjacent to $\rho_{-(a,b)}$. In particular, any other such semi-leader permutation x would differ from $\tau_1 \cdot \rho_{-(a,b)}$ in at most 4 positions, and would therefore have to agree with $\tau_1 \cdot \rho_{-(a,b)}$ in at least one position. Whenever two semi-leader permutations of the same length agree in at least one position, they must be the same permutation, meaning that x actually equals $\tau_1 \cdot \rho_{-(a,b)}$.

If τ_2 operates only on elements of $\rho_{-(a,b)}$ (i.e., τ_2 does not involve a), we can conclude that $\tau_2 = \tau_1$. In particular, $\tau_2 \cdot \rho_{-(a,b)}$ is a semi-leader permutation because $\tau_2 \cdot \rho_{-(b)}$ is a leader permutation. Thus $\tau_1 \cdot \rho_{-(a,b)} = \tau_2 \cdot \rho_{-(a,b)}$, so $\tau_1 = \tau_2$.

However, if $\tau_1 = \tau_2$ then τ_1 takes both $\rho_{-(b)}$ and $\rho_{-(a)}$ to leader permutations. We claim that this then implies that ρ is also a leader permutation, a contradiction. This is because when we add in b to $\rho_{-(b)}$ to get ρ , b is not adjacent to a (recall that $a \not\equiv b \pm 1 \pmod{k}$), so the characters that b is adjacent to in $\rho_{-(a)}$ remain adjacent to b in ρ (although their values may be shifted by 1), ensuring that ρ is a leader permutation.

It remains to consider the case where τ_2 operates on a . In this case, we define $\tau_2 \cdot \rho_{-(a,b)}$ to be the standardization of $\tau_2 \cdot \rho_{-(b)}$ with a removed. Now τ_2 moves only one letter in $\rho_{-(a,b)}$ and results in a semi-leader permutation (as $\tau_2 \cdot \rho_{-(a,b)}$ is contained in $\tau_2 \cdot \rho_{-(b)}$, which is a semi-leader permutation). Thus there are two semi-leader permutations $x = \tau_1 \cdot \rho_{-(a,b)}$ and $y = \tau_2 \cdot \rho_{-(a,b)}$, each of length $k - 2$, such that we can get from x to y by swapping two letters (call this transposition τ') and then moving one letter. We place each letter of the permutation on a circle in clockwise order starting with the first letter. Observe that, up to rotation, all semi-leader permutations are equivalent when put on a circle in this way. Hence, on the circle x and y are equivalent up to rotation. Let τ' swap letters j, k . Note that, as τ' does not swap adjacent letters, neither j nor k are adjacent to any of the same letters as they were before τ' . However, moving one letter can remove at most three adjacencies (2 containing the moved letter, 1 for the letters it moved between), so one of the four new adjacencies with j or k remains. Hence, after performing transposition τ' and moving one letter, the circle is not the same up to rotation, a contradiction. \square

Lemma 7. *There are exactly two **primary classes** in S_n for $n \geq 7$.*

Proof. Let $n \geq 7$. Define $\Pi_n = 123 \cdots n$ and $\Psi_n = 123 \cdots (n-3)(n-2)n(n-1)$. First of all, note that the permutations Π_n and Ψ_n are both primary permutations. In particular, for $n \geq 7$, no transposition takes Π or Ψ to a leader permutation because for any transposition τ , both $\tau \cdot \Pi$ and $\tau \cdot \Psi$ contain 123 patterns. Further, the parity of the number of inversions is even in Π and odd for Ψ , so they are in two distinct primary classes by Lemma 5. This proves that 2 is a lower bound on the number of primary classes.

To proceed, we use induction to show 2 is an upper bound on the number of primary classes.

The base cases of $n = 7, 8, 9, 10$ are verifiable by computer.

For the inductive step, we assume that Lemma 7 holds for an $n = k - 1$ with $k - 1 \geq 9$ and prove that Lemma 7 holds for $n = k$. To do so, we show that any primary permutation in S_k is equivalent to one of Π_k and Ψ_k . Let $\rho \in S_k$ be an arbitrary primary permutation; we will show that ρ is equivalent to either Π_k or Ψ_k depending on the parity of the number of inversions ρ contains. Because primary classes are nontrivial, we can find some 1234 or 3412 pattern in ρ . Take an arbitrary such pattern p . As $k \geq 10$, we can now find some number a inside of p such that $a \neq 1$, $a \neq k$, a is neither the first nor last number in p , and p does not contain a . Now, we look at the sub-standardization $\rho' \in S_{k-1}$ of ρ that excludes a . As a was not in p , ρ' still contains p and thus is part of a nontrivial class. As a was neither n nor the first number in ρ , ρ' does not start with n . Similarly, ρ' does not end with 1. We proceed with casework on whether ρ' is adjacent to a leader permutation.

Case 1: ρ' is a primary permutation. In this case, by the inductive hypothesis, ρ' is equivalent to either Π_{k-1} or Ψ_{k-1} . Hence, in ρ , we can use the equivalence relation to re-order the elements excluding a to form Π_{k-1} or Ψ_{k-1} . Call this new permutation after the re-ordering ρ_1 . Now, observe that because a was not the first number in ρ but both Π_{k-1}

and Ψ_{k-1} begin with 1, 1 is the first number in ρ_1 . Moreover, the second number in ρ_1 is either 2 or a , neither of which equals n .

Consider the sub-standardization ρ_1' created by excluding the 1 in ρ_1 . Note that ρ_1' neither starts with n nor ends with 1. Additionally, we attain a $1234 \cdots (k-4)$ pattern in ρ_1' by ignoring the a and the final two numbers of ρ_1' . Such a pattern clearly must contain a 1234 pattern and can never occur in a permutation adjacent to a leader permutation as $k-4 \geq 5$, so ρ_1' is primary. Since ρ_1' is primary, by our inductive hypothesis it is equivalent to either Ψ_{k-1} or Π_{k-1} . In either case rearranging the sub-permutation of ρ_1 that ρ_1' corresponds to yields that ρ_1 , and hence ρ , must be equivalent to either Ψ_k or Π_k , as desired.

Case 2: ρ' is not primary. In this case, consider some letter b in ρ such that

1. if the second letter of ρ is k , b is not the first letter in ρ
2. if the second to last letter of ρ is 1, b is not the last letter in ρ
3. if the last letter in ρ is 2, $b \neq 1$
4. if the first letter in ρ is $k-1$, $b \neq k$.

Next we show that, as long as $k \geq 11$, it is possible to select b satisfying conditions (1), (2), (3), and (4) as well as the conditions of 6 (which we will reiterate shortly). Let τ be the transposition taking ρ' to a leader permutation. Note that these conditions imply that the sub-standardization $\rho^* \in S_{k-1}$ attained by removing b from ρ neither ends in 1 nor starts with k . Further, note that if the if-conditions for (2) and (3) both take effect, this would force ρ to end with 12, and either the 1 or the 2 would be operated on by τ as no leader permutation ends with 12. Similarly, if the if-conditions for (1) and (4) both take effect, ρ would start with $(k-1)k$, and either $(k-1)$ or k would be operated on by τ . Also, it is easy to see that both letters operated by τ must be a part of any 1234 or 3412 pattern in ρ' , because the 3-letter subsequences of these patterns do not appear in leader permutations. Hence, we if we take b satisfying (1), (2), (3), and (4) and we require b to satisfy the conditions of Lemma 6 (meaning that b is not contained in the pattern p ; b is not operated on by τ ; $b \neq a$; $b \not\equiv a \pm 1 \pmod{k}$; and if τ swaps two letters separated by only one letter, b is not that letter) then in total we exclude at most 10 elements from the possible choices (in particular, (1), (2), (3), and (4) exclude at most two elements not already excluded as part of τ). This means that as long as $k \geq 11$, we can find such a b in ρ .

Then, by Lemma 6, the permutation pattern formed by removing b from ρ must be a primary permutation, so this case reduces to Case 1.

These two cases complete the induction, establishing that the number of primary classes is exactly 2. □

5 Putting the proof together

We now finish proving the theorem by putting the lemmas proved in the previous sections together.

Theorem (Theorem 1 restated). *For $n \geq 7$, the number of nontrivial equivalence classes of S_n under the $\{1234, 3412\}$ -equivalence is $\frac{n^3+6n^2-55n+54}{6}$.*

Proof. First of all, by Lemma 1, we see that it is sufficient to show that there are exactly $n+1$ classes of permutations in S_n not beginning with n or ending with 1. Every such permutation is in exactly one of two categories, the set of permutations adjacent to leader permutations and primary permutations. By Proposition 1 there are $n-1$ classes of permutations adjacent to a leader permutation, and by Lemma 7 there are exactly two primary classes. Thus, in total we find that permutations not beginning in n or ending in 1 fit into $n-1+2=n+1$ classes, as desired. By Lemma 1, this completes the proof. \square

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References

- [1] M. H. Albert, M. Elder, A. Rechnitzer, P. Westcott, and M. Zabrocki, On the Stanley–Wilf limit of 4231-avoiding permutations and a conjecture of Arratia, *Adv. in Appl. Math.* **36**(2) (2006), 96–105.
- [2] M. Bóna, New records in Stanley–Wilf limits, *European J. Combin* **28**(1) (2007), 75–85.
- [3] M. Bóna, The limit of a Stanley–Wilf sequence is not always rational, and layered patterns beat monotone patterns, preprint, 2004. Available at <https://arxiv.org/abs/math/0403502>.
- [4] A. Claesson, Generalized pattern avoidance, *European J. Combin* **22**(7) (2001), 961–971.
- [5] V. Fazel-Rezai, Equivalence Classes of Permutations Modulo Replacements Between 123 and Two-Integer Patterns, *Electron. J. Combin.* **21**(2) (2014), P2–47.
- [6] The International Conference on Permutation Patterns, 2020.
- [7] S. Kitaev, *Patterns in permutations and words*, Springer Science & Business Media, 2011.

- [8] D. Knuth, Permutations, matrices, and generalized Young tableaux, *Pacific J. Math.* **34**(3) (1970), 709–727.
- [9] W. Kuszmaul, Counting Permutations Modulo Pattern-Replacement Equivalences for Three-Letter Patterns, *Electron. J. Combin.* **20**(4) (2013), 10.
- [10] W. Kuszmaul and Z. Zhou, New results on families of pattern-replacement equivalences, *Discrete Math.* **343**(7) (2020), 111878.
- [11] J. B. Lewis, Pattern avoidance for alternating permutations and Young tableaux, *J. Combin. Theory Ser. A* **118**(4) (2011), 1436–1450.
- [12] S. Linton, J. Propp, T. Roby, and J. West, Equivalence classes of permutations under various relations generated by constrained transpositions, *J. Integer Seq.* **15**(2) (2012), 3.
- [13] M. Ma, New Results on Pattern-Replacement Equivalences: Generalizing a Classical Theorem and Revising a Recent Conjecture, 2017. Available at <https://math.mit.edu/research/highschool/primes/materials/2017/Ma.pdf>.
- [14] A. Marcus and G. Tardos, Excluded permutation matrices and the Stanley–Wilf conjecture, *J. Combin. Theory Ser. A* **107**(1) (2004), 153–160.
- [15] J.-C. Novelli and A. Schilling, The forgotten monoid, *RIMS Kôkyûroku Bessatsu* **8** (2008), 71–83.
- [16] A. Pierrot, D. Rossin, and J. West, Adjacent transformations in permutations, In *DMTCS Proceedings Vol. AO, 23rd International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2011)*, pp. 765–776, 2011.
- [17] R. Simion and F. W. Schmidt, Restricted permutations, *European J. Combin* **6**(4) (1985), 383–406.
- [18] N. J. A. Sloane et al., The On-line Encyclopedia of Integer Sequences, 2020.
- [19] R. P. Stanley, An equivalence relation on the symmetric group and multiplicity-free flag h-vectors, *J. Comb.* **3**(3) (2012), 277–298.

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(Concerned with sequence [A330395](#).)