

ON EMERGENT GEOMETRY OF THE GROMOV-WITTEN THEORY OF QUINTIC CALABI-YAU THREEFOLD

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Dedicated to the memory of Professor Boris Dubrovin

ABSTRACT. We carry out the explicit computations that are used to write down the integrable hierarchy associated with the quintic Calabi-Yau threefold. We also do the calculations for the geometric structures emerging in the Gromov-Witten theory of the quintic, such as the Frobenius manifold structure and the special Kähler structure.

1. INTRODUCTION

Gromov-Witten theory is the mathematical theory for the physical theory of topological sigma models coupled to two-dimensional topological gravity. The observables in a topological sigma models, called the *primary observable*, correspond to the cohomological classes of the underlying symplectic manifold, and they form a finite dimensional space called the *small phase spaces*. The coupling with the two-dimensional topological gravity associates to each primary observable a sequence of observables, called their *gravitational descendants*, which form an infinite-dimensional space called the *big phase space*. One assigns a coupling constant to each observable, so one gets infinitely many formal variables as linear coordinates on the big phase space. The goal of the Gromov-Witten theory is to study the partition functions and the free energy of the topological sigma model coupled to the two-dimensional topological gravity as formal power series in the infinitely many coupling constants.

There are two major paradigms in the study of Gromov-Witten theory. The first is the Witten Conjecture/Kontsevich Theorem [34, 26] which relates the two-dimensional topological gravity to the KdV hierarchy. Since it involves integrable hierarchy, I will refer to it as the *integrable hierarchy paradigm*. The second is the mirror symmetry [2] which identifies the Gromov-Witten theory of the quintic Calabi-Yau threefold with another more computable theory. I will refer to it as the

mirror symmetry paradigm. For the past thirty years, Gromov-Witten theory has been developed to provide the justifications and generalizations of these discoveries by physicists by mathematically rigorous methods.

More recently, the author has worked on a program to unify these two paradigms. This of course has to go in two directions: (a) Understand integrable hierarchies from the point of view of mirror symmetry; (b) Understand mirror symmetry from the point of view of integrable hierarchy. At first sight, there are some manifest differences between the two paradigms. The integrable hierarchy paradigm predicts that the partition function of the Gromov-Witten invariants of each symplectic manifold is the tau-function of a suitable integrable hierarchy, specified by suitable constraints. It naturally involves infinitely many variables and infinite-dimensional Lie algebras that describes the symmetries of the relevant integrable hierarchy. On the contrary, the hallmark example of the mirror symmetry of the quintic Calabi-Yau three-folds focuses on the computation of the free energy function of the Gromov-Witten invariants restricted to a one-dimensional subspace of the small phase space. But the key feature of mirror symmetry is that it predicts the equivalence with a different theory, called its *mirror theory*. Furthermore, in the physics literature the differential geometry of moduli spaces play an important role.

Therefore, a first step towards unification consists of three problems: Problem 1. Provide a construction of a mirror theory for each integrable hierarchy; Problem 2. Find the integrable hierarchy associated with the quintic Calabi-Yau threefold. Problem 3. Discover the geometric objects hidden behind the integrable hierarchies.

One of the purpose of this paper is to report our results on Problem 2 and Problem 3. However, our approach used in this work are based on our understandings obtained in our work on Problem 1. We will postpone a detailed review of some progress there in Section 9, here we will just mention that in our earlier work on Problem 1 we have borrowed some ideas from statistical physics and introduced the notion of emergent geometry.

It turns out that the idea of using statistical physics becomes the crucial point of departure for our work on Problem 2 and Problem 3 in this paper. More generally, one can consider the problem of finding the integrable hierarchy associated to the Gromov-Witten theory of any compact symplectic manifold.

Like ordinary mean field theory, the mean field theory of Gromov-Witten theory discovered by Dijkgraaf and Witten [4, 34] reduces a problem with an infinite degree of freedom to a problem with only

finite degree of freedom. More precisely, they introduced finitely many order parameters that encode all the information in genus zero. The dependence of these order parameters on the infinitely many coupling constants can be determined in two different ways. First they are the critical points of a Landau-Ginzburg potential functions. Secondly, they satisfy a sequence of evolution equations in the coupling constants. Furthermore, the integrable hierarchy is expected to be generalized to arbitrary genus.

We reinterpret Dubrovin's theory of Frobenius manifolds as part of emergent geometry of GW theory. The original goal of this theory is to reconstruct the whole theory from the genus zero part of the theory restricted to the small phase space. We reverse the direction. We start with the GW theory of a symplectic manifold and search for geometric structures that naturally emerge. Frobenius manifold structure is then an unavoidable candidate.

The rest of this paper consist of sections of two different natures. In some sections we review general theory reinterpreted from an emergent point of view, and in some sections we focus on the detailed computations for the quintic. More precisely, in Section 2 we review the mean field theory as developed by Dijkgraaf-Witten [4] and Witten [34]. The results for higher genera will be reviewed in Section 3 where we also present an operator formalism to compute the n -point functions in arbitrary genera. The corresponding concrete computations for the quintic are presented in Section 4 for genus zero and in Section 5 respectively. As our key results for these two Sections, we write down the Landau-Ginzburg potential function and the integrable hierarchy for the quintic. We briefly review some aspects of the Frobenius manifold theory useful for our purpose in Section 6, then we do the concrete computations for the quintic in Section 7. In Section 8 we discuss the emergent special geometry for the quintic. We make some conclusions and present some speculations for further investigations in the final Section 9.

2. MEAN FIELD THEORY OF GROMOV-WITTEN THEORY

In this Section we summarize the mean field theory for Gromov-Witten theory developed by Dijkgraaf-Witten [4] and Witten [34]. We emphasize that we want to understand their results as examples of emergent phenomena from the point of view of statistical physics. In the beginning we are working on an infinite-dimensional phase space, dealing with a formal power series with infinitely many formal variables

called the coupling constants. In the end we know that we need to only work with finitely many functions called the order parameters.

The dependence of the order parameters on the coupling constants can be determined in two different ways. First, the order parameters are the critical points of a Landau-Ginzburg potential function. Secondly, the order parameters are governed by a system of infinitely many evolutive equations, one for each coupling constant.

2.1. The Gromov-Witten invariants as correlators. Let M be a compact symplectic manifold of dimension $2m$. For simplicity, we will only consider its Gromov-Witten invariants that involve cohomology classes of even degrees. So the small phase space is

$$(1) \quad H^{ev}(M) := \bigoplus_{k=0}^m H^{2k}(M).$$

For $\alpha \in H^{2k}(M)$, set $\deg \alpha := k$. Fix a basis $\{\mathcal{O}_j\}_{j=0}^r$ of $H^{ev}(M)$ such that $\mathcal{O}_0 = 1$.

The big phase space is the space $H^{ev}(M)[[z]]$ of formal power series with coefficients in $H^{ev}(M)$. For $\omega \in H^{ev}(M)$, rewrite ωz^n as $\tau_n(\alpha)$. It is called the n -th gravitational descendant of ω .

For classes $\omega_1, \dots, \omega_n \in H^{ev}(M)$, and integers $a_1, \dots, a_n \geq 0$, the Gromov-Witten invariants are defined by

$$(2) \quad \langle \tau_{a_1}(\omega_1), \dots, \tau_{a_n}(\omega_n) \rangle_{g,n;\beta} := \int_{[\overline{\mathcal{M}}_{g,n}(M;\beta)]^{virt}} \omega_1 \psi_1^{a_1} \cdots \omega_n \psi_n^{a_n},$$

where $[\overline{\mathcal{M}}_{g,n}(M;\beta)]^{virt}$ is the virtual fundamental class of stable maps of genus g with n marked points in the homology class $\beta \in H_2(M; \mathbb{Z})$.

The degree of $[\overline{\mathcal{M}}_{g,n}(M;\beta)]^{virt}$ is the virtual dimension of $\overline{\mathcal{M}}_{g,n}(M;\beta)$, given by the formula:

$$(3) \quad (m-3)(1-g) + n + \int_{\beta} c_1(M).$$

So the correlator $\langle \tau_{a_1}(\omega_1), \dots, \tau_{a_n}(\omega_n) \rangle_{g,n;\beta}$ vanishes unless the following *selection rule* is satisfied:

$$(4) \quad \sum_{i=1}^n (\deg \omega_i + a_i) = (m-3)(1-g) + n + \int_{\beta} c_1(M).$$

2.2. The free energy, the partition function, and the n -point correlators. Denote by t_n^α the coupling constant associated with $\tau_n(\mathcal{O}_\alpha)$.

The genus g free energy of the Gromov-Witten theory of M is defined by:

$$(5) \quad F_g = \sum_{n \geq 1} \sum_{\alpha_1, \dots, \alpha_n = 0}^r \sum_{\beta \in H_2(M; \mathbb{Z})} \frac{\prod_{j=1}^n t_{a_j}^{\alpha_j}}{n!} q^\beta \langle \tau_{a_1}(\mathcal{O}_{\alpha_1}) \cdots \tau_{a_n}(\mathcal{O}_{\alpha_n}) \rangle_{g, n; \beta}.$$

For $g = 0$ and $\beta = 0$, the summation starts at $n = 3$. The total free energy is defined by:

$$(6) \quad F_g := \sum_{g=0}^{\infty} \lambda^{2g-2} F_g.$$

The free energy contains all the information in the theory. From it one can define the n -point correlators in genus g :

$$(7) \quad \langle \langle \tau_{a_1}(\mathcal{O}_{\alpha_1}) \cdots \tau_{a_n}(\mathcal{O}_{\alpha_n}) \rangle \rangle_g := \frac{\partial^n F_g}{\partial t_{a_1}^{\alpha_1} \cdots \partial t_{a_n}^{\alpha_n}}.$$

The partition function of the Gromov-Witten theory is defined by

$$(8) \quad Z = \exp F.$$

The ultimate goal of Gromov-Witten theory is of course to have the ability to compute the free energy or the partition function or the n -point correlators in closed forms. Because they all involve infinitely many formal variables, this does not seem to be possible. As mentioned in the Introduction, Dijkgraaf and Witten [4] observed that one can borrow ideas from mean field theory in statistical physics to make this possible.

2.3. Universal relations among correlators. The reason that the approach of Dijkgraaf and Witten works is because of the topological nature of the Gromov-Witten theory, in other words, it is a topologically twisted $N = 2$ superconformal field theory coupled with 2d topological gravity. Such a theory is based on the cohomology theory of the Deligne-Mumford moduli spaces $\overline{\mathcal{M}}_{g,n}$ of algebraic curves. There are some universal relations among classes on $\overline{\mathcal{M}}_{g,n}$ called tautological relations (see e.g. [32] for references, see also [28]). They lead to some universal relations among the correlators in Gromov-Witten theory. The following three equations hold in all genera. The first is the *puncture equation*:

$$(9) \quad \begin{aligned} & \langle 1 \tau_{a_1}(\mathcal{O}_{\alpha_1}) \cdots \tau_{a_n}(\mathcal{O}_{\alpha_n}) \rangle_{g, n; \beta} \\ &= \sum_{i=1}^n \langle \tau_{a_1}(\mathcal{O}_{\alpha_1}), \dots, \tau_{a_{i-1}}(\mathcal{O}_{\alpha_i}), \dots, \tau_{a_n}(\mathcal{O}_{\alpha_n}) \rangle_{g, n; \beta}, \end{aligned}$$

with the exceptional case:

$$(10) \quad \langle 1\mathcal{O}_{\alpha_1}\mathcal{O}_{\alpha_2}\rangle_{0,2;0} = \langle \mathcal{O}_{\alpha_1}\mathcal{O}_{\alpha_2}\rangle,$$

where the right-hand side is the Poincaré pairing of the two classes \mathcal{O}_{α_1} with \mathcal{O}_{α_2} . The second is the *dilaton equation*:

$$(11) \quad \begin{aligned} & \langle \tau_1(1), \tau_{a_1}(\alpha_1), \dots, \tau_{a_n}(\alpha_n) \rangle_{g,n;\beta} \\ & = (2g - 2 + n) \cdot \langle \tau_{a_1}(\alpha_1), \dots, \tau_{a_{i-1}}(\alpha_i), \dots, \tau_{a_n}(\alpha_n) \rangle_{g,n;\beta}, \end{aligned}$$

with the exceptional case:

$$(12) \quad \langle \tau_1(1) \rangle_{1,1;0} = \frac{1}{24}\chi(M).$$

In terms of the free energy, these equations can be rewritten as follows:

$$(13) \quad \frac{\partial F_g}{\partial t_0^0} = \sum_{n=0}^{\infty} \sum_{\alpha=0}^r t_{n+1}^{\alpha} \frac{\partial F_g}{\partial t_n^{\alpha}} + \delta_{g,0} \frac{1}{2} \eta_{\alpha\beta} t_0^{\alpha} t_0^{\beta},$$

$$(14) \quad \frac{\partial F_g}{\partial t_1^0} = \sum_{n=0}^{\infty} \sum_{\alpha=0}^r t_n^{\alpha} \frac{\partial F_g}{\partial t_n^{\alpha}} + (2g - 2)F_g + \delta_{g,1} \frac{1}{24}\chi(M),$$

where $\eta_{\alpha\beta} = \langle \mathcal{O}_{\alpha}, \mathcal{O}_{\beta} \rangle_{0,2;0}$. The third is the *divisor equation*:

$$(15) \quad \begin{aligned} & \langle D\tau_{a_1}(\mathcal{O}_{\alpha_1}) \dots \tau_{a_n}(\mathcal{O}_{\alpha_n}) \rangle_{g,n;\beta} \\ & = \langle D, \beta \rangle \cdot \langle \tau_{a_1}(\alpha_1) \dots \tau_{a_n}(\alpha_n) \rangle_{g,n;\beta} \\ & + \sum_{i=1}^n \langle \tau_{a_1}(\mathcal{O}_{\alpha_1}), \dots, \tau_{a_{i-1}}(\mathcal{O}_{\alpha_i}D), \dots, \tau_{a_n}(\mathcal{O}_{\alpha_n}) \rangle_{g,n;\beta}. \end{aligned}$$

2.4. Topological recursion relations. The *topological recursion relation* (TRR) in genus zero is the following equation:

$$(16) \quad \frac{\partial^3 F_0}{\partial t_a^{\alpha} \partial t_b^{\beta} \partial t_c^{\gamma}} = \frac{\partial^2 F_0}{\partial t_{a-1}^{\alpha} \partial t^{\mu}} \eta^{\mu\nu} \frac{\partial^3 F_0}{\partial t^{\nu} \partial t_b^{\beta} \partial t_c^{\gamma}}.$$

Witten [34] has shown that it implies the generalized WDVV equations:

$$(17) \quad \frac{\partial^3 F_0}{\partial t_a^{\alpha} \partial t_b^{\beta} \partial t^{\mu}} \eta^{\mu\nu} \frac{\partial^3 F_0}{\partial t^{\nu} \partial t_c^{\gamma} \partial t_d^{\delta}} = \frac{\partial^3 F_0}{\partial t_a^{\alpha} \partial t_c^{\gamma} \partial t^{\mu}} \eta^{\mu\nu} \frac{\partial^3 F_0}{\partial t^{\nu} \partial t_b^{\beta} \partial t_d^{\delta}}.$$

When $a = b = c = d = 0$, this reduces to the WDVV equations:

$$(18) \quad \frac{\partial^3 F_0}{\partial t^{\alpha} \partial t^{\beta} \partial t^{\mu}} \eta^{\mu\nu} \frac{\partial^3 F_0}{\partial t^{\nu} \partial t^{\gamma} \partial t^{\delta}} = \frac{\partial^3 F_0}{\partial t^{\alpha} \partial t^{\gamma} \partial t^{\mu}} \eta^{\mu\nu} \frac{\partial^3 F_0}{\partial t^{\nu} \partial t^{\beta} \partial t^{\delta}}.$$

In genus one, Witten proposed the following TRR [34, (3.48)]:

$$(19) \quad \frac{\partial F_1}{\partial t_n^{\alpha}} = \frac{\partial^2 F_1}{\partial t_{n-1}^{\alpha} \partial t^{\beta}} \eta^{\beta\gamma} \frac{\partial F_1}{\partial t^{\gamma}} + \frac{1}{24} \frac{\partial^3 F_1}{\partial t_{n-1}^{\alpha} \partial t^{\beta} \partial t^{\gamma}}.$$

For its mathematical proof, see Getzler [20]. See also [21] for Getzler's TRRs in genus two.

2.5. Quantum cohomology. The WDVV equations have the following algebraic interpretation. For $\mathbf{t} \in H^{ev}(M)$, define

$$(20) \quad \frac{\partial}{\partial t^\alpha} \circ_{\mathbf{t}} \frac{\partial}{\partial t^\beta} := c_{\alpha\beta}^\gamma(\mathbf{t}) \frac{\partial}{\partial t^\gamma},$$

where the coefficients $c_{\alpha\beta}^\gamma(\mathbf{t})$ are defined by:

$$(21) \quad c_{\alpha\beta}^\gamma(\mathbf{t}) := \frac{\partial^3 F_0(\mathbf{t})}{\partial t^\alpha \partial t^\beta \partial t^\delta} \eta^{\delta\gamma}.$$

Then $\circ_{\mathbf{t}}$ is a commutative multiplication on $H^e(M)$. By WDVV equations, it is also associative. Furthermore, because

$$(22) \quad \frac{\partial^3 F_0(\mathbf{t})}{\partial t^0 \partial t^\alpha \partial t^\beta} = \eta_{\alpha\beta},$$

$\frac{\partial}{\partial t^0}$ is a unit for all $\circ_{\mathbf{t}}$, i.e.,

$$(23) \quad \frac{\partial}{\partial t^0} \circ_{\mathbf{t}} \frac{\partial}{\partial t^\alpha} = \frac{\partial}{\partial t^\alpha}.$$

For simplicity of notations we will often write \circ for $\circ_{\mathbf{t}}$.

2.6. Order parameters in GW theory. Following Dijkgraaf and Witten [4], choose the order parameters in GW theory to be:

$$(24) \quad u_\alpha := \frac{\partial^2 F_0}{\partial t^1 \partial t^\alpha} = \langle\langle \mathcal{O}_0 \mathcal{O}_\alpha \rangle\rangle_0.$$

The genus zero free energy F_0 consists of two parts: The classical part $F_{classical}$ consists of 3-point correlators

$$(25) \quad \begin{aligned} F_{classical} &= \sum_{\alpha_1, \alpha_2, \alpha_3=0}^r \frac{t^{\alpha_1} t^{\alpha_2} t^{\alpha_3}}{3!} \langle \mathcal{O}(\alpha_1), \mathcal{O}(\alpha_2), \mathcal{O}(\alpha_3) \rangle_{0,3;0} \\ &= \sum_{\alpha_1, \alpha_2, \alpha_3=0}^r \frac{t^{\alpha_1} t^{\alpha_2} t^{\alpha_3}}{3!} \int_M \mathcal{O}_{\alpha_1} \mathcal{O}_{\alpha_2} \mathcal{O}_{\alpha_3}, \end{aligned}$$

and $F_{quantum}$ consists of n -point correlators for $n > 3$. And so

$$(26) \quad u_\alpha = \eta_{\alpha\beta} t^\beta + \dots,$$

where \dots involves correlators of the form:

$$\langle 1, \mathcal{O}_\alpha, \tau_{a_1}(\mathcal{O}_{\alpha_1}), \dots, \tau_{a_{n-2}}(\mathcal{O}_{\alpha_{n-2}}) \rangle_{0,n;\beta}$$

for $n > 3$. Note by the puncture equation (9),

$$\langle 1, \mathcal{O}_\alpha, \mathcal{O}_{\alpha_1}, \dots, \mathcal{O}_{\alpha_{n-2}} \rangle_0$$

for $n > 3$. Therefore, on the small phase space, i.e., after setting $t_n^\alpha = 0$ for all $n > 0$ and all $\alpha = 0, \dots, r$, we have:

$$(27) \quad u_\alpha = \eta_{\alpha\beta} t^\beta.$$

Equivalently, on the small phase space, t^β is equal to $\eta^{\alpha\beta} u_\alpha$.

2.7. Constitutive relations. Let us now recall a crucial observation of Dijkgraaf and Witten [4]. On the small phase space $\frac{\partial^2 F_0}{\partial t_a^\alpha \partial t_b^\beta}$ is a formal power series in $\{t^\alpha\}_{\alpha=0}^r$, so one can write:

$$(28) \quad \left. \frac{\partial^2 F_0}{\partial t_a^\alpha \partial t_b^\beta} \right|_{t_n^\gamma=0, n>0, \gamma=0, \dots, r} = R_{\alpha, a; \beta, b}(t^0, \dots, t^r).$$

Then Dijkgraaf and Witten [4] showed that on the big phase space

$$(29) \quad \frac{\partial^2 F_0}{\partial t_a^\alpha \partial t_b^\beta} = R_{\alpha, a; \beta, b}(u^0, \dots, u^r),$$

where $u^\alpha = \eta^{\alpha\beta} u_\beta$. They call these the *constitutive relations*. Their proof is based on the TRR in genus zero (16).

2.8. Mean field theory of GW theory. Let us recall how Dijkgraaf and Witten derived the Landau-Ginzburg equations for the order parameters, based on the puncture equation (13) and the constitutive relation (29) derived by TRR in genus zero (16).

Take $\frac{\partial}{\partial t^\alpha}$ on both sides of (13) to get:

$$\frac{\partial^2 F_0}{\partial t^0 \partial t^\alpha} = \sum_{n=0}^{\infty} \sum_{\beta=0}^r t_{n+1}^\beta \frac{\partial^2 F_0}{\partial t^\alpha \partial t_n^\beta} + \eta^{\alpha\beta} t^\beta,$$

now plug in the constitutive relations (29) to get:

$$(30) \quad u_\alpha = t_\alpha + \sum_{n=0}^{\infty} \sum_{\beta=0}^r t_{n+1}^\beta R_{\alpha, 0; \beta, n}(u_0, \dots, u_r),$$

for $\alpha = 0, 1, \dots, r$. These are the *Landau-Ginzburg equations* for the order parameters u_0, \dots, u_r . When $R_{\alpha, 0; \beta, n}$'s have been computed, one can solve for u_α as formal power series in $\{t_n^\alpha\}$.

2.9. The Landau-Ginzburg potential. This line of ideas were further developed in Witten [34]. Recall that on the small phase space we have:

$$R_{\alpha, 0; \beta, n}(t^0, \dots, t^r) = \left. \frac{\partial^2 F_0}{\partial t^\alpha \partial t_n^\beta} \right|_{t_n^\gamma=0, n>0, \gamma=0, \dots, r},$$

Therefore, if we write

$$(31) \quad R_{\beta,n}(t^0, \dots, t^r) = \frac{\partial F_0}{\partial t_n^\beta} \Big|_{t_n^\gamma=0, n>0, \gamma=0, \dots, r}.$$

Then we have the following relation:

$$\begin{aligned} R_{\alpha,0;\beta,n}(t_0, \dots, t^R) &= \frac{\partial}{\partial t^\alpha} \left(\frac{\partial F_0}{\partial t_n^\beta} \Big|_{t_n^\gamma=0, n>0, \gamma=0, \dots, r} \right) \\ &= \frac{\partial}{\partial t^\alpha} R_{\beta;n}(t^0, \dots, t^n). \end{aligned}$$

Therefore,

$$(32) \quad R_{\alpha,0;\beta,n}(u^0, \dots, u^r) = \frac{\partial}{\partial u^\alpha} R_{\beta;n}(u^0, \dots, u^n).$$

Plug this into the Landau-Ginzburg equations (30):

$$(33) \quad u_\alpha = t_\alpha + \sum_{n=0}^{\infty} \sum_{\beta=0}^r t_{n+1}^\beta \frac{\partial}{\partial u^\alpha} R_{\beta;n}(u^0, \dots, u^n).$$

Therefore, if one sets (cf. Witten [34, (3.40)]):

$$(34) \quad W = -\frac{1}{2} U_\alpha U^\alpha + t^\beta U_\beta + \sum_{n=0}^{\infty} \sum_{\beta=0}^r t_{n+1}^\beta R_{\beta;n}(U^0, \dots, U^r),$$

then the equations for the critical point of W are exactly the Landau-Ginzburg equations (30).

Remark 2.1. If we set

$$R_{\beta,-1}(U^0, \dots, U^r) = U_\beta,$$

and make the dilaton shift:

$$(35) \quad \tilde{t}_n^\beta = t_n^\beta - \delta_{\beta,0} \delta_{n,1},$$

then the Landau-Ginzburg potential can be written in a more compact form:

$$(36) \quad W = \sum_{n=-1}^{\infty} \sum_{\beta=0}^r \tilde{t}_{n+1}^\beta R_{\beta;n}(U^0, \dots, U^r).$$

Therefore, if one sets

$$(37) \quad R_\beta(U; z) := \sum_{n=-1}^{\infty} R_{\beta,n}(U) z^{-n-1}$$

and introduces the *source operators*:

$$(38) \quad J^\beta(z) := \sum_{n=0}^{\infty} \tilde{t}_n^\beta z^n,$$

then one rewrite (34) as a residue:

$$(39) \quad W(U) = \sum_{\beta=0}^r \operatorname{res}_{z=0} (J^\beta(z) R_\beta(U; z) \frac{dz}{z}).$$

2.10. Integrable hierarchy for genus zero GW invariants. The next observation in Witten [34] is the emergence of an integrable hierarchy for genus zero GW invariants:

$$\frac{\partial}{\partial t^\alpha} u_\beta = \frac{\partial}{\partial t^\alpha} \left(\frac{\partial^2 F_0}{\partial t^0 \partial t^\beta} \right) = \frac{\partial}{\partial t^0} \left(\frac{\partial^2 F_0}{\partial t^\alpha \partial t_n^\beta} \right).$$

Therefore, by the constitutive relations (29) he obtained a sequence of equations:

$$(40) \quad \frac{\partial u_\beta}{\partial t_n^\alpha} = \frac{\partial}{\partial t^0} R_{\alpha,0;\beta,n}(u^0, \dots, u^r).$$

Witten [34] gave the following Hamiltonian formulation for this integrable hierarchy. Regard u_α as functions of $x = t^0$, and define the following Poisson bracket on the loop space of the space of order parameters:

$$(41) \quad \{u_\alpha(x), u_\beta(y)\} = \eta_{\alpha\beta} \partial_x \delta(x - y).$$

Then (40) is the system of Hamiltonian of motion

$$(42) \quad \frac{\partial u_\alpha}{\partial t_n^\beta} = \{u_\alpha, H_{\beta,n}\},$$

where $H_{\beta,n}$ is the Hamiltonian functional

$$(43) \quad H_{\beta,n} = \int dx R_{\beta,n}(u^0, \dots, u^r).$$

3. RENORMALIZATION THEORY, NEW COORDINATES ON THE BIG PHASE SPACE AND OPERATOR FORMALISM OF GROMOV-WITTEN THEORY

We will present in this Section a method that computes the n -point correlation functions on the big phase space based on some works of Eguchi and his collaborators. It is a continuation of the development of the mean field theory of GW theory to general n -point functions in arbitrary genera. We will use an operator formalism that leads to an

interpretation of the partition function as an element in a bosonic Fock space.

3.1. Mean field theory for higher genera. Dijkgraaf and Witten computed the genus one free energy of the two-dimensional topological gravity by TRR in genus one. In general, Witten [34] claimed that if one sets

$$(44) \quad M_{\alpha\beta} = \frac{\partial^3 F_0}{\partial t^0 \partial t^\alpha \partial t^\beta} = \frac{\partial}{\partial t^0} R_{\alpha,\beta}(u_0, \dots, u_r),$$

then

$$(45) \quad F_1 = \frac{1}{24} \log \det(M_{\alpha\beta}) + E(u_0, \dots, u_r)$$

for some function E depending only on u_α . See Eguchi-Getzler-Xiong [14] for a proof.

For higher genera, Witten [34] introduced

$$(46) \quad u_{\alpha,n} := \frac{\partial^n}{\partial (t^0)^n} u_\alpha,$$

and set $\deg u_{\alpha,n} = n$, then he conjectured that there are functions $R_{\alpha,m;\beta,n}^{(g)}$ of degree $2g$ in $\{u_{\alpha,n}\}$ such that

$$(47) \quad \frac{\partial^2 F_g}{\partial t_m^\alpha \partial t_n^\beta} = R_{\alpha,m;\beta,n}^{(g)}(\{u_{\alpha,n}\}).$$

This conjecture implies that one can associate an integrable hierarchy of differential equations associated to every compact symplectic manifold via the GW theory. One simply sets

$$(48) \quad U_\alpha = \sum_{g=0}^{\infty} \lambda^{2g} u_\alpha^{(g)},$$

where $u_\alpha^{(g)}$ is defined by:

$$(49) \quad u_\alpha^{(g)} = \frac{\partial^2 F_g}{\partial t^0 \partial t^\alpha},$$

then one gets a system of equations:

$$(50) \quad \frac{\partial U_\alpha}{\partial t_n^\beta} = \partial_x \mathcal{R}_{\alpha,0;\beta,n}$$

where

$$(51) \quad \mathcal{R}_{\alpha,m;\beta,n} = \sum_{g=0}^{\infty} \lambda^{2g} R_{\alpha,m;\beta,n}^{(g)}.$$

3.2. Renormalized coupling constants as new coordinate systems on the big phase space. In the case of two-dimensional topological gravity, Itzykson and Zuber [24] noted that the following ansatz is compatible with the KdV hierarchy:

$$(52) \quad F_1 = \frac{1}{24} \log \frac{1}{1 - I_1},$$

$$(53) \quad F_g = \sum_{\sum_{2 \leq k \leq 3g-2} (k-1)l_k = 3g-3} \frac{\langle \tau_2^{l_2} \tau_3^{l_3} \cdots \tau_{3g-2}^{l_{3g-2}} \rangle_g I_2^{l_2} I_3^{l_3} \cdots I_{3g-2}^{l_{3g-2}}}{(1 - I_1)^{2(g-1)} l_2! l_3! \cdots l_{3g-2}!},$$

where I_n are defined by:

$$(54) \quad I_k = \sum_{p=0}^{\infty} t_{k+p} \frac{u_0^p}{p!},$$

and u_0 is determined by:

$$(55) \quad u_0 = \sum_{p=0}^{\infty} t_p \frac{u_0^p}{p!}.$$

Eguchi, Yamada and Yang [15] showed that there exist formulas of the form:

$$(56) \quad \begin{aligned} F_g &= \sum_{\sum_k (k-1)l_k = 3g-3} a_{l_2 \dots l_{3g-2}} \frac{u_2^{l_2} \cdots u_{3g-2}^{l_{3g-2}}}{u_1^{2(1-g) + \sum k l_k}} \\ &= \sum_{\sum_k (k-1)l_k = 3g-3} b_{l_2 \dots l_{3g-2}} \frac{I_2^{l_2} \cdots I_{3g-2}^{l_{3g-2}}}{u_1^{2(1-g) + \sum k l_k}} \end{aligned}$$

for $g > 1$, where $u_k = \frac{\partial^k u_0}{\partial x^k}$.

They also discussed the $O(N)$ vector model [30] and noted results similar to that of Itzykson-Zuber in the case of 2D topological gravity also hold. This was further studied by the author in [37]. The new interpretation in that work is that starting with an action:

$$(57) \quad S = -\frac{1}{2}x^2 + \sum_{n \geq 1} t_{n-1} \frac{x^n}{n!},$$

one can perform a sequence of renormalizations of the coupling constants to reach in the limit:

$$(58) \quad S = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} (I_k + \delta_{k,1}) I_0^{k+1} - \frac{1}{2}(x - I_0)^2 + \sum_{n=1}^{\infty} I_n \frac{(x - I_0)^{n+1}}{(n+1)!}.$$

So the series $\{I_k\}$ can be interpreted as renormalized coupling constants.

The Landau-Ginzburg equation (55) has been explicitly solved in [37] using Lagrange inversion formula:

$$(59) \quad u_0 = I_0 = \sum_{k=1}^{\infty} \frac{1}{k} \sum_{p_1+\dots+p_k=k-1} \frac{t_{p_1}}{p_1!} \dots \frac{t_{p_k}}{p_k!}.$$

Plugging this into (54), one can express each I_k as a formal power series in $\{t_n\}_{n \geq 0}$. Conversely, we also showed in [37] that

$$(60) \quad t_k = \sum_{n=0}^{\infty} \frac{(-1)^n I_0^n}{n!} I_{n+k}.$$

Therefore, the renormalized coupling constants can be used as new coordinates on the big phase space in this case. It follows that one can express the Virasoro operators in the case of 2D topological gravity in terms of these new coordinates. In a joint work with Qisheng Zhang [33], we have shown that one can use such expressions to prove the Itzykson-Zuber Ansatz and to find effective algorithms to compute F_g in terms of I_k 's.

3.3. Jet variables as new coordinate systems on the big phase space. In general, Eguchi and Xiong [16] conjectured that: If one sets

$$(61) \quad u_{\alpha_1 \alpha_2 \dots \alpha_n} = \frac{\partial^{n+1} F_0}{\partial t^0 \partial t^{\alpha_1} \dots \partial t^{\alpha_n}},$$

then for $g \geq 1$,

$$(62) \quad F_g = F_g(u_\alpha, u_{\alpha_1 \alpha_2}, \dots, u_{\alpha_1 \alpha_2 \dots \alpha_{3g-1}}), \quad g \geq 1.$$

Note by the constitutive relations and the integrable hierarchy (40),

$$\begin{aligned} u_{\alpha_1 \alpha_2 \dots \alpha_n} &= \frac{\partial}{\partial t^{\alpha_n}} \dots \frac{\partial}{\partial t^{\alpha_2}} u_{\alpha_1} \\ &= \frac{\partial}{\partial t^{\alpha_n}} \dots \frac{\partial}{\partial t^{\alpha_3}} (R_{\alpha_1, 0; \alpha_2, 0}(u))' \end{aligned}$$

$u_{\alpha_1 \alpha_2 \dots \alpha_n}$ can be expressed in terms of $\{u_{\alpha, k} \mid \alpha = 0, \dots, r, k = 0, \dots, n-1\}$. So F_g can be expressed in terms of $\{u_{\alpha, k} \mid \alpha = 0, \dots, r, k = 0, \dots, 3g-2\}$. This is called the $(3g-2)$ -conjecture. In Eguchi-Getzler-Xiong [14], it was shown that $\{u_{\alpha, k} \mid \alpha = 0, \dots, r, k \geq 0\}$ can be used as new coordinates on the big phase space. The $(3g-2)$ -conjecture has been proved in Dubrovin-Zhang [13] under a technical condition. When one compares with the results in the case of two-dimensional topological gravity in §3.2, it is desirable to see whether these jet variables can be interpreted in terms of renormalized coupling constants.

3.4. **The loop operators.** Introduce the *loop operators*:

$$(63) \quad D_{\beta,z} := \sum_{n=0}^{\infty} z^{-n-1} \frac{\partial}{\partial t_n^\beta}.$$

Then the system of the hierarchy can be written in the following compact form:

$$(64) \quad D_{\beta,z} u_\alpha = (R_{\alpha,0;\beta}(u_0, \dots, u_r; z))',$$

where

$$(65) \quad R_{\alpha,0;\beta}(u_0, \dots, u_r; z) = \eta_{\alpha\beta} + \sum_{n=0}^{\infty} R_{\alpha,0;\beta,n}(u_0, \dots, u_r) z^{-n-1}.$$

The reason for adding the extra term $\eta_{\alpha\beta}$ will be clear below.

Now we define the genus zero two-point function by:

$$(66) \quad \mathcal{V}_{\alpha,\beta}(z_1, z_2) = \sum_{m,n=0}^{\infty} \frac{\partial^2 F_0}{\partial t_m^\alpha \partial t_n^\beta} z_1^{-m-1} z_2^{-n-1} = D_{\alpha,z_1} D_{\beta,z_2} F_0.$$

By the constitutive relations,

$$(67) \quad \mathcal{V}_{\alpha,\beta}(z_1, z_2) = \sum_{m,n=0}^{\infty} R_{\alpha,m;\beta,n}(u_0, \dots, u_r) z_1^{-m-1} z_2^{-n-1}.$$

To compute $\mathcal{V}_{\alpha,\beta}(z_1, z_2)$, one can proceed as follows:

$$\begin{aligned} \frac{\partial}{\partial t^0} \mathcal{V}_{\alpha,\beta}(z_1, z_2) &= \sum_{m,n=0}^{\infty} \frac{\partial^2 F_0}{\partial t^0 \partial t_m^\alpha \partial t_n^\beta} z_1^{-m-1} z_2^{-n-1} \\ &= D_{\alpha,z_1} \sum_{n=0}^{\infty} \frac{\partial^2 F_0}{\partial t^0 \partial t_n^\beta} z_2^{-n-1} \\ &= D_{\alpha,z_1} \sum_{n=0}^{\infty} R_{0,0;\beta,n}(u_0, \dots, u_r) z_2^{-n-1}, \end{aligned}$$

where in the last equality we have used the constitutive relation again.

Now we can use (64) to get:

$$\begin{aligned} &\frac{\partial}{\partial t^0} \mathcal{V}_{\alpha,\beta}(z_1, z_2) \\ &= \sum_{n=0}^{\infty} \sum_{\gamma} \frac{\partial R_{0,0;\beta,n}(u_0, \dots, u_r)}{\partial u_\gamma} z_2^{-n-1} \cdot D_{\alpha,z_1}(u_\gamma) \\ &= \sum_{n=0}^{\infty} \sum_{\gamma} \frac{\partial R_{0,0;\beta,n}(u_0, \dots, u_r)}{\partial u_\gamma} z_2^{-n-1} \cdot (R_{\gamma,0;\alpha}(u_0, \dots, u_r; z_1))'. \end{aligned}$$

To understand $\frac{\partial R_{0,0;\beta,n}(u_0,\dots,u_r)}{\partial u_\gamma}$, we go back to the small space to see that it is equal to

$$\frac{\partial}{\partial t_\gamma} \left(\frac{\partial^3 F_0}{\partial t^0 \partial t_n^\beta} \right) = \eta^{\gamma\delta} \frac{\partial^3 F_0}{\partial t^0 \partial t_n^\beta \partial t^\delta} = \eta^{\gamma\delta} \frac{\partial}{\partial t^0} \left(\frac{\partial^2 F_0}{\partial t_n^\beta \partial t^\delta} \right)$$

and so by string equation it is equal to $\eta^{\gamma\delta} \frac{\partial^2 F_0}{\partial t_{n-1}^\beta \partial t^\delta}$ restricted to the small phase space, so it is equal to $\eta^{\gamma\delta} R_{\delta,0;\beta,n-1}(u_0, \dots, u_r)$, for $n > 0$, and for $n = 0$, $\eta^{\gamma\delta} \eta_{\delta\beta}$, on the big phase space. (This explains the reason why $\eta_{\alpha\beta}$ appears in (65).) So we get

$$\begin{aligned} & \frac{\partial}{\partial t^0} \mathcal{V}_{\alpha,\beta}(z_1, z_2) \\ &= \sum_{\gamma,\delta} \eta^{\gamma\delta} \left(\eta_{\delta\beta} z^{-1} + \sum_{n=1}^{\infty} R_{\delta,0;\beta,n-1}(u) z_2^{-n-1} \right) \cdot (R_{\gamma,0;\alpha}(u; z_1))' \\ &= z_2^{-1} \cdot \frac{\partial}{\partial t_0} R_{\gamma,0;\alpha}(u; z_1) \cdot \eta^{\gamma\delta} \cdot R_{\delta,0;\beta}(u; z_2). \end{aligned}$$

In the same fashion we also have:

$$\frac{\partial}{\partial t^0} \mathcal{V}_{\alpha,\beta}(z_1, z_2) = z_1^{-1} \cdot R_{\gamma,0;\alpha}(u; z_1) \cdot \eta^{\gamma\delta} \cdot \frac{\partial}{\partial t_0} R_{\delta,0;\beta}(u; z_2).$$

From the above two relations one easily deduce the following equation:

$$(68) \quad \mathcal{V}_{\alpha,\beta}(z_1, z_2) = \frac{1}{z_1 + z_2} \left(R_{\gamma,0;\alpha}(u; z_1) \cdot \eta^{\gamma\alpha} \cdot R_{\delta,0;\beta}(u; z_2) - \eta_{\alpha\beta} \right).$$

See Dubrovin [5] for a different derivation.

3.5. Source operators and closed formula for genus zero free energy. We now recall a result proved in Dubrovin [5, Proposition 3.6]. See also Eguchi-Yamada-Yang [15, Proposition 1]. The free energy at $g = 0$ is given by

$$(69) \quad F_0(t) = \frac{1}{2} \sum_{m,n=0} R_{\alpha,m;\beta,n} \tilde{t}_m^\alpha \tilde{t}_n^\beta,$$

where $\tilde{t}_m^\alpha = t_m^\alpha - \delta_{m,1} \delta_{\alpha,0}$. Therefore, if one sets

$$(70) \quad \mathcal{V}_{\alpha,\beta}(z_1, z_2) = \sum_{m,n=0}^{\infty} R_{\alpha,m;\beta,n}(U) z_1^{-m-1} z_2^{-n-1},$$

then

$$(71) \quad F_0 = \frac{1}{2} \sum_{\alpha,\beta=0}^r \text{res}_{z_1=0} (\text{res}_{z_2=0} (J^\alpha(z_1) J^\beta(z_2) \mathcal{V}_{\alpha,\beta}(u; z_1, z_2) dz_2) dz_1).$$

If we note

$$(72) \quad D_{t^\alpha, z} J^\beta(w) = \delta_{\alpha\beta} \sum_{n=0}^{\infty} \frac{w^n}{z^{n+1}} = \frac{\delta_{\alpha\beta}}{z-w},$$

then the genus zero one-point function is very easy to find:

$$(73) \quad \begin{aligned} D_{t^\gamma, z} F_0 &= \sum_{\alpha=0}^r \text{res}_{z_1=0} (\text{res}_{z_2=0} (J^\alpha(z_1) \mathcal{V}_{\alpha, \gamma}(u; z_1, z_2) \frac{dz_2}{z-z_2})) dz_1 \\ &+ \frac{1}{2} \sum_{\alpha, \beta=0}^r \text{res}_{z_1=0} (\text{res}_{z_2=0} (J^\alpha(z_1) J^\beta(z_2) D_{t^\gamma, z} \mathcal{V}_{\alpha, \beta}(u; z_1, z_2) dz_2) dz_1). \end{aligned}$$

3.6. Computations of n -point functions by operator formalism.

Now we present an algorithm to compute the n -point correlation functions of GW invariants in genus g when $2g - 2 + n > 0$. When $g = 0$, this requires that $n \geq 3$. We have already seen that the 2-point functions in genus zero can be expressed in terms of the order parameters. So for $n \geq 3$, one can simply apply the loop operators repeatedly on the two-point functions and apply the constitutive relations (29) and the integrable hierarchy (64):

$$(74) \quad D_{\alpha_1, z_1} \cdots D_{\alpha_n, z_n} F_0 = D_{\alpha_1, z_1} \cdots D_{\alpha_{n-2}, z_{n-2}} \mathcal{V}_{\alpha_{n-1}, \alpha_n}(u; z_{n-1}, z_n).$$

Similarly, for $g \geq 1$, suppose that F_g is expressed in terms of the jet variables $\{u_{\alpha, n}\}$, then the n -point functions can be obtained by applying the loop operators repeatedly on such expressions with the help of (64).

3.7. Emergent conformal field theory. Inspired by conformal field theory, it is natural to regard the source operators as the generating series of the creation operators, and the loop operators as the generating series of annihilation operators, and combine them into bosonic fields of operators:

$$(75) \quad \varphi_\alpha(z) := \eta_{\alpha\beta} J^\beta(z) + D_{\beta, z} = \eta_{\alpha\beta} \sum_{n=0}^{\infty} \tilde{t}_n^\beta z^n + \sum_{n=0}^{\infty} z^{-n-1} \frac{\partial}{\partial t_n^\alpha}.$$

It is very easy to see that one has the following operator product expansion:

$$(76) \quad \varphi_\alpha(z) \varphi_\beta(w) = \frac{\eta_{\alpha\beta}}{z-w} + : \varphi_\alpha(z) \varphi_\beta(w) :,$$

where $: \varphi_\alpha(z) \varphi_\beta(w) :$ is the normally ordered product of $\varphi_\alpha(z)$ and $\varphi_\beta(w)$. So one can regard the GW theory as associating a vector $|X\rangle$

in the bosonic Fock space of the systems of bosonic fields $\{\varphi_\alpha\}_{\alpha=0}^r$, and one can consider the n -point functions:

$$(77) \quad \langle 0 | \varphi_{\alpha_1}(z_1) \cdots \varphi_{\alpha_n}(z_n) | X \rangle.$$

One can also apply the boson-fermion correspondence to transform $|X\rangle$ into a vector in $|X\rangle^F$ in the fermionic Fock space.

This naive construction leads to an interesting point of view, but in order to obtain interesting results, we will need a mysterious Laplace transform suggested by the construction of the Virasoro constraints in GW theory.

4. MEAN FIELD THEORY OF THE GENUS ZERO GROMOV-WITTEN INVARIANTS OF QUINTIC CALABI-YAU THREEFOLD

In this Section and the next Section we explicitly carry out the computations mentioned in the above two Sections for the quintic CY threefold. We will focus on the genus zero part in this Section and treat the higher genera computation in the next Section.

Our main result in this Section is that we can get explicit formulas for the Landau-Ginzburg potential function and for the genus zero partition functions and the n -points functions for $n \geq 1$ in the case of the quintic. We note that the selection rules for the CY threefolds play a crucial role in this case.

4.1. Selection rules for the GW theory of the quintic. For the quintic Calabi-Yau threefold $M \subset \mathbb{P}^4$, $H^{ev}(M)$ is four dimensional. It is spanned by four primary operators $P = 1$, $Q = j^*H$, $R = \frac{1}{5}j^*H$, $S = \frac{1}{5}j^*H$, where $j : M \hookrightarrow \mathbb{P}^4$ is the inclusion map, and H is the hyperplane class in $H^2(\mathbb{P}^4)$.

Because $\deg_{\mathbb{R}} P = 0$, $\deg_{\mathbb{R}} Q = 2$, $\deg_{\mathbb{R}} R = 4$, $\deg_{\mathbb{R}} S = 6$, and the real dimension of M is 6, and by the selection rules (4), if one assigns

$$(78) \quad \deg t_n^P = n - 1, \quad \deg t_n^Q = n, \quad \deg t_n^R = n + 1, \quad \deg t_n^S = n + 2,$$

then F_g is homogeneous of degree 0:

$$(79) \quad \mathcal{X}F_g = 0,$$

where

$$(80) \quad \begin{aligned} \mathcal{X} = & \sum_{n=0}^{\infty} (n-1)t_n^P \frac{\partial}{\partial t_n^P} + \sum_{n=0}^{\infty} nt_n^Q \frac{\partial}{\partial t_n^Q} \\ & + \sum_{n=0}^{\infty} (n+1)t_n^R \frac{\partial}{\partial t_n^R} + \sum_{n=0}^{\infty} (n+2)t_n^S \frac{\partial}{\partial t_n^S}. \end{aligned}$$

4.2. The genus zero free energy on the small phase space. By the selection rules (4), for the quintic,

$$\langle \mathcal{O}_\alpha \mathcal{O}_\beta \mathcal{O}_\gamma \rangle_{0;d=0} \neq 0$$

only if $\deg \mathcal{O}_\alpha + \deg \mathcal{O}_\beta + \deg \mathcal{O}_\gamma = 3$, so if $\deg \mathcal{O}_\alpha \leq \deg \mathcal{O}_\beta \leq \deg \mathcal{O}_\gamma$, then there are only three possibilities:

$$(81) \quad \langle PPS \rangle_{0;0} = 1, \quad \langle PQR \rangle_{0;0} = 1, \quad \langle QQQ \rangle_{0;0} = 5.$$

So on the small phase space, the classical part of F_0^{small} is

$$(82) \quad F_{0,classical}^{small} = \frac{(t^P)^2 t^S}{2} + t^P t^Q t^R + \frac{5}{6} (t^Q)^3.$$

For the quantum part of F_0^{small} , we need to consider the intersection numbers on the moduli spaces $\overline{\mathcal{M}}_{0,n}(X; d)$ for $d > 0$ or $d = 0$ and $n > 3$. The moduli spaces have expect dimensions n , and so

$$(83) \quad \langle P^{m_0} Q^{m_1} R^{m_2} S^{m_3} \rangle_{0;d} \neq 0$$

only if:

$$(84) \quad m_1 + 2m_2 + 3m_3 = m_0 + m_1 + m_2 + m_3,$$

i.e., $m_0 = m_1 + 2m_2$. But by the puncture equation, such correlators vanish when $m_0 > 0$. So we have only $\langle Q^{m_1} \rangle_{0;d}$ to consider. By the divisor equation,

$$\langle Q^{m_1} \rangle_{0;d} = d \cdot \langle Q^{m_1-1} \rangle_{0;d} = d^{m_1} \langle 1 \rangle_{0;d} = d^{m_1} N_{0,d},$$

where

$$(85) \quad N_{0,d} := \int_{[\overline{\mathcal{M}}_{0,0}(X;d)]^{virt}} 1.$$

So the instanton correction to F_0^{small} is

$$\begin{aligned} F_{0,instanton}^{small} &= \sum_{d=1}^{\infty} \sum_{m_1=0}^{\infty} \frac{(t^Q)^{m_1}}{m_1!} q^d \langle Q^{m_1} \rangle_{0;d} \\ &= \sum_{d=1}^{\infty} \sum_{m_1=0}^{\infty} \frac{(t^Q)^{m_1}}{m_1!} q^d N_{0,d} d^{m_1} = \sum_{d=1}^{\infty} N_{0,d} e^{dt^Q} q^d. \end{aligned}$$

So the genus zero free energy on the small phase in this case is

$$(86) \quad \begin{aligned} F_0^{small} &= F_{0,classical}^{small} + F_{0,instanton}^{small} \\ &= \frac{1}{2} (t^P)^2 t^S + t^P t^Q t^R + \frac{5}{6} (t^Q)^3 + \sum_{m=1}^{\infty} N_{0,m} e^{mt^Q} q^m. \end{aligned}$$

For simplicity of notations, we will set:

$$(87) \quad f_0(x) = \frac{5}{6}x^3 + \sum_{d=1}^{\infty} N_{0,d} e^{dx} q^d.$$

So we have

$$(88) \quad F_0^{small} = \frac{1}{2}(t^P)^2 t^S + t^P t^Q t^R + f_0(t^Q).$$

The metric matrix in this case:

$$(89) \quad (\eta_{\lambda\mu}) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The gradient of F_0^{small} is

$$\begin{aligned} \frac{\partial F_0^{small}}{\partial t^P} &= t^P t^S + t^Q t^R, \\ \frac{\partial F_0^{small}}{\partial t^Q} &= t^P t^R + f_0'(t^Q), \\ \frac{\partial F_0^{small}}{\partial t^R} &= t^P t^Q, \\ \frac{\partial F_0^{small}}{\partial t^S} &= \frac{1}{2}(t^P)^2. \end{aligned}$$

The entropy is given by:

$$(90) \quad G_0 = t^P \frac{\partial F_0^{small}}{\partial t^P} + t^Q \frac{\partial F_0^{small}}{\partial t^Q} + t^R \frac{\partial F_0^{small}}{\partial t^R} + t^S \frac{\partial F_0^{small}}{\partial t^S} - F_0^{small} \\ = (t^P)^2 t^S + t^P t^Q t^R + t^Q f_0'(t^Q) - f_0(t^Q).$$

The Hessian of F_0^{small} is:

$$(91) \quad Hess(F_0^{small}) = \begin{pmatrix} t^S & t^R & t^Q & t^P \\ t^R & f_0''(t^Q) & t^P & 0 \\ t^Q & t^P & 0 & 0 \\ t^P & 0 & 0 & 0 \end{pmatrix}.$$

The Euler vector field is given by:

$$(92) \quad E := t^P \frac{\partial}{\partial t^P} - t^R \frac{\partial}{\partial t^R} - 2t^S \frac{\partial}{\partial t^S}.$$

It is clear that

$$(93) \quad EF_0^{small} = 0.$$

The explicit formula for $N_{0,d}$ is given by the mirror formula discovered in [2] and proved in [22, 27]:

$$(94) \quad \frac{5}{6}(t^Q)^3 + \sum_{d=1}^{\infty} N_{0,d} e^{dt^Q} q^d = \frac{5}{2} \left(\frac{\omega_3}{\omega_0} - \frac{f_1 \omega_2}{f_0 \omega_0} \right).$$

Here $\omega_0, \dots, \omega_3$ are the four solutions of hypergeometric equation

$$(95) \quad L\omega = 0,$$

where L is the differential operator

$$(96) \quad L = \theta^4 - \alpha \prod_{k=1}^4 \left(\theta + \frac{k}{5} \right),$$

where $\alpha = 5^5 e^t$ and $\theta = \alpha \frac{\partial}{\partial \alpha}$. The variable t is related to t^Q via the mirror formula:

$$(97) \quad t^Q = \frac{\omega_1}{\omega_0}.$$

Furthermore,

$$(98) \quad (\omega_0, \omega_1, \omega_2, \omega_3) = \omega_0 \cdot (1, t^Q, f_0'(t^Q), t^Q f_0'(t^Q) - 2f_0(t^Q)).$$

Even though such structure from special geometry originally appear on only the tiny phase space with parameter t^Q , we will show later that they natural appear in the emergent geometry on the small phase space.

4.3. Quantum cohomology ring and WDVV equation. Recall the quantum multiplication over the small phase space is defined by:

$$(99) \quad \frac{\partial}{\partial t^\alpha} \circ \frac{\partial}{\partial t^\beta} = \frac{\partial^3 F_0^{small}}{\partial t^\alpha \partial t_\beta \partial t^\lambda} \eta^{\lambda\mu} \frac{\partial}{\partial t^\mu}.$$

where $\eta_{\lambda\mu} = \frac{\partial^2 F_0}{\partial t^\alpha \partial t^\beta}$ and $(\eta^{\lambda\mu}) = (\eta_{\lambda\mu})^{-1}$. In our case

$$(100) \quad (\eta^{\lambda\mu}) = (\eta_{\lambda\mu})^{-1} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

and so we can give the explicit formula for the quantum multiplications:

$$\begin{aligned}
\frac{\partial}{\partial t^P} \circ \begin{pmatrix} \frac{\partial}{\partial t^P} \\ \frac{\partial}{\partial t^Q} \\ \frac{\partial}{\partial t^R} \\ \frac{\partial}{\partial t^S} \end{pmatrix} &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial}{\partial t^P} \\ \frac{\partial}{\partial t^Q} \\ \frac{\partial}{\partial t^R} \\ \frac{\partial}{\partial t^S} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial}{\partial t^P} \\ \frac{\partial}{\partial t^Q} \\ \frac{\partial}{\partial t^R} \\ \frac{\partial}{\partial t^S} \end{pmatrix}, \\
\frac{\partial}{\partial t^Q} \circ \begin{pmatrix} \frac{\partial}{\partial t^P} \\ \frac{\partial}{\partial t^Q} \\ \frac{\partial}{\partial t^R} \\ \frac{\partial}{\partial t^S} \end{pmatrix} &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & f_0'''(t^Q) & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial}{\partial t^P} \\ \frac{\partial}{\partial t^Q} \\ \frac{\partial}{\partial t^R} \\ \frac{\partial}{\partial t^S} \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & f_0'''(t^Q) & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial}{\partial t^P} \\ \frac{\partial}{\partial t^Q} \\ \frac{\partial}{\partial t^R} \\ \frac{\partial}{\partial t^S} \end{pmatrix}, \\
\frac{\partial}{\partial t^R} \circ \begin{pmatrix} \frac{\partial}{\partial t^P} \\ \frac{\partial}{\partial t^Q} \\ \frac{\partial}{\partial t^R} \\ \frac{\partial}{\partial t^S} \end{pmatrix} &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial}{\partial t^P} \\ \frac{\partial}{\partial t^Q} \\ \frac{\partial}{\partial t^R} \\ \frac{\partial}{\partial t^S} \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial}{\partial t^P} \\ \frac{\partial}{\partial t^Q} \\ \frac{\partial}{\partial t^R} \\ \frac{\partial}{\partial t^S} \end{pmatrix}, \\
\frac{\partial}{\partial t^S} \circ \begin{pmatrix} \frac{\partial}{\partial t^P} \\ \frac{\partial}{\partial t^Q} \\ \frac{\partial}{\partial t^R} \\ \frac{\partial}{\partial t^S} \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial}{\partial t^P} \\ \frac{\partial}{\partial t^Q} \\ \frac{\partial}{\partial t^R} \\ \frac{\partial}{\partial t^S} \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial}{\partial t^P} \\ \frac{\partial}{\partial t^Q} \\ \frac{\partial}{\partial t^R} \\ \frac{\partial}{\partial t^S} \end{pmatrix}.
\end{aligned}$$

At a first sight, the effect of quantum multiplication on the cohomology ring seems to be minor: It only modifies the multiplication of $\frac{\partial}{\partial t^Q}$ with $\frac{\partial}{\partial t^Q}$. Furthermore, in this case, the original cohomology ring is not

semisimple, after the quantum deformation over the small phase space, it is still not semisimple. So the reconstruction theory of Dubrovin and Zhang [13] based on semisimple Frobenius manifolds cannot be applied here directly.

One can also check that the associativity of the quantum multiplication, i.e., the WDVV equations, does not impose any constraints on the function f_0 .

4.4. The genus zero one-point functions on the small phase space. In this Subsection, we will show how to compute the genus zero one-point correlators of the quintic Calabi-Yau threefold from F_0^{small} by the selection rules.

We will use $\langle\langle \cdot \cdot \cdot \rangle\rangle$ to denote the correlator on the small phase space. For example, $\langle\langle \tau_n(S) \rangle\rangle_0$ means $\frac{\partial F_0}{\partial t_n^Q}$ restricted to the small phase space.

Proposition 4.1. *One can obtain by applications of the selection rules to get the following formulas for quintic CY threefolds from the genus zero free energy on the small phase space:*

$$(101) \quad \begin{aligned} t^S + \sum_{n \geq 0} z^{-n-1} \langle\langle \tau_n(P) \rangle\rangle_0 \\ = e^{t^P/z} t^S + \frac{1}{z} e^{t^P/z} t^Q t^R + \frac{1}{z^2} e^{t^P/z} (t^Q f_0'(t^Q) - f_0(t^Q)). \end{aligned}$$

$$(102) \quad t^R + \sum_{n=0}^{\infty} z^{-n-1} \langle\langle \tau_n(Q) \rangle\rangle_0 = e^{t^P/z} t^R + \frac{e^{t^P/z}}{z} f_0'(t^Q).$$

$$(103) \quad t^Q + \sum_{n \geq 0} z^{-n-1} \langle\langle \tau_n(R) \rangle\rangle_0 = e^{t^P/z} t^Q.$$

$$(104) \quad t^P + \sum_{n \geq 0} z^{-n-1} \langle\langle \tau_n(S) \rangle\rangle_0 = z \left(e^{t^P/z} - 1 \right).$$

Proof. From the formula (86), we have

$$(105) \quad \begin{aligned} & \langle \tau_0(P)^{m_0} \cdots \tau_0(S)^{m_3} \rangle_0 \\ & = \begin{cases} 1, & \text{if } (m_0, \dots, m_3) = (2, 0, 0, 1), \\ 1, & \text{if } (m_0, \dots, m_3) = (1, 1, 1, 0), \\ 5\delta_{m_1,3} + \sum_{d=1}^{\infty} N_d d^{m_1} q^d, & \text{if } (m_0, \dots, m_3) = (0, m_1, 0, 0), \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

To compute $\langle\langle\tau_n(P)\rangle\rangle_0$ we need to compute $\langle\tau_n(P)\tau_0(P)^{m_0}\cdots\tau_0(S)^{m_3}\rangle_0$.
By the selection rule,

$$n + m_1 + 2m_2 + 3m_3 = 1 + m_0 + m_1 + m_2 + m_3,$$

and so we get:

$$(106) \quad m_0 = n - 1 + m_2 + 2m_3.$$

For $n = 0$, by (105) one easily sees that:

$$\langle\langle\tau_0(P)\rangle\rangle_0 = t^P t^S + t^Q t^R.$$

For $n = 1$, we use dilaton equation to get:

$$\begin{aligned} & \langle\tau_1(P)\tau_0(P)^{m_0}\cdots\tau_0(S)^{m_3}\rangle_0 \\ &= (m_0 + \cdots + m_3 - 2) \cdot \langle\tau_0(P)^{m_0}\cdots\tau_0(S)^{m_3}\rangle_0 \\ &= \begin{cases} 1, & \text{if } (m_0, \dots, m_3) = (2, 0, 0, 1), \\ 1, & \text{if } (m_0, \dots, m_3) = (1, 1, 1, 0), \\ (m_1 - 2) \left(5\delta_{m_1,3} + \sum_{d=1}^{\infty} N_d d^{m_1} q^d \right), & \text{if } (m_0, \dots, m_3) = (0, m_1, 0, 0), \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

For $n \geq 2$,

$$\begin{aligned} & \langle\tau_n(P)\tau_0(P)^{m_0}\cdots\tau_0(S)^{m_3}\rangle_0 \\ &= \langle\tau_n(P)\tau_0(P)^{n-1+m_2+2m_3}\tau_0(Q)^{m_1}\cdots\tau_0(S)^{m_3}\rangle_0 \\ &= \langle\tau_0(P)^{m_2+2m_3}\tau_1(P)\tau_0(Q)^{m_1}\tau_0(R)^{m_2}\tau_0(S)^{m_3}\rangle_0. \end{aligned}$$

If $m_2 = m_3 = 0$, then by the dilaton equation one has

$$\langle\tau_1(P)\tau_0(Q)^{m_1}\rangle_0 = (m_1 - 2)\langle\tau_0(Q)^{m_1}\rangle_0 = \delta_{m_1,3}5 + (m_1 - 2) \sum_{d \geq 1} N_d d^{m_1} q^d.$$

If $m_2 > 0$ or $m_3 > 0$, then

$$\begin{aligned} & \langle\tau_0(P)^{m_2+2m_3}\tau_1(P)\tau_0(Q)^{m_1}\tau_0(R)^{m_2}\tau_0(S)^{m_3}\rangle_0 \\ &= \langle\tau_0(P)^{m_2+2m_3-1}\tau_0(P)\tau_0(Q)^{m_1}\tau_0(R)^{m_2}\tau_0(S)^{m_3}\rangle_0 \\ &= \langle\tau_0(P)^{m_2+2m_3}\tau_0(Q)^{m_1}\tau_0(R)^{m_2}\tau_0(S)^{m_3}\rangle_0, \end{aligned}$$

which is nonvanishing only when $(m_1, m_2, m_3) = (1, 1, 0)$ or $(0, 0, 1)$.

Now we can compute the generating series of

$$\langle\langle\tau_n(P)\rangle\rangle_0 = \sum_{m_0, m_1, m_2, m_3 \geq 0} \frac{t_{0,P}^{m_0}}{m_0!} \cdots \frac{t_{0,S}^{m_3}}{m_3!} \cdot \langle\tau_n(P)\tau_0(P)^{m_0}\cdots\tau_0(S)^{m_3}\rangle_0.$$

as follows:

$$\begin{aligned}
& \sum_{n \geq 0} z^{-n-1} \langle \langle \tau_n(P) \rangle \rangle_0 \\
&= \frac{1}{z} (t^P t^S + t^Q t^R) + \frac{1}{z^2} \left(\frac{(t^P)^2}{2} t^S + t^P t^Q t^R \right. \\
&\quad \left. + \sum_{m_1 \geq 0} \frac{(t^Q)^{m_1}}{m_1!} (m_1 - 2) \left(5\delta_{m_1,3} + \sum_{d=1}^{\infty} N_d d^{m_1} q^d \right) \right) \\
&+ \sum_{n \geq 2} z^{-n-1} \sum_{m_1 \geq 0} \frac{t_{0,P}^{n-1}}{(n-1)!} \frac{t_{0,Q}^{m_1}}{m_1!} \cdot \langle \tau_n(P) \tau_0(P)^n \tau_0(Q)^{m_1} \rangle_0 \\
&+ \sum_{n \geq 2} z^{-n-1} \langle \tau_n(P) \tau_0(P)^n \tau_0(Q) \tau_0(R) \rangle_0 \frac{t_{0,P}^n}{n!} t_{0,Q} t_{0,R} \\
&+ \sum_{n \geq 2} z^{-n-1} \langle \tau_n(P) \tau_0(P)^{n+1} \tau_0(S) \rangle_0 \frac{t_{0,P}^{n+1}}{(n+1)!} t_{0,S} \\
&= \sum_{n \geq 1} z^{-n-1} \sum_{m_1 \geq 0} \frac{(t^P)^{n-1}}{(n-1)!} \frac{(t^Q)^{m_1}}{m_1!} \cdot (\delta_{m_1,3} 5 + (m_1 - 2) \sum_{d \geq 1} N_d d^{m_1} q^d) \\
&+ \frac{1}{z} e^{t^P/z} t^Q t^R + (e^{t^P/z} - 1) t^S \\
&= \frac{1}{z^2} \frac{5(t^Q)^3}{6} e^{t^P/z} + \frac{1}{z^2} e^{t^P/z} \sum_{m_1 \geq 0} \frac{(t^Q)^{m_1}}{m_1!} (m_1 - 2) \sum_{d \geq 1} N_d d^{m_1} q^d \\
&+ \frac{1}{z} e^{t^P/z} t^Q t^R + (e^{t^P/z} - 1) t^S \\
&= (e^{t^P/z} - 1) t^S + \frac{1}{z} e^{t^P/z} t^Q t^R + \frac{1}{z^2} e^{t^P/z} \left(\frac{5(t^Q)^3}{6} \right. \\
&\quad \left. + \sum_{d \geq 1} N_d (d t^Q - 2) e^{d t^Q} q^d \right).
\end{aligned}$$

The other three formulas can be proved in the same way. \square

Remark 4.2. Here we recover some formulas in Example 5.3 in Dubrovin [5] by selection rules. We do this without solving any differential equations, only using the selection rules.

4.5. Explicit formulas for Landau-Ginzburg potential and Landau-Ginzburg equations. By changing t^P to u_S , t^Q to u_R , t^R to u_Q and

t^S to u_P in (101)-104, we then get:

$$(107) \quad \begin{aligned} & \sum_{n=-1}^{\infty} z^{-n-1} R_{P,n}(u_P, u_Q, u_R, u_S) \\ &= e^{u_S/z} u_P + \frac{1}{z} e^{u_S/z} u_Q u_R + \frac{1}{z^2} e^{u_S/z} (u_R f'_0(u_R) - 2f_0(u_R)). \end{aligned}$$

$$(108) \quad \sum_{n=-1}^{\infty} z^{-n-1} R_{Q,n}(u_P, \dots, u_S) = e^{u_S/z} u_Q + \frac{e^{u_S/z}}{z} f'_0(u_R).$$

$$(109) \quad \sum_{n=-1}^{\infty} z^{-n-1} R_{R,n}(u_P, \dots, u_S) = e^{u_S/z} u_R.$$

$$(110) \quad \sum_{n=-1}^{\infty} z^{-n-1} R_{S,n}(u_0, \dots, u_S) = z \left(e^{U_S/z} - 1 \right).$$

With these formulas, we can write down the Landau-Ginzburg equation for the GW theory of quintic CY threefold explicitly as follows:

Proposition 4.3. *The mean field theory of the GW theory of quintic CY threefold has the following Landau-Ginzburg potential:*

$$(111) \quad \begin{aligned} W &= -U_P U_S - U_Q U_R \\ &+ \sum_{n=0}^{\infty} t_n^P \left(U_P \frac{U_S^n}{n!} + \frac{U_Q U_R U_S^{n-1}}{(n-1)!} \right. \\ &\quad \left. + (U_R f'_0(U_R) - 2f_0(U_R)) \frac{U_S^{n-2}}{(n-2)!} \right) \\ &+ \sum_{n=0}^{\infty} t_n^Q \left(U_Q \frac{U_S^n}{n!} + \frac{U_S^{n-1}}{(n-1)!} f'_0(U_R) \right) \\ &+ \sum_{n=0}^{\infty} t_n^R U_R \frac{U_S^n}{n!} + \sum_{n=0}^{\infty} t_n^S \frac{U_S^{n+1}}{(n+1)!}. \end{aligned}$$

We also write W in the following form:

$$\begin{aligned}
(112) \quad W &= U_P \left(-U_S + \sum_{n=0}^{\infty} t_n^P \frac{U_S^n}{n!} \right) \\
&+ U_Q \left(-U_R + \sum_{n=0}^{\infty} t_n^Q \frac{U_S^n}{n!} + U_R \sum_{n=1}^{\infty} t_n^P \frac{U_S^{n-1}}{(n-1)!} \right) \\
&+ U_R \sum_{n=0}^{\infty} t_n^R \frac{U_S^n}{n!} + f'_0(U_R) \sum_{n=1}^{\infty} t_n^Q \frac{U_S^{n-1}}{(n-1)!} \\
&+ g_0(U_R) \sum_{n=2}^{\infty} t_n^P \frac{U_S^{n-2}}{(n-2)!} + \sum_{n=0}^{\infty} t_n^S \frac{U_S^{n+1}}{(n+1)!},
\end{aligned}$$

where $g_0(U_R) = U_R f'_0(U_R) - 2f_0(U_R)$. By computing the gradient of W , the Landau-Ginzburg equations can be written down explicitly as follows:

$$(113) \quad u_S = \sum_{n=0}^{\infty} t_n^P \frac{u_S^n}{n!},$$

$$(114) \quad u_R = \sum_{n=0}^{\infty} t_n^Q \frac{u_S^n}{n!} + u_R \sum_{n=1}^{\infty} t_n^P \frac{u_S^{n-1}}{(n-1)!}.$$

$$\begin{aligned}
(115) \quad u_Q &= \sum_{n=0}^{\infty} t_n^R \frac{u_S^n}{n!} + u_Q \sum_{n=1}^{\infty} t_n^P \frac{u_S^{n-1}}{(n-1)!} \\
&+ f''_0(u_R) \sum_{n=1}^{\infty} t_n^Q \frac{u_S^{n-1}}{(n-1)!} + g'_0(u_R) \sum_{n=2}^{\infty} t_n^P \frac{u_S^{n-2}}{(n-2)!}.
\end{aligned}$$

$$\begin{aligned}
(116) \quad u_P &= u_P \sum_{n=1}^{\infty} t_n^P \frac{u_S^{n-1}}{(n-1)!} \\
&+ u_Q \left(\sum_{n=1}^{\infty} t_n^Q \frac{u_S^{n-1}}{(n-1)!} + u_R \sum_{n=2}^{\infty} t_n^P \frac{u_S^{n-2}}{(n-2)!} \right) \\
&+ u_R \sum_{n=1}^{\infty} t_n^R \frac{u_S^{n-1}}{(n-1)!} + f'_0(u_R) \sum_{n=2}^{\infty} t_n^Q \frac{u_S^{n-2}}{(n-2)!} \\
&+ g_0(u_R) \sum_{n=3}^{\infty} t_n^P \frac{u_S^{n-3}}{(n-3)!} + \sum_{n=0}^{\infty} t_n^S \frac{u_S^n}{n!}.
\end{aligned}$$

4.6. Solutions of the Landau-Ginzburg equations. These equation can be easily solved. As we have mentioned in §3.2, equation (113) has been explicitly solved in [37]:

$$(117) \quad u_S = \sum_{k=1}^{\infty} \frac{1}{k} \sum_{p_1+\dots+p_k=k-1} \frac{t_{p_1}^P}{p_1!} \cdots \frac{t_{p_k}^P}{p_k!}.$$

We now give another formula which express it as a summation over partitions of nonnegative integers. A partition of $N \geq 0$ can be written as $1^{m_1} \cdots N^{m_N}$, where m_1, \dots, m_N are nonnegative integers such that:

$$(118) \quad m_1 1 + \cdots + m_N N = N.$$

Then we have

$$(119) \quad u_S = \sum_{N=0}^{\infty} \sum_{\sum_i i m_i = N} \frac{N! (t^P)^{N - \sum_i m_i + 1}}{m_1! \cdots m_N! \cdot (N - \sum_i m_i + 1)!} \prod_{i=1}^N \left(\frac{t_i^P}{i!} \right)^{m_i}.$$

The following are the first few terms:

$$\begin{aligned} u_S &= t^P + t^P t_1^P + \left((t^P)^2 \frac{t_2^P}{2!} + t^P (t_1^P)^2 \right) \\ &+ \left((t^P)^3 \frac{t_3^P}{3!} + 3(t^P)^2 \frac{t_2^P}{2!} t_1^P + t^P (t_1^P)^3 \right) \\ &+ \left((t^P)^4 \frac{t_4^P}{4!} + 4(t^P)^3 \frac{t_3^P}{3!} t_1^P + 2(t^P)^3 \left(\frac{t_2^P}{2!} \right)^2 + 6(t^P)^2 \frac{t_2^P}{2!} (t_1^P)^2 + t^P (t_1^P)^4 \right) + \cdots \end{aligned}$$

The coefficients $(1), (1, 1), (1, 3, 1), (1, 4, 2, 6, 1)$ are integer sequence A134264 on [31].

Taking $\frac{\partial}{\partial t^P}$ on both sides of (113) one can get:

$$(120) \quad u_{S,1} = \left(1 - \sum_{n=1}^{\infty} t_n^P \frac{u_S^{n-1}}{(n-1)!} \right)^{-1}.$$

From (114) we get:

$$(121) \quad u_R = \left(1 - \sum_{n=1}^{\infty} t_n^P \frac{u_S^{n-1}}{(n-1)!} \right)^{-1} \cdot \sum_{n=0}^{\infty} t_n^Q \frac{u_S^n}{n!} = u_{S,1} \sum_{n=0}^{\infty} t_n^Q \frac{u_S^n}{n!}.$$

From this we get:

$$(122) \quad \sum_{n=0}^{\infty} t_n^Q \frac{u_S^n}{n!} = \frac{u_R}{u_{S,1}}.$$

Proposition 4.4. *The following formula for u_R holds:*

$$(123) \quad u_R = \sum_{k=1}^{\infty} \sum_{p_1+\dots+p_k=k-1} \frac{t_{p_1}^Q t_{p_2}^P \dots t_{p_k}^P}{p_1! p_2! \dots p_k!}.$$

Proof. Consider an operator

$$(124) \quad D := \sum_{n=0}^{\infty} t_n^Q \frac{\partial}{\partial t_n^P}.$$

Claim. The action of this operator on u_S changes it to u_R :

$$(125) \quad Du_S = u_R.$$

This can be proved as follows. We will actually use some results from §4.12 below. From

$$D_{P,z}(u_S) = \sum_{n \geq 0} z^{-n-1} \frac{\partial u_P}{\partial t_{n,P}} = \left[e^{u_S/z} \right]'$$

we get

$$(126) \quad \frac{\partial u_S}{\partial t_n^P} = \frac{u_S^n}{n!} u_{S,1},$$

Therefore, we have

$$Du_S = \sum_{n=0}^{\infty} t_n^Q \frac{\partial u_S}{\partial t_n^P} = u_{S,1} \sum_{n=0}^{\infty} t_n^Q \frac{u_S^n}{n!} = u_R.$$

In other words, we have shown that

$$(127) \quad u_R = \sum_{n=0}^{\infty} t_n^Q \frac{\partial u_S}{\partial t_n^P}.$$

Now (123) follows from (117). □

From the proof we also get the following formula:

$$(128) \quad u_{S,1} \frac{u_S^n}{n!} = \frac{\partial u_S}{\partial t_n^P} = \frac{1}{n!} \sum_{k=n}^{\infty} \sum_{p_1+\dots+p_k=k-n} \frac{t_{p_1}^P \dots t_{p_k}^P}{p_1! \dots p_k!}.$$

It can also be written as:

$$\frac{\partial u_S}{\partial t_n^P} = \frac{1}{n!} \sum_{N=n}^{\infty} \sum_{\sum_i i m_i = N-n} \frac{N! (t^P)^{N-\sum_i m_i}}{m_1! \dots m_N! \cdot (N - \sum_i m_i)!} \prod_{i=1}^N \left(\frac{t_i^P}{i!} \right)^{m_i}.$$

For example,

$$\begin{aligned}
u_{S,1} &= \frac{\partial u_S}{\partial t^P} \\
&= 1 + t_1^P + \left(2t^P \frac{t_2^P}{2!} + (t_1^P)^2 \right) \\
&+ \left(3(t^P)^2 \frac{t_3^P}{3!} + 6t^P \frac{t_2^P}{2!} t_1^P + (t_1^P)^3 \right) \\
&+ \left(4(t^P)^3 \frac{t_4^P}{4!} + 12(t^P)^2 \frac{t_3^P}{3!} t_1^P + 6(t^P)^2 \left(\frac{t_2^P}{2!} \right)^2 + 12t^P \frac{t_2^P}{2!} (t_1^P)^2 + (t_1^P)^4 \right) + \dots,
\end{aligned}$$

where the coefficients 1, (2, 1), (3, 6, 1), (4, 1, 2, 6, 12, 1), ... are the sequence A035206 on [31].

$$\begin{aligned}
\frac{\partial u_S}{\partial t_1^P} &= t^P + 2t^P t_1^P + \left(3(t^P)^2 \frac{t_2^P}{2!} + 3t^P (t_1^P)^2 \right) \\
&+ \left(4(t^P)^3 \frac{t_3^P}{3!} + 12(t^P)^2 \frac{t_2^P}{2!} t_1^P + 4t^P (t_1^P)^3 \right) + \dots.
\end{aligned}$$

where the coefficients 1, 2, (3, 3), (4, 12, 4), ... are not on [31], instead, after rescaling we get 1, 1, (1, 1), (1, 3, 1), ... which are the sequence A134264 on [31] again. We also have:

$$\begin{aligned}
\frac{\partial u_S}{\partial t_2^P} &= \frac{1}{2!} \left[(t^P)^2 + 3(t^P)^2 t_1^P + \left(4(t^P)^3 \frac{t_2^P}{2!} + 6(t^P)^2 (t_1^P)^2 \right) + \dots \right], \\
\frac{\partial u_S}{\partial t_3^P} &= \frac{1}{3!} \left((t^P)^3 + 4(t^P)^3 t_1^P + \dots \right).
\end{aligned}$$

From these we can write the first few terms of u_R :

$$\begin{aligned}
u_R &= t^Q \frac{\partial u_S}{\partial t^P} + t_1^Q \frac{\partial u_S}{\partial t_1^P} + t_2^Q \frac{\partial u_S}{\partial t_2^P} + t_3^Q \frac{\partial u_S}{\partial t_3^P} + \dots \\
&= t^Q + (t^Q t_1^P + t_1^Q t^P) \\
&+ \left(t^Q \left(2t^P \frac{t_2^P}{2!} + (t_1^P)^2 \right) + 2t_1^Q \cdot t^P t_1^P + \frac{t_2^Q}{2!} \cdot (t^P)^2 \right) + \dots.
\end{aligned}$$

From (115) and (120) we get:

$$\begin{aligned}
(129) \quad u_Q &= u_{S,1} \cdot \left(\sum_{n=0}^{\infty} t_n^R \frac{u_S^n}{n!} + f_0''(u_R) \sum_{n=1}^{\infty} t_n^Q \frac{u_S^{n-1}}{(n-1)!} \right. \\
&\quad \left. + g_0'(u_R) \sum_{n=2}^{\infty} t_n^P \frac{u_S^{n-2}}{(n-2)!} \right).
\end{aligned}$$

Similarly, from (116) and (120) we get:

$$\begin{aligned}
(130) \quad u_P &= u_{S,1} \cdot \left[u_Q \left(\sum_{n=1}^{\infty} t_n^Q \frac{u_S^{n-1}}{(n-1)!} + u_R \sum_{n=2}^{\infty} t_n^P \frac{u_S^{n-2}}{(n-2)!} \right) \right. \\
&\quad + u_R \sum_{n=1}^{\infty} t_n^R \frac{u_S^{n-1}}{(n-1)!} + f'_0(u_R) \sum_{n=2}^{\infty} t_n^Q \frac{u_S^{n-2}}{(n-2)!} \\
&\quad \left. + g_0(u_R) \sum_{n=3}^{\infty} t_n^P \frac{u_S^{n-3}}{(n-3)!} + \sum_{n=0}^{\infty} t_n^S \frac{u_S^n}{n!} \right].
\end{aligned}$$

So all the order parameters have been determined.

4.7. The jet variables as new coordinates. Note u_R is linear in t_n^Q 's, u_Q is linear in t_n^R 's, and u_P is linear in t_n^S 's. For example,

$$\begin{aligned}
u_R &= t^Q \left(1 + t_1^P + (t^P t_2^P + (t_1^P)^2) + (3t_2^P t^P t_1^P + \frac{1}{2} t_3^P (t^P)^2 + (t_1^P)^3) + \dots \right) \\
&\quad + t_1^Q \left(t^P + 2t^P t_1^P + 3t^P (t_1^P)^2 + \frac{3}{2} (t^P)^2 t_2^P + \dots \right) \\
&\quad + \frac{1}{2} t_2^Q \left((t^P)^2 + 3(t^P)^2 t_1^P + \dots \right) + \frac{1}{6} t_3^Q \left((t^P)^3 + \dots \right) + \dots,
\end{aligned}$$

after repeatedly taking $\frac{\partial}{\partial t^P}$, we have:

$$\begin{aligned}
u_{R,1} &= t^Q \left(t_2^P + (3t_2^P t_1^P + t_3^P t^P) + \dots \right) \\
&\quad + t_1^Q \left(1 + 2t_1^P + 3(t_1^P)^2 + 3t^P t_2^P + \dots \right) \\
&\quad + t_2^Q \left(t^P + 3t^P t_1^P + \dots \right) + \frac{1}{2} t_3^Q \left((t^P)^2 + \dots \right) + \dots,
\end{aligned}$$

$$\begin{aligned}
u_{R,2} &= t^Q \left(t_3^P + \dots \right) + t_1^Q \left(3t_2^P + \dots \right) \\
&\quad + t_2^Q \left(1 + 3t_1^P + \dots \right) + t_3^Q \left(t^P + \dots \right) + \dots,
\end{aligned}$$

etc. One can easily see that $u_{S,n}$, $u_{R,s}$, $u_{Q,n}$ and $u_{P,n}$ are equal to t_n^P , t_n^Q , t_n^R and t_n^S respectively when restricted to the small phase space, and so they can be used as new coordinates.

4.8. Renormalized coupling constants. Now we generalize an idea in [37]. Write $U_S = \tilde{U}_S + u_s$, etc, and plug them into the Landau-Ginzburg potential. Let us focus on the following part of W and treat the rest of W as perturbation:

$$\begin{aligned}
(131) \quad W &= U_P \left(-U_S + \sum_{n=0}^{\infty} t_n^P \frac{U_S^n}{n!} \right) \\
&+ U_Q \left(-U_R + \sum_{n=0}^{\infty} t_n^Q \frac{U_S^n}{n!} + U_R \sum_{n=1}^{\infty} t_n^P \frac{U_S^{n-1}}{(n-1)!} \right) \\
&+ U_R \sum_{n=0}^{\infty} t_n^R \frac{U_S^n}{n!} + \sum_{n=0}^{\infty} t_n^S \frac{U_S^{n+1}}{(n+1)!} + \dots .
\end{aligned}$$

Then we get:

$$\begin{aligned}
W &= (\hat{U}_P + u_P) \left(-(\hat{U}_S + u_S) + \sum_{n=0}^{\infty} t_n^P \frac{(\hat{U}_S + u_S)^n}{n!} \right) \\
&+ (\hat{U}_Q + u_Q) \left(-(\hat{U}_R + u_R) + \sum_{n=0}^{\infty} t_n^Q \frac{(\hat{U}_S + u_S)^n}{n!} \right) \\
&+ (\hat{U}_R + u_R) \sum_{n=1}^{\infty} t_n^P \frac{(\hat{U}_S + u_S)^{n-1}}{(n-1)!} \\
&+ (\hat{U}_R + u_R) \sum_{n=0}^{\infty} t_n^R \frac{(\hat{U}_S + u_S)^n}{n!} + \sum_{n=0}^{\infty} t_n^S \frac{(\hat{U}_S + u_S)^{n+1}}{(n+1)!} + \dots .
\end{aligned}$$

so we get:

$$\begin{aligned}
(132) \quad W &= W_0 + \hat{U}_P \left(-\hat{U}_S + \sum_{n=1}^{\infty} \hat{t}_n^P \frac{U_S^n}{n!} \right) \\
&+ U_Q \left(-U_R + \sum_{n=0}^{\infty} t_n^Q \frac{U_S^n}{n!} + U_R \sum_{n=1}^{\infty} t_n^P \frac{U_S^{n-1}}{(n-1)!} \right) \\
&+ U_R \sum_{n=0}^{\infty} t_n^R \frac{U_S^n}{n!} + \sum_{n=0}^{\infty} t_n^S \frac{U_S^{n+1}}{(n+1)!} + \dots ,
\end{aligned}$$

where W_0 is independent of \hat{U}_α , and \hat{t}^α are deformations of t_n^α , for example,

$$\begin{aligned}\hat{t}_n^P &= \sum_{k=0}^{\infty} t_{n+k}^P \frac{u_S^k}{k!}, \\ \hat{t}_n^Q &= \sum_{l=0}^{\infty} (t_{n+l}^Q + u_R t_{n+l+1}^P) \frac{u_S^l}{l!}, \\ \hat{t}_n^R &= \sum_{l=0}^{\infty} (t_{n+l}^R + u_Q t_{n+l+1}^P) \frac{u_S^l}{l!}, \\ \hat{t}_n^S &= \sum_{l=0}^{\infty} (t_{n+l}^S + u_R t_{n+l+1}^R + u_Q (t_{n+l+1}^Q + u_R t_{n+l+2}^P) + u_P t_{n+l+1}^P) \frac{u_S^l}{l!}.\end{aligned}$$

We refer to \hat{t}_n^α as the renormalized coupling constants. They can also be used as new coordinate system on the big phase space.

4.9. The constitutive relations for two-point functions in genus zero. I. Taking $\frac{\partial}{\partial t^P}, \dots, \frac{\partial}{\partial t^S}$ on both sides of (101)-(104), we get:

$$\begin{aligned}S_{P,P}(z) &= e^{t^P/z} \left(\frac{t^S}{z} + \frac{t^Q t^R}{z^2} + \frac{1}{z^3} (t^Q f_0'(t^Q) - 2f_0(t^Q)) \right), \\ S_{Q,P}(z) &= \frac{1}{z} e^{t^P/z} t^R + \frac{1}{z^2} e^{t^P/z} (t^Q f_0''(t^Q) - f_0'(t^Q)), \\ S_{R,P}(z) &= \frac{1}{z} e^{t^P/z} t^Q, \\ S_{S,P}(z) &= e^{t^P/z}.\end{aligned}$$

$$\begin{aligned}S_{P,Q}(z) &= \frac{e^{t^P/z}}{z^2} f_0'(t^Q) + \frac{1}{z} e^{t^P/z} t^R, \\ S_{Q,Q}(z) &= \frac{e^{t^P/z}}{z} f_0''(t^Q), \\ S_{R,Q}(z) &= e^{t^P/z}, \\ S_{S,Q}(z) &= 0.\end{aligned}$$

$$S_{P,R}(z) = \frac{1}{z} e^{t^P/z} t^Q, \quad S_{Q,R}(z) = e^{t^P/z}, \quad S_{R,R}(z) = 0, \quad S_{S,R}(z) = 0.$$

$$S_{P,S}(z) = e^{t^P/z}, \quad S_{Q,S}(z) = 0, \quad S_{R,S}(z) = 0, \quad S_{S,S}(z) = 0.$$

Here we have used the following notation:

$$(133) \quad S_{\alpha,\beta} = \eta_{\alpha\beta} + \sum_{n \geq 0} z^{-n-1} \langle \langle \tau_0(\alpha) \tau_n(\beta) \rangle \rangle_0.$$

The first couple of terms of the expansion of the matrix $(S_{\alpha,\beta})_{\alpha,\beta=P,Q,R,S}$ are

$$\begin{aligned} (S_{\alpha\beta}(z)) &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} + \frac{1}{z} \begin{pmatrix} t^S & t^R & t^Q & t^P \\ t^R & f_0''(t^Q) & t^P & 0 \\ t^Q & t^P & 0 & 0 \\ t^P & 0 & 0 & 0 \end{pmatrix} \\ &+ \frac{1}{z^2} \begin{pmatrix} t^S t^P + t^Q t^R & t^R t^P + t^Q f_0'(t^Q) - 2f(t^Q) & t^Q t^P & \frac{(t^P)^2}{2!} \\ t^R t^P + f_0'(t^Q) & t^P f_0''(t^Q) & \frac{(t^P)^2}{2!} & 0 \\ t^Q t^P & \frac{(t^P)^2}{2!} & 0 & 0 \\ \frac{(t^P)^2}{2!} & 0 & 0 & 0 \end{pmatrix} + \dots \end{aligned}$$

The leading term and the subleading term are the metric matrix and the Hessian matrix in §4.2.

Next we will check some other properties of the matrix $(S_{\alpha,\beta})_{\alpha,\beta=P,Q,R,S}$.

4.10. Quantum differential equations. It is straightforward to check that the following equations are satisfied by $S_{\alpha,\beta}$ on the small phase space:

$$(134) \quad z \frac{\partial}{\partial t^P} \begin{pmatrix} S_{P,\gamma} \\ S_{Q,\gamma} \\ S_{R,\gamma} \\ S_{S,\gamma} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} S_{P,\gamma} \\ S_{Q,\gamma} \\ S_{R,\gamma} \\ S_{S,\gamma} \end{pmatrix},$$

$$(135) \quad z \frac{\partial}{\partial t^Q} \begin{pmatrix} S_{P,\gamma} \\ S_{Q,\gamma} \\ S_{R,\gamma} \\ S_{S,\gamma} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & f_0'''(t^Q) & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} S_{P,\gamma} \\ S_{Q,\gamma} \\ S_{R,\gamma} \\ S_{S,\gamma} \end{pmatrix},$$

$$(136) \quad z \frac{\partial}{\partial t^R} \begin{pmatrix} S_{P,\gamma} \\ S_{Q,\gamma} \\ S_{R,\gamma} \\ S_{S,\gamma} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} S_{P,\gamma} \\ S_{Q,\gamma} \\ S_{R,\gamma} \\ S_{S,\gamma} \end{pmatrix},$$

$$(137) \quad z \frac{\partial}{\partial t^S} \begin{pmatrix} S_{P,\gamma} \\ S_{Q,\gamma} \\ S_{R,\gamma} \\ S_{S,\gamma} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} S_{P,\gamma} \\ S_{Q,\gamma} \\ S_{R,\gamma} \\ S_{S,\gamma} \end{pmatrix},$$

where the matrices on the right-hand sides of these equations are computed in §4.3.

4.11. The genus zero two-point function. One can also check the the following orthogonality relations:

$$(138) \quad S_{\delta,\alpha}(z)\eta^{\delta\epsilon}S_{\epsilon,\beta}(-z) = \eta_{\alpha,\beta}.$$

Let

$$(139) \quad V_{\alpha,\beta}(z_1, z_2) = \frac{1}{z_1 + z_2} \left(\sum_{\delta,\epsilon} \eta^{\delta\epsilon} S_{\delta,\alpha}(z_1) S_{\epsilon,\beta}(z_2) - \eta_{\alpha\beta} \right).$$

It is clear that

$$(140) \quad V_{\alpha,\beta}(z_1, z_2) = V_{\beta,\alpha}(z_2, z_1).$$

One easily get:

$$\begin{aligned} V_{P,P}(z_1, z_2) &= \frac{e^{t^P/z_1+t^P/z_2}}{z_1 z_2} \left(t^S + \left(\frac{1}{z_1} + \frac{1}{z_2} \right) t^Q t^R \right. \\ &\quad + \left(\frac{1}{z_1^2} - \frac{1}{z_1 z_2} + \frac{1}{z_2^2} \right) (t^Q f'(t^Q) - 2f(t^Q)) \\ &\quad \left. + \frac{1}{z_1 z_2} t^Q (t^Q f_0''(t^Q) - f_0'(t^Q)) \right), \end{aligned}$$

$$V_{P,Q}(z_1, z_2) = \frac{e^{t^P/z_1+t^P/z_2}}{z_1 z_2} \left(t^R + \frac{1}{z_1} t^Q f_0''(t^Q) + \left(\frac{1}{z_2} - \frac{1}{z_1} \right) f_0'(t^Q) \right),$$

$$V_{P,R}(z_1, z_2) = \frac{e^{t^P/z_1+t^P/z_2}}{z_1 z_2} t^Q,$$

$$V_{P,S}(z_1, z_2) = \frac{e^{t^P/z_1+t^P/z_2} - 1}{z_1 + z_2}.$$

$$V_{Q,Q}(z_1, z_2) = \frac{e^{t^P/z_1+t^P/z_2}}{z_1 z_2} f_0''(t^Q).$$

$$V_{Q,R}(z_1, z_2) = \frac{e^{t^P/z_1+t^P/z_2} - 1}{z_1 + z_2}.$$

See also Dubrovin [5, Example 5.2]. By changing t^P to u_S , t^Q to u_R , t^R to u_Q and t^S to u_P , one then get the following formula for two-point

function of GW invariants of quintic CY threefold in genus zero on the big phase space:

$$\begin{aligned} \mathcal{V}_{P,P}(z_1, z_2) &= \frac{e^{u_S/z_1+u_S/z_2}}{z_1 z_2} \left(u_P + \left(\frac{1}{z_1} + \frac{1}{z_2} \right) u_Q u_R \right. \\ &\quad + \left(\frac{1}{z_1^2} - \frac{1}{z_1 z_2} + \frac{1}{z_2^2} \right) (u_R f'(u_R) - 2f(u_R)) \\ &\quad \left. + \frac{1}{z_1 z_2} u_R (u_R f_0''(u_R) - f_0'(u_R)) \right), \end{aligned}$$

$$\mathcal{V}_{P,Q}(z_1, z_2) = \frac{e^{u_S/z_1+u_S/z_2}}{z_1 z_2} \left(u_Q + \frac{1}{z_1} u_R f_0''(u_R) + \left(\frac{1}{z_2} - \frac{1}{z_1} \right) f_0'(u_R) \right),$$

$$\mathcal{V}_{P,R}(z_1, z_2) = \frac{e^{u_S/z_1+u_S/z_2}}{z_1 z_2} u_R,$$

$$\mathcal{V}_{P,S}(z_1, z_2) = \frac{e^{u_R/z_1+u_R/z_2} - 1}{z_1 + z_2}.$$

$$\mathcal{V}_{Q,Q}(z_1, z_2) = \frac{e^{u_S/z_1+u_S/z_2}}{z_1 z_2} f_0''(u_R).$$

$$\mathcal{V}_{Q,R}(z_1, z_2) = \frac{e^{u_S/z_1+u_S/z_2} - 1}{z_1 + z_2}.$$

4.12. The integrable hierarchy in genus zero. Recall the system of the hierarchy in genus zero can be written in the following compact form:

$$(141) \quad D_{\beta,z} u_\alpha = (R_{\alpha,0;\beta}(u_0, \dots, u_r; z))',$$

where

$$(142) \quad R_{\alpha,0;\beta}(u_0, \dots, u_r; z) = \eta_{\alpha\beta} + \sum_{n=0}^{\infty} R_{\alpha,0;\beta,n}(u_0, \dots, u_r) z^{-n-1}.$$

and $D_{\beta,z}$ is the *loop operators*:

$$(143) \quad D_{\beta,z} := \sum_{n=0}^{\infty} z^{-n-1} \frac{\partial}{\partial t_n^\beta}.$$

Therefore, using the explicit formula is §4.9, we get:

$$\begin{aligned} R_{P,0;P}(z) &= e^{u_R/z} \left(\frac{u_P}{z} + \frac{u_Q u_R R}{z^2} + \frac{1}{z^3} (u_R f'_0(u_R) - 2f_0(u_R)) \right), \\ R_{Q,0;P}(z) &= \frac{1}{z} e^{u_S/z} u_Q + \frac{1}{z^2} e^{u_S/z} (u_R f''_0(u_R) - f'_0(u_R)), \\ R_{R,0;P}(z) &= \frac{1}{z} e^{u_S/z} u_R, \\ R_{S,P}(z) &= e^{u_S/z}. \end{aligned}$$

$$\begin{aligned} R_{P,0;Q}(z) &= \frac{e^{u_S/z}}{z^2} f'(u_R) + \frac{1}{z} e^{u_S/z} u_Q, \\ R_{Q,0;Q}(z) &= \frac{e^{u_S/z}}{z} f''_0(u_R), \\ R_{R,0;Q}(z) &= e^{u_S/z}, \\ R_{S,0;Q}(z) &= 0. \end{aligned}$$

$$R_{P,0;R}(z) = \frac{1}{z} e^{u_S/z} u_R, \quad R_{Q,0;R}(z) = e^{u_S/z}, \quad R_{R,0;R}(z) = 0, \quad R_{S,0;R}(z) = 0.$$

$$R_{P,0;S}(z) = e^{u_S/z}, \quad R_{Q,0;S}(z) = 0, \quad R_{R,0;S}(z) = 0, \quad R_{S,0;S}(z) = 0.$$

Therefore, the integrable hierarchy in genus zero can be written down explicitly as operations of the loop operators on the order parameters as follows.

$$\begin{aligned} D_{P,z}(u_P) &= \sum_{n \geq 0} z^{-n-1} \frac{\partial u_S}{\partial t_{n,P}} \\ &= \left[\frac{e^{u_S/z}}{z} u_P + \frac{e^{u_S/z}}{z^2} u_Q u_R + \frac{e^{u_S/z}}{z^3} (u_R f'_0(u_R) - f_0(u_R)) \right]'. \end{aligned}$$

$$\begin{aligned} D_{P,z}(u_Q) &= \sum_{n \geq 0} z^{-n-1} \frac{\partial u_R}{\partial t_{n,P}} \\ &= \left[\frac{1}{z} e^{u_S/z} u_Q + \frac{1}{z^2} e^{u_S/z} (u_R f''_0(u_R) - f'_0(u_R)) \right]'. \end{aligned}$$

$$D_{P,z}(u_R) = \sum_{n \geq 0} z^{-n-1} \frac{\partial u_Q}{\partial t_{n,P}} = \left[\frac{1}{z} e^{u_S/z} u_R \right]'.$$

$$D_{P,z}(u_S) = \sum_{n \geq 0} z^{-n-1} \frac{\partial u_P}{\partial t_{n,P}} = \left[e^{u_S/z} \right]'$$

$$D_{Q,z}(u_P) = \sum_{n \geq 0} z^{-n-1} \frac{\partial u_S}{\partial t_{n,Q}} = \left[\frac{1}{z} e^{u_S/z} u_Q + \frac{e^{u_S/z}}{z^2} f_0'(u_R) \right]'$$

$$D_{Q,z}(u_Q) = \sum_{n=0}^{\infty} z^{-n-1} \frac{\partial u_R}{\partial t_{n,Q}} = \left[\frac{e^{u_S/z}}{z} f_0''(u_R) \right]'$$

$$D_{Q,z}(u_R) = \sum_{n=0}^{\infty} z^{-n-1} \frac{\partial u_Q}{\partial t_{n,Q}} = (e^{u_S/z})'$$

$$D_{Q,z}(u_S) = \sum_{n=0}^{\infty} z^{-n-1} \frac{\partial u_P}{\partial t_{n,Q}} = 0.$$

$$D_{R,z}(u_P) = \sum_{n=0}^{\infty} z^{-n-1} \frac{\partial u_S}{\partial t_{n,R}} = \left[\frac{1}{z} e^{u_S/z} u_R \right]'$$

$$D_{R,z}(u_Q) = \sum_{n=0}^{\infty} z^{-n-1} \frac{\partial u_R}{\partial t_{n,R}} = \left[e^{u_S/z} \right]'$$

$$D_{R,z}(u_R) = \sum_{n=0}^{\infty} z^{-n-1} \frac{\partial u_Q}{\partial t_{n,R}} = 0.$$

$$D_{R,z}(u_S) = \sum_{n=0}^{\infty} z^{-n-1} \frac{\partial u_P}{\partial t_{n,R}} = 0.$$

$$D_{S,z}(u_P) = \sum_{n=0}^{\infty} z^{-n-1} \frac{\partial u_S}{\partial t_{n,S}} = \left[e^{u_S/z} \right]'$$

$$D_{S,z}(u_Q) = \sum_{n=0}^{\infty} z^{-n-1} \frac{\partial u_R}{\partial t_{n,S}} = 0.$$

$$D_{S,z}(u_R) = \sum_{n=0}^{\infty} z^{-n-1} \frac{\partial u_Q}{\partial t_{n,S}} = 0.$$

$$D_{S,z}(u_S) = \sum_{n=0}^{\infty} z^{-n-1} \frac{\partial u_P}{\partial t_{n,S}} = 0.$$

4.13. Three-point functions. We now show how to combine the results of the preceding two Subsections to obtain a recursive algorithm to compute n -point functions of genus zero GW invariants of the quintic defined as follows: For $n \geq 3$, define:

$$(144) \quad \begin{aligned} \mathcal{V}_{\alpha_1, \dots, \alpha_n}(z_1, \dots, z_n) &:= D_{\alpha_1, z_1} \cdots D_{\alpha_n, z_n} F_0 \\ &= \sum_{m_1, \dots, m_n \geq 0} z_1^{-m_1-1} \cdots z_n^{-m_n-1} \frac{\partial^n F_0}{\partial t_{m_1}^{\alpha_1} \cdots \partial t_{m_n}^{\alpha_n}}. \end{aligned}$$

Their computation is based on the following observation. When $n \geq 3$, the n -point functions can be computed by applying the loop operators repeatedly on the two-point functions:

$$(145) \quad \mathcal{V}_{\alpha_1, \dots, \alpha_n}(z_1, \dots, z_n) = D_{\alpha_1, z_1} \cdots D_{\alpha_{n-2}, z_{n-2}} \mathcal{V}_{\alpha_{n-1}, \alpha_n}(z_{n-1}, z_n).$$

For example,

$$\begin{aligned} \mathcal{V}_{P,Q,R}(z_1, z_2, z_3) &= D_{P, z_1} \mathcal{V}_{Q,R}(z_2, z_3) \\ &= D_{P, z_1} \frac{e^{u_S/z_2 + u_S/z_3} - 1}{z_2 + z_3} \\ &= \frac{e^{u_S/z_2 + u_S/z_3}}{z_2 + z_3} \cdot \left(\frac{1}{z_2} + \frac{1}{z_3} \right) D_{P, z_1} u_S \\ &= \frac{e^{u_S/z_2 + u_S/z_3}}{z_2 + z_3} \cdot \left(\frac{1}{z_2} + \frac{1}{z_3} \right) \cdot \left[e^{u_S/z_1} \right]'. \end{aligned}$$

After simplification we get:

$$\mathcal{V}_{P,Q,R}(z_1, z_2, z_3) = \frac{e^{u_S/z_1}}{z_1} \cdot \frac{e^{u_S/z_2}}{z_2} \cdot \frac{e^{u_S/z_3}}{z_3} \cdot u'_S.$$

In this way we get the other nonvanishing three-point functions (up to permutations):

$$\begin{aligned} \mathcal{V}_{Q,Q,Q} &= \frac{e^{u_S/z_1}}{z_1} \cdot \frac{e^{u_S/z_2}}{z_2} \cdot \frac{e^{u_S/z_3}}{z_3} \cdot u'_S f_0'''(u_R). \\ \mathcal{V}_{P,Q,Q}(z_1, z_2, z_3) &= \frac{e^{u_S/z_1}}{z_1} \cdot \frac{e^{u_S/z_2}}{z_2} \cdot \frac{e^{u_S/z_3}}{z_3} \\ &\quad \cdot \left(u'_R f_0'''(u_R) + \frac{u_R f_0'''(u_R) u'_S}{z_1} + \frac{f_0''(u_R) u'_S}{z_2} + \frac{f_0''(u_R) u'_S}{z_3} \right). \end{aligned}$$

$$\mathcal{V}_{S,P,S}(z_1, z_2, z_3) = \mathcal{V}_{R,P,S}(z_1, z_2, z_3) = \mathcal{V}_{Q,P,S}(z_1, z_2, z_3) = 0.$$

$$\mathcal{V}_{P,P,S}(z_1, z_2, z_3) = \frac{e^{u_S/z_1}}{z_1} \cdot \frac{e^{u_S/z_2}}{z_2} \cdot \frac{e^{u_S/z_3}}{z_3} \cdot u'_S.$$

$$\mathcal{V}_{P,P,R}(z_1, z_2, z_3) = \frac{e^{u_S/z_1}}{z_1} \cdot \frac{e^{u_S/z_2}}{z_2} \cdot \frac{e^{u_S/z_3}}{z_3} \cdot \left[u'_R + \left(\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} \right) u_R u'_S \right].$$

$$\begin{aligned} \mathcal{V}_{P,P,Q}(z_1, z_2, z_3) &= \frac{e^{u_S/z_1}}{z_1} \cdot \frac{e^{u_S/z_2}}{z_2} \cdot \frac{e^{u_S/z_3}}{z_3} \cdot \left[u'_Q + \left(\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} \right) u_Q u'_S \right. \\ &+ \left. \left(\frac{1}{z_3^2} - \frac{1}{z_1^2} - \frac{1}{z_2^2} \right) u'_S f'_0(u_R) \right. \\ &+ \left. \left(\frac{u'_R}{z_3} + \left(\frac{1}{z_1 z_3} + \frac{1}{z_2 z_3} + \frac{1}{z_1^2} + \frac{1}{z_2^2} \right) u'_S u_R \right) f''_0(u_R) \right. \\ &+ \left. \left(\left(\frac{1}{z_1} + \frac{1}{z_2} \right) u_R u'_R + \frac{u_R^2}{z_1 z_2} \right) f'''_0(u_R) \right]. \end{aligned}$$

We omit a similar formula for $\mathcal{V}_{P,P,P}$.

4.14. A formula for general n -point functions. By repeatedly applying the loop operators on the nonvanishing three-point functions, one gets n -point functions. Among them, we are particularly interested in n -point functions of the form

$$\begin{aligned} &V_{P,\dots,P,Q,R}(z_1, \dots, z_n, z_{n+1}, z_{n+2}) \\ &= V_{P,\dots,P,P,S}(z_1, \dots, z_n, z_{n+1}, z_{n+2}). \end{aligned}$$

To see why we expect some nice formulas for them, let us first do the calculations for $n = 2$ and $n = 3$.

For $n = 2$, we have

$$\begin{aligned}
V_{P,P,Q,R}(z_1, z_2, z_3, z_4) &= D_{P,z_1} \left(\frac{e^{u_S/z_1}}{z_1} \cdot \frac{e^{u_S/z_2}}{z_2} \cdot \frac{e^{u_S/z_3}}{z_3} \cdot u'_S \right) \\
&= \prod_{i=1}^3 \frac{e^{u_S/z_i}}{z_i} \cdot \left(\sum_{i=1}^3 \frac{1}{z_i} \cdot D_{P,z_1} u_S \cdot u'_S + (D_{P,z_1} u_S)' \right) \\
&= \prod_{i=1}^3 \frac{e^{u_S/z_i}}{z_i} \cdot \left(\sum_{i=1}^3 \frac{1}{z_i} \cdot (e^{u_S/z_1})' + (e^{u_S/z_1})'' \right) \\
&= \prod_{i=1}^3 \frac{e^{u_S/z_i}}{z_i} \cdot \left(\sum_{i=1}^3 \frac{1}{z_i} \cdot \frac{e^{u_S/z_1}}{z_1} u'_S + \frac{e^{u_S/z_1}}{z_1^2} (u'_S)^2 + \frac{e^{u_S/z_1}}{z_1} u''_S \right) \\
&= \prod_{i=1}^4 \frac{e^{u_S/z_i}}{z_i} \cdot \left(u''_S + \sum_{i=1}^4 \frac{1}{z_i} (u'_S)^2 \right).
\end{aligned}$$

For $n = 3$, one can similarly find that

$$\begin{aligned}
&V_{P,P,P,Q,R}(z_1, \dots, z_5) \\
&= \prod_{i=1}^5 \frac{e^{u_S/z_i}}{z_i} \cdot \left(u'''_S + 3 \cdot \sum_{i=1}^5 \frac{1}{z_i} \cdot u'_S u''_S + \left(\sum_{i=1}^5 \frac{1}{z_i} \right)^2 (u'_S)^3 \right).
\end{aligned}$$

The situation is similar to Section 7.6 in [37]. One can show that for $n \geq 1$,

$$\begin{aligned}
(146) \quad &\mathcal{V}_{P,\dots,P,Q,R}(z_1, \dots, z_{n+2}) = \mathcal{V}_{P,\dots,P,P,S}(z_1, \dots, z_{n+2}) \\
&= \prod_{i=1}^{n+2} \frac{e^{u_S/z_i}}{z_i} \cdot n! \sum_{\substack{m_1, \dots, m_n \geq 0 \\ \sum_{j=1}^n m_j = n}} \left(\sum_{i=1}^{n+2} \frac{1}{z_i} \right)^{m_1 + \dots + m_n - 1} \prod_{j=1}^n \frac{u_{S,j}^{m_j}}{(j!)^{m_j} m_j!}.
\end{aligned}$$

The coefficients $(1), (1, 1), (1, 3, 1), (1, 4, 3, 6, 1)$ are integer sequence A036040 or A080575 on [31]. They are related to the complete Bell polynomials:

$$(147) \quad \exp \sum_{j=1}^{\infty} \frac{x_j}{j!} t^j = \sum_{n=0}^{\infty} t^n \sum_{\substack{m_1, \dots, m_n \geq 0 \\ \sum_{j=1}^n m_j = n}} \prod_{j=1}^n \frac{x_j^{m_j}}{(j!)^{m_j} m_j!}.$$

A more compact formula is as follows:

$$\begin{aligned}
(148) \quad &\mathcal{V}_{P,\dots,P,Q,R}(z_1, \dots, z_{n+2}) = \mathcal{V}_{P,\dots,P,P,S}(z_1, \dots, z_{n+2}) \\
&= \left(\sum_{i=1}^{n+2} \frac{1}{z_i} \right)^{-1} \cdot \left(\frac{\partial}{\partial t_0^P} \right)^n \left(\prod_{i=1}^{n+2} \frac{e^{u_S/z_i}}{z_i} \right).
\end{aligned}$$

In the same fashion we also get:

$$\begin{aligned}
& \mathcal{V}_{P,\dots,P,R}(z_1, \dots, z_{n+2}) \\
&= \prod_{i=1}^{n+2} \frac{e^{u_S/z_i}}{z_i} \cdot n! \sum_{\substack{l, m_1, \dots, m_n \geq 0 \\ l + \sum_{j=1}^n m_j = n}} \left(\sum_{i=1}^{n+2} \frac{1}{z_i} \right)^{m_1 + \dots + m_n - 1} \\
& \quad \cdot \frac{u_{R,l}}{l!} \prod_{j=1}^n \frac{u_{S,j}^{m_j}}{(j!)^{m_j} m_j!}.
\end{aligned} \tag{149}$$

A more compact formula is as follows:

$$\mathcal{V}_{P,\dots,P,P,R}(z_1, \dots, z_{n+2}) = \left(\sum_{i=1}^{n+2} \frac{1}{z_i} \right)^{-1} \cdot \left(\frac{\partial}{\partial t_0^P} \right)^n \left(u_R \prod_{i=1}^{n+2} \frac{e^{u_S/z_i}}{z_i} \right). \tag{150}$$

For example,

$$\begin{aligned}
& \mathcal{V}_{P,P,R}(z_1, z_2, z_3) = \prod_{i=1}^3 \frac{e^{u_S/z_i}}{z_i} \cdot \left[u'_R + \sum_{i=1}^3 \frac{1}{z_i} u'_S \right]. \\
& \mathcal{V}_{P,P,P,R}(z_1, z_2, z_3, z_4) \\
&= \prod_{i=1}^4 \frac{e^{u_S/z_i}}{z_i} \cdot \left[u_{R,2} + \sum_{i=1}^4 \frac{1}{z_i} (2u_{S,1}u_{R,1} + u_{S,2}u_R) + \left(\sum_{i=1}^4 \frac{1}{z_i} \right)^2 u_{S,1}^2 u_R \right]. \\
& \mathcal{V}_{P,\dots,P,R}(z_1, \dots, z_5) \\
&= \prod_{i=1}^5 \frac{e^{u_S/z_i}}{z_i} \cdot \left[u_{R,3} + \sum_{i=1}^5 \frac{1}{z_i} \cdot (3u_{S,1}u_{R,2} + 3u_{S,2}u_{R,1} + u_{S,3}u_R) \right. \\
& \quad \left. + \left(\sum_{i=1}^5 \frac{1}{z_i} \right)^2 (3u_{S,1}^2 u_{R,1} + 3u_{S,1}u_{S,2}u_R) + \left(\sum_{i=1}^5 \frac{1}{z_i} \right)^3 \cdot u_{S,1}^3 u_R \right].
\end{aligned}$$

4.15. Genus zero free energy of the quintic. Let us now compute the genus zero partition function of the quintic on the big phase space. Recall the general formula

$$F_0(t) = \frac{1}{2} \sum_{m,n=0} R_{\alpha,m;\beta,n} \tilde{t}_m^\alpha \tilde{t}_n^\beta, \tag{151}$$

where $\tilde{t}_m^\alpha = t_m^\alpha - \delta_{m,1} \delta_{\alpha,0}$, and

$$\mathcal{V}_{\alpha,\beta}(z_1, z_2) = \sum_{m,n=0}^{\infty} R_{\alpha,m;\beta,n}(U) z_1^{-m-1} z_2^{-n-1}. \tag{152}$$

Since for the quintic $\mathcal{V}_{\alpha,\beta}(z_1, z_2)$ is already explicitly known from §4.11, so it is possible to explicitly compute F_0 on the big phase space for the quintic. The result is the following:

Theorem 4.5. *We have the following formulas for F_0 of the quintic on the big phase space:*

$$(153) \quad F_0 = \frac{f_0(u_R)}{u_{S,1}^2} + \sum_{m,n=0}^{\infty} \frac{u_S^{m+n+1}}{m+n+1} \frac{\tilde{t}_m^P t_n^S}{m! n!} + \sum_{m,n=0}^{\infty} \frac{u_S^{m+n+1}}{m+n+1} \frac{t_m^Q t_n^R}{m! n!}.$$

$$(154) \quad F_0 = \frac{1}{u_{S,1}^2} \left(f_0(u_R) + \sum_{m,n=0}^{\infty} \frac{\tilde{t}_m^P t_n^S + t_m^Q t_n^R}{m+n+1} u_S \frac{\partial u_S}{\partial t_m^P} \frac{\partial u_S}{\partial t_n^R} \right).$$

Proof. From the expansion:

$$\begin{aligned} & \mathcal{V}_{P,P}(z_1, z_2) \\ &= \frac{e^{u_S/z_1 + u_S/z_2}}{z_1 z_2} \left(u_P + \left(\frac{1}{z_1} + \frac{1}{z_2} \right) u_Q u_R + \left(\frac{1}{z_1^2} - \frac{1}{z_1 z_2} + \frac{1}{z_2^2} \right) (u_R f'(u_R) - 2f(u_R)) \right. \\ & \quad \left. + \frac{1}{z_1 z_2} u_R (u_R f_0''(u_R) - f_0'(u_R)) \right) \\ &= \sum_{m,n=0}^{\infty} \frac{u_S^m u_S^n}{m! n!} z_1^{-m-1} z_2^{-n-1} \left(u_P + \left(\frac{1}{z_1} + \frac{1}{z_2} \right) u_Q u_R \right. \\ & \quad \left. + \left(\frac{1}{z_1^2} - \frac{1}{z_1 z_2} + \frac{1}{z_2^2} \right) (u_R f'(u_R) - 2f(u_R)) \right. \\ & \quad \left. + \frac{1}{z_1 z_2} u_R (u_R f_0''(u_R) - f_0'(u_R)) \right) \\ &= \sum_{m,n=0}^{\infty} z_1^{-m-1} z_2^{-n-1} \left(u_P \frac{u_S^m u_S^n}{m! n!} + u_Q u_R \frac{u_S^{m-1} u_S^n}{(m-1)! n!} \right. \\ & \quad \left. + u_Q u_R \frac{u_S^m u_S^{n-1}}{m! (n-1)!} + (u_R f'(u_R) - 2f(u_R)) \frac{u_S^{m-2} u_S^n}{(m-2)! n!} \right. \\ & \quad \left. - (u_R f'(u_R) - 2f(u_R)) \frac{u_S^{m-1} u_S^{n-1}}{(m-1)! (n-1)!} \right. \\ & \quad \left. + (u_R f'(u_R) - 2f(u_R)) \frac{u_S^m u_S^{n-2}}{m! (n-2)!} \right. \\ & \quad \left. + (u_R^2 f_0''(u_R) - u_R f_0'(u_R)) \frac{u_S^{m-1} u_S^{n-1}}{(m-1)! (n-1)!} \right), \end{aligned}$$

we get:

$$\begin{aligned}
& \sum_{m,n=0}^{\infty} R_{P,m;P,n} \tilde{t}_m^P \tilde{t}_n^P \\
= & u_P \sum_{m=0}^{\infty} \tilde{t}_m^P \frac{u_S^m}{m!} \sum_{n=0}^{\infty} \tilde{t}_n^P \frac{u_S^n}{n!} + u_Q u_R \sum_{m=1}^{\infty} \tilde{t}_m^P \frac{u_S^{m-1}}{(m-1)!} \sum_{n=0}^{\infty} \tilde{t}_n^P \frac{u_S^n}{n!} \\
& + u_Q u_R \sum_{m=0}^{\infty} \tilde{t}_m^P \frac{u_S^m}{m!} \sum_{n=0}^{\infty} \tilde{t}_n^P \frac{u_S^{n-1}}{(n-1)!} \\
& + (u_R f'(u_R) - 2f(u_R)) \sum_{m=2}^{\infty} \tilde{t}_m^P \frac{u_S^{m-2}}{(m-2)!} \sum_{n=0}^{\infty} \tilde{t}_n^P \frac{u_S^n}{n!} \\
& - (u_R f'(u_R) - 2f(u_R)) \sum_{m=1}^{\infty} \tilde{t}_m^P \frac{u_S^{m-1}}{(m-1)!} \sum_{n=1}^{\infty} \tilde{t}_n^P \frac{u_S^{n-1}}{(n-1)!} \\
& + (u_R f'(u_R) - 2f(u_R)) \sum_{m=0}^{\infty} \tilde{t}_m^P \frac{u_S^m}{m!} \sum_{n=2}^{\infty} \tilde{t}_n^P \frac{u_S^{n-2}}{(n-2)!} \\
& + (u_R^2 f_0''(u_R) - u_R f_0'(u_R)) \sum_{m=1}^{\infty} \tilde{t}_m^P \frac{u_S^{m-1}}{(m-1)!} \sum_{n=1}^{\infty} \tilde{t}_n^P \frac{u_S^{n-1}}{(n-1)!} \\
= & \frac{1}{u_{S,1}^2} \left(u_R^2 f_0''(u_R) - 2u_R f_0'(u_R) + 2f_0(u_R) \right).
\end{aligned}$$

Here we have used (113) and (120). From the following expansion:

$$\begin{aligned}
\mathcal{V}_{P,Q}(z_1, z_2) &= \frac{e^{u_S/z_1 + u_S/z_2}}{z_1 z_2} \left(u_Q + \frac{1}{z_1} u_R f_0''(u_R) + \left(\frac{1}{z_2} - \frac{1}{z_1} \right) f_0'(u_R) \right) \\
&= \sum_{m,n=0}^{\infty} \frac{u_S^m}{m!} \frac{u_S^n}{n!} z_1^{-m-1} z_2^{-n-1} \left(u_Q + \frac{1}{z_1} u_R f_0''(u_R) + \left(\frac{1}{z_2} - \frac{1}{z_1} \right) f_0'(u_R) \right) \\
&= \sum_{m,n=0}^{\infty} z_1^{-m-1} z_2^{-n-1} \left(u_Q \frac{u_S^m}{m!} \frac{u_S^n}{n!} + \frac{u_S^{m-1}}{(m-1)!} \frac{u_S^n}{n!} u_R f_0''(u_R) \right. \\
&\quad \left. - f_0'(u_R) \frac{u_S^{m-1}}{(m-1)!} \frac{u_S^n}{n!} + f_0'(u_R) \frac{u_S^m}{m!} \frac{u_S^{n-1}}{(n-1)!} \right),
\end{aligned}$$

we get:

$$\begin{aligned}
& \sum_{m,n=0}^{\infty} R_{P,m;Q,n} \tilde{t}_m^P \tilde{t}_n^Q \\
&= \sum_{m,n=0}^{\infty} \tilde{t}_m^P t_n^Q \left(u_Q \frac{u_S^m u_S^n}{m! n!} + \frac{u_S^{m-1} u_S^n}{(m-1)! n!} u_R f_0''(u_R) \right. \\
&+ \left. f_0'(u_R) \frac{u_S^{m-1} u_S^n}{(m-1)! n!} - f_0'(u_R) \frac{u_S^m u_S^{n-1}}{m! (n-1)!} \right) \\
&= u_Q \sum_{m=0}^{\infty} \tilde{t}_m^P \frac{u_S^m}{m!} \cdot \sum_{n=0}^{\infty} t_n^Q \frac{u_S^n}{n!} + u_R f_0''(u_R) \sum_{m=1}^{\infty} \tilde{t}_m^P \frac{u_S^{m-1}}{(m-1)!} \cdot \sum_{n=0}^{\infty} t_n^Q \frac{u_S^n}{n!} \\
&- f_0'(u_R) \sum_{m=1}^{\infty} \tilde{t}_m^P \frac{u_S^{m-1}}{(m-1)!} \cdot \sum_{n=0}^{\infty} t_n^Q \frac{u_S^n}{n!} - f_0'(u_R) \sum_{m=1}^{\infty} \tilde{t}_m^P \frac{u_S^m}{m!} \cdot \sum_{n=1}^{\infty} t_n^Q \frac{u_S^{n-1}}{(n-1)!} \\
&= \frac{1}{u_{S,1}^2} \left(-u_R^2 f_0''(u_R) + u_R f_0'(u_R) \right).
\end{aligned}$$

Here we have used (113) and (121). From the following expansion:

$$\mathcal{V}_{P,R}(z_1, z_2) = \frac{e^{u_S/z_1 + u_S/z_2}}{z_1 z_2} u_R = u_R \sum_{m,n=0}^{\infty} \frac{u_S^m u_S^n}{m! n!} z_1^{-m-1} z_2^{-n-1},$$

we get:

$$\sum_{m,n=0}^{\infty} R_{P,m;R,n} \tilde{t}_m^P \tilde{t}_n^R = u_R \sum_{m,n=0}^{\infty} \frac{u_S^m u_S^n}{m! n!} \tilde{t}_m^P t_n^R = u_R \sum_{m=0}^{\infty} \tilde{t}_m^P \frac{u_S^m}{m!} \cdot \sum_{n=0}^{\infty} t_n^R \frac{u_S^n}{n!} = 0.$$

From the following expansion:

$$\begin{aligned}
\mathcal{V}_{P,S}(z_1, z_2) &= \frac{e^{u_R/z_1 + u_R/z_2} - 1}{z_1 + z_2} = \frac{1}{z_1 z_2} \sum_{M=1}^{\infty} \left(\frac{1}{z_1} + \frac{1}{z_2} \right)^{M-1} \frac{u_S^M}{M!} \\
&= \sum_{m,n=0}^{\infty} \frac{u_S}{m+n+1} \frac{u_S^m u_S^n}{m! n!} z_1^{-m-1} z_2^{-n-1},
\end{aligned}$$

we get:

$$\sum_{m,n=0}^{\infty} R_{P,m;S,n} \tilde{t}_m^P \tilde{t}_n^S = \sum_{m,n=0}^{\infty} \frac{u_S^{m+n+1}}{m+n+1} \frac{\tilde{t}_m^P t_n^S}{m! n!}.$$

From the following expansion:

$$\mathcal{V}_{Q,Q}(z_1, z_2) = \frac{e^{u_S/z_1 + u_S/z_2}}{z_1 z_2} f_0''(u_R) = f_0''(u_R) \sum_{m,n=0}^{\infty} \frac{u_S^m u_S^n}{m! n!} z_1^{-m-1} z_2^{-n-1},$$

we get:

$$\begin{aligned} \sum_{m,n=0}^{\infty} R_{Q,m;Q,n} \tilde{t}_m^Q \tilde{t}_n^Q &= \sum_{m,n=0}^{\infty} \frac{u_S^m u_S^n}{m! n!} f_0''(u_R) t_m^Q t_n^Q \\ &= f_0''(u_R) \left(\sum_{m=0}^{\infty} t_m^Q \frac{u_S^m}{m!} \right)^2 = \frac{1}{u_{S,1}^2} u_R^2 f_0''(u_R). \end{aligned}$$

Here we have used (121). From the following expansion:

$$\mathcal{V}_{Q,R}(z_1, z_2) = \frac{e^{u_S/z_1 + u_S/z_2} - 1}{z_1 + z_2} = \sum_{m,n=0}^{\infty} \frac{u_S}{m+n+1} \frac{u_S^m}{m!} \frac{u_S^n}{n!},$$

we get:

$$\sum_{m,n=0}^{\infty} R_{Q,m;R,n} \tilde{t}_m^Q \tilde{t}_n^R = \sum_{m,n=0}^{\infty} \frac{u_S^{m+n+1}}{m+n+1} \frac{t_m^Q}{m!} \frac{t_n^R}{n!}.$$

The proof is finished by putting all these together.

$$\begin{aligned} F_0 &= \frac{1}{2} \frac{1}{u_{S,1}^2} \left(u_R^2 f_0''(u_R) - 2u_R f_0'(u_R) + 2f_0(u_R) \right) \\ &+ \frac{1}{u_{S,1}^2} \left(-u_R^2 f_0''(u_R) + u_R f_0'(u_R) \right) + \frac{1}{2} \frac{1}{u_{S,1}^2} u_R^2 f_0''(u_R) \\ &+ \sum_{m,n=0}^{\infty} \frac{u_S^{m+n+1}}{m+n+1} \frac{\tilde{t}_m^P}{m!} \frac{t_n^S}{n!} + \sum_{m,n=0}^{\infty} \frac{u_S^{m+n+1}}{m+n+1} \frac{t_m^Q}{m!} \frac{t_n^R}{n!} \\ &= \frac{f_0(u_R)}{u_{S,1}^2} + \sum_{m,n=0}^{\infty} \frac{u_S^{m+n+1}}{m+n+1} \frac{\tilde{t}_m^P}{m!} \frac{t_n^S}{n!} + \sum_{m,n=0}^{\infty} \frac{u_S^{m+n+1}}{m+n+1} \frac{t_m^Q}{m!} \frac{t_n^R}{n!}. \end{aligned}$$

This proves the first formula. The second formula is proved by applying (128). \square

4.16. Genus zero one-point functions.

Theorem 4.6. *For the quintic the following formulas for the one-point functions in genus zero hold:*

$$\begin{aligned} (155) \quad D_{P,z} F_0 &= \sum_{m,n=0}^{\infty} \frac{u_S^{m+n+1}}{m+n+1} \frac{t_m^S}{m!} \frac{z^{-n-1}}{n!} \\ &+ \frac{e^{u_S/z}}{z} \cdot \left[\frac{u_Q u_R}{u_{S,1}} - \frac{u_R f_0''(u_R) - f_0'(u_R)}{u_{S,1}^2} u_{R,1} \right. \\ &\quad \left. + \frac{u_R f_0'(u_R) - 2f_0(u_R)}{u_{S,1}} \left(\frac{u_{S,2}}{u_{S,1}^2} + \frac{1}{z} \right) \right]. \end{aligned}$$

$$(156) \quad D_{Q,z}F_0 = \frac{f'_0(u_R)}{u_{S,1}} \cdot \frac{e^{u_S/z}}{z} + \sum_{m,n=0}^{\infty} \frac{u_S^{m+n+1}}{m+n+1} \frac{t_m^R}{m!} \frac{z^{-n-1}}{n!}.$$

$$(157) \quad D_{R,z}F_0 = \sum_{m,n=0}^{\infty} \frac{u_S^{m+n+1}}{m+n+1} \frac{t_m^Q}{m!} \frac{z^{-n-1}}{n!}.$$

$$(158) \quad D_{S,z}F_0 = \sum_{m,n=0}^{\infty} \frac{u_S^{m+n+1}}{m+n+1} \frac{\tilde{t}_m^P}{m!} \frac{z^{-n-1}}{n!}.$$

Proof. Note we have

$$\begin{aligned} D_{S,z}u_S &= D_{S,z}u_R = 0, \\ D_{S,z}\tilde{t}_n^P &= D_{S,z}t_n^Q = D_{S,z}t_n^R = 0, & D_{S,z}t_n^S &= z^{-n-1}. \end{aligned}$$

So we get:

$$\begin{aligned} D_{S,z}F_0 &= D_{S,z} \left(\frac{f_0(u_R)}{u_{S,1}^2} + \sum_{m,n=0}^{\infty} \frac{u_S^{m+n+1}}{m+n+1} \frac{\tilde{t}_m^P}{m!} \frac{t_n^S}{n!} + \sum_{m,n=0}^{\infty} \frac{u_S^{m+n+1}}{m+n+1} \frac{t_m^Q}{m!} \frac{t_n^R}{n!} \right) \\ &= \sum_{m,n=0}^{\infty} \frac{u_S^{m+n+1}}{m+n+1} \frac{\tilde{t}_m^P}{m!} \frac{z^{-n-1}}{n!}. \end{aligned}$$

Similar, we have

$$\begin{aligned} D_{R,z}u_S &= D_{R,z}u_R = 0, \\ D_{R,z}\tilde{t}_n^P &= D_{R,z}t_n^Q = D_{R,z}t_n^S = 0, & D_{R,z}t_n^R &= z^{-n-1}. \end{aligned}$$

So we get

$$\begin{aligned} D_{R,z}F_0 &= D_{R,z} \left(\frac{f_0(u_R)}{u_{S,1}^2} + \sum_{m,n=0}^{\infty} \frac{u_S^{m+n+1}}{m+n+1} \frac{\tilde{t}_m^P}{m!} \frac{t_n^S}{n!} + \sum_{m,n=0}^{\infty} \frac{u_S^{m+n+1}}{m+n+1} \frac{t_m^Q}{m!} \frac{t_n^R}{n!} \right) \\ &= \sum_{m,n=0}^{\infty} \frac{u_S^{m+n+1}}{m+n+1} \frac{t_m^Q}{m!} \frac{z^{-n-1}}{n!}. \end{aligned}$$

Similar, we have

$$\begin{aligned} D_{Q,z}u_S &= 0, & D_{Q,z}u_R &= \frac{e^{u_S/z}}{z} u_{S,1}, \\ D_{Q,z}\tilde{t}_n^P &= D_{R,z}t_n^R = D_{R,z}t_n^S = 0, & D_{Q,z}t_n^Q &= z^{-n-1}. \end{aligned}$$

So we get

$$\begin{aligned} D_{Q,z}F_0 &= D_{Q,z}\left(\frac{f_0(u_R)}{u_{S,1}^2} + \sum_{m,n=0}^{\infty} \frac{u_S^{m+n+1}}{m+n+1} \frac{\tilde{t}_m^P t_n^S}{m! n!} + \sum_{m,n=0}^{\infty} \frac{u_S^{m+n+1}}{m+n+1} \frac{t_m^Q t_n^R}{m! n!}\right) \\ &= \frac{f_0'(u_R)}{u_{S,1}} \cdot \frac{e^{u_S/z}}{z} + \sum_{m,n=0}^{\infty} \frac{u_S^{m+n+1}}{m+n+1} \frac{t_m^R}{m!} \frac{z^{-n-1}}{n!}. \end{aligned}$$

Finally, we have:

$$\begin{aligned} D_{P,z}u_S &= \frac{e^{u_S/z}}{z} u_{S,1}, & D_{P,z}u_R &= \frac{e^{u_S/z}}{z} (u_{R,1} + u_R \frac{u_{S,1}}{z}), \\ D_{P,z}t_n^Q &= D_{R,z}t_n^R = D_{R,z}t_n^S = 0, & D_{P,z}\tilde{t}_n^P &= z^{-n-1}. \end{aligned}$$

So we get

$$\begin{aligned} &D_{P,z}F_0 \\ &= D_{P,z}\left(\frac{f_0(u_R)}{u_{S,1}^2} + \sum_{m,n=0}^{\infty} \frac{u_S^{m+n+1}}{m+n+1} \frac{\tilde{t}_m^P t_n^S}{m! n!} + \sum_{m,n=0}^{\infty} \frac{u_S^{m+n+1}}{m+n+1} \frac{t_m^Q t_n^R}{m! n!}\right) \\ &= \frac{f_0'(u_R)}{u_{S,1}^2} \cdot \frac{e^{u_S/z}}{z} (u_{R,1} + u_R \frac{u_{S,1}}{z}) - 2 \frac{f_0(u_R)}{u_{S,1}^3} \cdot \left(\frac{e^{u_S/z}}{z} u_{S,1}\right)' \\ &+ \sum_{m,n=0}^{\infty} \frac{u_S^{m+n+1}}{m+n+1} \frac{t_m^S}{m!} \frac{z^{-n-1}}{n!} \\ &+ \sum_{m,n=0}^{\infty} u_S^{m+n} \cdot \frac{e^{u_S/z_1}}{z_1} u_{S,1} \cdot \frac{\tilde{t}_m^P t_n^S}{m! n!} + \sum_{m,n=0}^{\infty} u_S^{m+n} \cdot \frac{e^{u_S/z_1}}{z_1} u_{S,1} \cdot \frac{t_m^Q t_n^R}{m! n!} \\ &= \sum_{m,n=0}^{\infty} \frac{u_S^{m+n+1}}{m+n+1} \frac{t_m^S}{m!} \frac{z^{-n-1}}{n!} \\ &+ \frac{e^{u_S/z}}{z} \left(\frac{f_0'(u_R)}{u_{S,1}^2} \cdot u_{R,1} - 2 \frac{f_0(u_R)}{u_{S,1}^3} \cdot u_{S,2} + \frac{1}{z} \frac{u_R f_0'(u_R) - 2f_0(u_R)}{u_{S,1}} \right) \\ &+ \frac{e^{u_S/z}}{z} u_{S,1} \cdot \sum_{m=0}^{\infty} \frac{t_m^Q}{m!} u_S^m \cdot \sum_{n=0}^{\infty} \frac{t_n^R}{n!} u_S^n. \end{aligned}$$

The summations on the third line of the right-hand side of the last equality can be performed as follows. By (121) we have

$$(159) \quad \sum_{n=0}^{\infty} \frac{t_n^Q}{n!} u_S^n = \frac{u_R}{u_{S,1}}.$$

The summations on the third line of the right-hand side of the last equality can be performed as follows. By (129) we have

$$\sum_{n=0}^{\infty} \frac{t_n^Q}{n!} u_S^n = \frac{u_Q}{u_{S,1}} - f_0''(u_R) \sum_{n=1}^{\infty} t_n^Q \frac{u_S^{n-1}}{(n-1)!} - g_0'(u_R) \sum_{n=2}^{\infty} t_n^P \frac{u_S^{n-2}}{(n-2)!}.$$

Take $\frac{\partial}{\partial t^P}$ on both sides of eqrefeqn:Sum-t-Qn, we get:

$$\sum_{n=1}^{\infty} \frac{t_n^Q}{(n-1)!} u_S^{n-1} \cdot u_{S,1} = \frac{u_{R,1}}{u_{S,1}} - \frac{u_R}{u_{S,1}^2} \cdot u_{S,2},$$

and so

$$(160) \quad \sum_{n=1}^{\infty} \frac{t_n^Q}{(n-1)!} u_S^{n-1} = \frac{u_{R,1}}{u_{S,1}^2} - \frac{u_R}{u_{S,1}^3} \cdot u_{S,2}.$$

From (120) we get:

$$(161) \quad \sum_{n=1}^{\infty} t_n^P \frac{u_S^{n-1}}{(n-1)!} = 1 - \frac{1}{u_{S,1}}.$$

After taking $\frac{\partial}{\partial t^P}$ and dividing by $u_{S,1}$ on both sides we get:

$$(162) \quad \sum_{n=2}^{\infty} t_n^P \frac{u_S^{n-2}}{(n-2)!} = \frac{u_{S,2}}{u_{S,1}^3}.$$

Also note:

$$g_0'(u_R) = u_R f_0''(u_R) - f_0'(u_R),$$

therefore,

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{t_n^Q}{n!} u_S^n \\ &= \frac{u_Q}{u_{S,1}} - f_0''(u_R) \cdot \left(\frac{u_{R,1}}{u_{S,1}^2} - \frac{u_R}{u_{S,1}^3} \cdot u_{S,2} \right) - (u_R f_0''(u_R) - f_0'(u_R)) \frac{u_{S,2}}{u_{S,1}^3} \\ &= \frac{u_Q}{u_{S,1}} - f_0''(u_R) \cdot \frac{u_{R,1}}{u_{S,1}^2} + f_0'(u_R) \frac{u_{S,2}}{u_{S,1}^3}. \end{aligned}$$

Now we can resume the computation of $D_{P,z}F_0$:

$$\begin{aligned}
& D_{P,z}F_0 \\
&= \sum_{m,n=0}^{\infty} \frac{u_S^{m+n+1}}{m+n+1} \frac{t_m^S}{m!} \frac{z^{-n-1}}{n!} \\
&+ \frac{e^{u_S/z}}{z} \left(\frac{f'_0(u_R)}{u_{S,1}^2} \cdot u_{R,1} - 2 \frac{f_0(u_R)}{u_{S,1}^3} \cdot u_{S,2} + \frac{1}{z} \frac{u_R f'_0(u_R) - 2f_0(u_R)}{u_{S,1}} \right) \\
&+ \frac{e^{u_S/z}}{z} \cdot u_R \cdot \left(\frac{u_Q}{u_{S,1}} - \frac{f''_0(u_R)u_{R,1}}{u_{S,1}^2} + \frac{f'_0(u_R)}{u_{S,1}^3} u_{S,2} \right) \\
&= \sum_{m,n=0}^{\infty} \frac{u_S^{m+n+1}}{m+n+1} \frac{t_m^S}{m!} \frac{z^{-n-1}}{n!} \\
&+ \frac{e^{u_S/z}}{z} \cdot \left[\frac{u_Q u_R}{u_{S,1}} - \frac{u_R f''_0(u_R) - f'_0(u_R)}{u_{S,1}^2} u_{R,1} \right. \\
&+ \left. \frac{u_R f'_0(u_R) - 2f_0(u_R)}{u_{S,1}} \left(\frac{u_{S,2}}{u_{S,1}^2} + \frac{1}{z} \right) \right].
\end{aligned}$$

□

Note for $m \geq 0$,

$$\sum_{n=0}^{\infty} \frac{u_S^{n+1}}{m+n+1} \frac{z^{-n-1}}{n!} = e^{u_S/z} \sum_{j=0}^m (-1)^j \frac{m!}{(m-j)!} \left(\frac{z}{u_S} \right)^j - (-1)^m m! \frac{z^m}{u_S^m},$$

so we get:

$$\begin{aligned}
D_{S,z}F_0 &= \sum_{m=0}^{\infty} \frac{\tilde{t}_m^P u_S^m}{m!} \left(e^{u_S/z} \sum_{j=0}^m (-1)^j \frac{m!}{(m-j)!} \left(\frac{z}{u_S} \right)^j - (-1)^m m! \frac{z^m}{u_S^m} \right) \\
&= e^{u_S/z} \sum_{m=0}^{\infty} \tilde{t}_m^P \sum_{j=0}^m (-1)^j \frac{u_S^{m-j}}{(m-j)!} z^j - \sum_{m=0}^{\infty} (-1)^m \tilde{t}_m^P z^m \\
&= e^{u_S/z} \sum_{j=0}^{\infty} (-z)^j \sum_{k=0}^{\infty} \tilde{t}_{j+k}^P \frac{u_S^k}{k!} - \sum_{m=0}^{\infty} (-1)^m \tilde{t}_m^P z^m.
\end{aligned}$$

We can also rewrite other $\frac{\partial F_0}{\partial t^\alpha}$ in the same fashion.

5. MEAN FIELD THEORY COMPUTATIONS FOR GW THEORY OF THE QUINTIC IN GENUS $g \geq 1$

In this Section we extend the discussions to higher genera. Again we will see that the selection rules play a crucial role.

We will also write down the integrable hierarchy associated with the GW theory of the quintic in all genera.

5.1. Degree zero part of the free energy vs. the instanton part of the free energy. We will split the free energy in genus $g \geq 0$ into two parts:

$$(163) \quad F_h = F_{g,0} + F_{g,inst},$$

where $F_{g,0}$ is the total contributions of the GW invariants of degree zero (called the degree zero part of the free energy), and $F_{g,inst}$ is the total contributions of the GW invariants of positive degrees (called the instanton part of the free energy).

The degree 0 gravitational descendent invariants of X are the integrals:

$$(164) \quad \begin{aligned} & \langle \tau_{k_1}(\mathcal{O}_{\alpha_1}) \cdots \tau_{k_n}(\mathcal{O}_{\alpha_n}) \rangle_{g,n;0}^X \\ &= \int_{X \times \overline{\mathcal{M}}_{g,n}} \mathcal{O}_{\alpha_1} \cdots \mathcal{O}_{\alpha_n} \psi_1^{k_1} \cdots \psi_n^{k_n} \cup e(TX \boxtimes \mathbb{E}^\vee), \end{aligned}$$

where $\mathbb{E} \rightarrow \overline{\mathcal{M}}_{g,n}$ is the Hodge bundle. When $g = 0$, \mathbb{E} is trivial, and the only degree zero contributions are given by:

$$(165) \quad \langle \tau_{k_1}(\mathcal{O}_{\alpha_1}) \cdots \tau_{k_n}(\mathcal{O}_{\alpha_n}) \rangle_{0,n;0}^X = \int_X \mathcal{O}_{\alpha_1} \cdots \mathcal{O}_{\alpha_n} \cdot \int_{\overline{\mathcal{M}}_{0,n}} \psi_1^{k_1} \cdots \psi_n^{k_n}.$$

By Theorem 4.5 we have seen that:

$$(166) \quad \begin{aligned} F_{0,0} &= \frac{1}{u_{S,1}^2} \cdot \frac{5}{6} u_R^3 + \sum_{m,n=0}^{\infty} \frac{u_S^{m+n+1}}{(m+n+1)m!n!} (\tilde{t}_m^P t_n^S + t_m^Q t_n^R) \\ &= \frac{1}{u_{S,1}^2} \left(\frac{5}{6} u_R^3 + \sum_{m,n=0}^{\infty} \frac{\tilde{t}_m^P t_n^S + t_m^Q t_n^R}{m+n+1} u_S \frac{\partial u_S}{\partial t_m^P} \frac{\partial u_S}{\partial t_n^R} \right), \end{aligned}$$

and

$$(167) \quad F_{0,inst} = \frac{1}{u_{S,1}^2} \left(f_0(u_R) - \frac{5}{6} u_R^3 \right).$$

5.2. Degree zero contribution to genus one free energy. The $g = 1$ case is also exceptional:

$$(168) \quad e(TX \boxtimes \mathbb{E}^\vee) = c_3(X) - c_2(X)\lambda_1,$$

and for $g > 1$,

$$(169) \quad e(TX \boxtimes \mathbb{E}^\vee) = \frac{(-1)^g}{2} (c_3(X) - c_2(X)c_1(X)) \lambda_{g-1}^3.$$

For the quintic, the total Chern class is given by:

$$c(X) = (1 + H)^5(1 + 5H)^{-1}|_X = (1 + 10H^2 - 40H^3)|_X.$$

and so

$$c_1(X) = 0, \quad c_2(X) = 10H^2|_X, \quad c_3(X) = -40H^3|_X.$$

So in genus one we have

$$\begin{aligned} & \langle \tau_{k_1}(\mathcal{O}_{\alpha_1}) \cdots \tau_{k_n}(\mathcal{O}_{\alpha_n}) \rangle_{1,n;0}^X \\ &= \int_{X \times \overline{\mathcal{M}}_{1,n}} \mathcal{O}_{\alpha_1} \cdots \mathcal{O}_{\alpha_n} \psi_1^{k_1} \cdots \psi_n^{k_n} \cup (-40H^3 + 10H^2\lambda_1) \\ &= - \int_X \mathcal{O}_{\alpha_1} \cdots \mathcal{O}_{\alpha_n} \cup 40H^3 \cdot \int_{\overline{\mathcal{M}}_{1,n}} \psi_1^{k_1} \cdots \psi_n^{k_n} \\ &+ \int_X \mathcal{O}_{\alpha_1} \cdots \mathcal{O}_{\alpha_n} \cup 10H^2 \cdot \int_{\overline{\mathcal{M}}_{1,n}} \psi_1^{k_1} \cdots \psi_n^{k_n} \lambda_1. \end{aligned}$$

So either

$$\begin{aligned} \deg \mathcal{O}_{\alpha_1} + \cdots + \deg \mathcal{O}_{\alpha_n} &= 0, \\ k_1 + \cdots + k_n &= n, \end{aligned}$$

or

$$\begin{aligned} \deg \mathcal{O}_{\alpha_1} + \cdots + \deg \mathcal{O}_{\alpha_n} &= 1, \\ k_1 + \cdots + k_n &= n - 1, \end{aligned}$$

Restricted to the small phase space, i.e., when $k_1 = \cdots = k_n = 0$, we must have $n = 1$ and $\deg \mathcal{O}_{\alpha_1} = 1$, this gives us

$$(170) \quad \langle Q \rangle_{1,1;0}^X = \int_X Q \cup 10H^2 \cdot \int_{\overline{\mathcal{M}}_{1,1}} \lambda_1 = 50 \cdot \frac{1}{24} = \frac{25}{12},$$

and so there is a contribution of $\frac{25}{12}t^Q$ to F_1^{small} .

On the big phase space we have

$$\begin{aligned} & \langle \tau_{k_1}(Q) \tau_{k_2}(P) \cdots \tau_{k_n}(P) \rangle_{1,0}^X = \int_X H \cup 10H^2 \cdot \int_{\overline{\mathcal{M}}_{1,n}} \psi_1^{k_1} \cdots \psi_n^{k_n} \lambda_1 \\ &= \frac{50}{24} \binom{n-1}{k_1, \dots, k_n}. \end{aligned}$$

Their total contributions to $F_{1,0}$ is:

$$\begin{aligned} & \sum_{n=1}^{\infty} \sum_{k_1+\dots+k_n=n-1} \frac{50}{24} \binom{n-1}{k_1, \dots, k_n} t_{k_1}^Q \frac{1}{(n-1)!} t_{k_2}^P \cdots t_{k_n}^P \\ &= \frac{50}{24} \sum_{n=1}^{\infty} \sum_{k_1+\dots+k_n=n-1} \frac{t_{k_1}^Q}{k_1!} \cdot \frac{t_{k_2}^P}{k_2!} \cdots \frac{t_{k_n}^P}{k_n!} = \frac{25}{12} u_R. \end{aligned}$$

For the other case,

$$\begin{aligned} & \langle \tau_{k_1}(P) \cdots \tau_{k_n}(P) \rangle_{1,0}^X = - \int_X 40H^3 \cdot \int_{\overline{\mathcal{M}}_{1,n}} \psi_1^{k_1} \cdots \psi_n^{k_n} \\ &= -200 \int_{\overline{\mathcal{M}}_{1,n}} \psi_1^{k_1} \cdots \psi_n^{k_n}. \end{aligned}$$

It vanishes unless

$$(171) \quad k_1 + \cdots + k_n = n.$$

Their total contributions to $F_{1,0}$ is:

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{t_{k_1}^P \cdots t_{k_n}^P}{n!} \langle \tau_{k_1}(P) \cdots \tau_{k_n}(P) \rangle_{1,0}^X \\ &= -200 \sum_{n=1}^{\infty} \frac{t_{k_1}^P \cdots t_{k_n}^P}{n!} \int_{\overline{\mathcal{M}}_{1,n}} \psi_1^{k_1} \cdots \psi_n^{k_n} \\ &= -200 \cdot \frac{1}{24} \log u_{S,1}. \end{aligned}$$

So we get for the quintic the following formula for the degree zero part $F_{1,0}$ of the free energy in genus one:

$$(172) \quad F_{1,0} = -\frac{25}{3} \log u_{S,1} + \frac{25}{12} u_R.$$

5.3. Degree zero contribution to free energy in genus $g > 1$.

Now by (169), when $g > 1$,

$$\begin{aligned} & \langle \tau_{k_1}(\mathcal{O}_{\alpha_1}) \cdots \tau_{k_n}(\mathcal{O}_{\alpha_n}) \rangle_{g,0}^X \\ &= \int_{X \times \overline{\mathcal{M}}_{g,n}} \mathcal{O}_{\alpha_1} \cdots \mathcal{O}_{\alpha_n} \psi_1^{k_1} \cdots \psi_n^{k_n} \cup \frac{(-1)^g}{2} (-40H^3) \lambda_{g-1}^3 \\ &= - \int_X \mathcal{O}_{\alpha_1} \cdots \mathcal{O}_{\alpha_n} \cup 40H^3 \cdot \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1} \cdots \psi_n^{k_n} \lambda_{g-1}^3. \end{aligned}$$

Therefore, we can only have $\mathcal{O}_{\alpha_1} = \cdots = \mathcal{O}_{\alpha_n} = P$, and

$$\begin{aligned} \langle \tau_{k_1}(P) \cdots \tau_{k_n}(P) \rangle_{g,0}^X &= - \int_X 40H^3 \cdot \int_{\overline{\mathcal{M}}_{1,n}} \psi_1^{k_1} \cdots \psi_n^{k_n} \lambda_{g-1}^3 \\ &= -200 \int_{\overline{\mathcal{M}}_{1,n}} \psi_1^{k_1} \cdots \psi_n^{k_n} \lambda_{g-1}^3. \end{aligned}$$

Therefore, for $g \geq 2$,

$$\begin{aligned} F_{g,0} &= \sum_{n=1}^{\infty} \frac{t_{k_1}^P \cdots t_{k_n}^P}{n!} \langle \tau_{k_1}(P) \cdots \tau_{k_n}(P) \rangle_{g,0}^X \\ &= -200 \sum_{n=1}^{\infty} \frac{t_{k_1}^P \cdots t_{k_n}^P}{n!} \int_{\overline{\mathcal{M}}_{1,n}} \psi_1^{k_1} \cdots \psi_n^{k_n} \lambda_{g-1}^3 \\ &= -200 \cdot \frac{1}{2(2g-2)!} \frac{|B_{2g-2}|}{2g-2} \frac{|B_{2g}|}{2g} u_{S,1}^{2g-2}. \end{aligned}$$

Here in the second equality we have used a result in Dubrovin-Liu-Yang-Zhang [12, Cor. 3.10].

5.4. Instanton part of the free energy in genus ≥ 1 . We will first consider $F_{g, \text{instanton}}$ restricted to the small phase space for $g \geq 1$. Then we need to consider the correlators

$$(173) \quad \langle P^{m_0} Q^{m_1} R^{m_2} S^{m_3} \rangle_{g, m_0 + \cdots + m_3; d}$$

By the selection rules (4), we have

$$(174) \quad m_0 = m_2 + 2m_3.$$

By the string equation we must have $m_0 = m_2 = m_3 = 0$ to get a nonzero correlator

$$(175) \quad \langle Q^{m_1} \rangle_{g, m_1; d} = d^{m_1} \langle 1 \rangle_{g, 0; d} = d^{m_1} N_{g, d},$$

where

$$(176) \quad N_{g, d} := \int_{[\overline{\mathcal{M}}_{g,0}(X, d)]^{\text{virt}}} 1.$$

So the instanton part of the free energy in genus $g \geq 1$ restricted to the small phase space is given by

$$(177) \quad \sum_{d=1}^{\infty} q^d \sum_{m_1=0}^{\infty} \frac{(t^Q)^{m_1}}{m_1!} \langle Q^{m_1} \rangle_{g, m_1; d} = \sum_{d=1}^{\infty} q^d N_{g, d} e^{dt^Q}.$$

Next we will show that the selection rules (78) and (79) implies the following:

Proposition 5.1. *For $g \geq 1$,*

$$(178) \quad F_{g, \text{instanton}} = u_{S,1}^{2g-2} \sum_{d=1}^{\infty} q^d N_{g,d} e^{du_R}.$$

Proof. First recall $F_{g, \text{instanton}}$ as a formal power series in $\{t_n^\alpha\}$ is homogeneous of degree 0, where the degree of t_n^α is given by (78). By changing to the jet variables $\{u_{\alpha,n}\}$, it is also homogeneous of degree zero, where

$$(179) \quad \deg u_{S,n} = n - 1, \quad \deg u_{R,n} = n,$$

$$(180) \quad \deg u_{Q,n} = n + 1, \quad \deg u_{P,n} = n + 2.$$

In Eguchi-Getzler-Xiong [14] it has been shown that one can rewrite the puncture operator as

$$(181) \quad L_{-1} = -\frac{\partial}{\partial u_S}.$$

So we know that F_g does not depend on u_S when $g \geq 1$, and it depends only on the two degree zero jet variables $u_{S,1}$ and u_R . To find the explicit expression, let us first consider the correlators of the form:

$$(182) \quad \langle P^{m_0} \tau_1(P)^n Q^{m_1} R^{m_2} S^{m_3} \rangle_{g, n+m_0+\dots+m_3; d}.$$

By the selection rules (4),

$$(183) \quad m_0 = m_2 + 2m_3.$$

By the string equation we must have $m_0 = m_2 = m_3 = 0$ to get a nonzero correlator of the form:

$$\begin{aligned} & \langle \tau_1(P)^n Q^{m_1} \rangle_{g, m_1; d} = (2g - 2 + m_1 + n - 1) \langle \tau_1(P)^{n-1} Q^{m_1} \rangle_{g, m_1; d} \\ & = \dots = (2g - 2 + m_1 + n - 1) \dots (2g - 2 + m_1) \cdot \langle Q^{m_1} \rangle_{g, m_1; d} \\ & = (2g - 2 + m_1 + n - 1) \dots (2g - 2 + m_1) \cdot d^{m_1} N_{g,d}. \end{aligned}$$

Their contribution to the free energy is

$$\begin{aligned}
& \sum_{d=1}^{\infty} q^d \sum_{n, m_1 \geq 0} \frac{(t_1^P)^n (t^Q)^{m_1}}{n! m_1!} \prod_{j=0}^{n-1} (2g-2+m_1+j) \cdot d^{m_1} N_{g,d} \\
&= \sum_{d=1}^{\infty} q^d N_{g,d} \sum_{m_1 \geq 0} \frac{(t^Q)^{m_1}}{m_1!} d^{m_1} \sum_{n \geq 0} \frac{(t_1^P)^n}{n!} \prod_{j=0}^{n-1} (2g-2+m_1+j) \\
&= \sum_{d=1}^{\infty} q^d N_{g,d} \sum_{m_1 \geq 0} \frac{(t^Q)^{m_1}}{m_1!} d^{m_1} \cdot \frac{1}{(1-t_1^P)^{2g-2+m_1}} \\
&= \frac{1}{(1-t_1^P)^{2g-2}} \cdot \sum_{d=1}^{\infty} q^d N_{g,d} e^{t^Q/(1-t_1^P)}.
\end{aligned}$$

By (120) we have Recall that

$$u_{S,1} = \left(1 - \sum_{n=1}^{\infty} t_n^P \frac{u_S^{n-1}}{(n-1)!} \right)^{-1} = \frac{1}{1-t_1^P},$$

and by (121),

$$u_R = \left(1 - \sum_{n=1}^{\infty} t_n^P \frac{u_S^{n-1}}{(n-1)!} \right)^{-1} \cdot \sum_{n=0}^{\infty} t_n^Q \frac{u_S^n}{n!} = \frac{t^Q}{1-t_1^P}.$$

□

Remark 5.2. One can use the same method to treat the degree zero part of F_g . A similar argument yields:

$$F_{g,0} = -200 u_{S,1}^{2g-2} \int_{\mathcal{M}_g} \lambda_{g-1}^3 = -200 u_{S,1}^{2g-2} \cdot \frac{1}{2(2g-2)!} \frac{|B_{2g-2}|}{2g-2} \frac{|B_{2g}|}{2g}.$$

Here in the second equality we have used the results predicted by Mariño and Moore [29] and Gopakumar and Vafa [23] and proved by Faber and Pandharipande [18].

To summarize, we have

Theorem 5.3. *For the quintic, the free energy in genus $g \geq 1$ on the big phase space has the following form:*

$$(184) \quad F_1 = -200 \cdot \frac{1}{24} \log u_{S,1} + \frac{25}{12} u_R + \sum_{d=1}^{\infty} N_{1,d} e^{du_R} q^d$$

and for $g \geq 2$,

$$(185) \quad F_g = -200 \cdot \frac{1}{2(2g-2)!} \frac{|B_{2g-2}|}{2g-2} \frac{|B_{2g}|}{2g} u_{S,1}^{2g-2} + u_{S,1}^{2g-2} \sum_{d=1}^{\infty} q^d N_{g,d} e^{du_R}.$$

As in the genus zero case, one can apply the loop operators repeatedly to get the n -point functions in genus g .

5.5. Deformation of the order parameter. Define

$$(186) \quad U_P = \sum_{g=0}^{\infty} \lambda^{2g} \frac{\partial^2 F_g}{\partial t^P \partial t^P},$$

and similarly define U_Q , U_R and U_S .

Write

$$(187) \quad F_g = -\frac{25}{3} \delta_{g,1} \log u_{S,1} + u_{S,1}^{2g-2} f_g(u_R).$$

Their first derivatives are:

$$(188) \quad \frac{\partial F_g}{\partial t^P} = -\frac{25}{3} \frac{u_{S,2}}{u_{S,1}} \delta_{g,1} + (2g-2) u_{S,1}^{2g-3} u_{S,2} f_g(u_R) + u_{S,1}^{2g-2} f'_g(u_R) u_{R,1}.$$

$$(189) \quad \frac{\partial F_g}{\partial t^Q} = u_{S,1}^{2g-2} f'_g(u_R) \cdot \frac{\partial u_R}{\partial t^Q} = u_{S,1}^{2g-1} f'_g(u_R),$$

$$(190) \quad \frac{\partial F_g}{\partial t^R} = \frac{\partial F_g}{\partial t^S} = 0.$$

Taking $\frac{\partial}{\partial t^P}$ we get:

$$(191) \quad \begin{aligned} \frac{\partial^2 F_g}{\partial t^P \partial t^P} &= -\frac{25}{3} \frac{u_{S,3}}{u_{S,1}} \delta_{g,1} + \frac{25}{3} \frac{u_{S,2}^2}{u_{S,1}^2} \delta_{g,1} \\ &+ (2g-2)(2g-3) u_{S,1}^{2g-4} u_{S,2}^2 f_g(u_R) \\ &+ 2(2g-2) u_{S,1}^{2g-3} u_{S,2} f'_g(u_R) u_{R,1} \\ &+ (2g-2) u_{S,1}^{2g-3} u_{S,3} f_g(u_R) \\ &+ u_{S,1}^{2g-2} f'_g(u_R) u_{R,2} + u_{S,1}^{2g-2} f''_g(u_R) u_{R,1}^2. \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 F_g}{\partial t^P \partial t^Q} &= \frac{\partial}{\partial t^P} \left(u_{S,1}^{2g-2} f'_g(u_R) \frac{\partial u_R}{\partial t^Q} \right) = \frac{\partial}{\partial t^P} \left(u_{S,1}^{2g-1} f'_g(u_R) \right) \\ &= (2g-1) u_{S,1}^{2g-2} u_{S,2} f'_g(u_R) + u_{S,1}^{2g-1} u_{R,1} f''_g(u_R). \end{aligned}$$

$$\frac{\partial^F}{\partial t^P \partial t^R} = \frac{\partial^2 F_g}{\partial t^P \partial t^S} = 0.$$

So we get:

$$\begin{aligned}
(192) \quad U_P &= u_P - \frac{25\lambda^2}{3} \left(\frac{u_{S,3}}{u_{S,1}} - \frac{u_{S,2}^2}{u_{S,1}^2} \right) \\
&+ \sum_{g=1}^{\infty} \lambda^{2g} \left(u_{S,1}^{2g-2} f_g''(u_R) u_{R,1}^2 + u_{S,1}^{2g-2} f_g'(u_R) u_{R,2} \right. \\
&+ (4g-4) u_{S,1}^{2g-3} u_{S,2} u_{R,1} f_g'(u_R) + (2g-2) u_{S,1}^{2g-3} u_{S,3} f_g(u_R) \\
&\left. + (2g-2)(2g-3) u_{S,1}^{2g-4} u_{S,2}^2 f_g(u_R) \right).
\end{aligned}$$

$$(193) \quad U_Q = u_Q + \sum_{g=1}^{\infty} \lambda^{2g} \left((2g-1) u_{S,1}^{2g-2} u_{S,2} f_g'(u_R) + u_{S,1}^{2g-1} u_{R,1} f_g''(u_R) \right).$$

$$(194) \quad U_R = u_R, \quad U_S = u_S.$$

5.6. Deformation of the flow equations. The flow equations for u_S in genus zero are:

$$\frac{\partial u_S}{\partial t_n^P} = \left(\frac{u_S^{n+1}}{(n+1)!} \right)', \quad \frac{\partial u_S}{\partial t_n^Q} = 0, \quad \frac{\partial u_S}{\partial t_n^R} = 0, \quad \frac{\partial u_S}{\partial t_n^S} = 0,$$

they simply become

$$\frac{\partial U_S}{\partial t_n^P} = \left(\frac{U_S^{n+1}}{(n+1)!} \right)', \quad \frac{\partial U_S}{\partial t_n^Q} = 0, \quad \frac{\partial U_S}{\partial t_n^R} = 0, \quad \frac{\partial U_S}{\partial t_n^S} = 0,$$

The flow equations for u_R in genus zero are:

$$\frac{\partial u_R}{\partial t_n^P} = \left(\frac{u_R u_S^n}{(n)!} \right)', \quad \frac{\partial u_R}{\partial t_n^Q} = \left(\frac{u_S^{n+1}}{(n+1)!} \right)', \quad \frac{\partial u_R}{\partial t_n^R} = 0, \quad \frac{\partial u_R}{\partial t_n^S} = 0,$$

they simply become

$$\frac{\partial U_R}{\partial t_n^P} = \left(\frac{U_R U_S^n}{n!} \right)', \quad \frac{\partial U_R}{\partial t_n^Q} = \left(\frac{U_S^{n+1}}{(n+1)!} \right)', \quad \frac{\partial U_R}{\partial t_n^R} = 0, \quad \frac{\partial U_R}{\partial t_n^S} = 0,$$

So we will focus on the flow equations of U_P and U_Q . Since they involve only u_S and u_R , we have

$$\frac{\partial U_Q}{\partial t_n^S} = \frac{\partial u_Q}{\partial t_n^S} = 0,$$

$$\frac{\partial U_Q}{\partial t_n^R} = \frac{\partial u_Q}{\partial t_n^R} = \left(\frac{u_S^{n+1}}{(n+1)!} \right)' = \left(\frac{U_S^{n+1}}{(n+1)!} \right)',$$

$$\frac{\partial U_P}{\partial t_n^S} = \frac{\partial u_P}{\partial t_n^S} \left(\frac{u_S^{n+1}}{(n+1)!} \right)' = \left(\frac{U_S^{n+1}}{(n+1)!} \right)',$$

$$\frac{\partial U_P}{\partial t_n^R} = \frac{\partial u_P}{\partial t_n^R} = \left(u_R \frac{u_S^n}{n!} \right)' = \left(\frac{U_R U_S^n}{n!} \right)'.$$

So we will focus on the equations for $\frac{\partial U_P}{\partial t_n^P}$, $\frac{\partial U_P}{\partial t_n^Q}$, $\frac{\partial U_Q}{\partial t_n^P}$ and $\frac{\partial U_Q}{\partial t_n^Q}$. For $n = 0$, we need to consider the deformations of

$$(195) \quad \frac{\partial u_P}{\partial t^P} = u'_P, \quad \frac{\partial u_P}{\partial t^Q} = u'_Q,$$

and

$$(196) \quad \frac{\partial u_Q}{\partial t^P} = u'_Q, \quad \frac{\partial u_Q}{\partial t^Q} = (f_0''(u_R))'.$$

The first three equations are linear, it is clear that they get deformed to:

$$\frac{\partial U_P}{\partial t^P} = U'_P, \quad \frac{\partial U_P}{\partial t^Q} = U'_Q, \quad \frac{\partial U_Q}{\partial t^P} = U'_Q,$$

respectively. Now let us examine the deformation of the second equation in (196). By (189) we get:

$$\frac{\partial^2 F_g}{\partial t^Q \partial t^Q} = \frac{\partial}{\partial t^Q} (u_{S,1}^{2g-1} f'_g(u_R)) = u_{S,1}^{2g} f''_g(u_R) = U_{S,1}^{2g} f''_g(U_R).$$

It follows that

$$\begin{aligned} \frac{\partial U_Q}{\partial t^Q} &= \frac{\partial}{\partial t^P} \left(\frac{\partial^2 F_0}{\partial t^Q \partial t^Q} + \lambda^2 \frac{\partial^2 F_1}{\partial t^Q \partial t^Q} + \dots \right) \\ &= (f_0''(U_R) + \lambda^2 U_{S,1}^2 f_1''(U_R) + \dots)', \end{aligned}$$

and so we get:

$$(197) \quad \frac{\partial U_Q}{\partial t^Q} = \left(\sum_{g=0}^{\infty} U_{S,1}^{2g} f_g''(U_R) \right)'.$$

For $n = 1$ we need to consider the deformation of the following equations:

$$(198) \quad \frac{\partial u_P}{\partial t_1^P} = (u_P u_S + u_Q u_R)', \quad \frac{\partial u_P}{\partial t_1^Q} = (u_S u_Q + f_0'(u_R))',$$

$$(199) \quad \frac{\partial u_Q}{\partial t_1^P} = (u_S u_Q + u_R f_0''(u_R) - f_0'(u_R))', \quad \frac{\partial u_Q}{\partial t_1^Q} = (u_S f_0''(u_R))'.$$

Let us begin with the second equation in (199). Its treatment is similar to the case of $\frac{\partial u_Q}{\partial t^Q}$. First note:

$$\begin{aligned}\frac{\partial^2 F_g}{\partial t^Q \partial t_1^Q} &= \frac{\partial}{\partial t_1^Q} (u_{S,1}^{2g-1} f'_g(u_R)) = u_{S,1}^{2g-1} f''_g(u_R) \cdot \frac{\partial u_R}{\partial t_1^Q} \\ &= u_{S,1}^{2g-1} f''_g(u_R) \cdot u_S u_{S,1} = U_S U_{1,1}^{2g} f''_g(U_R).\end{aligned}$$

Then from

$$\frac{\partial U_Q}{\partial t_1^Q} = \frac{\partial}{\partial t^P} \left(\sum_{g=0}^{\infty} \lambda^{2g} \frac{\partial^2 F_g}{\partial t^Q \partial t_1^Q} \right),$$

we get

$$(200) \quad \frac{\partial U_Q}{\partial t_1^Q} = \left(U_S \sum_{g=0}^{\infty} \lambda^{2g} U_{S,1}^{2g} f''_g(u_R) \right)'$$

In the same fashion we see that for general n ,

$$(201) \quad \frac{\partial U_Q}{\partial t_n^Q} = \left(\frac{U_S^n}{n!} \sum_{g=0}^{\infty} \lambda^{2g} U_{S,1}^{2g} f''_g(u_R) \right)'$$

Let us now examine the deformation of the second equation in (198).

$$\begin{aligned}\frac{\partial^2 F_g}{\partial t^P \partial t_1^Q} &= \frac{\partial}{\partial t^P} \left(u_{S,1}^{2g-2} f'_g(u_R) \frac{\partial u_R}{\partial t_1^Q} \right) = \frac{\partial}{\partial t^P} \left(u_S u_{S,1}^{2g-1} f'_g(u_R) \right) \\ &= u_{S,1}^{2g} f'_g(u_R) + (2g-1) u_S u_{S,1}^{2g-2} u_{S,2} f'_g(u_R) + u_S u_{S,1}^{2g-1} u_{R,1} f''_g(u_R).\end{aligned}$$

So we have

$$\begin{aligned}\frac{\partial U_P}{\partial t_1^Q} &= \lambda^2 \frac{\partial}{\partial t^P} \left(\frac{\partial^2 F}{\partial t^P \partial t_1^Q} \right) \\ &= \left[u_S u_Q + f'_0(u_R) + \sum_{g=1}^{\infty} \lambda^{2g} \left(u_{S,1}^{2g} f'_g(u_R) \right. \right. \\ &\quad \left. \left. + (2g-1) u_S u_{S,1}^{2g-2} u_{S,2} f'_g(u_R) + u_S u_{S,1}^{2g-1} u_{R,1} f''_g(u_R) \right) \right]' \\ &= \left[u_S \left(u_Q + \sum_{g=1}^{\infty} \lambda^{2g} \left((2g-1) u_{S,1}^{2g-2} u_{S,2} f'_g(u_R) + u_{S,1}^{2g-1} u_{R,1} f''_g(u_R) \right) \right) \right. \\ &\quad \left. + f'_0(u_R) + \sum_{g=1}^{\infty} \lambda^{2g} u_{S,1}^{2g} f'_g(u_R) \right]'\end{aligned}$$

Therefore, by (193) and (194) we get:

$$(202) \quad \frac{\partial U_P}{\partial t_1^Q} = \left(U_S U_Q + \sum_{g=0}^{\infty} \lambda^{2g} U_{S,1}^{2g} f'_g(U_R) \right)'.$$

For general n ,

$$(203) \quad \frac{\partial U_P}{\partial t_n^Q} = \left(\frac{U_S^n}{n!} U_Q + \frac{U_S^{n-1}}{(n-1)!} \sum_{g=0}^{\infty} \lambda^{2g} U_{S,1}^{2g} f'_g(u_R) \right)'.$$

Now we consider the deformation of the first equation in (199). We differentiate (189) to get:

$$\begin{aligned} \frac{\partial^2 F_g}{\partial t_1^P \partial t^Q} &= \frac{\partial}{\partial t_1^P} \left(u_{S,1}^{2g-2} f'_g(u_R) \frac{\partial u_R}{\partial t^Q} \right) = \frac{\partial}{\partial t_1^P} \left(u_{S,1}^{2g-1} f'_g(u_R) \right) \\ &= (2g-1) u_{S,1}^{2g-2} \frac{\partial u_{S,1}}{\partial t_1^P} f'_g(u_R) + u_{S,1}^{2g-1} f''_g(u_R) \frac{\partial u_R}{\partial t_1^P} \\ &= (2g-1) u_{S,1}^{2g-2} (u_S u_{S,2} + u_{S,1}^2) f'_g(u_R) \\ &\quad + u_{S,1}^{2g-1} (u_S u_{R,1} + u_R u_{S,1}) f''_g(u_R). \end{aligned}$$

From this we get:

$$\begin{aligned} \frac{\partial U_Q}{\partial t_1^P} &= \lambda^2 \frac{\partial}{\partial t^P} \left(\frac{\partial^2 F}{\partial t_1^P \partial t^Q} \right) \\ &= \left[u_S u_Q + u_R f''_0(u_R) - f'_0(u_R) \right. \\ &\quad \left. + \sum_{g=1}^{\infty} \lambda^{2g} \left((2g-1) u_{S,1}^{2g-2} (u_S u_{S,2} + u_{S,1}^2) f'_g(u_R) \right. \right. \\ &\quad \left. \left. + u_{S,1}^{2g-1} (u_S u_{R,1} + u_R u_{S,1}) f''_g(u_R) \right) \right]' \\ &= \left[u_S \left(u_Q + \sum_{g=1}^{\infty} \lambda^{2g} \left((2g-1) u_{S,1}^{2g-2} u_{S,2} f'_g(u_R) + u_{S,1}^{2g-1} u_{R,1} f''_g(u_R) \right) \right) \right. \\ &\quad \left. + u_R \left(f''_0(u_R) + \sum_{g=1}^{\infty} \lambda^{2g} u_{S,1}^{2g} f''_g(u_R) \right) \right. \\ &\quad \left. - f'_0(u_R) + \sum_{g=1}^{\infty} (2g-1) \lambda^{2g} u_{S,1}^{2g} f'_g(u_R) \right]', \end{aligned}$$

so using (193) and (194) we get:

$$(204) \quad \frac{\partial U_Q}{\partial t_1^P} = \left[U_Q U_S + U_R \sum_{g=0}^{\infty} \lambda^{2g} U_{S,1}^{2g} f_g''(U_R) + \sum_{g=0}^{\infty} (2g-1) \lambda^{2g} U_{S,1}^{2g} f_g'(U_R) \right]'$$

In general we have:

$$(205) \quad \frac{\partial U_Q}{\partial t_1^P} = \left[U_Q \frac{U_S^n}{n!} + \frac{U_S^{n-1}}{(n-1)!} U_R \sum_{g=0}^{\infty} \lambda^{2g} U_{S,1}^{2g} f_g''(U_R) + \frac{U_S^{n-1}}{(n-1)!} \sum_{g=0}^{\infty} (2g-1) \lambda^{2g} U_{S,1}^{2g} f_g'(U_R) \right]'$$

Now we consider the deformation of the first equation in (199). We begin with:

$$\frac{\partial U_P}{\partial t_1^P} = \frac{\partial}{\partial t^P} \frac{\partial}{\partial t_1^P} \frac{\partial F}{\partial t^P}.$$

Then we take $\frac{\partial}{\partial t_1^P}$ on both sides of (188) to get:

$$\begin{aligned}
& \frac{\partial}{\partial t_1^P} \frac{\partial F}{\partial t^P} \\
&= u_P u_S + u_Q u_R - \frac{25\lambda^2}{3} \frac{\partial}{\partial t_1^P} \left(\frac{u_{S,2}}{u_{S,1}} \right) \\
&+ \sum_{g=1}^{\infty} \lambda^{2g} \frac{\partial}{\partial t_1^P} \left(u_{S,1}^{2g-2} f'_g(u_R) u_{R,1} + (2g-2) u_{S,1}^{2g-3} u_{S,2} f_g(u_R) \right) \\
&= u_P u_S + u_Q u_R - \frac{25\lambda^2}{3} \left(\frac{\frac{\partial u_{S,2}}{\partial t_1^P}}{u_{S,1}} - \frac{u_{S,2}}{u_{S,1}^2} \frac{\partial u_{S,1}}{\partial t_1^P} \right) \\
&+ \sum_{g=1}^{\infty} \lambda^{2g} \left(u_{S,1}^{2g-2} f''_g(u_R) \frac{\partial u_R}{\partial t_1^P} u_{R,1} + u_{S,1}^{2g-2} f'_g(u_R) \frac{\partial u_{R,1}}{\partial t_1^P} \right. \\
&+ (2g-2) u_{S,1}^{2g-2} \frac{\partial u_{S,1}}{\partial t_1^P} f'_g(u_R) u_{R,1} + (2g-2) u_{S,1}^{2g-3} u_{S,2} f'_g(u_R) \frac{\partial u_R}{\partial t_1^P} \\
&+ \left. (2g-2) u_{S,1}^{2g-3} \frac{\partial u_{S,2}}{\partial t_1^P} f_g(u_R) + (2g-2)(2g-3) u_{S,1}^{2g-4} \frac{\partial u_{S,1}}{\partial t_1^P} u_{S,2} f_g(u_R) \right) \\
&= u_P u_S + u_Q u_R - \frac{25\lambda^2}{3} \left(\frac{(u_S^2/2!)'''}{u_{S,1}} - \frac{u_{S,2}}{u_{S,1}^2} (u_S^2/2!)'' \right) \\
&+ \sum_{g=1}^{\infty} \lambda^{2g} \left(u_{S,1}^{2g-2} f''_g(u_R) (u_S u_R)' u_{R,1} + u_{S,1}^{2g-2} f'_g(u_R) (u_S u_R)'' \right. \\
&+ (2g-2) u_{S,1}^{2g-2} (u_S^2/2!)'' f'_g(u_R) u_{R,1} + (2g-2) u_{S,1}^{2g-3} u_{S,2} f'_g(u_R) (u_S u_R)' \\
&+ \left. (2g-2) u_{S,1}^{2g-3} (u_S^2/2!)''' f_g(u_R) + (2g-2)(2g-3) u_{S,1}^{2g-4} (u_S^2/2!)'' u_{S,2} f_g(u_R) \right) \\
&= u_P u_S + u_Q u_R - \frac{25\lambda^2}{3} \left(u_S \left(\frac{u_{S,3}}{u_{S,1}} - \frac{u_{S,2}^2}{u_{S,1}^2} \right) + 2u_{S,2} \right) \\
&+ \sum_{g=1}^{\infty} \lambda^{2g} \left(u_{S,1}^{2g-2} f''_g(u_R) (u_{S,1} u_R + u_S u_{R,1}) u_{R,1} \right. \\
&+ u_{S,1}^{2g-2} f'_g(u_R) (u_{S,2} u_R + 2u_{S,1} u_{R,1} + u_S u_{R,2}) \\
&+ (2g-2) u_{S,1}^{2g-3} (u_S u_{S,2} + u_{S,1}^2) f'_g(u_R) u_{R,1} \\
&+ (2g-2) u_{S,1}^{2g-3} u_{S,2} f'_g(u_R) (u_{S,1} u_R + u_S u_{R,1}) \\
&+ (2g-2) u_{S,1}^{2g-3} (u_S u_{S,3} + 3u_{S,1} u_{S,2}) f_g(u_R) \\
&+ \left. (2g-2)(2g-3) u_{S,1}^{2g-4} (u_S u_{S,2} + u_{S,1}^2) u_{S,2} f_g(u_R) \right).
\end{aligned}$$

We rewrite the right-hand side of the last equality as follows:

$$\begin{aligned}
\frac{\partial}{\partial t_1^P} \frac{\partial F}{\partial t^P} &= u_S \left(u_P - \frac{25\lambda^2}{3} \left(\frac{u_{S,3}}{u_{S,1}} - \frac{u_{S,2}^2}{u_{S,1}^2} \right) \right. \\
&+ \sum_{g=1}^{\infty} \lambda^{2g} \left(u_{S,1}^{2g-2} f_g''(u_R) u_{R,1}^2 + u_{S,1}^{2g-2} f_g'(u_R) u_{R,2} \right) \\
&+ (4g-4) u_{S,1}^{2g-3} u_{S,2} f_g'(u_R) u_{R,1} + (2g-2) u_{S,1}^{2g-3} u_{S,3} f_g(u_R) \\
&+ \left. (2g-2)(2g-3) u_{S,1}^{2g-4} u_{S,2}^2 f_g(u_R) \right) \\
&+ u_R \left(u_Q + \sum_{g=1}^{\infty} \lambda^{2g} \left(u_{S,1}^{2g-1} f_g''(u_R) u_{R,1} + (2g-1) u_{S,1}^{2g-2} f_g'(u_R) u_{S,2} \right) \right) \\
&- \frac{50\lambda^2}{3} u_{S,2} + \sum_{g=1}^{\infty} \lambda^{2g} \left(2g u_{S,1}^{2g-1} f_g'(u_R) u_{R,1} + 2g(2g-2) u_{S,1}^{2g-2} u_{S,2} f_g(u_R) \right).
\end{aligned}$$

Now we use (192)-(194) to get

$$\begin{aligned}
(206) \quad \frac{\partial U_P}{\partial t_1^P} &= U_P U_S + U_Q U_R - \frac{50\lambda^2}{3} U_{S,2} \\
&+ \sum_{g=1}^{\infty} \lambda^{2g} \left(2g U_{S,1}^{2g-1} f_g'(U_R) U_{R,1} + 2g(2g-2) U_{S,1}^{2g-2} U_{S,2} f_g(U_R) \right).
\end{aligned}$$

In general, we have:

$$\begin{aligned}
& \frac{\partial}{\partial t_n^P} \frac{\partial F}{\partial t^P} \\
= & u_P \frac{u_S^n}{n!} + u_Q u_R \frac{u_S^{n-1}}{(n-1)!} + \frac{u_S^{n-2}}{(n-2)!} (u^R f_0'(u^Q) - 2f_0(u_R)) - \frac{25\lambda^2}{3} \frac{\partial}{\partial t_n^P} \left(\frac{u_{S,2}}{u_{S,1}} \right) \\
& + \sum_{g=1}^{\infty} \lambda^{2g} \frac{\partial}{\partial t_n^P} (u_{S,1}^{2g-2} f_g'(u_R) u_{R,1} + (2g-2) u_{S,1}^{2g-3} u_{S,2} f_g(u_R)) \\
= & u_P \frac{u_S^n}{n!} + u_Q u_R \frac{u_S^{n-1}}{(n-1)!} + \frac{u_S^{n-2}}{(n-2)!} (u^R f_0'(u^Q) - 2f_0(u_R)) \\
& - \frac{25\lambda^2}{3} \left(\frac{\frac{\partial u_{S,2}}{\partial t_n^P}}{u_{S,1}} - \frac{u_{S,2}}{u_{S,1}^2} \frac{\partial u_{S,1}}{\partial t_n^P} \right) \\
& + \sum_{g=1}^{\infty} \lambda^{2g} \left(u_{S,1}^{2g-2} f_g''(u_R) \frac{\partial u_R}{\partial t_n^P} u_{R,1} + u_{S,1}^{2g-2} f_g'(u_R) \frac{\partial u_{R,1}}{\partial t_n^P} \right. \\
& + (2g-2) u_{S,1}^{2g-3} \frac{\partial u_{S,1}}{\partial t_n^P} f_g'(u_R) u_{R,1} + (2g-2) u_{S,1}^{2g-3} u_{S,2} f_g'(u_R) \frac{\partial u_R}{\partial t_n^P} \\
& \left. + (2g-2) u_{S,1}^{2g-3} \frac{\partial u_{S,2}}{\partial t_n^P} f_g(u_R) + (2g-2)(2g-3) u_{S,1}^{2g-4} \frac{\partial u_{S,1}}{\partial t_n^P} u_{S,2} f_g(u_R) \right) \\
= & u_P \frac{u_S^n}{n!} + u_Q u_R \frac{u_S^{n-1}}{(n-1)!} + \frac{u_S^{n-2}}{(n-2)!} (u^R f_0'(u^Q) - 2f_0(u_R)) \\
& - \frac{25\lambda^2}{3} \left(\frac{(u_S^{n+1}/(n+1)!)'''}{u_{S,1}} - \frac{u_{S,2}}{u_{S,1}^2} (u_S^{n+1}/(n+1)!)'' \right) \\
& + \sum_{g=1}^{\infty} \lambda^{2g} \left(u_{S,1}^{2g-2} f_g''(u_R) \left(\frac{u_S^n}{n!} u_R \right)' u_{R,1} + u_{S,1}^{2g-2} f_g'(u_R) \left(\frac{u_S^n}{n!} u_R \right)'' \right. \\
& + (2g-2) u_{S,1}^{2g-3} (u_S^{n+1}/(n+1)!)'' f_g'(u_R) u_{R,1} + (2g-2) u_{S,1}^{2g-3} u_{S,2} f_g'(u_R) \left(\frac{u_S^n}{n!} u_R \right)' \\
& \left. + (2g-2) u_{S,1}^{2g-3} (u_S^{n+1}/(n+1)!)''' f_g(u_R) + (2g-2)(2g-3) u_{S,1}^{2g-4} (u_S^{n+1}/(n+1)!)'' u_{S,2} f_g(u_R) \right)
\end{aligned}$$

$$\begin{aligned}
& \frac{\partial}{\partial t_n^P} \frac{\partial F}{\partial t^P} \\
= & u_P \frac{u_S^n}{n!} + u_Q u_R \frac{u_S^{n-1}}{(n-1)!} + \frac{u_S^{n-2}}{(n-2)!} (u^R f_0'(u^Q) - 2f_0(u_R)) \\
& - \frac{25\lambda^2}{3} \left(\frac{u_{S,3} u_S^n / n! + 3u_{S,2} u_{S,1} u_S^{n-1} / (n-1)! + u_{S,1}^3 u_S^{n-2} / (n-2)!}{u_{S,1}} \right) \\
& - \frac{u_{S,2}}{u_{S,1}^2} \left(u_{S,2} \frac{u_S^n}{n!} + u_{S,1}^2 \frac{u_S^{n-1}}{(n-1)!} \right) \\
& + \sum_{g=1}^{\infty} \lambda^{2g} \left(u_{S,1}^{2g-2} f_g''(u_R) \left(\frac{u_S^n}{n!} u_{R,1} + \frac{u_S^{n-1}}{(n-1)!} u_{S,1} u_R \right) u_{R,1} \right. \\
& + u_{S,1}^{2g-2} f_g'(u_R) \left(\frac{u_S^n}{n!} u_{R,1} + 2 \frac{u_S^{n-1}}{(n-1)!} u_{S,1} u_{R,1} + \frac{u_S^{n-1}}{(n-1)!} u_{S,2} u_R + \frac{u_S^{n-2}}{(n-2)!} u_{S,1}^2 u_R \right) \\
& + (2g-2) u_{S,1}^{2g-3} \left(u_{S,2} \frac{u_S^n}{n!} + u_{S,1}^2 \frac{u_S^{n-1}}{(n-1)!} \right) f_g'(u_R) u_{R,1} \\
& + (2g-2) u_{S,1}^{2g-3} u_{S,2} f_g'(u_R) \left(\frac{u_S^n}{n!} u_{R,1} + \frac{u_S^{n-1}}{(n-1)!} u_{S,1} u_R \right) \\
& + (2g-2) u_{S,1}^{2g-3} \left(u_{S,3} \frac{u_S^n}{n!} + 3u_{S,2} u_{S,1} \frac{u_S^{n-1}}{(n-1)!} + u_{S,1}^3 \frac{u_S^{n-2}}{(n-2)!} \right) f_g(u_R) \\
& \left. + (2g-2)(2g-3) u_{S,1}^{2g-4} \left(u_{S,2} \frac{u_S^n}{n!} + u_{S,1}^2 \frac{u_S^{n-1}}{(n-1)!} \right) u_{S,2} f_g(u_R) \right)
\end{aligned}$$

$$\begin{aligned}
& \frac{\partial}{\partial t_n^P} \frac{\partial F}{\partial t^P} \\
&= \frac{u_S^n}{n!} \left[u_P - \frac{25\lambda^2}{3} \left(\frac{u_{S,3}}{u_{S,1}} - \frac{u_{S,2}^2}{u_{S,1}^2} \right) \right. \\
&+ \sum_{g=1}^{\infty} \lambda^{2g} \left(u_{S,1}^{2g-2} f_g''(u_R) u_{R,1}^2 + u_{S,1}^{2g-2} f_g'(u_R) u_{R,1} + 2(2g-2) u_{S,1}^{2g-3} u_{S,2} f_g'(u_R) u_{R,1} \right. \\
&+ \left. \left. (2g-2) u_{S,1}^{2g-3} u_{S,3} f_g(u_R) + (2g-2)(2g-3) u_{S,1}^{2g-4} u_{S,2}^2 f_g(u_R) \right) \right] \\
&+ \frac{u_S^{n-1}}{(n-1)!} u_R \left[u_Q + \sum_{g=1}^{\infty} \lambda^{2g} \left(u_{S,1}^{2g-2} f_g''(u_R) u_{S,1} u_R u_{R,1} + (2g-1) u_{S,1}^{2g-2} f_g'(u_R) u_{S,2} \right) \right] \\
&+ \frac{u_S^{n-2}}{(n-2)!} \left[(u_R f_0'(u_R) - 2f_0(u_R)) + \sum_{g=1}^{\infty} \lambda^{2g} u_{S,1}^{2g} (u_R f_g'(u_R) + (2g-2) f_g(u_R)) \right] \\
&- \frac{25\lambda^2}{3} \left(2u_{S,2} \frac{u_S^{n-1}}{(n-1)!} + u_{S,1}^2 \frac{u_S^{n-2}}{(n-2)!} \right) \\
&+ \frac{u_S^{n-1}}{(n-1)!} \sum_{g=1}^{\infty} \lambda^{2g} \left(2g u_{S,1}^{2g-1} f_g'(u_R) u_{R,1} + 2g(2g-2) u_{S,1}^{2g-2} u_{S,2} f_g(u_R) \right).
\end{aligned}$$

To summarize, we have proved the following:

Theorem 5.4. *For the quintic, the deformed order parameters satisfy the following system of integrable hierarchy:*

$$\begin{aligned}
\frac{\partial U_S}{\partial t_n^P} &= \left(\frac{U_S^{n+1}}{(n+1)!} \right)', & \frac{\partial U_S}{\partial t_n^Q} &= 0, & \frac{\partial U_S}{\partial t_n^R} &= 0, & \frac{\partial U_S}{\partial t_n^S} &= 0, \\
\frac{\partial U_R}{\partial t_n^P} &= \left(\frac{U_R U_S^n}{n!} \right)', & \frac{\partial U_R}{\partial t_n^Q} &= \left(\frac{U_S^{n+1}}{(n+1)!} \right)', & \frac{\partial U_R}{\partial t_n^R} &= 0, & \frac{\partial U_R}{\partial t_n^S} &= 0, \\
\frac{\partial U_Q}{\partial t_n^S} &= 0, & \frac{\partial U_Q}{\partial t_n^R} &= \left(\frac{U_S^{n+1}}{(n+1)!} \right)'. \\
(207) \quad \frac{\partial U_Q}{\partial t_n^Q} &= \left(\frac{U_S^n}{n!} \sum_{g=0}^{\infty} \lambda^{2g} U_{S,1}^{2g} f_g''(u_R) \right)'.
\end{aligned}$$

$$(208) \quad \frac{\partial U_Q}{\partial t_1^P} = \left[U_Q \frac{U_S^n}{n!} + \frac{U_S^{n-1}}{(n-1)!} U_R \sum_{g=0}^{\infty} \lambda^{2g} U_{S,1}^{2g} f_g''(U_R) \right. \\ \left. + \frac{U_S^{n-1}}{(n-1)!} \sum_{g=0}^{\infty} (2g-1) \lambda^{2g} U_{S,1}^{2g} f_g'(U_R) \right]'$$

$$(209) \quad \frac{\partial U_P}{\partial t_n^S} = \left(\frac{U_S^{n+1}}{(n+1)!} \right)', \quad \frac{\partial U_P}{\partial t_n^R} = \left(\frac{U_R U_S^n}{n!} \right)'$$

$$(210) \quad \frac{\partial U_P}{\partial t_n^Q} = \left(\frac{U_S^n}{n!} U_Q + \frac{U_S^{n-1}}{(n-1)!} \sum_{g=0}^{\infty} \lambda^{2g} U_{S,1}^{2g} f_g'(u_R) \right)'$$

$$\frac{\partial U_P}{\partial t_n^P} = \left[\frac{U_S^n}{n!} U_P + \frac{u_S^{n-1}}{(n-1)!} U_R U_Q \right. \\ + \frac{U_S^{n-2}}{(n-2)!} \left[(U_R f_0'(U_R) - 2f_0(U_R)) + \sum_{g=1}^{\infty} \lambda^{2g} u_{S,1}^{2g} (U_R f_g'(U_R) \right. \\ \left. \left. + (2g-2)f_g(U_R)) \right) \right] - \frac{25\lambda^2}{3} \left(2U_{S,2} \frac{U_S^{n-1}}{(n-1)!} + U_{S,1}^2 \frac{U_S^{n-2}}{(n-2)!} \right) \\ \left. + \frac{U_S^{n-1}}{(n-1)!} \sum_{g=1}^{\infty} \lambda^{2g} \left(2g U_{S,1}^{2g-1} f_g'(U_R) U_{R,1} + 2g(2g-2) U_{S,1}^{2g-2} u_{S,2} f_g(U_R) \right) \right]'$$

6. EMERGENT GEOMETRY OF GROMOV-WITTEN THEORY

In this Section we return to the general discussions for GW theory of all symplectic manifolds or Gromov-Witten type theories. As pointed out by Dubrovin [5], two types of integrable systems are hidden in such theories. The first one is the system of WDVV equations (18), and the second one is the integrable hierarchy (40) and its deformation by higher genus contributions. We will recall the geometric structures behind such integrable systems. We will again take an emergent point of view, i.e., we will try to understand how geometric structures emerge in the GW-type theories.

6.1. WDVV equations, Frobenius manifolds and Higgs systems. Dubrovin [5] introduced the notion of a Frobenius manifold to give a geometric reformulation of the system of WDVV equations (18). Let us recall how it naturally emerge in the setting of GW theory. Let \mathcal{M} denote the small phase space with linear coordinates $\{t^\alpha\}$, with t^0

corresponding to $\mathcal{O}_0 = 1 \in H^0(M)$. The Poincaré pairing gives \mathcal{M} a flat metric

$$(211) \quad \eta_{\alpha\beta} = \eta\left(\frac{\partial}{\partial t^\alpha}, \frac{\partial}{\partial t^\beta}\right) = \int_M \mathcal{O}_\alpha \cup \mathcal{O}_\beta,$$

Since this metric is given by a constant matrix, it is flat. In particular, its Levi-Civita connection is given by:

$$(212) \quad \nabla_{\frac{\partial}{\partial t^\alpha}}(s^\beta \frac{\partial}{\partial t^\beta}) = \frac{\partial s^\beta}{\partial t^\alpha} \frac{\partial}{\partial t^\beta}.$$

The quantum multiplications defined in §2.5 define a Higgs field $A \in \Omega^1(M, \text{End}(TM))$:

$$(213) \quad A\left(\frac{\partial}{\partial t^\alpha}\right) : TM \rightarrow TM, \quad A\left(\frac{\partial}{\partial t^\alpha}\right) \frac{\partial}{\partial t^\beta} := \frac{\partial}{\partial t^\alpha} \circ_t \frac{\partial}{\partial t^\beta}.$$

Dubrovin [5] showed that the system of the WDVV equations is equivalent to the flatness of the new connection

$$(214) \quad \tilde{\nabla}_X Y = \nabla_X Y + z \cdot X \circ Y.$$

Furthermore,

$$(215) \quad e := \frac{\partial}{\partial t^0}$$

is a flat section with respect to ∇ . Such structures make \mathcal{M} a Frobenius manifold. This gives a geometric reformulation of some of the results in §2.4 and §2.5.

6.2. The deformed flat coordinates. Let

$$(216) \quad x^\alpha = x^\alpha(t, z) = t^\alpha + \sum_{n=1}^{\infty} z^n v_n^\alpha(t)$$

be flat coordinates for the Dubrovin connection $\tilde{\nabla}$. They are specified by the condition of vanishing of the covariant hessian

$$(217) \quad \tilde{\nabla}_{\frac{\partial}{\partial t^\alpha}} \tilde{\nabla}_{\frac{\partial}{\partial t^\beta}} \left(x^\gamma \frac{\partial}{\partial t^\gamma}\right) = 0,$$

or, equivalently, by the system

$$(218) \quad \frac{\partial}{\partial t^\alpha} \frac{\partial}{\partial t^\beta} x^\gamma = z c_{\alpha\beta}^\lambda \frac{\partial}{\partial t^\lambda} x^\gamma.$$

After plugging in (216), we get the following sequences of equations:

$$\begin{aligned} \frac{\partial^2 v_1^\gamma}{\partial t^\alpha \partial t^\beta} &= \frac{\partial^3 F_0}{\partial t^\alpha \partial t^\beta \partial t^\mu} \eta^{\gamma\mu}, \\ \frac{\partial^2 v_n^\gamma}{\partial t^\alpha \partial t^\beta} &= \frac{\partial^3 F_0}{\partial t^\alpha \partial t^\beta \partial t^\mu} \eta^{\lambda\mu} \frac{\partial v_{n-1}^\gamma}{\partial t^\lambda}, \quad n \geq 2. \end{aligned}$$

One can check that by the TRR in genus zero

$$(219) \quad v_1^\gamma = \frac{\partial F_0}{\partial t_{n-1}^\mu} \eta^{\gamma\mu}.$$

See Givental [22]. We recall that the WDVV equations can be derived from the TRR in genus zero. So we have seen that both the WDVV equations and their solutions are consequences of the TRR in genus zero.

6.3. The geometric structure behind the integrable hierarchy (40). As pointed out by Witten [34, §3d], the system of equations (40) is a system of Hamiltonian equations with the following Poisson bracket:

$$(220) \quad \{u_\alpha(x), u_\beta(y)\} = \eta_{\alpha\beta} \delta'(x - y),$$

with the Hamiltonians the genus zero one-point functions which just appear in last subsection:

$$(221) \quad H_{n,\alpha} = \int dx R_{\alpha,n}(u^0, \dots, u^r).$$

As remarked by Witten [34], the KdV hierarchy is bi-Hamiltonian, i.e., the KdV flows are Hamiltonian with respect to two Poisson structures. The second Poisson structure is related to the Virasoro algebra, the recursion for the Hamiltonians of the KdV hierarchy can be constructed using this bi-Hamiltonian structure. He remarked that: “We do not know of any evidence for a second symplectic structure playing a role for general M .”

As pointed out by [5], the system of equations (40) is a system of Hamiltonian equations of hydrodynamic type studied in Dubrovin-Novikov [6]. hydrodynamic type. Let \mathcal{M} be any manifold and u^0, \dots, u^r any local coordinates on \mathcal{M} . Recall that

$$(222) \quad \{u^a(x), u^b(y)\} = g^{ab}(u(x)) \delta'(x - y) + b_c^{ab}(u(x)) \frac{\partial u^c}{\partial x} \delta(x - y)$$

determines a Poisson bracket on the loop space $L\mathcal{M}$ consisting of smooth functions $u^a : S^1 \rightarrow \mathbb{C}$ (Poisson brackets of hydrodynamic type) iff the tensor field $g_{ab} du^a du^b$ is a flat metric on \mathcal{M} , where $(g_{ab}) = (g^{ab})^{-1}$, and the coefficients $b_c^{ab}(u)$ can be determined from g_{ab} as follows:

$$(223) \quad b_c^{ab}(u) = -g^{ad}(u) \Gamma_{dc}^b(u),$$

where Γ_{dc}^b are the Christoffel symbols of the Levi-Civita connection for the metric η . Such a Poisson structure is called a Poisson structure

of the hydrodynamic type. A hamiltonian of hydrodynamic type is a functional of the form:

$$(224) \quad H = \int h(u(x))dx,$$

where the density $h = h(u)$ does not depend on derivatives. Any such function $h(u)$ on \mathcal{M} determines a hamiltonian system on $L\mathcal{M}$ of the following form:

$$(225) \quad \partial_t u^a(x) = \{u^a(x), \int h(u(y))dy\} = w_b^a(u)\partial_x u^b,$$

where $w_b^a(u) = \nabla^a \nabla_a h(u)$. This is called a hamiltonian system of hydrodynamic type.

6.4. Conformal Frobenius manifolds and the second Poisson structures. By the selection rules (78) and (79), the Frobenius manifolds associated with the GW theory of Calabi-Yau threefolds are conformal invariant Frobenius manifold. They correspond to solutions of the WDVV equations self-similar with respect to some scaling transformations:

$$\begin{aligned} t^\alpha &\mapsto c^{1-q_\alpha} t^\alpha, \alpha = 0, 1, \dots, r, \\ \eta &= \eta_{\alpha\beta} dt^\alpha dt^\beta \mapsto c^{2-d} \eta, \\ c_{\alpha\beta\gamma} &\mapsto c^{q_\alpha + q_\beta + q_\gamma - d} c_{\alpha\beta\gamma}, \\ F_0(\{t^\alpha\}) &\mapsto c^{3-d} F_0(\{t^\alpha\}), \end{aligned}$$

for some $q_0 = 0, q_2, \dots, q_r, d$. A pair (α, n) is resonant if

$$(226) \quad \frac{d+1}{2} = q_\alpha - n.$$

If all pairs are nonresonant, then the Frobenius manifold is said to be nonresonant.

For a Calabi-Yau threefold M , fix a basis $\{\mathcal{O}_\alpha\}_{\alpha=0,\dots,r}$ of $H^{ev}(M)$,

$$(227) \quad q_\alpha = 1 - \deg \mathcal{O}_\alpha,$$

then $d = 3$. It is easy to see that Calabi-Yau threefolds give rise to resonant conformal invariant Frobenius manifolds.

As shown in Dubrovin [7, Theorem 3.2], for a conformal invariant Frobenius manifold, the formula

$$(228) \quad \begin{aligned} &\{t^\alpha(x), t^\beta(y)\}_1 \\ &= [(\frac{d+1}{2} - q_\alpha)F^{\alpha\beta}(t(x)) + (\frac{d+1}{2} - q_\beta)F^{\alpha\beta}(t(y))] \delta'(x-y) \end{aligned}$$

determines a Poisson bracket compatible with the Poisson bracket (222), where

$$(229) \quad F^{\alpha\beta}(t) = \eta^{\alpha\alpha'} \eta^{\beta\beta'} \partial_{\alpha'} \partial_{\beta'} F(t)$$

I.e., any linear combination of them again is a Poisson bracket.

6.5. Compatible Poisson structures and pencils of flat metrics.

By the theory of Dubrovin-Novikov [6] on Poisson brackets of hydrodynamic type, the pencil of Poisson structures $\{, \}_1 - \lambda \{, \}_2$ corresponds to a pencil of flat metrics:

$$(230) \quad g^{\alpha\beta} - \lambda \eta^{\alpha,\beta},$$

where

$$(231) \quad g^{\alpha\beta} = i_E(dt^\alpha \circ dt^\beta),$$

where E is Euler vector field:

$$(232) \quad E = \sum (1 - q_\alpha) t^\alpha \frac{\partial}{\partial t^\alpha}.$$

Given λ , the subset of \mathcal{M} where $g^{\alpha\beta} - \lambda \eta^{\alpha,\beta}$ is degenerate is denoted by Σ_λ , it is called the discriminant locus.

Recall that two metrics g_1 and g_2 form a flat pencil if:

- 1) The linear combination $g_1 - \lambda g_2$ is a flat metric for any λ .
- 2) The Levi-Civita connection of this linear combination has the form

$$\Gamma_k^{ij} = \Gamma_{1k}^{ij} - \lambda \Gamma_{2k}^{ij}.$$

The flat pencil of metrics is quasihomogeneous of the degree d if there exists a function τ such that the vector fields

$$\begin{aligned} E &:= g_1^{is} \frac{\partial \tau}{\partial t^s} \frac{\partial}{\partial t^i}, \\ e &:= g_2^{is} \frac{\partial \tau}{\partial t^s} \frac{\partial}{\partial t^i} \end{aligned}$$

satisfy the following properties

$$\begin{aligned} [e, E] &= e, \\ L_E g_1 &= (d-1)g_1, \\ L_e g_1 &= g_2, \\ L_e g_2 &= 0. \end{aligned}$$

If the flat pencil comes from a Frobenius manifold, then the function τ can be taken to be

$$(233) \quad \tau = \eta_{0\alpha} t^\alpha.$$

A quasihomogeneous flat pencil is said to be regular if the $(1, 1)$ -tensor

$$(234) \quad R_i^j = \frac{d-1}{2} \delta_i^j + \nabla_{2i} E^j$$

does not degenerate. We have seen that from a Frobenius manifold one can associate a flat pencil of metrics. Conversely, given a regular quasihomogeneous flat pencil one can reconstruct a Frobenius manifold (cf. e.g. [9, Theorem 2.1]).

6.6. The λ -periods and the dual flat coordinates. A function $p = p(t; \lambda)$ is called λ -period of the Frobenius manifold if it satisfies

$$(235) \quad (\nabla_1 - \lambda \nabla) dp = 0.$$

If one chooses the flat coordinates $p^0(t), \dots, p^r(t)$ of the intersection form, then

$$(236) \quad (dp^a, dp^b) = G^{ab}$$

are constants. Furthermore, we obtain a reduction of the Poisson pencil to the canonical form

$$(237) \quad \{p^i, p^j\}_\lambda = G^{ij} \delta'(x - y).$$

If $p^i(t)$ are chosen such that

$$(238) \quad L_E p^i = \frac{1-d}{2} p^i,$$

then

$$(239) \quad \tau = \eta_{0\alpha} t^\alpha = \frac{1-d}{4} G_{ab} p^a p^b,$$

where $(G_{ab}) = (G^{ab})^{-1}$. See [10, Lemma 2.6].

6.7. The dual multiplications. The metric $(,)$ and \langle, \rangle are related by multiplication by Euler vector field:

$$(240) \quad (E \circ u, v) = \langle u, v \rangle$$

See [8, Exercise 3.3]. It follows that:

$$(241) \quad (E \circ u \circ v, w) = \langle u \circ v, w \rangle.$$

It turns out that E is an invertible element in the quantum cohomology ring.

$$(242) \quad (u \circ v, w) = \langle E^{-1} \circ u \circ v, w \rangle.$$

Dubrovin [10] defined a new multiplication $u * v$ by

$$(243) \quad u * v := E^{-1} \circ u \circ v.$$

Then one has:

$$(244) \quad (u \circ v, w) = \langle u * v, w \rangle .$$

6.8. The dual potential function. As proved in Dubrovin [10], the multiplication (243) together with the intersection form (\cdot, \cdot) , the unity $=$ the Euler vector field $= E$ satisfies all the axioms of Frobenius manifold except for the unity is not parallel. Furthermore, let $p^1(t), \dots, p^n(t)$ be a system of local flat coordinates of the intersection form. Then there exists a function $F_*(p)$ such that

$$(245) \quad \frac{\partial^3 F_*(p)}{\partial p^i \partial p^j \partial p^c} = G_{ia} G_{jb} \frac{\partial t^\gamma}{\partial p^k} \frac{\partial p^a}{\partial t^\alpha} \frac{\partial p^b}{\partial t^\beta} c_{\gamma}^{\alpha\beta}(t).$$

This can be rewritten in the following way

$$(246) \quad d\left(\frac{\partial^2 F_*}{\partial p^a \partial p^b}\right) = dp^a \circ dp^b.$$

The function $F_*(p)$ satisfies the following associativity equations

$$(247) \quad \frac{\partial^3 F_*(p)}{\partial p^i \partial p^j \partial p^a} G^{ab} \frac{\partial^3 F_*(p)}{\partial p^b \partial p^k \partial p^l} = \frac{\partial^3 F_*(p)}{\partial p^l \partial p^j \partial p^a} G^{ab} \frac{\partial^3 F_*(p)}{\partial p^b \partial p^k \partial p^i},$$

$i, i, j, k, l = 1, \dots, n$. For $d \neq 1$, $F_*(p)$ satisfies the homogeneity condition

$$(248) \quad \sum_a p^a \frac{\partial F_*}{\partial p^a} = 2F_* + \frac{1}{1-d} \sum G_{ab} p^a p^b.$$

6.9. The deformed dual flat coordinates. Let

$$(249) \quad p^\alpha = p^\alpha(t, z) = p^\alpha + \sum_{n=1}^{\infty} z^n v_n^\alpha(t)$$

be dual flat coordinates for the Dubrovin connection $\tilde{\nabla}$. They are specified by the condition of vanishing of the covariant hessian

$$(250) \quad \tilde{\nabla}_{\frac{\partial}{\partial t^\alpha}} \tilde{\nabla}_{\frac{\partial}{\partial t^\beta}} (x^\gamma \frac{\partial}{\partial t^\gamma}) = 0,$$

or, equivalently, by the system

$$(251) \quad \frac{\partial}{\partial t^\alpha} \frac{\partial}{\partial t^\beta} x^\gamma = z c_{\alpha\beta}^\lambda \frac{\partial}{\partial t^\lambda} x^\gamma.$$

After plugging in (216), we get the following sequences of equations:

$$\begin{aligned} \frac{\partial^2 v_1^\gamma}{\partial t^\alpha \partial t^\beta} &= \frac{\partial^3 F_0}{\partial t^\alpha \partial t^\beta \partial t^\mu} \eta^{\gamma\mu}, \\ \frac{\partial^2 v_n^\gamma}{\partial t^\alpha \partial t^\beta} &= \frac{\partial^3 F_0}{\partial t^\alpha \partial t^\beta \partial t^\mu} \eta^{\lambda\mu} \frac{\partial v_{n-1}^\gamma}{\partial t^\lambda}, \quad n \geq 2. \end{aligned}$$

One can check that by the TRR in genus zero

$$(252) \quad v_1^\gamma = \frac{\partial F_0}{\partial t_{n-1}^\mu} \eta^{\gamma\mu}.$$

See Givental [22]. We recall that the WDVV equations can be derived from the TRR in genus zero. So we have seen that both the WDVV equations and their solutions are consequences of the TRR in genus zero.

In the case of semisimple Frobenius manifolds, Dubrovin and Zhang [13] showed that the deformed flat coordinates and the deformed dual flat coordinates are related by Laplace transform. We will verify that this also holds for the quintic in §7.9.

6.10. The reconstruction program. In [7] Dubrovin launched a program to reconstruct a complete GW-type theory from a Frobenius manifold. He focused on the case of semisimple Frobenius manifolds. This program was developed by Dubrovin and Zhang [13] and many of their subsequent works, based on the bihamiltonian systems. For a survey of the current status of this program, see Dubrovin [11].

7. FROBENIUS MANIFOLD ASSOCIATED WITH THE QUINTIC

In this Section we carry out the explicit computations in the theory of Frobenius manifolds associated with the quintic. We will rewrite the genus zero free energy on the small phase space in terms of the dual flat coordinates. This will reveal some similarity to the formulas in mirror symmetry computations in [2].

7.1. Induced multiplications on the cotangent bundle and the intersection form. Using the quantum multiplications computed in §4.3, we get the following induced multiplications on the cotangent bundle of the small phase space:

$$dt^P \circ \begin{pmatrix} dt^P \\ dt^Q \\ dt^R \\ dt^S \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} dt^P \\ dt^Q \\ dt^R \\ dt^S \end{pmatrix},$$

$$dt^Q \circ \begin{pmatrix} dt^P \\ dt^Q \\ dt^R \\ dt^S \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} dt^P \\ dt^Q \\ dt^R \\ dt^S \end{pmatrix},$$

$$dt^R \circ \begin{pmatrix} dt^P \\ dt^Q \\ dt^R \\ dt^S \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & f_0'''(t^Q) & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} dt^P \\ dt^Q \\ dt^R \\ dt^S \end{pmatrix},$$

$$dt^S \circ \begin{pmatrix} dt^P \\ dt^Q \\ dt^R \\ dt^S \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} dt^P \\ dt^Q \\ dt^R \\ dt^S \end{pmatrix},$$

Therefore, after taking inner product with the Euler vector field

$$(253) \quad E = t^P \frac{\partial}{\partial t^P} - t^R \frac{\partial}{\partial t^R} - 2t^S \frac{\partial}{\partial t^S},$$

we get the intersection form:

$$(dt^\alpha, dt^\beta) = \begin{pmatrix} 0 & 0 & 0 & t^P \\ 0 & 0 & t^P & 0 \\ 0 & t^P & 0 & -t^R \\ t^P & 0 & -t^R & -2t^S \end{pmatrix}.$$

One can check that

$$(254) \quad (dt^\alpha, dt^\beta) = R \cdot (F^{\alpha\beta}) + (F^{\alpha\beta}) \cdot R,$$

where the matrix R is

$$(255) \quad R = \frac{d-1}{2} + \text{diag}(d_\alpha) = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

where $d = \dim X = 3$,

$$(256) \quad d_P = 1, \quad d_Q = 0, \quad d_R = -1, \quad d_S = -2,$$

i.e., $d_\alpha = 1 - \deg \alpha$. A related matrix is

$$(257) \quad \mathcal{V} = R - \frac{1}{2}I = \begin{pmatrix} \frac{3}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{3}{2} \end{pmatrix}$$

Note

$$(258) \quad \mathcal{V} \cdot (\eta_{\alpha\beta}) = -(\eta_{\alpha\beta}) \cdot \mathcal{V},$$

it defines a symplectic structure on the small phase space.

Recall that

$$(259) \quad F^{\alpha\beta} = \eta^{\alpha\alpha'} \eta^{\beta\beta'} \frac{\partial^2 F_0}{\partial t^{\alpha'} \partial t^{\beta'}}$$

Because we have

$$\langle dt^\alpha, dt^\beta \rangle = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

and

$$(260) \quad (F_{\alpha\beta}) = \begin{pmatrix} t^S & t^R & t^Q & t^P \\ t^R & f_0''(t^Q) & t^P & 0 \\ t^Q & t^P & 0 & 0 \\ t^P & 0 & 0 & 0 \end{pmatrix},$$

so explicitly we have:

$$(261) \quad (F^{\alpha\beta}) = \begin{pmatrix} 0 & 0 & 0 & t^P \\ 0 & 0 & t^P & t^Q \\ 0 & t^P & f_0''(t^Q) & t^R \\ t^P & t^Q & t^R & t^S \end{pmatrix}.$$

7.2. Flat pencil of metrics. Now we have:

$$(dt^\alpha, dt^\beta) - \lambda \langle dt^\alpha, dt^\beta \rangle = \begin{pmatrix} 0 & 0 & 0 & t^P - \lambda \\ 0 & 0 & t^P - \lambda & 0 \\ 0 & t^P - \lambda & 0 & -t^R \\ t^P - \lambda & 0 & -t^R & -2t^S \end{pmatrix}$$

So the discriminant locus is given by:

$$(262) \quad \Sigma_\lambda = \{t^P = \lambda\}.$$

The inverse matrix

$$g_{\alpha\beta}(\lambda) = \frac{1}{(t^P - \lambda)^2} \begin{pmatrix} 2t^S & t^R & 0 & t^P - \lambda \\ t^R & 0 & t^P - \lambda & 0 \\ 0 & t^P - \lambda & 0 & 0 \\ t^P - \lambda & 0 & 0 & 0 \end{pmatrix}$$

defines a flat metric on $M \setminus \Sigma_\lambda$. Using the formula for Christoffel symbols:

$$(263) \quad \Gamma_{\beta\gamma}^\alpha(\lambda) = \frac{1}{2} g^{\alpha\rho}(\lambda) (\partial_\beta g_{\gamma\rho}(\lambda) + \partial_\gamma g_{\beta\rho}(\lambda) - \partial_\rho g_{\beta\gamma}(\lambda)),$$

one can find the nonzero components of the Christoffel symbols of $g(\lambda)$ are:

$$\begin{aligned}\Gamma_{PP}^P(\lambda) &= -\frac{2}{t^P - \lambda}, \Gamma_{PQ}^Q(\lambda) = \Gamma_{QP}^Q(\lambda) = -\frac{1}{t^P - \lambda}, \\ \Gamma_{PP}^S(\lambda) &= \frac{2t^S}{(t^P - \lambda)^2}, \Gamma_{PQ}^S(\lambda) = \Gamma_{QP}^S(\lambda) = \frac{t^R}{(t^P - \lambda)^2}, \\ \Gamma_{PS}^S(\lambda) &= \Gamma_{SP}^S(\lambda) = \frac{1}{t^P - \lambda}, \Gamma_{QR}^S(\lambda) = \Gamma_{RQ}^S(\lambda) = \frac{1}{t^P - \lambda}.\end{aligned}$$

One can verify that

$$(264) \quad \Gamma_{\beta\gamma}^\alpha(\lambda) = -g_{\beta\rho}(\lambda) \cdot \Gamma_\gamma^{\rho\alpha},$$

where

$$\begin{aligned}(\Gamma_P^{\alpha\beta}) &= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{pmatrix}, & (\Gamma_Q^{\alpha\beta}) &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \\ (\Gamma_R^{\alpha\beta}) &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & (\Gamma_S^{\alpha\beta}) &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.\end{aligned}$$

One can also check that:

$$(265) \quad \Gamma_\gamma^{\alpha\beta} = c_\gamma^{\alpha\epsilon} R_\epsilon^\beta.$$

7.3. The periods. We now solve the system of equations:

$$(266) \quad \partial_\alpha \partial_\beta x - \Gamma_{\alpha\beta}^\gamma(\lambda) \partial_\gamma x = 0.$$

They are explicitly given by:

$$\partial_P \begin{pmatrix} \partial_P x \\ \partial_Q x \\ \partial_R x \\ \partial_S x \end{pmatrix} = \begin{pmatrix} -\frac{2}{t^P - \lambda} & 0 & 0 & \frac{2t^S}{(t^P - \lambda)^2} \\ 0 & -\frac{1}{t^P - \lambda} & 0 & \frac{t^R}{(t^P - \lambda)^2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{t^P - \lambda} \end{pmatrix} \cdot \begin{pmatrix} \partial_P x \\ \partial_Q x \\ \partial_R x \\ \partial_S x \end{pmatrix}$$

$$\partial_Q \begin{pmatrix} \partial_P x \\ \partial_Q x \\ \partial_R x \\ \partial_S x \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{t^P - \lambda} & 0 & \frac{t^R}{(t^P - \lambda)^2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{t^P - \lambda} \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \partial_P x \\ \partial_Q x \\ \partial_R x \\ \partial_S x \end{pmatrix}$$

$$\partial_R \begin{pmatrix} \partial_P x \\ \partial_Q x \\ \partial_R x \\ \partial_S x \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{t^P - \lambda} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \partial_P x \\ \partial_Q x \\ \partial_R x \\ \partial_S x \end{pmatrix}$$

$$\partial_S \begin{pmatrix} \partial_P x \\ \partial_Q x \\ \partial_R x \\ \partial_S x \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{t^P - \lambda} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \partial_P x \\ \partial_Q x \\ \partial_R x \\ \partial_S x \end{pmatrix}$$

The equations for $\partial_S x$ are

$$\partial_P(\partial_S x) = \frac{1}{t^P - \lambda} \partial_S x, \quad \partial_Q(\partial_S x) = \partial_R(\partial_S x) = \partial_S(\partial_S x) = 0,$$

so we get:

$$\partial_S x = C_1 \cdot (t^P - \lambda).$$

The equations for $\partial_R x$ are

$$\begin{aligned} \partial_Q(\partial_R x) &= \frac{1}{t^P - \lambda} \partial_S x = C_1, \\ \partial_P(\partial_R x) &= \partial_R(\partial_R x) = \partial_S(\partial_R x) = 0, \end{aligned}$$

so we get:

$$(267) \quad \partial_R x = C_1 \cdot t^Q + C_2.$$

The equations for $\partial_Q x$ are

$$\begin{aligned} \partial_P(\partial_Q x) &= -\frac{1}{t^P - \lambda} \partial_Q x + \frac{t^R}{(t^P - \lambda)^2} \partial_S x = -\frac{1}{t^P - \lambda} \partial_Q x + C_1 \frac{t^R}{t^P - \lambda}, \\ \partial_Q(\partial_Q x) &= 0, \\ \partial_R(\partial_Q x) &= \frac{1}{t^P - \lambda} \partial_S x = C_1, \\ \partial_S(\partial_Q x) &= 0, \end{aligned}$$

so we get:

$$(268) \quad \partial_Q x = C_1 t^R + \frac{C_3}{t^P - \lambda}.$$

The equations for $\partial_P x$ are

$$\begin{aligned}\partial_P(\partial_P x) &= -\frac{2}{t^P - \lambda} \partial_P x + \frac{2t^S}{(t^P - \lambda)^2} \partial_S x = -\frac{2}{t^P - \lambda} \partial_P x + C_1 \frac{2t^S}{t^P - \lambda}, \\ \partial_Q(\partial_P x) &= -\frac{1}{t^P - \lambda} \partial_Q x + \frac{t^R}{(t^P - \lambda)^2} \partial_S x = -\frac{C_3}{(t^P - \lambda)^2}, \\ \partial_R(\partial_P x) &= 0, \\ \partial_S(\partial_P x) &= \frac{1}{t^P - \lambda} \partial_S x = C_1,\end{aligned}$$

so we get:

$$(269) \quad \partial_P x = C_1 t^S - C_3 \frac{t^Q}{(t^P - \lambda)^2} - C_4 \frac{1}{(t^P - \lambda)^2}.$$

Therefore,

$$(270) \quad x = C_0 + C_1((t^P - \lambda)t^S + t^Q t^R) + C_2 t^R + C_3 \frac{t^Q}{t^P - \lambda} + C_4 \frac{1}{t^P - \lambda}.$$

From this we get the following

Proposition 7.1. *For the quintic we can take the following system of dual flat coordinates for the flat pencil:*

$$(271) \quad p^4 = (t^P - \lambda)t^S + t^Q t^R, p^3 = t^R, p^2 = \frac{t^Q}{t^P - \lambda}, p^1 = \frac{1}{t^P - \lambda},$$

Note

$$(272) \quad \deg p^i(\lambda) = -1$$

if we set

$$(273) \quad \deg t^P = \deg \lambda = 1, \deg t^Q = 0, \deg t^R = -1, \deg t^S = -2.$$

7.4. The free energy in the dual flat coordinates. From (271) we get:

$$(274) \quad t^P - \lambda = \frac{1}{p^1}, t^Q = \frac{p^2}{p_1}, t^R = p^3, t^S = p^1 p^4 - p^2 p^3.$$

Recall that for the quintic the free energy in genus zero has the following form:

$$(275) \quad \begin{aligned}F_0 &= \frac{1}{2}(t^P)^2 t^S + t^P t^Q t^R + \frac{5}{6}(t^Q)^3 + \sum_{m=1}^{\infty} N_m e^{mt^Q} q^m \\ &= \frac{1}{2}(t^P)^2 t^S + t^P t^Q t^R + \frac{5}{2} \left(\frac{\omega_1 \omega_2}{\omega_0^2} - \frac{\omega_3}{\omega_0} \right).\end{aligned}$$

Therefore, taking $\lambda = 0$ we have:

$$\begin{aligned} F_0 &= \frac{1}{2} \frac{p^1 p^4 - p^2 p^3}{(p^1)^2} + \frac{p^2 p^3}{(p^1)^2} + \frac{5}{2} \left(\frac{\omega_1 \omega_2}{\omega_0^2} - \frac{\omega_3}{\omega_0} \right) \\ &= \frac{1}{2} \frac{1}{(p^1 \omega_0)^2} ((p^1 \omega_0)(p^4 \omega_0 - 5p^1 \omega_3) + \frac{1}{2}(p^2 \omega_0)(p^3 \omega_0 + 5p^1 \omega_2)), \end{aligned}$$

where in the second equality we have used the equality

$$(276) \quad p^1 \omega_1 = p^2 \omega_0$$

derived from

$$(277) \quad t^Q = \frac{p^2}{p^1} = \frac{\omega_1(t)}{\omega_0(t)}.$$

Therefore, if we set

$$\begin{aligned} \hat{p}^1 &= p^1 \omega_0, \\ \hat{p}^2 &= p^2 \omega_0 = p^1 \omega_1, \\ \hat{p}^3 &= p^3 \omega_0 + 5p^1 \omega_2, \\ \hat{p}^4 &= p^4 \omega_0 - 5p^1 \omega_3, \end{aligned}$$

then

$$(278) \quad F_0 = \frac{1}{2} \frac{\hat{p}^1 \hat{p}^4 + \hat{p}^2 \hat{p}^3}{(\hat{p}^1)^2}.$$

7.5. The metric G^{ij} . A constant symmetric nondegenerate matrix (G_{ij}) is defined by

$$(279) \quad G_{ij} = \frac{\partial p^i}{\partial t^\alpha} g_{\alpha\beta} \frac{\partial p^j}{\partial t^\beta}.$$

From

$$(dt^\alpha, dt^\beta) = \begin{pmatrix} 0 & 0 & 0 & t^P \\ 0 & 0 & t^P & 0 \\ 0 & t^P & 0 & -t^R \\ t^P & 0 & -t^R & -2t^S \end{pmatrix}$$

a computation gives us:

$$(280) \quad (dp^i, dp^j) = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

7.6. The almost dual quantum multiplication. The quantum multiplication by the Euler vector field is given by:

$$(281) \quad E \circ \begin{pmatrix} \frac{\partial}{\partial t^P} \\ \frac{\partial}{\partial t^Q} \\ \frac{\partial}{\partial t^R} \\ \frac{\partial}{\partial t^S} \end{pmatrix} = \begin{pmatrix} t^P & 0 & -t^R & -2t^S \\ 0 & t^P & 0 & -t^R \\ 0 & 0 & t^P & 0 \\ 0 & 0 & 0 & t^P \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial}{\partial t^P} \\ \frac{\partial}{\partial t^Q} \\ \frac{\partial}{\partial t^R} \\ \frac{\partial}{\partial t^S} \end{pmatrix}$$

$$(282) \quad \mathcal{U} = \begin{pmatrix} t^P & 0 & -t^R & -2t^S \\ 0 & t^P & 0 & -t^R \\ 0 & 0 & t^P & 0 \\ 0 & 0 & 0 & t^P \end{pmatrix}$$

The inverse of E can be found in the following fashion: Let $E^{-1} = A \frac{\partial}{\partial t^P} + B \frac{\partial}{\partial t^Q} + C \frac{\partial}{\partial t^R} + D \frac{\partial}{\partial t^S}$. Recall that $E = t^P \frac{\partial}{\partial t^P} - t^R \frac{\partial}{\partial t^R} - 2t^S \frac{\partial}{\partial t^S}$.

$$\begin{aligned} & \left(t^P \frac{\partial}{\partial t^P} - t^R \frac{\partial}{\partial t^R} - 2t^S \frac{\partial}{\partial t^S} \right) \cdot \left(A \frac{\partial}{\partial t^P} + B \frac{\partial}{\partial t^Q} + C \frac{\partial}{\partial t^R} + D \frac{\partial}{\partial t^S} \right) \\ &= At^P \frac{\partial}{\partial t^P} + Bt^P \frac{\partial}{\partial t^Q} + Ct^P \frac{\partial}{\partial t^R} + Dt^P \frac{\partial}{\partial t^S} - At^R \frac{\partial}{\partial t^R} - Bt^R \frac{\partial}{\partial t^S} - 2At^S \frac{\partial}{\partial t^S}, \end{aligned}$$

and so $A = \frac{1}{t^P}$, $B = 0$, $C = \frac{t^R}{(t^P)^3}$, $D = \frac{2t^S}{(t^P)^2}$, i.e.,

$$(283) \quad E^{-1} = \frac{1}{t^P} \frac{\partial}{\partial t^P} + \frac{t^R}{(t^P)^2} \frac{\partial}{\partial t^R} + \frac{2t^S}{(t^P)^2} \frac{\partial}{\partial t^S}.$$

So we get the following explicit formula for dual multiplications:

$$\begin{aligned} \frac{\partial}{\partial t^P} * \begin{pmatrix} \frac{\partial}{\partial t^P} \\ \frac{\partial}{\partial t^Q} \\ \frac{\partial}{\partial t^R} \\ \frac{\partial}{\partial t^S} \end{pmatrix} &= E^{-1} \circ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial}{\partial t^P} \\ \frac{\partial}{\partial t^Q} \\ \frac{\partial}{\partial t^R} \\ \frac{\partial}{\partial t^S} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial}{\partial t^P} \\ \frac{\partial}{\partial t^Q} \\ \frac{\partial}{\partial t^R} \\ \frac{\partial}{\partial t^S} \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial t^Q} * \begin{pmatrix} \frac{\partial}{\partial t^P} \\ \frac{\partial}{\partial t^Q} \\ \frac{\partial}{\partial t^R} \\ \frac{\partial}{\partial t^S} \end{pmatrix} &= E^{-1} \circ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 5 + \sum_{m=1}^{\infty} K_m m^3 e^{mt^Q} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial}{\partial t^P} \\ \frac{\partial}{\partial t^Q} \\ \frac{\partial}{\partial t^R} \\ \frac{\partial}{\partial t^S} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 5 + \sum_{m=1}^{\infty} K_m m^3 e^{mt^Q} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial}{\partial t^P} \\ \frac{\partial}{\partial t^Q} \\ \frac{\partial}{\partial t^R} \\ \frac{\partial}{\partial t^S} \end{pmatrix} \end{aligned}$$

$$\frac{\partial}{\partial t^R} * \begin{pmatrix} \frac{\partial}{\partial t^P} \\ \frac{\partial}{\partial t^Q} \\ \frac{\partial}{\partial t^R} \\ \frac{\partial}{\partial t^S} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial}{\partial t^P} \\ \frac{\partial}{\partial t^Q} \\ \frac{\partial}{\partial t^R} \\ \frac{\partial}{\partial t^S} \end{pmatrix},$$

$$\frac{\partial}{\partial t^S} * \begin{pmatrix} \frac{\partial}{\partial t^P} \\ \frac{\partial}{\partial t^Q} \\ \frac{\partial}{\partial t^R} \\ \frac{\partial}{\partial t^S} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial}{\partial t^P} \\ \frac{\partial}{\partial t^Q} \\ \frac{\partial}{\partial t^R} \\ \frac{\partial}{\partial t^S} \end{pmatrix}.$$

7.7. The almost dual potential function. We use

$$(284) \quad d\left(\frac{\partial^2 F_*}{\partial p^a \partial p^b}\right) = dp_a \circ dp_b$$

to compute F_* . First note:

$$\begin{aligned} dp^1 &= -\frac{dt^P}{(t^P)^2}, & dp^2 &= -\frac{t^Q dt^P}{(t^P)^2} + \frac{dt^Q}{t^P}, \\ dp^3 &= dt^R, & dp^4 &= t^S dt^P + t^R dt^Q + t^Q dt^R + t^P dt^S. \end{aligned}$$

$$\begin{aligned} dt^P &= -\frac{dp^1}{(p^1)^2}, & dt^Q &= -\frac{p^2 dp^1}{(p^1)^2} + \frac{dp^2}{p^1}, \\ dt^R &= dp^3, & dt^S &= p^4 dp^1 - p^3 dp^2 - p^2 dp^3 + p^1 dp^4. \end{aligned}$$

So we get:

$$\begin{aligned} dp^1 \circ \begin{pmatrix} dp^1 \\ dp^2 \\ dp^3 \\ dp^4 \end{pmatrix} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{t^P} & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} dt^P \\ dt^Q \\ dt^R \\ dt^S \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{p^1} & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} dp^1 \\ dp^2 \\ dp^3 \\ dp^4 \end{pmatrix}, \end{aligned}$$

therefore,

$$(285) \quad \frac{\partial^2 F_*}{\partial p^4 \partial p^4} = \frac{\partial^2 F_*}{\partial p^4 \partial p^3} = \frac{\partial^2 F_*}{\partial p^4 \partial p^2} = 0, \quad \frac{\partial^2 F_*}{\partial p^4 \partial p^1} = \log(p^1).$$

Similarly,

$$\begin{aligned} dp^2 \circ \begin{pmatrix} dp^1 \\ dp^2 \\ dp^3 \\ dp^4 \end{pmatrix} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{t^P} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} dt^P \\ dt^Q \\ dt^R \\ dt^S \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{p^1} & 0 & 0 & 0 \\ -\frac{p^2}{(p^1)^2} & \frac{1}{p^1} & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} dp^1 \\ dp^2 \\ dp^3 \\ dp^4 \end{pmatrix} \end{aligned}$$

so we get:

$$(286) \quad \frac{\partial^2 F_*}{\partial p^3 \partial p^4} = \frac{\partial^2 F_*}{\partial p^3 \partial p^3} = 0, \quad \frac{\partial^2 F_*}{\partial p^3 \partial p^2} = -\log(p^1), \quad \frac{\partial^2 F_*}{\partial p^3 \partial p^1} = -\frac{p^2}{p^1}.$$

From

$$dp^3 \circ \begin{pmatrix} dp^1 \\ dp^2 \\ dp^3 \\ dp^4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{t^P} & 0 & 0 & 0 \\ 0 & f_0'''(t^Q) & 0 & 0 \\ t^R & t^Q f_0'''(t^Q) & t^P & 0 \end{pmatrix} \cdot \begin{pmatrix} dt^P \\ dt^Q \\ dt^R \\ dt^S \end{pmatrix},$$

we know that

$$\begin{aligned}
dp_2 \cdot dp_4 &= 0, \\
dp_2 \cdot dp_3 &= \frac{dt^P}{t^P} = d \log t^P = -d \log p^1, \\
dp_2 \cdot dp_2 &= f_0'''(t^Q) dt^Q = df''(t^Q) = df''\left(\frac{p^2}{p^1}\right), \\
dp_2 \cdot dp_1 &= t^R dt^P + t^Q f_0'''(t^Q) dt^Q + t^P dt^R = d(t^P t^R + t^Q f_0''(t^Q) - f_0'(t^Q)) \\
&= d\left(\frac{p^3}{p^1} + \frac{p^2}{p^1} f_0''\left(\frac{p^2}{p^1}\right) - f_0'\left(\frac{p^2}{p^1}\right)\right).
\end{aligned}$$

so we get:

$$\begin{aligned}
(287) \quad \frac{\partial^2 F_*}{\partial p^2 \partial p^4} &= 0, \\
\frac{\partial^2 F_*}{\partial p^2 \partial p^3} &= -\log p^1, \\
\frac{\partial^2 F_*}{\partial p^2 \partial p^2} &= f''\left(\frac{p^2}{p^1}\right), \\
\frac{\partial^2 F_*}{\partial p^2 \partial p^1} &= -\left(\frac{p^3}{p^1} + \frac{p^2}{p^1} f_0''\left(\frac{p^2}{p^1}\right) - f_0'\left(\frac{p^2}{p^1}\right)\right).
\end{aligned}$$

$$dp^4 \cdot \begin{pmatrix} dp^1 \\ dp^2 \\ dp^3 \\ dp^4 \end{pmatrix} = \begin{pmatrix} -\frac{1}{t^P} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ t^R & t^Q f_0'''(t^Q) & t^P & 0 \\ 2t^P t^S + 2t^Q t^R & 2t^P t^R + (t^Q)^2 f_0'''(t^Q) & 2t^P t^Q & (t^P)^2 \end{pmatrix} \cdot \begin{pmatrix} dt^P \\ dt^Q \\ dt^R \\ dt^S \end{pmatrix},$$

From this we know that

$$\begin{aligned}
dp_1 \cdot dp_4 &= -\frac{dt^P}{t^P} = -d \log t^P = d \log p^1, \\
dp_1 \cdot dp_3 &= -dt^Q = -d \frac{p^2}{p^1}, \\
dp_1 \cdot dp_2 &= -(t^R dt^P + t^Q f_0'''(t^Q) dt^Q + t^P dt^R) \\
&= -d(t^P t^R + t^Q f_0''(t^Q) - f_0'(t^Q)) \\
&= -d\left(\frac{p^3}{p^1} + \frac{p^2}{p^1} f_0''\left(\frac{p^2}{p^1}\right) - f_0'\left(\frac{p^2}{p^1}\right)\right), \\
dp_1 \cdot dp_1 &= (2t^P t^S + 2t^Q t^R) dt^P + (2t^P t^R + (t^Q)^2 f_0'''(t^Q)) dt^Q \\
&\quad + 2t^P t^Q dt^R + (t^P)^2 dt^S \\
&= d\left((t^P)^2 t^S + 2t^P t^Q t^R + 2f_0(t^Q) - 2t^Q f_0'(t^Q) + (t^Q)^2 f_0''(t^Q)\right).
\end{aligned}$$

Note $2F_0$ appears on the right-hand side of the last equality. So we have:

$$\begin{aligned}\frac{\partial^2 F_*}{\partial p^1 \partial p^4} &= \log p^1, \\ \frac{\partial^2 F_*}{\partial p^1 \partial p^3} &= \frac{p^2}{p^1}, \\ \frac{\partial^2 F_*}{\partial p^1 \partial p^2} &= \frac{p^3}{p^1} + \frac{p^2}{p^1} f_0''\left(\frac{p^2}{p^1}\right) - f_0'\left(\frac{p^2}{p^1}\right), \\ \frac{\partial^2 F_*}{\partial p^1 \partial p^1} &= (t^P)^2 t^S + 2t^P t^Q t^R + 2f_0(t^Q) - 2t^Q f_0'(t^Q) + (t^Q)^2 f_0''(t^Q) \\ &= \frac{p^1 p^4 + p^2 p^3}{(p^1)^2} + 2f\left(\frac{p^2}{p^1}\right) - 2\frac{p^2}{p^1} f_0'\left(\frac{p^2}{p^1}\right) + \left(\frac{p^2}{p^1}\right)^2 f_0''\left(\frac{p^2}{p^1}\right).\end{aligned}$$

Proposition 7.2. *For the quintic we have*

$$(288) \quad F_* = (p^1 p^4 - p^2 p^3) \log p^1 - p^1 p^4 + (p^1)^2 f_0\left(\frac{p^2}{p^1}\right).$$

Note we can verify that

$$(289) \quad \sum_a p^a \frac{\partial F_*}{\partial p^a} = 2F_* + \frac{1}{1-d} \sum_{a,b} G_{ab} p^a p^b.$$

Also note:

$$(290) \quad E = \frac{\partial}{\partial t^P} - \frac{\partial}{\partial t^R} - 2\frac{\partial}{\partial t^S} = -p^1 \frac{\partial}{\partial p^1} - p^2 \frac{\partial}{\partial p^2} - p^3 \frac{\partial}{\partial p^3} - p^4 \frac{\partial}{\partial p^4},$$

$$(291) \quad e = \frac{\partial}{\partial t^P} = -(p^1)^2 \frac{\partial}{\partial p^1} - p^1 p^2 \frac{\partial}{\partial p^2}.$$

7.8. The almost dual flat coordinates. The equations for the almost dual flat coordinates (also called the *twisted periods*) are:

$$(292) \quad \partial_\alpha \xi \cdot \mathcal{U} = \xi \cdot (\mathcal{V} + \nu - \frac{1}{2}) C_\alpha.$$

Explicitly we get:

$$\begin{aligned}
& \overline{\partial_P(\partial_P x, \partial_Q x, \partial_R x, \partial_S x)} \\
= & (\partial_P x, \partial_Q x, \partial_R x, \partial_S x) \cdot \begin{pmatrix} -2 + \nu & 0 & 0 & 0 \\ 0 & -1 + \nu & 0 & 0 \\ 0 & 0 & 0 + \nu & 0 \\ 0 & 0 & 0 & 1 + \nu \end{pmatrix} \\
& \cdot \frac{1}{(t^P)^2} \begin{pmatrix} t^P & 0 & 0 & 0 \\ 0 & t^P & 0 & 0 \\ t^R & 0 & t^P & 0 \\ 2t^S & t^R & 0 & t^P \end{pmatrix} \\
= & (\partial_P x, \partial_Q x, \partial_R x, \partial_S x) \\
& \cdot \frac{1}{(t^P)^2} \begin{pmatrix} (-2 + \nu)t^P & 0 & 0 & 0 \\ 0 & (-1 + \nu)t^P & 0 & 0 \\ \nu t^R & 0 & \nu t^P & 0 \\ 2(1 + \nu)t^S & (1 + \nu)t^R & 0 & (1 + \nu)t^P \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
& \partial_Q(\partial_P x, \partial_Q x, \partial_R x, \partial_S x) \\
= & (\partial_P x, \partial_Q x, \partial_R x, \partial_S x) \cdot \begin{pmatrix} -2 + \nu & 0 & 0 & 0 \\ 0 & -1 + \nu & 0 & 0 \\ 0 & 0 & 0 + \nu & 0 \\ 0 & 0 & 0 & 1 + \nu \end{pmatrix} \\
& \cdot \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & f_0'''(t^Q) & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \cdot \frac{1}{(t^P)^2} \begin{pmatrix} t^P & 0 & 0 & 0 \\ 0 & t^P & 0 & 0 \\ t^R & 0 & t^P & 0 \\ 2t^S & t^R & 0 & t^P \end{pmatrix} \\
= & (\partial_P x, \partial_Q x, \partial_R x, \partial_S x) \cdot \frac{1}{(t^P)^2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ (-1 + \nu)t^P & 0 & 0 & 0 \\ 0 & \nu t^P f_0'''(t^Q) & 0 & 0 \\ (1 + \nu)t^R & 0 & (1 + \nu)t^P & 0 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
& \partial_R(\partial_P x, \partial_Q x, \partial_R x, \partial_S x) \\
= & (\partial_P x, \partial_Q x, \partial_R x, \partial_S x) \cdot \begin{pmatrix} -2 + \nu & 0 & 0 & 0 \\ 0 & -1 + \nu & 0 & 0 \\ 0 & 0 & 0 + \nu & 0 \\ 0 & 0 & 0 & 1 + \nu \end{pmatrix} \\
& \cdot \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \cdot \frac{1}{(t^P)^2} \begin{pmatrix} t^P & 0 & 0 & 0 \\ 0 & t^P & 0 & 0 \\ t^R & 0 & t^P & 0 \\ 2t^S & t^R & 0 & t^P \end{pmatrix} \\
= & (\partial_P x, \partial_Q x, \partial_R x, \partial_S x) \cdot \frac{1}{(t^P)^2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \nu t^P & 0 & 0 & 0 \\ 0 & (1 + \nu)t^P & 0 & 0 \end{pmatrix} \\
& \partial_S(\partial_P x, \partial_Q x, \partial_R x, \partial_S x) \\
= & (\partial_P x, \partial_Q x, \partial_R x, \partial_S x) \cdot \begin{pmatrix} -2 + \nu & 0 & 0 & 0 \\ 0 & -1 + \nu & 0 & 0 \\ 0 & 0 & 0 + \nu & 0 \\ 0 & 0 & 0 & 1 + \nu \end{pmatrix} \\
& \cdot \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \cdot \frac{1}{(t^P)^2} \begin{pmatrix} t^P & 0 & 0 & 0 \\ 0 & t^P & 0 & 0 \\ t^R & 0 & t^P & 0 \\ 2t^S & t^R & 0 & t^P \end{pmatrix} \\
= & (\partial_P x, \partial_Q x, \partial_R x, \partial_S x) \cdot \frac{1}{(t^P)^2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ (1 + \nu)t^P & 0 & 0 & 0 \end{pmatrix}
\end{aligned}$$

The equations for $\partial_S x$ are

$$\partial_P(\partial_S x) = \frac{1 + \nu}{t^P - \lambda} \partial_S x, \quad \partial_Q(\partial_S x) = \partial_R(\partial_S x) = \partial_S(\partial_S x) = 0,$$

so we get:

$$\partial_S x = C_1 \cdot (t^P - \lambda)^{1+\nu}.$$

The equations for $\partial_R x$ are

$$\begin{aligned}
\partial_P(\partial_R x) &= \frac{\nu}{t^P - \lambda} \partial_R x, \\
\partial_Q(\partial_R x) &= \frac{1 + \nu}{t^P - \lambda} \partial_S x = C_1 \cdot (1 + \nu)(t^P - \lambda)^\nu, \\
\partial_R(\partial_R x) &= \partial_S(\partial_R x) = 0,
\end{aligned}$$

so we get:

$$(293) \quad \partial_R x = C_1 \cdot t^Q \cdot (1 + \nu)(t^P - \lambda)^\nu + C_2(t^P - \lambda)^\nu.$$

The equations for $\partial_Q x$ are

$$\begin{aligned} \partial_P(\partial_Q x) &= \frac{(-1 + \nu)}{t^P - \lambda} \partial_Q x + \frac{(1 + \nu)t^R}{(t^P - \lambda)^2} \partial_S x \\ &= \frac{(-1 + \nu)}{t^P - \lambda} \partial_Q x + C_1(1 + \nu)t^R(t^P - \lambda)^{\nu-1}, \\ \partial_Q(\partial_Q x) &= \frac{\nu}{t^P - \lambda} f_0'''(t^Q) \partial_R x \\ &= \frac{\nu}{t^P - \lambda} f_0'''(t^Q) \cdot (C_1 \cdot t^Q \cdot (1 + \nu)(t^P - \lambda)^\nu + C_2(t^P - \lambda)^\nu), \\ \partial_R(\partial_Q x) &= \frac{1 + \nu}{t^P - \lambda} \partial_S x = C_1 \cdot (1 + \nu)(t^P - \lambda)^\nu, \\ \partial_S(\partial_Q x) &= 0, \end{aligned}$$

so we get:

$$(294) \quad \begin{aligned} \partial_Q x &= C_1 \cdot (1 + \nu) \left(t^R(t^P - \lambda)^\nu + \nu(t^P - \lambda)^{\nu-1}(t^Q f_0''(t^Q) - f_0'(t^Q)) \right) \\ &\quad + C_2 \nu(t^P - \lambda)^{\nu-1} f_0''(t^Q) + C_3(t^P - \lambda)^{\nu-1}. \end{aligned}$$

The equations for $\partial_P x$ are

$$\begin{aligned}
\partial_P(\partial_P x) &= \frac{-2 + \nu}{t^P - \lambda} \partial_P x + \frac{2(1 + \nu)t^S}{(t^P - \lambda)^2} \partial_S x \\
&= \frac{(-2 + \nu)}{t^P - \lambda} \partial_P x + C_1 \cdot 2(1 + \nu)t^S (t^P - \lambda)^{\nu-1}, \\
\partial_Q(\partial_P x) &= \frac{-1 + \nu}{t^P - \lambda} \partial_Q x + \frac{(1 + \nu)t^R}{(t^P - \lambda)^2} \partial_S x \\
&= \frac{\nu - 1}{t^P - \lambda} \left(C_1 \cdot (1 + \nu) \left(t^R (t^P - \lambda)^\nu + \nu (t^P - \lambda)^{\nu-1} (t^Q f_0''(t^Q) - f_0'(t^Q)) \right) \right) \\
&\quad + C_2 \cdot \nu (t^P - \lambda)^{\nu-1} f_0''(t^Q) + C_3 (t^P - \lambda)^{\nu-1} \Big) + C_1 \cdot (\nu + 1) t^R (t^P - \lambda)^{\nu-1} \\
&= C_1 \left((\nu + 1) \nu t^R (t^P - \lambda)^{\nu-1} + (\nu + 1) \nu (\nu - 1) (t^P - \lambda)^{\nu-2} (t^Q f_0''(t^Q) - f_0'(t^Q)) \right) \\
&\quad + C_2 \cdot \nu (\nu - 1) (t^P - \lambda)^{\nu-2} f_0''(t^Q) + C_3 \cdot (\nu - 1) (t^P - \lambda)^{\nu-2}, \\
\partial_R(\partial_P x) &= \frac{\nu}{t^P - \lambda} \partial_R x \\
&= C_1 \cdot t^Q \cdot (1 + \nu) \nu (t^P - \lambda)^{\nu-1} + C_2 \cdot \nu (t^P - \lambda)^{\nu-1}, \\
\partial_S(\partial_P x) &= \frac{1 + \nu}{t^P - \lambda} \partial_S x = C_1 \cdot (\nu + 1) (t^P - \lambda)^\nu,
\end{aligned}$$

so we get:

$$\begin{aligned}
\partial_P x &= C_1 \cdot \left(t^S \cdot (\nu + 1) (t^P - \lambda)^\nu + (\nu + 1) \nu (t^P - \lambda)^{\nu-1} t^Q t^R \right. \\
(295) \quad &\quad \left. + (\nu + 1) \nu (\nu - 1) (t^P - \lambda)^{\nu-2} (t^Q f_0'(t^Q) - f_0(t^Q)) \right) \\
&\quad + C_2 \cdot \nu (\nu - 1) (t^P - \lambda)^{\nu-2} f_0'(t^Q) \\
&\quad + C_3 \cdot (\nu - 1) (t^P - \lambda)^{\nu-2} t^Q + C_4 \cdot (\nu - 1) (t^P - \lambda)^{\nu-2}.
\end{aligned}$$

The general solution is:

$$\begin{aligned}
x &= C_0 + C_1 \left((t^P - \lambda)^{\nu+1} t^S + (\nu + 1) (t^P - \lambda)^\nu t^Q t^R \right. \\
&\quad \left. + (\nu + 1) \nu (t^P - \lambda)^{\nu-1} (t^Q f_0'(t^Q) - f_0(t^Q)) \right) \\
(296) \quad &\quad + C_2 \cdot (t^R (t^P - \lambda)^\nu + \nu (t^P - \lambda)^{\nu-1} f_0'(t^Q)) \\
&\quad + C_3 \cdot (t^P - \lambda)^{\nu-1} t^Q \\
&\quad + C_4 \cdot (t^P - \lambda)^{\nu-1}.
\end{aligned}$$

Proposition 7.3. *For the quintic one can take the following deformed dual flat coordinates:*

$$\begin{aligned}
p^4(\nu) &= (t^P - \lambda)^{\nu+1} t^S + (\nu + 1)(t^P - \lambda)^\nu t^Q t^R \\
&\quad + (\nu + 1)\nu(t^P - \lambda)^{\nu-1}(t^Q f'_0(t^Q) - f_0(t^Q)), \\
(297) \quad p^3(\nu) &= t^R(t^P - \lambda)^\nu + \nu(t^P - \lambda)^{\nu-1} f'_0(t^Q), \\
p^2(\nu) &= (t^P - \lambda)^{\nu-1} t^Q \\
p^1(\nu) &= (t^P - \lambda)^{\nu-1}.
\end{aligned}$$

Note the following property of the deformed dual flat coordinates:

$$\begin{aligned}
(298) \quad \frac{\partial p^1(\nu)}{\partial t^P} &= (\nu - 1) \cdot p^1(\nu - 1), \\
\frac{\partial p^2(\nu)}{\partial t^P} &= (\nu - 1) \cdot p^2(\nu - 1), \\
\frac{\partial p^3(\nu)}{\partial t^P} &= \nu \cdot p^3(\nu - 1), \\
\frac{\partial p^4(\nu)}{\partial t^P} &= (\nu + 1) \cdot p^1(\nu - 1).
\end{aligned}$$

The inverse map of (297) is given by:

$$\begin{aligned}
(299) \quad t^P - \lambda &= (p^1(\nu))^{\nu-1}, \\
t^Q &= \frac{p^2(\nu)}{p^1(\nu)}, \\
t^R &= \frac{p^3(\nu) - \nu p^1(\nu) f'_0\left(\frac{p^2(\nu)}{p^1(\nu)}\right)}{(p^1(\nu))^{\nu/(\nu-1)}},
\end{aligned}$$

$$\begin{aligned}
t^S &= (p^1)^{-(\nu+1)/(\nu-1)} \left(p^4 - (\nu + 1)(p^1)^{\nu/(\nu-1)} \cdot \frac{p^2}{p^1} \cdot \frac{p^3 - \nu p^1 f'\left(\frac{p^2}{p^1}\right)}{(p^1)^{\nu/(\nu-1)}} \right. \\
&\quad \left. + (\nu + 1)\nu p^1 \left(\frac{p^2}{p^1} f'_0\left(\frac{p^2}{p^1}\right) - f_0\left(\frac{p^2}{p^1}\right) \right) \right) \\
&= (p^1)^{-(\nu+1)/(\nu-1)} \left(p^4 - (\nu + 1) \cdot \frac{p^2 \cdot p^3}{p^1} - (\nu + 1)\nu p^1 f_0\left(\frac{p^2}{p^1}\right) \right).
\end{aligned}$$

According to Dubrovin [10, Proposition 5.11], if one set $f = (-1 - \nu)t^S$. Then the intersection form is given by the Hessian of f :

$$(300) \quad \left(\frac{\partial}{\partial p^i}, \frac{\partial}{\partial p^j} \right) = \frac{\partial^2 f}{\partial p^i \partial p^j}.$$

7.9. Deformed dual flat coordinates as Laplace transform of the deformed flat coordinates. As observed by Givental [22], the deformed flat coordinates are given by the genus zero one-point functions on the small phase space. For the quintic, such functions have been computed in §4.4 and have been verify to satisfy the quantum differential equations in §4.10.

Using notations of §6.2, we can rewrite the results in §4.4 as follows:

$$\begin{aligned}
x^P(z) &= t^P + \sum_{n \geq 0} z^{n+1} \langle \langle \tau_n(S) \rangle \rangle_0 = \frac{e^{tPz} - 1}{z}, \\
x^Q(z) &= t^Q + \sum_{n \geq 0} z^{n+1} \langle \langle \tau_n(R) \rangle \rangle_0 = e^{tPz} t^Q, \\
x^R(z) &= t^R + \sum_{n=0}^{\infty} z^{n+1} \langle \langle \tau_n(Q) \rangle \rangle_0 = e^{tPz} t^R + z e^{tPz} f'_0(t^Q), \\
x^S(z) &= t^S + \sum_{n \geq 0} z^{n+1} \langle \langle \tau_n(P) \rangle \rangle_0 \\
&= e^{tPz} t^S + z e^{tPz} t^Q t^R + z^2 e^{tPz} (t^Q f'_0(t^Q) - f_0(t^Q)).
\end{aligned}$$

Proposition 7.4. *For the quintic, the deformed dual flat coordinates are given by the Laplace transform of the deformed flat coordinates. More precisely,*

$$(301) \quad p^1(\nu) = \frac{(-1)^{1-\nu}}{\Gamma(1-\nu)} \int_0^\infty e^{-\lambda z} \cdot z^{1-\nu} (x^P(z) + \frac{1}{z}) dz,$$

$$(302) \quad p^2(\nu) = \frac{(-1)^{1-\nu}}{\Gamma(1-\nu)} \int_0^\infty e^{-\lambda z} \cdot z^{0-\nu} x^Q(z) dz,$$

$$(303) \quad p^3(\nu) = \frac{(-1)^{-\nu}}{\Gamma(-\nu)} \int_0^\infty e^{-\lambda z} \cdot z^{-1-\nu} x^R(z) dz,$$

$$(304) \quad p^4(\nu) = \frac{(-1)^{-1-\nu}}{\Gamma(-1-\nu)} \int_0^\infty e^{-\lambda z} \cdot z^{-2-\nu} x^S(z) dz.$$

Proof. By the definition of the Gamma-function we have:

$$\begin{aligned}
& \int_0^\infty e^{-\lambda z} \cdot z^{1-\nu} \left(x^P(z) + \frac{1}{z}\right) dz = \int_0^\infty e^{-\lambda z} \cdot z^{1-\nu} \frac{e^{t^P z}}{z} dz \\
&= \int_0^\infty e^{-(\lambda - t^P)z} \cdot z^{(1-\nu)-1} dz \\
&= (\lambda - t^P)^{\mu-1} \cdot \int_0^\infty e^{-z} \cdot z^{(1-\nu)-1} dz \\
&= \Gamma(1 - \nu) \cdot (\lambda - t^P)^{\nu-1}.
\end{aligned}$$

This proves (301). The other three identities are proved in the same fashion. \square

8. EMERGENT KÄHLER GEOMETRY ON THE SMALL PHASE SPACE OF THE QUINTIC

We now present the special geometry on the small phase space of the quintic.

8.1. The flat sections for the Dubrovin connection. By the results in §4.9 and §4.10, we know that the following are flat sections for the Dubrovin connection:

$$\begin{aligned}
s_S &= e^{-t^P} \frac{\partial}{\partial t^S}, \\
s_R &= e^{-t^P} \left(\frac{\partial}{\partial t^R} - t^Q \frac{\partial}{\partial t^S} \right), \\
s_Q &= e^{-t^P} \left[\frac{\partial}{\partial t^Q} - \frac{\partial^2 f_0(t^Q)}{\partial (t^Q)^2} \frac{\partial}{\partial t^R} + \left(\frac{\partial f_0(t^Q)}{\partial t^Q} - t^R \right) \frac{\partial}{\partial t^S} \right], \\
s_P &= e^{-t^P} \left[\frac{\partial}{\partial t^P} - t^Q \frac{\partial}{\partial t^Q} + \left(t^Q \frac{\partial^2 f_0(t^Q)}{\partial (t^Q)^2} - \frac{\partial f_0(t^Q)}{\partial t^Q} - t^R \right) \frac{\partial}{\partial t^R} \right. \\
&\quad \left. - \left(t^Q \frac{\partial f_0(t^Q)}{\partial t^Q} - 2f_0(t^Q) + t^S - t^Q t^R \right) \frac{\partial}{\partial t^S} \right].
\end{aligned}$$

They are given by the formulas for $S_{\alpha,\beta}(z)$. For example,

$$s_P = S_{S,P}(-1) \frac{\partial}{\partial t^P} + S_{R,P}(-1) \frac{\partial}{\partial t^Q} + S_{Q,P}(-1) \frac{\partial}{\partial t^R} + S_{P,P}(-1) \frac{\partial}{\partial t^S}.$$

8.2. The complex symplectic structure. We introduce a complex symplectic structure such that:

$$(305) \quad \Omega(s_P, s_S) = -\Omega(s_S, s_P) = -\Omega(s_Q, s_R) = \Omega(s_R, s_Q) = 1,$$

and all other $\omega(s_\alpha, s_\beta) = 0$. Because we have:

$$\begin{aligned}\frac{\partial}{\partial t^S} &= e^{t^P} s_S, \\ \frac{\partial}{\partial t^R} &= e^{t^P} \left[s_R + t^Q s_S \right], \\ \frac{\partial}{\partial t^Q} &= e^{t^P} \left[s_Q + \frac{\partial^2 f_0(t^Q)}{\partial (t^Q)^2} s_R + \left(t^Q \frac{\partial^2 f_0(t^Q)}{\partial (t^Q)^2} - \frac{\partial f_0(t^Q)}{\partial t^Q} + t^R \right) s_S \right], \\ \frac{\partial}{\partial t^P} &= e^{t^P} \left[s_P + t^Q s_Q + \left(\frac{\partial f_0(t^Q)}{\partial t^Q} + t^R \right) s_R \right. \\ &\quad \left. + \left(t^Q \frac{\partial f_0(t^Q)}{\partial t^Q} - 2f_0(t^Q) + t^S + t^Q t^R \right) s_S \right],\end{aligned}$$

we get:

$$\begin{aligned}\Omega\left(\frac{\partial}{\partial t^P}, \frac{\partial}{\partial t^Q}\right) &= 2e^{2t^P} t^R, \\ \Omega\left(\frac{\partial}{\partial t^P}, \frac{\partial}{\partial t^R}\right) &= 0, \\ \Omega\left(\frac{\partial}{\partial t^P}, \frac{\partial}{\partial t^S}\right) &= e^{2t^P}, \\ \Omega\left(\frac{\partial}{\partial t^Q}, \frac{\partial}{\partial t^R}\right) &= -e^{2t^P}, \\ \Omega\left(\frac{\partial}{\partial t^Q}, \frac{\partial}{\partial t^S}\right) &= 0, \\ \Omega\left(\frac{\partial}{\partial t^R}, \frac{\partial}{\partial t^S}\right) &= 0.\end{aligned}$$

Therefore, in the flat coordinates t^P, t^Q, t^R, t^S on the small phase space of the quintic, we have:

$$(306) \quad \Omega = 2e^{2t^P} t^R dt^P \wedge dt^Q + e^{2t^P} dt^P \wedge dt^S - e^{2t^P} dt^Q \wedge dt^R.$$

It is clear that

$$(307) \quad d\Omega = 0.$$

I.e., Ω is a holomorphic symplectic structure on the small phase space. Note we can find Darboux coordinates for this symplectic structure:

$$(308) \quad \Omega = d(e^{t^P} t^R) \wedge d(e^{t^P} t^Q) + d(e^{t^P}) \wedge d(e^{t^P} t^S).$$

Remark 8.1. When $t^P = t^R = t^S = 0$, we have

$$(309) \quad \frac{\partial}{\partial t^P} = s_P + t^Q s_Q + \frac{\partial f_0(t^Q)}{\partial t^Q} s_R + \left(t^Q \frac{\partial f_0(t^Q)}{\partial t^Q} - 2f_0(t^Q) \right) s_S.$$

The coefficients on the right-hand side are

$$(1, t^Q, f_0'(t^Q), t^Q f_0'(t^Q) - 2f_0(t^Q)).$$

These also appear in (98) which describe the special geometry of the four periods of the Picard-Fuchs equation of the quintic.

8.3. The real structure. We define a real structure τ on the small phase of the quintic by requiring:

$$(310) \quad \tau(s_\alpha) = s_\alpha, \quad \alpha = P, Q, R, S.$$

The action of τ on the basis $\frac{\partial}{\partial t^\alpha}$ is computed as follows:

$$\begin{aligned}
\frac{\partial}{\partial t^S} &\mapsto s_S = \frac{\partial}{\partial t^S}, \\
\frac{\partial}{\partial t^R} &\mapsto s_Q + \overline{t^Q} s_S \\
&= \frac{\partial}{\partial t^R} + (\overline{t^Q} - t^Q) \frac{\partial}{\partial t^S}, \\
\frac{\partial}{\partial t^Q} &\mapsto s_Q + \frac{\overline{\partial^2 f_0(t^Q)}}{\partial(t^Q)^2} s_R + \left(\frac{\overline{\partial^2 f_0(t^Q)}}{\partial(t^Q)^2} - \frac{\overline{\partial f_0(t^Q)}}{\partial t^Q} \right) s_S \\
&= \frac{\partial}{\partial t^Q} + \left(\frac{\overline{\partial^2 f_0(t^Q)}}{\partial(t^Q)^2} - \frac{\partial^2 f_0(t^Q)}{\partial(t^Q)^2} \right) \frac{\partial}{\partial t^R} \\
&\quad + \left((\overline{t^Q} - t^Q) \frac{\overline{\partial^2 f_0(t^Q)}}{\partial(t^Q)^2} + \frac{\partial f_0(t^Q)}{\partial t^Q} - \frac{\overline{\partial f_0(t^Q)}}{\partial t^Q} \right) \frac{\partial}{\partial t^S}, \\
\frac{\partial}{\partial t^P} &\mapsto s_P + \overline{t^Q} s_Q + \frac{\overline{\partial f_0(t^Q)}}{\partial t^Q} s_R + \left(\frac{\overline{\partial f_0(t^Q)}}{\partial t^Q} - 2\overline{f_0(t^Q)} \right) s_S \\
&= \frac{\partial}{\partial t^P} - t^Q \frac{\partial}{\partial t^Q} + \left(t^Q \frac{\partial^2 f_0(t^Q)}{\partial(t^Q)^2} - \frac{\partial f_0(t^Q)}{\partial t^Q} \right) \frac{\partial}{\partial t^R} \\
&\quad - \left(t^Q \frac{\partial f_0(t^Q)}{\partial t^Q} - 2f_0(t^Q) \right) \frac{\partial}{\partial t^S}. \\
&\quad + \overline{t^Q} \left(\frac{\partial}{\partial t^Q} - \frac{\partial^2 f_0(t^Q)}{\partial(t^Q)^2} \frac{\partial}{\partial t^R} + \frac{\partial f_0(t^Q)}{\partial t^Q} \frac{\partial}{\partial t^S} \right) \\
&\quad + \frac{\overline{\partial f_0(t^Q)}}{\partial t^Q} \left(\frac{\partial}{\partial t^R} - t^Q \frac{\partial}{\partial t^S} \right) + \left(\frac{\overline{\partial f_0(t^Q)}}{\partial t^Q} - 2\overline{f_0(t^Q)} \right) \frac{\partial}{\partial t^S} \\
&= \frac{\partial}{\partial t^P} - (t^Q - \overline{t^Q}) \frac{\partial}{\partial t^Q} \\
&\quad + \left((t^Q - \overline{t^Q}) \frac{\partial^2 f_0(t^Q)}{\partial(t^Q)^2} - \frac{\partial f_0(t^Q)}{\partial t^Q} + \frac{\overline{\partial f_0(t^Q)}}{\partial t^Q} \right) \frac{\partial}{\partial t^R} \\
&\quad - \left[(t^Q - \overline{t^Q}) \left(\frac{\partial f_0(t^Q)}{\partial t^Q} + \frac{\overline{\partial f_0(t^Q)}}{\partial t^Q} \right) - 2(f_0(t^Q) - \overline{f_0(t^Q)}) \right] \frac{\partial}{\partial t^S}.
\end{aligned}$$

8.4. The pseudo-Hermitian metric. Using the holomorphic symplectic form and the real structure defined the preceding two Subsections we now define:

$$(311) \quad h\left(\frac{\partial}{\partial t^\alpha}, \frac{\partial}{\partial t^\beta}\right) = i\Omega\left(\frac{\partial}{\partial t^\alpha}, \tau\left(\frac{\partial}{\partial t^\beta}\right)\right).$$

Explicitly, we have:

$$\begin{aligned}
h\left(\frac{\partial}{\partial t^P}, \frac{\partial}{\partial t^P}\right) &= i\Omega\left(\frac{\partial}{\partial t^P}, \tau\left(\frac{\partial}{\partial t^P}\right)\right) \\
&= ie^{t^P + \bar{t}^P} \left[-\left(t^Q \frac{\partial f_0(t^Q)}{\partial t^Q} - 2f_0(t^Q) + t^S + t^Q t^R\right) \right. \\
&\quad + \left. \left(\overline{t^Q} \frac{\partial \overline{f_0(t^Q)}}{\partial t^Q} - 2\overline{f_0(t^Q)} + \bar{t}^S + \overline{t^Q t^R}\right) \right. \\
&\quad + \left. \overline{t^Q} \left(\frac{\partial f_0(t^Q)}{\partial t^Q} + t^R\right) - t^Q \left(\frac{\partial \overline{f_0(t^Q)}}{\partial t^Q} + \bar{t}^R\right) \right] \\
&= ie^{t^P + \bar{t}^P} \left[-(t^Q - \overline{t^Q}) \left(\frac{\partial f_0(t^Q)}{\partial t^Q} + \frac{\partial \overline{f_0(t^Q)}}{\partial t^Q}\right) \right. \\
&\quad + \left. 2\left(f_0(t^Q) - \overline{f_0(t^Q)}\right) - (t^S - \bar{t}^S) - (t^Q - \overline{t^Q})(t^R + \bar{t}^R) \right].
\end{aligned}$$

$$\begin{aligned}
h\left(\frac{\partial}{\partial t^P}, \frac{\partial}{\partial t^Q}\right) &= i\Omega\left(\frac{\partial}{\partial t^P}, \tau\left(\frac{\partial}{\partial t^Q}\right)\right) \\
&= ie^{t^P + \bar{t}^P} \left[\left(\overline{t^Q} \frac{\partial^2 f_0(t^Q)}{\partial (t^Q)^2} - \frac{\partial f_0(t^Q)}{\partial t^Q} + \bar{t}^R\right) - t^Q \frac{\partial^2 \overline{f_0(t^Q)}}{\partial (t^Q)^2} + \left(\frac{\partial f_0(t^Q)}{\partial t^Q} + t^R\right) \right] \\
&= ie^{t^P + \bar{t}^P} \left[-\left(t^Q - \overline{t^Q}\right) \frac{\partial^2 f_0(t^Q)}{\partial (t^Q)^2} + \left(\frac{\partial f_0(t^Q)}{\partial t^Q} - \frac{\partial \overline{f_0(t^Q)}}{\partial t^Q}\right) + \left(t^R + \bar{t}^R\right) \right].
\end{aligned}$$

$$h\left(\frac{\partial}{\partial t^P}, \frac{\partial}{\partial t^R}\right) = i\Omega\left(\frac{\partial}{\partial t^P}, \tau\left(\frac{\partial}{\partial t^R}\right)\right) = -ie^{t^P + \bar{t}^P} \left[t^Q - \overline{t^Q}\right].$$

$$h\left(\frac{\partial}{\partial t^P}, \frac{\partial}{\partial t^S}\right) = i\Omega\left(\frac{\partial}{\partial t^P}, \tau\left(\frac{\partial}{\partial t^S}\right)\right) = ie^{t^P + \bar{t}^P}.$$

$$h\left(\frac{\partial}{\partial t^Q}, \frac{\partial}{\partial t^Q}\right) = i\Omega\left(\frac{\partial}{\partial t^Q}, \tau\left(\frac{\partial}{\partial t^Q}\right)\right) = ie^{t^P + \bar{t}^P} \left[\frac{\partial^2 f_0(t^Q)}{\partial (t^Q)^2} - \frac{\partial^2 \overline{f_0(t^Q)}}{\partial (t^Q)^2}\right].$$

$$h\left(\frac{\partial}{\partial t^Q}, \frac{\partial}{\partial t^R}\right) = i\Omega\left(\frac{\partial}{\partial t^Q}, \tau\left(\frac{\partial}{\partial t^R}\right)\right) = -ie^{t^P + \bar{t}^P}.$$

$$h\left(\frac{\partial}{\partial t^Q}, \frac{\partial}{\partial t^S}\right) = i\Omega\left(\frac{\partial}{\partial t^Q}, \tau\left(\frac{\partial}{\partial t^S}\right)\right) = 0.$$

$$h\left(\frac{\partial}{\partial t^R}, \frac{\partial}{\partial t^R}\right) = i\Omega\left(\frac{\partial}{\partial t^R}, \tau\left(\frac{\partial}{\partial t^R}\right)\right) = 0.$$

$$h\left(\frac{\partial}{\partial t^R}, \frac{\partial}{\partial t^S}\right) = i\Omega\left(\frac{\partial}{\partial t^R}, \tau\left(\frac{\partial}{\partial t^S}\right)\right) = 0.$$

$$h\left(\frac{\partial}{\partial t^S}, \frac{\partial}{\partial t^S}\right) = i\Omega\left(\frac{\partial}{\partial t^S}, \tau\left(\frac{\partial}{\partial t^S}\right)\right) = 0.$$

With these explicit expressions, we can directly check the following:

Proposition 8.2. *The Kähler potential K for h can be taken to be:*

$$\begin{aligned} h\left(\frac{\partial}{\partial t^P}, \frac{\partial}{\partial t^P}\right) &= ie^{t^P + \bar{t}^P} \left[-(t^Q - \bar{t}^Q) \left(\frac{\partial f_0(t^Q)}{\partial t^Q} + \frac{\partial \overline{f_0(t^Q)}}{\partial t^Q} \right) \right. \\ &\quad \left. + 2 \left(f_0(t^Q) - \overline{f_0(t^Q)} \right) - (t^S - \bar{t}^S) - (t^Q - \bar{t}^Q)(t^R + \bar{t}^R) \right] \\ &= i \left[e^{t^P} \cdot \overline{e^{t^P} \left(t^Q \frac{\partial f_0(t^Q)}{\partial t^Q} - 2f_0(t^Q) + t^S + t^Q t^R \right)} \right. \\ &\quad - e^{t^P} \left(t^Q \frac{\partial f_0(t^Q)}{\partial t^Q} - 2f_0(t^Q) + t^S + t^Q t^R \right) \cdot e^{\bar{t}^P} \\ &\quad - \left(e^{t^P} t^Q \cdot \overline{e^{t^P} \left(\frac{\partial f_0(t^Q)}{\partial t^Q} + t^R \right)} \right. \\ &\quad \left. \left. - \overline{e^{t^P} t^Q} \cdot e^{t^P} \left(\frac{\partial f_0(t^Q)}{\partial t^Q} + t^R \right) \right) \right]. \end{aligned}$$

8.5. Hyperkähler potential. From the explicit formula for the Kähler potential K , we introduce:

$$(312) \quad u = e^{t^P}, \quad v = e^{t^P} t^Q, \quad w = e^{t^P} t^R, \quad x = e^{t^P} (t^S + t^Q t^R).$$

We also introduce

$$(313) \quad \mathcal{F} = -u^2 f_0\left(\frac{v}{u}\right) + ux - vw.$$

Note we have

$$\begin{aligned} \frac{\partial \mathcal{F}}{\partial u} &= e^{t^P} t^Q \frac{\partial f_0(t^Q)}{\partial t^Q} - 2e^{t^P} f_0(t^Q) + e^{t^P} (t^S + t^Q t^R) \\ &= v f_0'\left(\frac{v}{u}\right) - 2u f_0\left(\frac{v}{u}\right) + x, \\ \frac{\partial \mathcal{F}}{\partial v} &= -(e^{t^P} \frac{\partial f_0(t^Q)}{\partial t^Q} + e^{t^P} t^R) = -u f_0'\left(\frac{v}{u}\right) - w, \end{aligned}$$

and so

$$\begin{aligned}
d\frac{\partial\mathcal{F}}{\partial u} &= \left(-\frac{v^2}{u^2}f_0''\left(\frac{v}{u}\right) + \frac{2v}{u}f_0'\left(\frac{v}{u}\right) - 2f\left(\frac{v}{u}\right)\right)du \\
&\quad + \left(\frac{v}{u}f_0''\left(\frac{v}{u}\right) - f_0'\left(\frac{v}{u}\right)\right)dv + dx, \\
d\frac{\partial\mathcal{F}}{\partial v} &= \left(\frac{v}{u}f_0''\left(\frac{v}{u}\right) - f_0'\left(\frac{v}{u}\right)\right)du - f_0''\left(\frac{v}{u}\right)dv - dw.
\end{aligned}$$

Now we easily check the following:

Proposition 8.3. *In the new coordinates u, v, w and x , the holomorphic symplectic form and the Kähler potential can be written as:*

$$(314) \quad \Omega = du \wedge d\frac{\partial\mathcal{F}}{\partial u} + dv \wedge d\frac{\partial\mathcal{F}}{\partial v} = du \wedge dx - dv \wedge dw.$$

$$(315) \quad K = i \left[u \frac{\overline{\partial\mathcal{F}}}{\partial u} + v \frac{\overline{\partial\mathcal{F}}}{\partial v} - \bar{u} \frac{\partial\mathcal{F}}{\partial u} - \bar{v} \frac{\partial\mathcal{F}}{\partial v} \right]$$

Proposition 8.4. *The holomorphic symplectic form Ω is parallel with respect to the Levi-Civita connection of h .*

Proof. Let us first compute the pseudo-Hermitian metric in the new coordinates u, v, w, x :

$$\begin{aligned}
\frac{\partial K}{\partial u} &= i \left[\frac{\overline{\partial\mathcal{F}}}{\partial u} - \bar{u} \frac{\partial^2\mathcal{F}}{\partial u^2} - \bar{v} \frac{\partial^2\mathcal{F}}{\partial u\partial v} \right], \\
\frac{\partial K}{\partial v} &= i \left[\frac{\overline{\partial\mathcal{F}}}{\partial v} - \bar{u} \frac{\partial^2\mathcal{F}}{\partial u\partial v} - \bar{v} \frac{\partial^2\mathcal{F}}{\partial v^2} \right], \\
\frac{\partial K}{\partial w} &= i\bar{v}, \\
\frac{\partial K}{\partial x} &= -i\bar{u},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 K}{\partial u \partial \bar{u}} &= i \left[\frac{\overline{\partial^2 \mathcal{F}}}{\partial u^2} - \frac{\partial^2 \mathcal{F}}{\partial u^2} \right], & \frac{\partial^2 K}{\partial u \partial \bar{v}} &= i \left[\frac{\overline{\partial^2 \mathcal{F}}}{\partial u \partial v} - \frac{\partial^2 \mathcal{F}}{\partial u \partial v} \right], \\
\frac{\partial^2 K}{\partial u \partial \bar{w}} &= 0, & \frac{\partial^2 K}{\partial u \partial \bar{x}} &= i, \\
\frac{\partial^2 K}{\partial v \partial \bar{v}} &= i \left[\frac{\overline{\partial^2 \mathcal{F}}}{\partial v^2} - \frac{\partial^2 \mathcal{F}}{\partial v^2} \right], & \frac{\partial^2 K}{\partial v \partial \bar{w}} &= -i, \\
\frac{\partial^2 K}{\partial v \partial \bar{x}} &= 0, & \frac{\partial^2 K}{\partial w \partial \bar{w}} &= 0, \\
\frac{\partial^2 K}{\partial w \partial \bar{x}} &= 0, & \frac{\partial^2 K}{\partial x \partial \bar{x}} &= 0.
\end{aligned}$$

So we get the matrix for the metric:

$$h = \begin{pmatrix} i \left[\frac{\overline{\partial^2 \mathcal{F}}}{\partial u^2} - \frac{\partial^2 \mathcal{F}}{\partial u^2} \right] & i \left[\frac{\overline{\partial^2 \mathcal{F}}}{\partial u \partial v} - \frac{\partial^2 \mathcal{F}}{\partial u \partial v} \right] & 0 & i \\ i \left[\frac{\overline{\partial^2 \mathcal{F}}}{\partial u \partial v} - \frac{\partial^2 \mathcal{F}}{\partial u \partial v} \right] & i \left[\frac{\overline{\partial^2 \mathcal{F}}}{\partial v^2} - \frac{\partial^2 \mathcal{F}}{\partial v^2} \right] & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}$$

and its inverse matrix:

$$h^{-1} = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & -i \left[\frac{\overline{\partial^2 \mathcal{F}}}{\partial v^2} - \frac{\partial^2 \mathcal{F}}{\partial v^2} \right] & i \left[\frac{\overline{\partial^2 \mathcal{F}}}{\partial u \partial v} - \frac{\partial^2 \mathcal{F}}{\partial u \partial v} \right] \\ -i & 0 & i \left[\frac{\overline{\partial^2 \mathcal{F}}}{\partial u \partial v} - \frac{\partial^2 \mathcal{F}}{\partial u \partial v} \right] & -i \left[\frac{\overline{\partial^2 \mathcal{F}}}{\partial u^2} - \frac{\partial^2 \mathcal{F}}{\partial u^2} \right] \end{pmatrix}.$$

Next we compute the Levi-Civita connection. First we have

$$\partial_u h = \begin{pmatrix} -i \frac{\partial^3 \mathcal{F}}{\partial u^3} & -i \frac{\partial^3 \mathcal{F}}{\partial u^2 \partial v} & 0 & 0 \\ -i \frac{\partial^3 \mathcal{F}}{\partial u^2 \partial v} & -i \frac{\partial^3 \mathcal{F}}{\partial u \partial v^2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

then we get:

$$(316) \quad \partial_u h \cdot h^{-1} = \begin{pmatrix} 0 & 0 & -\frac{\partial^3 \mathcal{F}}{\partial u^2 \partial v} & \frac{\partial^3 \mathcal{F}}{\partial u^3} \\ 0 & 0 & -\frac{\partial^3 \mathcal{F}}{\partial u \partial v^2} & \frac{\partial^3 \mathcal{F}}{\partial u^2 \partial v} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

This implies

$$(317) \quad \nabla_{\frac{\partial}{\partial u}} \begin{pmatrix} \frac{\partial}{\partial u} \\ \frac{\partial}{\partial v} \\ \frac{\partial}{\partial w} \\ \frac{\partial}{\partial x} \end{pmatrix} = \begin{pmatrix} 0 & 0 & -\frac{\partial^3 \mathcal{F}}{\partial u^2 \partial v} & \frac{\partial^3 \mathcal{F}}{\partial u^3} \\ 0 & 0 & -\frac{\partial^3 \mathcal{F}}{\partial u \partial v^2} & \frac{\partial^3 \mathcal{F}}{\partial u^2 \partial v} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial}{\partial u} \\ \frac{\partial}{\partial v} \\ \frac{\partial}{\partial w} \\ \frac{\partial}{\partial x} \end{pmatrix}.$$

and

$$(318) \quad \nabla_{\frac{\partial}{\partial u}} \begin{pmatrix} du \\ dv \\ dw \\ dx \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{\partial^3 \mathcal{F}}{\partial u^2 \partial v} & \frac{\partial^3 \mathcal{F}}{\partial u \partial v^2} & 0 & 0 \\ -\frac{\partial^3 \mathcal{F}}{\partial u^3} & -\frac{\partial^3 \mathcal{F}}{\partial u^2 \partial v} & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} du \\ dv \\ dw \\ dx \end{pmatrix}.$$

Therefore,

$$\begin{aligned} \nabla_{\frac{\partial}{\partial u}} \Omega &= \nabla_{\frac{\partial}{\partial u}} (du \wedge dx - dv \wedge dw) \\ &= \nabla_{\frac{\partial}{\partial u}} du \wedge dx + du \wedge \nabla_{\frac{\partial}{\partial u}} dx - \nabla_{\frac{\partial}{\partial u}} dv \wedge dw - dv \wedge \nabla_{\frac{\partial}{\partial u}} dw \\ &= 0 \wedge dx + du \wedge \left(-\frac{\partial^3 \mathcal{F}}{\partial u^3} du - \frac{\partial^3 \mathcal{F}}{\partial u^2 \partial v} dv \right) \\ &\quad - 0 \wedge dw - dv \wedge \left(\frac{\partial^3 \mathcal{F}}{\partial u^2 \partial v} du + \frac{\partial^3 \mathcal{F}}{\partial u \partial v^2} dv \right) \\ &= 0. \end{aligned}$$

The covariant derivative in v can be computed similarly:

$$\partial_v h = \begin{pmatrix} -i \frac{\partial^3 \mathcal{F}}{\partial u^2 \partial v} & -i \frac{\partial^3 \mathcal{F}}{\partial u \partial v^2} & 0 & 0 \\ -i \frac{\partial^3 \mathcal{F}}{\partial u \partial v^2} & -i \frac{\partial^3 \mathcal{F}}{\partial v^3} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\partial_v h \cdot h^{-1} = \begin{pmatrix} 0 & 0 & -\frac{\partial^3 \mathcal{F}}{\partial u \partial v^2} & \frac{\partial^3 \mathcal{F}}{\partial u^2 \partial v} \\ 0 & 0 & -\frac{\partial^3 \mathcal{F}}{\partial v^3} & \frac{\partial^3 \mathcal{F}}{\partial u \partial v^2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$(319) \quad \nabla_{\frac{\partial}{\partial v}} \begin{pmatrix} \frac{\partial}{\partial u} \\ \frac{\partial}{\partial v} \\ \frac{\partial}{\partial w} \\ \frac{\partial}{\partial x} \end{pmatrix} = \begin{pmatrix} 0 & 0 & -\frac{\partial^3 \mathcal{F}}{\partial u \partial v^2} & \frac{\partial^3 \mathcal{F}}{\partial u^2 \partial v} \\ 0 & 0 & -\frac{\partial^3 \mathcal{F}}{\partial v^3} & \frac{\partial^3 \mathcal{F}}{\partial u \partial v^2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial}{\partial u} \\ \frac{\partial}{\partial v} \\ \frac{\partial}{\partial w} \\ \frac{\partial}{\partial x} \end{pmatrix}.$$

$$(320) \quad \nabla_{\frac{\partial}{\partial v}} \begin{pmatrix} du \\ dv \\ dw \\ dx \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{\partial^3 \mathcal{F}}{\partial u \partial v^2} & \frac{\partial^3 \mathcal{F}}{\partial v^3} & 0 & 0 \\ -\frac{\partial^3 \mathcal{F}}{\partial u^2 \partial v} & -\frac{\partial^3 \mathcal{F}}{\partial u \partial v^2} & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} du \\ dv \\ dw \\ dx \end{pmatrix}.$$

$$\begin{aligned} \nabla_{\frac{\partial}{\partial v}} \Omega &= \nabla_{\frac{\partial}{\partial v}} (du \wedge dx - dv \wedge dw) \\ &= \nabla_{\frac{\partial}{\partial v}} du \wedge dx + du \wedge \nabla_{\frac{\partial}{\partial v}} dx - \nabla_{\frac{\partial}{\partial v}} dv \wedge dw - dv \wedge \nabla_{\frac{\partial}{\partial v}} dw \\ &= 0 \wedge dx + du \wedge \left(-\frac{\partial^3 \mathcal{F}}{\partial u^2 \partial v} du - \frac{\partial^3 \mathcal{F}}{\partial u \partial v^2} dv \right) \\ &\quad - 0 \wedge dw - dv \wedge \left(\frac{\partial^3 \mathcal{F}}{\partial u \partial v^2} du + \frac{\partial^3 \mathcal{F}}{\partial v^3} dv \right) \\ &= 0. \end{aligned}$$

Finally, from

$$\partial_w h = \partial_x h = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

we can compute the covariant derivatives in w and x . \square

The above two Propositions imply that there is a structure of pseudo-hyperkähler manifold on the small phase space of the quintic. (We call \mathcal{F} in the hyperkähler potential of this structure.) This gives us the emergent special Kähler structure on the small phase space of the quintic. For mathematical reference on special Kähler geometry, see Freed [19].

8.6. Compatibility with the Frobenius manifold structure. Since we have computed the covariant derivative in the coordinates u, v, w, x , we will also compute the quantum multiplications in these coordinates. Note

$$t^P = \log u, \quad t^Q = \frac{v}{u}, \quad t^R = \frac{w}{u}, \quad t^S = \frac{x}{u} - \frac{vw}{u^2}.$$

So we have

$$\begin{aligned}\frac{\partial}{\partial t^P} &= u \frac{\partial}{\partial u} + v \frac{\partial}{\partial v} + w \frac{\partial}{\partial w} + x \frac{\partial}{\partial x}, \\ \frac{\partial}{\partial t^Q} &= u \frac{\partial}{\partial v} + w \frac{\partial}{\partial x}, \\ \frac{\partial}{\partial t^R} &= u \frac{\partial}{\partial w} + v \frac{\partial}{\partial x}, \\ \frac{\partial}{\partial t^S} &= u \frac{\partial}{\partial x}.\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial u} &= \frac{1}{u} \frac{\partial}{\partial t^P} - \frac{v}{u^2} \frac{\partial}{\partial t^Q} - \frac{w}{u^2} \frac{\partial}{\partial t^R} - \left(\frac{x}{u^2} - \frac{2vw}{u^3} \right) \frac{\partial}{\partial t^S}, \\ \frac{\partial}{\partial v} &= \frac{1}{u} \frac{\partial}{\partial t^Q} - \frac{w}{u^2} \frac{\partial}{\partial t^S}, \\ \frac{\partial}{\partial w} &= \frac{1}{u} \frac{\partial}{\partial t^R} - \frac{v}{u^2} \frac{\partial}{\partial t^S}, \\ \frac{\partial}{\partial x} &= \frac{1}{u} \frac{\partial}{\partial t^S}.\end{aligned}$$

We get

$$\eta = \begin{pmatrix} -\frac{2x}{u^3} + \frac{6vw}{u^4} & -\frac{2w}{u^3} & -\frac{2v}{u^3} & \frac{1}{u^2} \\ -\frac{2w}{u^3} & 0 & \frac{1}{u^2} & 0 \\ -\frac{2v}{u^3} & \frac{1}{u^2} & 0 & 0 \\ \frac{1}{u^2} & 0 & 0 & 0 \end{pmatrix}$$

and

$$\eta^{-1} = \begin{pmatrix} 0 & 0 & 0 & u^2 \\ 0 & 0 & u^2 & 2uv \\ 0 & u^2 & 0 & 2uw \\ u^2 & 2uv & 2uw & 2(uv - vw) \end{pmatrix}$$

Using the formulas for quantum multiplications in §4.3, we get:

$$(321) \quad \frac{\partial}{\partial u} \circ \begin{pmatrix} \frac{\partial}{\partial u} \\ \frac{\partial}{\partial v} \\ \frac{\partial}{\partial w} \\ \frac{\partial}{\partial x} \end{pmatrix} = \begin{pmatrix} \frac{1}{u} & -\frac{v}{u^2} & -\frac{w}{u^2} + \frac{v^2}{u^3} f_0''' \left(\frac{v}{u} \right) & \frac{v^3}{u^4} f_0''' \left(\frac{v}{u} \right) - \frac{x}{u^2} + \frac{2vw}{u^3} \\ 0 & \frac{1}{u} & -\frac{v}{u^2} f_0''' \left(\frac{v}{u} \right) & -\frac{w}{u^2} - \frac{v^2}{u^3} f_0''' \left(\frac{v}{u} \right) \\ 0 & 0 & \frac{1}{u} & -\frac{v}{u^2} \\ 0 & 0 & 0 & \frac{1}{u} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial}{\partial u} \\ \frac{\partial}{\partial v} \\ \frac{\partial}{\partial w} \\ \frac{\partial}{\partial x} \end{pmatrix}.$$

$$(322) \quad \frac{\partial}{\partial v} \circ \begin{pmatrix} \frac{\partial}{\partial u} \\ \frac{\partial}{\partial v} \\ \frac{\partial}{\partial w} \\ \frac{\partial}{\partial x} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{u} & -\frac{v}{u^2} f_0'''(\frac{v}{u}) & -\frac{w}{u^2} - \frac{v^2}{u^3} f_0'''(\frac{v}{u}) \\ 0 & 0 & \frac{1}{u} f_0'''(\frac{v}{u}) & \frac{v}{u^2} f_0'''(\frac{v}{u}) \\ 0 & 0 & 0 & \frac{1}{u} \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial}{\partial u} \\ \frac{\partial}{\partial v} \\ \frac{\partial}{\partial w} \\ \frac{\partial}{\partial x} \end{pmatrix}.$$

$$(323) \quad \frac{\partial}{\partial w} \circ \begin{pmatrix} \frac{\partial}{\partial u} \\ \frac{\partial}{\partial v} \\ \frac{\partial}{\partial w} \\ \frac{\partial}{\partial x} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{1}{u} & -\frac{v}{u^2} \\ 0 & 0 & 0 & \frac{1}{u} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial}{\partial u} \\ \frac{\partial}{\partial v} \\ \frac{\partial}{\partial w} \\ \frac{\partial}{\partial x} \end{pmatrix}.$$

$$(324) \quad \frac{\partial}{\partial x} \circ \begin{pmatrix} \frac{\partial}{\partial u} \\ \frac{\partial}{\partial v} \\ \frac{\partial}{\partial w} \\ \frac{\partial}{\partial x} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{u} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial}{\partial u} \\ \frac{\partial}{\partial v} \\ \frac{\partial}{\partial w} \\ \frac{\partial}{\partial x} \end{pmatrix}.$$

Denote by C_u , C_v , C_w and C_x the 4×4 -matrices on the right-hand sides of these equalities respectively.

Theorem 8.5. *For the quintic we have*

$$(325) \quad [\nabla_{\frac{\partial}{\partial \alpha}}, C_\beta] = [\nabla_{\frac{\partial}{\partial \beta}}, C_\alpha]$$

for $\alpha, \beta = u, v, w, x$.

Proof. These can be verified by explicit computations below:

$$[\nabla_{\frac{\partial}{\partial w}}, C_x] = \frac{\partial}{\partial w} \begin{pmatrix} 0 & 0 & 0 & \frac{1}{u} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$[\nabla_{\frac{\partial}{\partial x}}, C_w] = \frac{\partial}{\partial x} \begin{pmatrix} 0 & 0 & \frac{1}{u} & -\frac{v}{u^2} \\ 0 & 0 & 0 & \frac{1}{u} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$[\nabla_{\frac{\partial}{\partial v}}, C_x] = \frac{\partial}{\partial v} \begin{pmatrix} 0 & 0 & 0 & \frac{1}{u} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{aligned}
[\nabla_{\frac{\partial}{\partial x}}, C_v] &= \frac{\partial}{\partial x} \begin{pmatrix} 0 & \frac{1}{u} & -\frac{v}{u^2} f_0'''(\frac{v}{u}) & -\frac{w}{u^2} - \frac{v^2}{u^3} f_0'''(\frac{v}{u}) \\ 0 & 0 & \frac{1}{u} f_0'''(\frac{v}{u}) & \frac{v}{u^2} f_0'''(\frac{v}{u}) \\ 0 & 0 & 0 & \frac{1}{u} \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\end{aligned}$$

$$[\nabla_{\frac{\partial}{\partial u}}, C_x] = \frac{\partial}{\partial u} \begin{pmatrix} 0 & 0 & 0 & \frac{1}{u} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{u^2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{aligned}
[\nabla_{\frac{\partial}{\partial x}}, C_u] &= \frac{\partial}{\partial x} \begin{pmatrix} \frac{1}{u} & -\frac{v}{u^2} & -\frac{w}{u^2} + \frac{v^2}{u^3} f_0'''(\frac{v}{u}) & \frac{v^3}{u^4} f_0'''(\frac{v}{u}) - \frac{x}{u^2} + \frac{2vw}{u^3} \\ 0 & \frac{1}{u} & -\frac{v}{u^2} f_0'''(\frac{v}{u}) & -\frac{w}{u^2} - \frac{v^2}{u^3} f_0'''(\frac{v}{u}) \\ 0 & 0 & \frac{1}{u} & -\frac{v}{u^2} \\ 0 & 0 & 0 & \frac{1}{u} \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{u^2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}$$

$$[\nabla_{\frac{\partial}{\partial v}}, C_w] = \frac{\partial}{\partial v} \begin{pmatrix} 0 & 0 & \frac{1}{u} & -\frac{v}{u^2} \\ 0 & 0 & 0 & \frac{1}{u} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{u^2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{aligned}
[\nabla_{\frac{\partial}{\partial w}}, C_v] &= \frac{\partial}{\partial w} \begin{pmatrix} 0 & \frac{1}{u} & -\frac{v}{u^2} f_0'''(\frac{v}{u}) & -\frac{w}{u^2} - \frac{v^2}{u^3} f_0'''(\frac{v}{u}) \\ 0 & 0 & \frac{1}{u} f_0'''(\frac{v}{u}) & \frac{v}{u^2} f_0'''(\frac{v}{u}) \\ 0 & 0 & 0 & \frac{1}{u} \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{u^2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\end{aligned}$$

$$\begin{aligned}
\nabla_{\frac{\partial}{\partial u}} C_w &= \frac{\partial}{\partial u} \begin{pmatrix} 0 & 0 & \frac{1}{u} & -\frac{v}{u^2} \\ 0 & 0 & 0 & \frac{1}{u} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -\frac{1}{u^2} & \frac{2v}{u^3} \\ 0 & 0 & 0 & -\frac{1}{u^2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
\nabla_{\frac{\partial}{\partial w}} C_u &= \frac{\partial}{\partial w} \begin{pmatrix} \frac{1}{u} & -\frac{v}{u^2} & -\frac{w}{u^2} + \frac{v^2}{u^3} f_0'''(\frac{v}{u}) & \frac{v^3}{u^4} f_0'''(\frac{v}{u}) - \frac{x}{u^2} + \frac{2vw}{u^3} \\ 0 & \frac{1}{u} & -\frac{v}{u^2} f_0'''(\frac{v}{u}) & -\frac{w}{u^2} - \frac{v^2}{u^3} f_0'''(\frac{v}{u}) \\ 0 & 0 & \frac{1}{u} & -\frac{v}{u^2} \\ 0 & 0 & 0 & \frac{1}{u} \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & -\frac{1}{u^2} & \frac{2v}{u^3} \\ 0 & 0 & 0 & -\frac{1}{u^2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \\
\nabla_{\frac{\partial}{\partial u}} C_v &= \frac{\partial}{\partial u} \begin{pmatrix} 0 & \frac{1}{u} & -\frac{v}{u^2} f_0'''(\frac{v}{u}) & -\frac{w}{u^2} - \frac{v^2}{u^3} f_0'''(\frac{v}{u}) \\ 0 & 0 & \frac{1}{u} f_0'''(\frac{v}{u}) & \frac{v}{u^2} f_0'''(\frac{v}{u}) \\ 0 & 0 & 0 & \frac{1}{u} \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -\frac{1}{u^2} & \frac{2v}{u^3} f_0'''(\frac{v}{u}) + \frac{v^2}{u^4} f_0^{(4)}(\frac{v}{u}) & \frac{2w}{u^3} + \frac{3v^2}{u^4} f_0'''(\frac{v}{u}) + \frac{v^3}{u^5} f_0^{(4)}(\frac{v}{u}) \\ 0 & 0 & -\frac{1}{u^2} f_0'''(\frac{v}{u}) - \frac{v}{u^3} f_0^{(4)}(\frac{v}{u}) & -\frac{2v}{u^3} f_0'''(\frac{v}{u}) - \frac{v^2}{u^4} f_0^{(4)}(\frac{v}{u}) \\ 0 & 0 & 0 & -\frac{1}{u^2} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
\nabla_{\frac{\partial}{\partial v}} C_u &= \frac{\partial}{\partial v} \begin{pmatrix} \frac{1}{u} & -\frac{v}{u^2} & -\frac{w}{u^2} + \frac{v^2}{u^3} f_0'''(\frac{v}{u}) & \frac{v^3}{u^4} f_0'''(\frac{v}{u}) - \frac{x}{u^2} + \frac{2vw}{u^3} \\ 0 & \frac{1}{u} & -\frac{v}{u^2} f_0'''(\frac{v}{u}) & -\frac{w}{u^2} - \frac{v^2}{u^3} f_0'''(\frac{v}{u}) \\ 0 & 0 & \frac{1}{u} & -\frac{v}{u^2} \\ 0 & 0 & 0 & \frac{1}{u} \end{pmatrix} \\
&= \begin{pmatrix} 0 & -\frac{1}{u^2} & \frac{2v}{u^3} f_0'''(\frac{v}{u}) + \frac{v^2}{u^4} f_0^{(4)}(\frac{v}{u}) & \frac{3v^2}{u^4} f_0'''(\frac{v}{u}) + \frac{v^3}{u^5} f_0^{(4)}(\frac{v}{u}) + \frac{2w}{u^3} \\ 0 & 0 & -\frac{1}{u^2} f_0'''(\frac{v}{u}) - \frac{v}{u^3} f_0^{(4)}(\frac{v}{u}) & -\frac{2v}{u^3} f_0'''(\frac{v}{u}) - \frac{v^2}{u^4} f_0^{(4)}(\frac{v}{u}) \\ 0 & 0 & 0 & -\frac{1}{u^2} \\ 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}$$

□

8.7. **The \bar{C} 's.** To understand the emergent tt^* -geometry, let us first compute the action of the real structure on the basis $\frac{\partial}{\partial u}$, $\frac{\partial}{\partial v}$, $\frac{\partial}{\partial w}$ and $\frac{\partial}{\partial x}$. Note

$$\frac{\partial}{\partial x} = \frac{1}{u} \frac{\partial}{\partial t^S} = \frac{1}{u} s_S,$$

and so

$$s_S = u \frac{\partial}{\partial x}.$$

Now we get:

$$(326) \quad \tau\left(\frac{\partial}{\partial x}\right) = \frac{1}{\bar{u}} s_S = \frac{u}{\bar{u}} \frac{\partial}{\partial x}.$$

Similarly,

$$\frac{\partial}{\partial w} = \frac{1}{u} \frac{\partial}{\partial t^R} - \frac{v}{u^2} \frac{\partial}{\partial t^S} = \frac{1}{u} (s_R + \frac{v}{u} s_S) - \frac{v}{u^2} s_S = \frac{1}{u} s_R,$$

so we get

$$(327) \quad s_R = u \frac{\partial}{\partial w}$$

and

$$(328) \quad \tau\left(\frac{\partial}{\partial w}\right) = \frac{1}{\bar{u}} s_R = \frac{u}{\bar{u}} \frac{\partial}{\partial w}.$$

The formula $\frac{\partial}{\partial v}$ involves more terms:

$$\begin{aligned} \frac{\partial}{\partial v} &= \frac{1}{u} \frac{\partial}{\partial t^Q} - \frac{w}{u^2} \frac{\partial}{\partial t^S} \\ &= \frac{1}{u} \left(s_Q + \frac{\partial^2 f_0(t^Q)}{\partial (t^Q)^2} s_R + \left(t^Q \frac{\partial^2 f_0(t^Q)}{\partial (t^Q)^2} - \frac{\partial f_0(t^Q)}{\partial t^Q} \right) s_S \right) - \frac{w}{u^2} s_S \\ &= \frac{1}{u} s_Q + \frac{1}{u} f_0''\left(\frac{v}{u}\right) s_R + \left(\frac{v}{u^2} f_0''\left(\frac{v}{u}\right) - \frac{1}{u} f_0'\left(\frac{v}{u}\right) - \frac{w}{u^2} \right) s_S, \end{aligned}$$

and so

$$\begin{aligned} s_Q &= u \frac{\partial}{\partial v} - f_0''\left(\frac{v}{u}\right) s_R - \left(\frac{v}{u} f_0''\left(\frac{v}{u}\right) - f_0'\left(\frac{v}{u}\right) - \frac{w}{u} \right) s_S \\ &= u \frac{\partial}{\partial v} - u f_0''\left(\frac{v}{u}\right) \frac{\partial}{\partial w} - \left(v f_0''\left(\frac{v}{u}\right) - u f_0'\left(\frac{v}{u}\right) - w \right) \frac{\partial}{\partial x}. \end{aligned}$$

$$\begin{aligned}
\tau\left(\frac{\partial}{\partial v}\right) &= \frac{1}{\bar{u}}s_Q + \frac{1}{\bar{u}}\overline{f_0''\left(\frac{v}{u}\right)}s_R + \left(\frac{\bar{v}}{\bar{u}^2}\overline{f_0''\left(\frac{v}{u}\right)} - \frac{1}{\bar{u}}\overline{f_0'\left(\frac{v}{u}\right)} - \frac{\bar{w}}{\bar{u}^2}\right)s_S \\
&= \frac{1}{\bar{u}}\left(u\frac{\partial}{\partial v} - uf_0''\left(\frac{v}{u}\right)\frac{\partial}{\partial w} - \left(vf_0''\left(\frac{v}{u}\right) - uf_0'\left(\frac{v}{u}\right) - w\right)\frac{\partial}{\partial x}\right) \\
&\quad + \frac{1}{\bar{u}}\overline{f_0''\left(\frac{v}{u}\right)} \cdot u\frac{\partial}{\partial w} + \left(\frac{\bar{v}}{\bar{u}^2}\overline{f_0''\left(\frac{v}{u}\right)} - \frac{1}{\bar{u}}\overline{f_0'\left(\frac{v}{u}\right)} - \frac{\bar{w}}{\bar{u}^2}\right)u\frac{\partial}{\partial x} \\
&= \frac{u}{\bar{u}}\frac{\partial}{\partial v} - \frac{u}{\bar{u}}\left(f_0''\left(\frac{v}{u}\right) - \overline{f_0''\left(\frac{v}{u}\right)}\right)\frac{\partial}{\partial w} \\
&\quad - \frac{u}{\bar{u}}\left(\frac{v}{u}f_0''\left(\frac{v}{u}\right) - f_0'\left(\frac{v}{u}\right) - \frac{w}{u} - \frac{\bar{v}}{u}\overline{f_0''\left(\frac{v}{u}\right)} - f_0'\left(\frac{v}{u}\right) - \frac{\bar{w}}{u}\right)\frac{\partial}{\partial x}.
\end{aligned}$$

Finally,

$$\begin{aligned}
\frac{\partial}{\partial u} &= \frac{1}{u}\frac{\partial}{\partial t^P} - \frac{v}{u^2}\frac{\partial}{\partial t^Q} - \frac{w}{u^2}\frac{\partial}{\partial t^R} - \left(\frac{x}{u^2} - \frac{2vw}{u^3}\right)\frac{\partial}{\partial t^S} \\
&= \frac{1}{u}\left(s_P + t^Qs_Q + \frac{\partial f_0(t^Q)}{\partial t^Q}s_R + \left(t^Q\frac{\partial f_0(t^Q)}{\partial t^Q} - 2f_0(t^Q)\right)s_S\right) \\
&\quad - \frac{v}{u^2}\left(s_Q + \frac{\partial^2 f_0(t^Q)}{\partial (t^Q)^2}s_R + \left(t^Q\frac{\partial^2 f_0(t^Q)}{\partial (t^Q)^2} - \frac{\partial f_0(t^Q)}{\partial t^Q}\right)s_S\right) \\
&\quad - \frac{w}{u^2}\left(s_R + t^Qs_S\right) - \left(\frac{x}{u^2} - \frac{2vw}{u^3}\right)\frac{\partial}{\partial t^S} \\
&= \frac{1}{u}s_P - \left(\frac{v}{u^2}f_0''\left(\frac{v}{u}\right) - \frac{1}{u}f_0'\left(\frac{v}{u}\right) + \frac{w}{u^2}\right)s_R \\
&\quad - \left(\frac{v^2}{u^3}f_0''\left(\frac{v}{u}\right) - \frac{2v}{u^2}f_0'\left(\frac{v}{u}\right) + \frac{2}{u}f_0\left(\frac{v}{u}\right) + \frac{x}{u^2} - \frac{vw}{u^3}\right)s_S.
\end{aligned}$$

$$\begin{aligned}
s_P &= u\left[\frac{\partial}{\partial u} + \left(\frac{v}{u^2}f_0''\left(\frac{v}{u}\right) - \frac{1}{u}f_0'\left(\frac{v}{u}\right) + \frac{w}{u^2}\right)s_R\right. \\
&\quad \left.+ \left(\frac{v^2}{u^3}f_0''\left(\frac{v}{u}\right) - \frac{2v}{u^2}f_0'\left(\frac{v}{u}\right) + \frac{2}{u}f_0\left(\frac{v}{u}\right) + \frac{x}{u^2} - \frac{vw}{u^3}\right)s_S\right] \\
&= u\left[\frac{\partial}{\partial u} + \left(\frac{v}{u^2}f_0''\left(\frac{v}{u}\right) - \frac{1}{u}f_0'\left(\frac{v}{u}\right) + \frac{w}{u^2}\right)u\frac{\partial}{\partial w}\right. \\
&\quad \left.+ \left(\frac{v^2}{u^3}f_0''\left(\frac{v}{u}\right) - \frac{2v}{u^2}f_0'\left(\frac{v}{u}\right) + \frac{2}{u}f_0\left(\frac{v}{u}\right) + \frac{x}{u^2} - \frac{vw}{u^3}\right)u\frac{\partial}{\partial x}\right].
\end{aligned}$$

$$\begin{aligned}
\tau\left(\frac{\partial}{\partial u}\right) &= \frac{1}{\bar{u}}s_P - \overline{\left(\frac{v}{u^2}f_0''\left(\frac{v}{u}\right) - \frac{1}{u}f_0'\left(\frac{v}{u}\right) + \frac{w}{u^2}\right)}s_R \\
&- \overline{\left(\frac{v^2}{u^3}f_0''\left(\frac{v}{u}\right) - \frac{2v}{u^2}f_0'\left(\frac{v}{u}\right) + \frac{2}{u}f_0\left(\frac{v}{u}\right) + \frac{x}{u^2} - \frac{vw}{u^3}\right)}s_S \\
&= \frac{u}{\bar{u}}\left[\frac{\partial}{\partial u} + \left(\frac{v}{u^2}f_0''\left(\frac{v}{u}\right) - \frac{1}{u}f_0'\left(\frac{v}{u}\right) + \frac{w}{u^2}\right)u\frac{\partial}{\partial w}\right. \\
&+ \left.\left(\frac{v^2}{u^3}f_0''\left(\frac{v}{u}\right) - \frac{2v}{u^2}f_0'\left(\frac{v}{u}\right) + \frac{2}{u}f_0\left(\frac{v}{u}\right) + \frac{x}{u^2} - \frac{vw}{u^3}\right)u\frac{\partial}{\partial x}\right] \\
&- \overline{\left(\frac{v}{u^2}f_0''\left(\frac{v}{u}\right) - \frac{1}{u}f_0'\left(\frac{v}{u}\right) + \frac{w}{u^2}\right)}u\frac{\partial}{\partial w} \\
&- \overline{\left(\frac{v^2}{u^3}f_0''\left(\frac{v}{u}\right) - \frac{2v}{u^2}f_0'\left(\frac{v}{u}\right) + \frac{2}{u}f_0\left(\frac{v}{u}\right) + \frac{x}{u^2} - \frac{vw}{u^3}\right)}u\frac{\partial}{\partial x}.
\end{aligned}$$

To summarize, we get:

$$\tau\begin{pmatrix} \frac{\partial}{\partial u} \\ \frac{\partial v}{\partial u} \\ \frac{\partial w}{\partial u} \\ \frac{\partial}{\partial x} \end{pmatrix} = M\begin{pmatrix} \frac{\partial}{\partial u} \\ \frac{\partial v}{\partial u} \\ \frac{\partial w}{\partial u} \\ \frac{\partial}{\partial x} \end{pmatrix} = \frac{u}{\bar{u}}\begin{pmatrix} 1 & 0 & a - \bar{a} & b - \bar{b} \\ 0 & 1 & c - \bar{c} & d - \bar{d} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial}{\partial u} \\ \frac{\partial v}{\partial u} \\ \frac{\partial w}{\partial u} \\ \frac{\partial}{\partial x} \end{pmatrix},$$

where

$$\begin{aligned}
a &= \frac{v}{u}f_0''\left(\frac{v}{u}\right) - f_0'\left(\frac{v}{u}\right) + \frac{w}{u}, \\
b &= \frac{v^2}{u^2}f_0''\left(\frac{v}{u}\right) - \frac{2v}{u}f_0'\left(\frac{v}{u}\right) + 2f_0\left(\frac{v}{u}\right) + \frac{x}{u} - \frac{vw}{u^2}, \\
c &= -f_0''\left(\frac{v}{u}\right), \\
d &= -\left(\frac{v}{u}f_0''\left(\frac{v}{u}\right) - f_0'\left(\frac{v}{u}\right) - \frac{w}{u}\right).
\end{aligned}$$

It is easy to check that:

$$(329) \quad \overline{M}M = I.$$

$$\begin{aligned} \tau C_x \tau \begin{pmatrix} \frac{\partial}{\partial y} \\ \frac{\partial}{\partial w} \\ \frac{\partial}{\partial x} \end{pmatrix} &= M \bar{C}_x \cdot \bar{M} \begin{pmatrix} \frac{\partial}{\partial y} \\ \frac{\partial}{\partial w} \\ \frac{\partial}{\partial x} \end{pmatrix} = \tau \begin{pmatrix} \frac{1}{u} \frac{\partial}{\partial x} \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{u} \cdot \frac{u}{u} \frac{\partial}{\partial x} \\ 0 \\ 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & \frac{1}{u} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial}{\partial y} \\ \frac{\partial}{\partial w} \\ \frac{\partial}{\partial x} \end{pmatrix}, \end{aligned}$$

$$(330) \quad \bar{C}_{\bar{x}} = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{\bar{u}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$(331) \quad \bar{C}_{\bar{w}} = \begin{pmatrix} 0 & 0 & \frac{1}{\bar{u}} & -\frac{\bar{v}}{\bar{u}^2} \\ 0 & 0 & 0 & \frac{1}{\bar{u}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$(332) \quad \bar{C}_{\bar{v}} = \begin{pmatrix} 0 & \frac{1}{\bar{u}} & -\frac{\bar{v}}{\bar{u}^2} \overline{f_0'''(\frac{v}{u})} - \frac{C}{\bar{u}} & -\frac{\bar{w}}{\bar{u}^2} - \frac{\bar{v}^2}{\bar{u}^3} \overline{f_0'''(\frac{v}{u})} + \frac{A-D}{\bar{u}} \\ 0 & 0 & \frac{1}{\bar{u}} \overline{f_0'''(\frac{v}{u})} & \frac{\bar{v}}{\bar{u}^2} \overline{f_0'''(\frac{v}{u})} + \frac{C}{\bar{u}} \\ 0 & 0 & 0 & \frac{1}{\bar{u}} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$(333) \quad \bar{C}_{\bar{u}} = \begin{pmatrix} \frac{1}{\bar{u}} & -\frac{\bar{v}}{\bar{u}^2} & -\frac{\bar{w}}{\bar{u}^2} + \frac{\bar{v}^2}{\bar{u}^3} \overline{f_0'''(\frac{v}{u})} + \frac{C\bar{v}}{\bar{u}^2} & -\frac{\bar{x}}{\bar{u}^2} + \frac{\bar{v}^3}{\bar{u}^4} \overline{f_0'''(\frac{v}{u})} + \frac{2\bar{v}\bar{w}}{\bar{u}^3} - \frac{A-D}{\bar{u}^2} \\ 0 & \frac{1}{\bar{u}} & -\frac{\bar{v}}{\bar{u}} \overline{f_0'''(\frac{v}{u})} & -\frac{\bar{w}}{\bar{u}^2} - \frac{\bar{v}^2}{\bar{u}^3} \overline{f_0'''(\frac{v}{u})} - \frac{C\bar{v}}{\bar{u}^2} \\ 0 & 0 & \frac{1}{\bar{u}} & -\frac{\bar{v}}{\bar{u}^2} \\ 0 & 0 & 0 & \frac{1}{\bar{u}} \end{pmatrix}$$

where $A = a - \bar{a}$, etc. One can check that

$$(334) \quad [C_\alpha, \bar{C}_\beta] = 0$$

except for

$$[C_v, \bar{C}_{\bar{v}}] = \frac{1}{|u|^2} \begin{pmatrix} 0 & 0 & \overline{f_0'''(\frac{v}{u})} - f_0'''(\frac{v}{u}) & \frac{v}{u} \overline{f_0'''(\frac{v}{u})} - \frac{v}{u} f_0'''(\frac{v}{u}) \\ 0 & 0 & 0 & f_0'''(\frac{v}{u}) - \overline{f_0'''(\frac{v}{u})} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and three other cases: $[C_v, \bar{C}_{\bar{u}}]$, $[C_u, \bar{C}_{\bar{v}}]$ and $[C_u, \bar{C}_{\bar{u}}]$.

8.8. The change of coordinate on the complexified Kähler moduli space. The variable t^Q is understood as the coordinate on the complexified Kähler moduli space. Introduce a new coordinate

$$(335) \quad x = \exp(2\pi i t^Q).$$

Then we have

$$(336) \quad \frac{\partial}{\partial t^Q} = x \frac{\partial}{\partial x}.$$

When one circles around $x = 0$ in the x -plane, t^Q is changed to $t^Q + 1$. The flat sections undergo the following changes:

$$\begin{aligned} \tilde{s}_S &= \frac{\partial}{\partial t^S} = s_S, \\ \tilde{s}_R &= \frac{\partial}{\partial t^R} - (t^Q + 1) \frac{\partial}{\partial t^S} = s_R - s_S, \\ \tilde{s}_Q &= \frac{\partial}{\partial t^Q} - \frac{\partial^2 f_0(t^Q + 1)}{\partial (t^Q)^2} \frac{\partial}{\partial t^R} + \frac{\partial f_0(t^Q + 1)}{\partial t^Q} \frac{\partial}{\partial t^S} \\ &= s_Q - 5 \frac{\partial}{\partial t^R} + \frac{5}{2} (2t^Q + 1) \frac{\partial}{\partial t^S} \\ &= s_Q - 5s_R + \frac{5}{2}s_S, \\ \tilde{s}_P &= \frac{\partial}{\partial t^P} - (t^Q + 1) \frac{\partial}{\partial t^Q} + \left((t^Q + 1) \frac{\partial^2 f_0(t^Q + 1)}{\partial (t^Q)^2} - \frac{\partial f_0(t^Q + 1)}{\partial t^Q} \right) \frac{\partial}{\partial t^R} \\ &\quad - \left((t^Q + 1) \frac{\partial f_0(t^Q + 1)}{\partial t^Q} - 2f_0(t^Q + 1) \right) \frac{\partial}{\partial t^S} \\ &= s_P - \frac{\partial}{\partial t^Q} + \left(5t^Q + \frac{5}{2} \right) \frac{\partial}{\partial t^R} - \left(\frac{5}{2}(t^Q)^2 + \frac{5}{2}t^Q + \frac{5}{6} \right) \frac{\partial}{\partial t^S} \\ &= s_P - s_Q + \frac{5}{2}s_R - \frac{5}{6}s_S. \end{aligned}$$

Written in matrix form:

$$(337) \quad \begin{pmatrix} \tilde{s}_S \\ \tilde{s}_R \\ \tilde{s}_Q \\ \tilde{s}_P \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ \frac{5}{2} & -5 & 1 & 0 \\ -\frac{5}{6} & \frac{5}{2} & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} s_S \\ s_R \\ s_Q \\ s_P \end{pmatrix}.$$

The monodromy matrix is the exponential of the following nilpotent matrix:

$$(338) \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

This matrix is the matrix of the cup product with $-Q$:

$$(339) \quad Q \cdot \begin{pmatrix} S \\ R \\ Q \\ P \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} S \\ R \\ Q \\ P \end{pmatrix}.$$

9. CONCLUSIONS AND SPECULATIONS

In our attempt to unify Witten Conjecture/Kontsevich Theorem with mirror symmetry of the quintic, we have found that methods of statistical physics to be very useful. By the mean field theory developed by Dijkgraaf, Eguchi and his collaborators, we have written down the integrable hierarchy associated with the GW theory of the quintic. This leads to the structure of Frobenius manifold developed by Dubrovin and his collaborators on the small phase space of the quintic. We also present the special Kähler geometry on the small phase. In carrying out the explicit computations, we have reduced to the computations of the free energy on the small phase space. Note we are working on the A -side side, but many geometric structures originally discovered on the B -theory side naturally emerge. So we feel it is natural to refer to them as emergent geometry.

Such results reveal the similarities and differences between the GW theory of a point and the GW theory of the quintic. And the similarities and differences suggest some directions for future investigations which we now present some speculations. In both cases we obtain integrable hierarchies. In case of a point the integrable hierarchy is the KdV hierarchy, which is a reduction of the KP hierarchy. This suggests that the integrable hierarchy for the quintic might be a reduction of the 4-component KP hierarchy. It will be interesting to check whether this is the case, because in case of a point one can identify the partition function as an element in the fermionic Fock space via the boson-fermion correspondence via the Kac-Schwarz operator [25]. An explicit formula for the affine coordinates of the corresponding element in the Sato Grassmannian has been obtained by the author [36] using

the Virasoro constraints. An alternative derivation using the KP hierarchy was later given by Balogh-Yang [1]. A formula for the n -point function in all genera based on the affine coordinates of an element in the Sato Grassmannian has been derived by the author [39]. In particular it can be applied to the Witten-Kontsevich tau-function. In a work in progress we will generalize this formula to n -component KP hierarchy.

Another direction of search for unification is the Eynard-Orantin topological recursions [17]. In [35] the author proved that the GW theory of a point satisfies the EO topological recursion, and in this case the recursion is equivalent to the DVV Virasoro constraints. Furthermore, the spectral curve in this case lies in a family of curves obtained by computing the genus zero one-point function together with a Laplace transform. The starting point of most of the computations in this paper is the genus zero one-point function and in the end we also take a Laplace transform to get the deformed dual flat coordinates. We speculate that this is not just a coincidence, it means that the EO topological recursion should also hold for the quintic.

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