On a Stirling-Whitney-Riordan triangle *

Bao-Xuan Zhu

School of Mathematics and Statistics, Jiangsu Normal University, Xuzhou 221116, PR China

Abstract

Motivated by the Stirling triangle of the second kind, the Whitney triangle of the second kind and a triangle of Riordan, we study a Stirling-Whitney-Riordan triangle $[T_{n,k}]_{n,k}$ satisfying the recurrence relation:

$$\begin{aligned} T_{n,k} &= (b_1k+b_2)T_{n-1,k-1}+[(2\lambda b_1+a_1)k+a_2+\lambda(b_1+b_2)]T_{n-1,k}+\\ &\quad \lambda(a_1+\lambda b_1)(k+1)T_{n-1,k+1}, \end{aligned}$$

where initial conditions $T_{n,k}=0$ unless $0\le k\le n$ and $T_{0,0}=1$. Let its row-generating function $T_n(q)=\sum_{k\geq 0}T_{n,k}q^k$ for $n\geq 0$.

We prove that the Stirling-Whitney-Riordan triangle $[T_{n,k}]_{n,k}$ is $\mathbf{x}$-totally positive with $\mathbf{x}=(a_1,a_2,b_1,b_2,\lambda)$. We show real rootedness and log-concavity of $T_n(q)$ and stability of the Turán-type polynomial $T_{n+1}(q)T_{n-1}(q)-T_n^2(q)$. We also present explicit formulae of $T_{n,k}$ and the exponential generating function of $T_n(q)$, and the ordinary generating function of $T_n(q)$ in terms of a Jacobi continued fraction expansion. Furthermore, we get the $\mathbf{x}$-Stieltjes moment property and 3-$\mathbf{x}$-log-convexity of $T_n(q)$ and that the triangular convolution $z_n=\sum_{i=0}^{n}T_{n,k}x_iy_{n-i}$ preserves Stieltjes moment property of sequences. Finally, for the first column $(T_{n,0})_{n\geq 0}$, we derive some similar properties to those of $(T_n(q))_{n\geq 0}$.

MSC: 05A20; 05A15; 11A55; 15B48; 26C10; 30B70; 44A60

Keywords: Recurrence relations; Total positivity; Real zeros; Jacobi continued fractions; Hakel matrices; $\mathbf{x}$-Stieltjes moment sequences; Convolutions; Row-generating functions; 3-$\mathbf{x}$-log-convexity; Stirling numbers; Whitney numbers

1 Introduction

1.1 Stirling numbers of the second kind

Let $\left\{ n \atop k \right\}$ denote the Stirling number of the second kind, which enumerates the number of partitions of a set with n elements consisting of k disjoint nonempty sets. It is well-known

*Supported partially by the National Natural Science Foundation of China (No. 11971206).

Email address: bxzhu@jsnu.edu.cn (B.-X. Zhu)

that the Stirling number of the second kind satisfies the recurrence relation

$$\left\{ \begin{array}{c} n \\ k \end{array} \right\} = k \left\{ \begin{array}{c} n-1 \\ k \end{array} \right\} + \left\{ \begin{array}{c} n-1 \\ k-1 \end{array} \right\},$$

where initial conditions $\left\{ \begin{array}{c} 0 \\ 0 \end{array} \right\} = 1$ and $\left\{ \begin{array}{c} 0 \\ k \end{array} \right\} = 0$ for $k \geq 1$ or $k < 0$. The triangular array $\left[\left\{ \begin{array}{c} n \\ k \end{array} \right\} \right]_{n,k \geq 0}$ is called *the Stirling triangle of the second kind*. Its row-generating function, *i.e., the Bell polynomial*, is defined to be

$$B_n(x) = \sum_{k=0}^{n} \left\{ \begin{array}{c} n \\ k \end{array} \right\} x^k.$$

There are many nice properties about the Stirling number and the Bell polynomial. For example:

(i) The following is a classical explicit formula of the Stirling number of the second kind

$$\left\{ \begin{array}{c} n \\ k \end{array} \right\} k! = \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} j^n \tag{1.1}$$

for $n, k \geq 1$, see [14] for instance.

(ii) Let $G_{n,k} = k! \left\{ \begin{array}{c} n \\ k \end{array} \right\}$, which counts the number of distinct ordered partitions of a set with n elements. Then the geometric polynomial $G_n(x) = \sum_{k=1}^{n} G_{n,k} x^k$ was studied by Tanny in [39] and $G_{n,k}$ satisfies the recurrence relation

$$G_{n,k} = kG_{n-1,k} + kG_{n-1,k-1}.$$

(iii) The Stirling triangle of the second kind $\left[\left\{ \begin{array}{c} n \\ k \end{array} \right\} \right]_{n,k \geq 0}$ is totally positive [8].

(iv) It is also well known that $B_n(x)$ has only real zeros and therefore is log-concave [42].

(v) The iterated polynomial $B_{n+1}(x)B_{n-1}(x) - B_n^2(x)$ has no zeros in the right half plane [17].

(vi) It is well known that $\sum_{n \geq 0} B_n(x) \frac{t^n}{n!} = e^{x(e^t - 1)}$ [14].

(vii) The Jacobi continued fraction expansion

$$\sum_{n \geq 0} B_n(x)\, t^n = \cfrac{1}{1 - s_0 t - \cfrac{r_1 t^2}{1 - s_1 t - \cfrac{r_2 t^2}{1 - s_2 t - \cdots}}},$$

where $s_n = n + x$ and $r_{n+1} = (n+1)x$ for $n \geq 0$ [18].

(viii) The sequence $(B_n(q))_{n \geq 0}$ is q-log-convex, strongly q-log-convex, 3-q-log-convex and q-Stieltjes moment, see [12, 25, 44, 46, 49], respectively.

We refer reader to [14] for more information of Stirling numbers and Bell polynomials.

1.2 Whitney numbers of the second kind

As a generalization of the partition lattice, the Dowling lattice $Q_n(G)$ is a class of geometric lattices based on finite groups introduced by Dowling [15]. The Whitney number of the second kind, denote by $W_m(n,k)$, is the number of elements of corank k of $Q_n(G)$, which satisfies the recurrence relation

$$W_m(n,k) = (mk+1)W_m(n-1,k) + W_m(n-1,k-1)$$

with $W_m(0,0) = 1$. Its row-generating function $D_n(m,x) = \sum_{k=0}^{n} W_m(n,k)x^k$ is called *the Dowling polynomial* by Benoumhani [2].

The Whitney number $W_m(n,k)$ (resp. the Dowling polynomial) has many similar properties to those of the Stirling number $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ (resp. the Bell polynomial). The associated Whitney number of the second kind $W_m(n,k)k!$, denoted by $W_m^{\diamond}(n,k)$, satisfies the recurrence relation

$$W_m^{\diamond}(n,k) = (mk+1)W_m^{\diamond}(n-1,k) + kW_m^{\diamond}(n-1,k-1).$$

Its row-generating function $F_m(n,x) = \sum_{k=0}^{n} W_m^{\diamond}(n,k)x^k$ is called *the Tanny-geometric polynomial* in [2]. See [2, 3] for many properties of Whitney numbers and Dowling polynomials such as explicit formulae, recurrence relations, log-concavity, real rootedness and generating functions. For the corresponding q-log-convexity, strong q-log-convexity, 3-q-log-convexity and q-Stieltjes moment properties, see [12, 25, 44, 46, 49], respectively. We also refer reader to [13] for more interesting properties for $W_m(n,k)$.

1.3 A triangle of Riordan

Let $a_{n,k}$ denote number of set partitions of $[n]$ in which exactly k of the blocks have been distinguished. It satisfies the recurrence relation

$$a_{n,k} = a_{n-1,k-1} + (k+1)a_{n-1,k} + (k+1)a_{n-1,k+1},$$

where initial conditions $a_{0,0} = 1$ and $a_{n,k} = 0$ unless $0 \le k \le n$, see [34, A049020]. The triangle $[a_{n,k}]_{n,k}$ first arises in Riordan's letter [31]. Its explicit formula can be written as

$$a_{n,k} = \sum_{i=0}^{n} \left\{ \begin{matrix} n \\ i \end{matrix} \right\} \binom{i}{k}.$$

Its row-generating function has the next Jacobi continued fraction expansion

$$\sum_{n\ge 0}\sum_{k=0}^{n} a_{n,k}x^k t^n = \cfrac{1}{1 - s_0 t - \cfrac{r_1 t^2}{1 - s_1 t - \cfrac{r_2 t^2}{1 - s_2 t - \cdots}}},$$

where $s_n = n+1+x$ and $r_{n+1} = (n+1)(x+1)$ for $n \ge 0$. We refer reader to [34, A049020] for more information of $a_{n,k}$.

1.4 Structure of this paper

As we know that Stirling numbers have many nice properties. In addition, various generalizations of Stirling numbers were considered, see [16, 22, 47] for instance. Motivated by Stirling numbers of the second kind, Whitney numbers of the second kind and that triangle of Riordan, the aim of this paper is to consider their generalization and get generalized properties in a unified manner.

Let $\mathbb{R}$ (resp. $\mathbb{R}^{\geq 0}$) denote the set of all (resp. nonnegative) real numbers. For $\{\lambda, a_1, a_2, b_1, b_2\} \subseteq \mathbb{R}$, define an array $[T_{n,k}]_{n,k}$, which satisfies the recurrence relation:

$$T_{n,k} = (b_1k + b_2)T_{n-1,k-1} + [(2\lambda b_1 + a_1)k + a_2 + \lambda(b_1 + b_2)]T_{n-1,k} + \lambda(a_1 + \lambda b_1)(k + 1)T_{n-1,k+1},$$

where $T_{0,0} = 1$ and $T_{n,k} = 0$ unless $0 \leq k \leq n$. Let its row-generating function $T_n(q) = \sum_{k\geq 0} T_{n,k}q^k$ for $n \geq 0$. Obviously, we have

- $T_{n,k} = \left\{ \begin{array}{c} n \\ k \end{array} \right\}$ if $a_1 = b_2 = 1$ and $a_2 = b_1 = \lambda = 0$;
- $T_{n,k} = G_{n,k}$ if $a_1 = b_1 = 1, a_2 = b_2 = \lambda = 0$;
- $T_{n,k} = W_m(n, k)$ if $a_1 = m, a_2 = b_2 = 1$ and $b_1 = \lambda = 0$;
- $T_{n,k} = W^{\diamond}_m(n, k)$ if $a_1 = m, a_2 = b_1 = 1$ and $b_2 = \lambda = 0$;
- $T_{n,k} = a_{n,k}$ if $a_1 = b_2 = \lambda = 1$ and $a_2 = b_1 = 0$;
- $T_{n,k} = \binom{n}{k}k!$ if $b_1 = a_2 = 1$ and $a_1 = b_2 = \lambda = 0$ ([34, A008279]);
- $T_{n,k}$ is A154602 in [34] if $a_1 = 2$, $b_2 = \lambda = 1$ and $a_2 = b_1 = 0$.

We call this array $[T_{n,k}]_{n,k}$ *a Stirling-Whitney-Riordan triangle*. The number $T_{n,k}$ can be interpreted in terms of weighted Motzkin paths due to Flajolet [18]. Let $u_k = b_1k + b_2 + b_1, v_k = [(2\lambda b_1 + a_1)k + a_2 + \lambda(b_1 + b_2)]$ and $w_k = \lambda(a_1 + \lambda b_1)n$ for $k \geq 0$. Then $T_{n,k}$ counts the number of weighted paths starting from the origin $(0, 0)$ never falling below the x-axis and ending at (n, k) with up diagonal steps $(1, 1)$ weighted u_{i-1}, down diagonal steps $(1, -1)$ weighted w_{i+1} and horizontal steps $(1, 0)$ weighted v_i on the line $y = i$.

In Section 2, using two different methods, we prove that the Stirling-Whitney-Riordan triangle $[T_{n,k}]_{n,k}$ is $\mathbf{x}$-totally positive with $\mathbf{x} = (a_1, a_2, b_1, b_2, \lambda)$. In Section 3, using the method of zeros interlacing, we show real rootedness and log-concavity of $T_n(q)$ and stability of the Turán-type polynomial $T_{n+1}(q)T_{n-1}(q) - T^2_n(q)$. In Section 4, we present explicit formulae of $T_{n,k}$ and the exponential generating function of $T_n(q)$, and in Section 5, using addition formulae of the Stieltjes-Rogers type, we get the ordinary generating function of $T_n(q)$ in terms of a Jacobi continued fraction expansion. Furthermore, we get the $\mathbf{x}$-Stieltjes moment property and 3-$\mathbf{x}$-log-convexity of $T_n(q)$ and that the triangular convolution $z_n = \sum_{k=0}^{n} T_{n,k}x_ky_{n-k}$ preserves Stieltjes moment property of sequences. Finally, in Section 6, for the first column $(T_{n,0})_{n\geq 0}$, we derive some similar properties to those of $(T_n(q))_{n\geq 0}$.

2 Total positivity of the Stirling-Whitney-Riordan triangle

Let $A = [a_{n,k}]_{n,k\geq 0}$ be a matrix of real numbers. It is called *totally positive* (*TP* for short) if all its minors are nonnegative. It is called TP_r if all minors of order $k \leq r$ are nonnegative. Let $\mathbf{x} = (x_i)_{i\in I}$ is a set of indeterminates. A matrix A with entries being polynomials in $\mathbb{R}[\mathbf{x}]$ is called **x-totally positive** (**x**-TP for short) if all its minors are polynomials with nonnegative coefficients in the indeterminates $\mathbf{x}$ and is called **x-totally positive of order** r ($\boldsymbol{x}$-TP_r for short) if all its minors of order $k \leq r$ are polynomials with nonnegative coefficients in the indeterminates $\mathbf{x}$. Total positivity of matrices is an important and powerful concept that arises often in various branches of mathematics, see the monographs [23, 29] for general details about total positivity. We also refer reader to [6, 8, 9, 10, 19, 36, 47, 51, 55] for total positivity in combinatorics. We present the total positivity of the Stirling-Whitney-Riordan triangle as follows.

Theorem 2.1. *Let $[T_{n,k}]_{n,k}$ be the generalized Stirling-Whitney-Riordan triangle. Then the lower triangular matrix $[T_{n,k}]_{n,k}$ is $\boldsymbol{x}$-TP with $\boldsymbol{x} = (a_1, a_2, b_1, b_2, \lambda)$.*

The first proof: Let $T = [T_{n,k}]_{n,k}$ and $\overline{T}$ denote the triangle obtained from T by deleting its first row. Assume that

$$J = \begin{bmatrix} s_0 & r_0 & & & \\ t_1 & s_1 & r_1 & & \\ & t_2 & s_2 & r_2 & \\ & & \ddots & \ddots & \ddots \end{bmatrix},$$

where $r_n = b_1 n + b_1 + b_2$, $s_n = (2\lambda b_1 + a_1)n + a_2 + \lambda(b_1 + b_2)$ and $t_n = \lambda(a_1 + \lambda b_1)n$. The recurrence relation:

$$T_{n,k} = (b_1 k + b_2)T_{n-1,k-1} + [(2\lambda b_1 + a_1)k + a_2 + \lambda(b_1 + b_2)]T_{n-1,k} + \lambda(a_1 + \lambda b_1)(k+1)T_{n-1,k+1}$$

implies that

$$\overline{T} = TJ.$$

It follows from [54, Theorem 2.1] that $\mathbf{x}$-total positivity of J implies that of T. In addition, J is $\mathbf{x}$-TP if and only if

$$J^* = \begin{bmatrix} s_0 & r_0^* & & & \\ t_1^* & s_1 & r_1^* & & \\ & t_2^* & s_2 & r_2^* & \\ & & \ddots & \ddots & \ddots \end{bmatrix}$$

is $\mathbf{x}$-TP, where $r_n^* = \lambda(b_1 n + b_1 + b_2)$, $s_n = (2\lambda b_1 + a_1)n + a_2 + \lambda(b_1 + b_2)$ and $t_n^* = (a_1 + \lambda b_1)n$. By [54, Proposition 3.3 (i)], we immediately get that J^* is $\mathbf{x}$-TP with $\mathbf{x} = (a_1, a_2, b_1, b_2, \lambda)$. Thus we show that the triangular array T is $\mathbf{x}$-TP with $\mathbf{x} = (a_1, a_2, b_1, b_2, \lambda)$.

The second proof: Assume that triangular arrays $[A_{n,k}]_{n,k\geq 0}$ and $[B_{n,k}]_{n,k\geq 0}$ respectively satisfy the recurrence relation

$$\begin{aligned} A_{n,k} &= (a_1 k + a_2)A_{n-1,k} + (b_1 k + b_2)A_{n-1,k-1}, \\ B_{n,k} &= \lambda B_{n-1,k} + B_{n-1,k-1}, \end{aligned} \tag{2.1}$$

where $A_{0,0}=1$ and $B_{0,0}=1$. We claim

$$[T_{n,k}]_{n,k}=[A_{n,k}]_{n,k}[B_{n,k}]_{n,k}.$$

It is obvious that

$$\begin{aligned}
B_{n,k} &= \binom{n}{k}\lambda^{n-k},\\
(k-i)B_{k,i} &= \lambda(i+1)B_{k,i+1}\\
(k-i)B_{k-1,i-1} &= \lambda i B_{k-1,i}.
\end{aligned}$$

Then we can prove $[T_{n,k}]_{n,k}=[A_{n,k}]_{n,k}[B_{n,k}]_{n,k}$ by induction on n, as follows:

$$\begin{aligned}
T_{n,i} &= \sum_{k\geq 0} A_{n,k}B_{k,i}\\
&= \sum_{k\geq 0}\left[(a_1k+a_2)A_{n-1,k}+(b_1k+b_2)A_{n-1,k-1}\right]B_{k,i}\\
&= (a_1i+a_2)T_{n-1,i}+\sum_{k\geq 0}a_1(k-i)A_{n-1,k}B_{k,i}+\sum_{k\geq 0}(b_1k+b_2)A_{n-1,k-1}B_{k,i}\\
&= (a_1i+a_2)T_{n-1,i}+\lambda a_1(i+1)\sum_{k\geq 0}A_{n-1,k}B_{k,i+1}+\sum_{k\geq 0}(b_1k+b_2)A_{n-1,k-1}(\lambda B_{k-1,i}+B_{k-1,i-1})\\
&= (a_1i+a_2)T_{n-1,i}+\lambda a_1(i+1)T_{n-1,i+1}+\lambda b_2T_{n-1,i}+b_2T_{n-1,i-1}\\
&\quad +\lambda b_1\sum_{k\geq 0}[(k-i-1)+(i+1)]A_{n-1,k-1}B_{k-1,i}+b_1\sum_{k\geq 0}[(k-i)+i]A_{n-1,k-1}B_{k-1,i-1}\\
&= (a_1i+a_2+\lambda b_2)T_{n-1,i}+\lambda a_1(i+1)T_{n-1,i+1}+b_2T_{n-1,i-1}\\
&\quad +\lambda b_1\sum_{k\geq 0}(k-i-1)A_{n-1,k-1}B_{k-1,i}+\lambda b_1\sum_{k\geq 0}(i+1)A_{n-1,k-1}B_{k-1,i}\\
&\quad +b_1\sum_{k\geq 0}(k-i)A_{n-1,k-1}B_{k-1,i-1}+b_1\sum_{k\geq 0}iA_{n-1,k-1}B_{k-1,i-1}\\
&= (a_1i+a_2+\lambda b_1(i+1)+ab_2)T_{n-1,i}+\lambda a_1(i+1)T_{n-1,i+1}+(b_1i+b_2)T_{n-1,i-1}\\
&\quad +\lambda b_1\sum_{k\geq 0}(k-i-1)A_{n-1,k-1}B_{k-1,i}+b_1\sum_{k\geq 0}(k-i)A_{n-1,k-1}B_{k-1,i-1}\\
&= (a_1i+a_2+\lambda b_1(i+1)+\lambda b_2)T_{n-1,i}+\lambda a_1(i+1)T_{n-1,i+1}+(b_1i+b_2)T_{n-1,i-1}\\
&\quad +\lambda^2b_1(i+1)\sum_{k\geq 0}A_{n-1,k-1}B_{k-1,i+1}+\lambda b_1i\sum_{k\geq 0}A_{n-1,k-1}B_{k-1,i}\\
&= [a_1i+a_2+\lambda b_1(2i+1)+\lambda b_2]T_{n-1,i}+\lambda(a_1+\lambda b_1)(i+1)T_{n-1,i+1}+(b_1i+b_2)T_{n-1,i-1}.
\end{aligned}$$

Using the classical Cauchy-Binet formula, in order to prove that $[T_{n,k}]_{n,k}$ is $\mathbf{x}$-TP, it suffices to show that both $[A_{n,k}]_{n,k}$ and $[B_{n,k}]_{n,k}$ are $\mathbf{x}$-TP. Note that $[B_{n,k}]_{n,k}$ is a special case of $[A_{n,k}]_{n,k}$. Thus we only need to demonstrate that $\mathbf{x}$-total positivity for $[A_{n,k}]_{n,k}$.

Let $A=[A_{n,k}]_{n,k}$ and $\overline{A}$ denote the matrix obtained from A by deleting its first row. Also let $\overline{A}_n$ and A_n denote the nth leading principal submatrices of $\overline{A}$ and A, respectively.

It follows from (2.1) that we have $\overline{A}_n = A_n J_n$, where

$$J_n = \begin{pmatrix} a_2 & b_1+b_2 & & & \\ & a_1+a_2 & 2b_1+b_2 & & \\ & & 2a_1+a_2 & \ddots & \\ & & & \ddots & b_1(n-1)+b_2 \\ & & & & a_1(n-1)+a_2 \end{pmatrix}.$$

Obviously, J_n is $\mathbf{x}$-TP with $\mathbf{x} = (a_1, a_2, b_1, b_2)$ for $n \geq 0$. In addition, by induction on n, applying the classical Cauchy-Binet formula to $\overline{A}_n = A_n J_n$, we get that A_n is $\mathbf{x}$-TP with $\mathbf{x} = (a_1, a_2, b_1, b_2, \lambda)$ for $n \geq 0$. This implies that the lower triangular matrix A is $\mathbf{x}$-TP. Thus, we get that T is $\mathbf{x}$-TP. The proof is complete. $\square$

3 Real rootedness and log-concavity of row-generating functions

Let $\alpha = (a_k)_{k\geq 0}$ be a sequence of nonnegative numbers. The sequence α is *log-concave* if $a_{i-1}a_{i+1} \leq a_i^2$ for $i \geq 1$. A basic approach to log-concavity problems is to use Newton's inequalities: Suppose that the polynomial $\sum_{k=0}^{n} a_k x^k$ has only real zeros. Then

$$a_k^2 \geq a_{k-1}a_{k+1}\left(1+\frac{1}{k}\right)\left(1+\frac{1}{n-k}\right), \quad k = 1, 2, \ldots, n-1,$$

and the sequence α is therefore log-concave (see Hardy, Littlewood and Pólya [21, p. 104]). Log-concave sequences and real-rooted polynomials often occur in combinatorics and have been extensively investigated. We refer the reader to Brenti [7], Stanely [35] and Wang and Yeh [43] for the log-concavity, Brändén [4, 5], Brenti [6], Liu and Wang [24], Wang and Yeh [42] for real rootedness.

Following Wagner [41], a real polynomial is said to be *standard* if either it is identically zero or its leading coefficient is positive. Assume that both polynomials f and g only have real zeros. Let $\{r_i\}$ and $\{s_j\}$ be all zeros of f and g in nondecreasing order respectively. We say that g *interlaces* f denoted by $g \preceq f$ if $\deg f = \deg g + 1 = n$ and

$$r_n \leq s_{n-1} \leq \cdots \leq s_2 \leq r_2 \leq s_1 \leq r_1. \tag{3.1}$$

For two interlacing polynomials, Fisk showed the following result.

Lemma 3.1. [17, Lemma 1.20] *Let both $f(x)$ and $g(x)$ be standard real polynomials with only real zeros. Assume that $\deg(f(x)) = n$ and all real zeros of $f(x)$ are $s_1, \ldots, s_n$. If $\deg(g) = n-1$ and we write*

$$g(x) = \sum_{i=1}^{n} \frac{c_i f(x)}{x - s_i},$$

then $g \preceq f$ if and only if all c_i are positive.

A real polynomial is *weakly (Hurwitz) stable* if all of its zeros lie in the closed left half of the complex plane. We refer to [26, Chapter 9] for deep surveys on the stability theory of polynomials. Let $L_n(x)$ be the nth Legendre polynomial. Turán-type inequalities [40] state that

$$L_n^2(x) - L_{n+1}(x)L_{n-1}(x) > 0 \text{ for } -1 < x < 1.$$

In 1948, Szegö gave four different proofs of the famous Turán-type inequality on Legendre polynomials [38]. Many important (orthogonal) polynomials and special functions were proved to satisfy some Turán-type inequalities, see [1] for instance.

We get the zeros distribution related to $T_n(x)$ as follows.

Theorem 3.2. *Let $T_n(x)$ be the row-generating function of the Stirling-Whitney-Riordan triangle $[T_{n,k}]_{n,k\geq 0}$. If $\{\lambda, a_1, a_2, b_1, b_2\} \subseteq \mathbb{R}^{\geq 0}$ and $a_1(b_1 + b_2) \geq b_1 a_2$, then*

- (i) *$T_n(x)$ has only simple real zeros in $(-\lambda - \frac{a_1}{b_1}, -\lambda)$* [1] *and $T_{n-1}(x) \preceq T_n(x)$ for $n \geq 1$. Therefore $T_n(x)$ is log-concave for $n \geq 1$.*
- (ii) *The Turán-type polynomial $T_{n+1}(x)T_{n-1}(x) - (T_n(x))^2$ is a weakly stable polynomial for $n \geq 1$.*

Proof. (i) It follows from the recurrence relation:

$$T_{n,k} = (b_1 k + b_2)T_{n-1,k-1} + [(2\lambda b_1 + a_1)k + a_2 + \lambda(b_1 + b_2)]T_{n-1,k} + \lambda(a_1 + \lambda b_1)(k+1)T_{n-1,k+1}$$

that

$$T_n(x) = [a_2 + (b_1 + b_2)(x + \lambda)]T_{n-1}(x) + (x + \lambda)[a_1 + b_1(x + \lambda)]T'_{n-1}(x) \tag{3.2}$$

for $n \geq 1$. We will prove that $T_n(x)$ has only simple real zeros in $(-\lambda - \frac{a_1}{b_1}, -\lambda)$ and $T_{n-1}(x) \preceq T_n(x)$ for $n \geq 1$ by induction on n. Clearly, $T_0(x) = 1$.

Case 1: Assume $b_1 \neq 0$. For $n = 1$, then

$$T_1(x) = a_2 + (b_1 + b_2)(x + \lambda).$$

Obviously, it follows from $a_1(b_1+b_2) \geq b_1 a_2$ that $T_1(x)$ has only real zero in $(-\lambda - \frac{a_1}{b_1}, -\lambda)$. Suppose for $n \geq 2$ that $T_{n-1}(x)$ has $n-1$ real zeros denoted by

$$-\lambda > s_1 > s_2 > \ldots > s_{n-1} > -\lambda - \frac{a_1}{b_1}.$$

Then by the recurrence relation (3.2), we have

$$sign[T_n(s_k)] = (-1)^k.$$

Note that coefficients of $T_n(x)$ are nonnegative. Then we get that $T_n(x)$ has n simple real zeros denoted by $r_1 > r_2 > \ldots > r_n$ such that

$$r_1 > s_1 > r_2 > s_2 > \ldots > s_{n-1} > r_n. \tag{3.3}$$

[1] If $b_1 = 0$, then $-\lambda - \frac{a_1}{b_1}$ means $-\infty$.

On the other hand, by the recurrence relation (3.2), we have

$$T_n(-\lambda) = a_2T_{n-1}(-\lambda) > 0, T_n(-\lambda - \frac{a_1}{b_1}) = \frac{a_2b_1 - a_1(b_1+b_2)}{b_1}T_{n-1}(-\lambda - \frac{a_1}{b_1}),$$

which imply $r_1 < -\lambda$ and $r_n > -\lambda - \frac{a_1}{b_1}$.

Case 2: Assume $b_1 = 0$. For $n = 1$, then

$$T_1(x) = a_2 + b_2(x + \lambda).$$

Obviously, it follows from $a_1b_2 \geq b_1a_2$ that $T_1(x)$ has only real zero in $(-\infty, -\lambda)$. Suppose for $n \geq 2$ that $T_{n-1}(x)$ has $n-1$ real zeros denoted by

$$-\lambda > s_1 > s_2 > \ldots > s_{n-1}.$$

Then by the recurrence relation (3.2), we have

$$sign[T_n(s_k)] = (-1)^k.$$

Then we get that $T_n(x)$ has n simple real zeros denoted by $r_1 > r_2 > \ldots > r_n$ such that

$$r_1 > s_1 > r_2 > s_2 > \ldots > s_{n-1} > r_n. \tag{3.4}$$

In addition, by the recurrence relation (3.2), we have

$$T_n(-\lambda) = a_2T_{n-1}(-\lambda) > 0,$$

which implies $r_1 < -\lambda$. This completes the proof of (i).

(ii) It follows from the recurrence relation (3.2) that we deduce that

$$\begin{aligned}
& T_{n+1}(x)T_{n-1}(x) - (T_n(x))^2 \\
= \; & [a_2 + (b_1+b_2)(x+\lambda)]T_n(x)T_{n-1}(x) + (x+\lambda)[a_1 + b_1(x+\lambda)]T_n'(x)T_{n-1}(x) - \\
& [a_2 + (b_1+b_2)(x+\lambda)]T_n(x)T_{n-1}(x) - (x+\lambda)[a_1 + b_1(x+\lambda)]T_{n-1}'(x)T_n(x) \\
= \; & (x+\lambda)[a_1 + b_1(x+\lambda)]\left[T_n'(x)T_{n-1}(x) - T_{n-1}'(x)T_n(x)\right] \\
= \; & -(x+\lambda)[a_1 + b_1(x+\lambda)][T_n(x)]^2\left(\frac{T_{n-1}(x)}{T_n(x)}\right)'. \qquad (3.5)
\end{aligned}$$

By (i), assume that $T_n(x)$ has n negative zeros as $r_1 > r_2 > \ldots > r_n$. It follows from $T_{n-1}^*(x) \preceq T_n^*(x)$ in (i) and Lemma 3.1 that we get

$$T_{n-1}(x) = T_n(x)\sum_{k=1}^{n}\frac{t_k}{x - r_k},$$

where $t_k > 0$ for $1 \leq k \leq n$. Then by (3.5), we get that the iterated Turán-type polynomial

$$T_{n+1}(x)T_{n-1}(x) - (T_n(x))^2 \;\; = \;\; (x+\lambda)[a_1 + b_1(x+\lambda)][T_n(x)]^2\sum_{k=1}^{n}\frac{t_k}{(x-r_k)^2}.$$

Obviously, $\sum_{k=1}^{n} \frac{t_k}{(x-r_k)^2} > 0$ for $x \geq 0$. In addition, let $x = u + vi$ with $u > 0$ and $v \neq 0$. Then we have

$$Im\left(\sum_{k=1}^{n} \frac{t_k}{(x-r_k)^2}\right) = -2v\sum_{k=1}^{n} \frac{t_k(u-r_k)}{(u-r_k)^2+v^2} \neq 0$$

since $\frac{t_k(u-r_k)}{(u-r_k)^2+v^2} > 0$ for each $k \in [1, n]$. In consequence, we get that

$$\sum_{k=1}^{n} \frac{t_k}{(x-r_k)^2}$$

has no zeros in the right half plane. So does the polynomial $T_{n+1}(x)T_{n-1}(x) - (T_n(x))^2$. We complete the proof of (ii). □

4 Exponential generating function and explicit formula

In this section, we will present the exponential generating function of $T_n(q)$ and an explicit formula of $T_{n,k}$ as follows.

Theorem 4.1. *Let $T_n(q)$ be the row-generating function of the Stirling-Whitney-Riordan triangle $[T_{n,k}]_{n,k\geq 0}$. Let $a_1^2 + b_1^2 \neq 0$.*

(i) *The exponential generating function of $T_n(q)$ is* [2]

$$\sum_{n\geq 0} T_n(q)\frac{t^n}{n!} = e^{a_2 t}\left[1 + \frac{b_1(q+\lambda)(1-e^{a_1 t})}{a_1}\right]^{-(1+\frac{b_2}{b_1})}.$$

(ii) *An explicit formula of $T_{n,k}$ is*

$$T_{n,k} = \begin{cases} \sum_{i\geq k} \frac{\prod_{j=1}^{i}(b_2+b_1 j)}{a_1^i} \times \binom{i}{k}\lambda^{i-k} \times \frac{1}{i!}\sum_{j=0}^{i}\binom{i}{j}(-1)^{i-j}(a_2+a_1 j)^n, & \text{for} \quad a_1 \neq 0 \\ \sum_{i\geq k}\prod_{j=1}^{i}(b_2+b_1 j) \times \binom{n}{i}\binom{i}{k}\lambda^{i-k}a_2^{n-i}, & \text{for} \quad a_1 = 0. \end{cases}$$

Proof. Let the exponential generating function

$$\mathcal{T}(q,t) = \sum_{n,k\geq 0} T_{n,k} q^k \frac{t^n}{n!} = \sum_{n\geq 0} T_n(q)\frac{t^n}{n!}.$$

---

[2] For $a_1 = 0$ or $b_1 = 0$, using continuity of functions, the corresponding formula for exponential generating function means its limits as follows:

$$\sum_{n\geq 0} T_n(q)\frac{t^n}{n!} = \begin{cases} e^{a_2 t}\left[1 - b_1(q+\lambda)t\right]^{-(1+\frac{b_2}{b_1})}, & \text{for} \quad a_1 = 0, b_1 \neq 0 \\ e^{a_2 t + \left[\frac{b_2(q+\lambda)(e^{a_1 t}-1)}{a_1}\right]}, & \text{for} \quad a_1 \neq 0, b_1 = 0. \end{cases}$$

Then by the recurrence relation:

$$T_{n,k}=(b_1k+b_2)T_{n-1,k-1}+[(2\lambda b_1+a_1)k+a_2+\lambda(b_1+b_2)]T_{n-1,k}+\lambda(a_1+\lambda b_1)(k+1)T_{n-1,k+1},$$

we have the next partial differential equation

$$\mathcal{T}_t(q,t)-(b_1q+a_1)(q+\lambda)\mathcal{T}_q(q,t)=[a_2+(b_1+b_2)(q+\lambda)]\mathcal{T}(q,t)$$

with the initial condition $\mathcal{T}(q,0)=1$. It is routine to check that

$$\mathcal{T}(q,t)=e^{a_2t}\left[1+\frac{b_1(q+\lambda)(1-e^{a_1t})}{a_1}\right]^{-(1+\frac{b_2}{b_1})} \tag{4.1}$$

is a solution of the above equation with the initial condition.

(ii) We will prove it by divide the proof into three cases.

Case 1: $a_1b_1\neq 0$. We have

$$\begin{aligned}
\sum_{n,k\geq 0}T_{n,k}q^k\frac{t^n}{n!} &= e^{a_2t}\left[1+\frac{b_1(q+\lambda)(1-e^{a_1t})}{a_1}\right]^{-(1+\frac{b_2}{b_1})}\\
&= e^{a_2t}\left[1-\frac{b_1(q+\lambda)(e^{a_1t}-1)}{a_1}\right]^{-(1+\frac{b_2}{b_1})}\\
&= e^{a_2t}\sum_{i\geq 0}\binom{\frac{b_2}{b_1}+i}{i}\left[\frac{b_1(q+\lambda)(e^{a_1t}-1)}{a_1}\right]^i\\
&= e^{a_2t}\sum_{i\geq 0}\binom{\frac{b_2}{b_1}+i}{i}\left(\frac{b_1(q+\lambda)}{a_1}\right)^i\sum_{j=0}^{i}\binom{i}{j}(-1)^{i-j}e^{a_1jt}\\
&= \sum_{i\geq 0}\binom{\frac{b_2}{b_1}+i}{i}\left(\frac{b_1(q+\lambda)}{a_1}\right)^i\sum_{j=0}^{i}\binom{i}{j}(-1)^{i-j}e^{(a_2+a_1j)t}\\
&= \sum_{i\geq 0}\frac{\prod_{j=1}^{i}(b_2+b_1j)}{i!}\times\frac{(q+\lambda)^i}{a_1^i}\sum_{j=0}^{i}\binom{i}{j}(-1)^{i-j}\sum_{n\geq 0}\frac{(a_2+a_1j)^nt^n}{n!},
\end{aligned}$$

which implies

$$T_{n,k}=\sum_{i\geq k}\frac{\prod_{j=1}^{i}(b_2+b_1j)}{a_1^i}\times\binom{i}{k}\lambda^{i-k}\times\frac{1}{i!}\sum_{j=0}^{i}\binom{i}{j}(-1)^{i-j}(a_2+a_1j)^n.$$

Case 2: $b_1 = 0, a_1 \neq 0$. We have

$$\begin{aligned}
\sum_{n,k\geq 0} T_{n,k} q^k \frac{t^n}{n!} &= e^{a_2 t + \left[\frac{b_2(q+\lambda)(e^{a_1 t}-1)}{a_1}\right]} \\
&= e^{a_2 t} \sum_{i\geq 0} \left[\frac{b_2(q+\lambda)(e^{a_1 t}-1)}{a_1}\right]^i \times \frac{1}{i!} \\
&= e^{a_2 t} \sum_{i\geq 0} \left(\frac{b_2(q+\lambda)}{a_1}\right)^i \times \frac{1}{i!} \sum_{j=0}^{i} \binom{i}{j} (-1)^{i-j} e^{a_1 j t} \\
&= \sum_{i\geq 0} \left(\frac{b_2}{a_1}\right)^i (q+\lambda)^i \times \frac{1}{i!} \sum_{j=0}^{i} \binom{i}{j} (-1)^{i-j} e^{(a_2+a_1 j)t} \\
&= \sum_{i\geq 0} \left(\frac{b_2}{a_1}\right)^i (q+\lambda)^i \times \frac{1}{i!} \sum_{j=0}^{i} \binom{i}{j} (-1)^{i-j} \sum_{n\geq 0} \frac{(a_2+a_1 j)^n t^n}{n!},
\end{aligned}$$

which implies

$$T_{n,k} = \sum_{i\geq k} \frac{b_2^i}{a_1^i} \binom{i}{k} \lambda^{i-k} \times \frac{1}{i!} \sum_{j=0}^{i} \binom{i}{j} (-1)^{i-j} (a_2 + a_1 j)^n.$$

Case 3: $a_1 = 0, b_1 \neq 0$. We have

$$\begin{aligned}
\sum_{n,k\geq 0} T_{n,k} q^k \frac{t^n}{n!} &= e^{a_2 t} \left[1 - b_1(q+\lambda)t\right]^{-(1+\frac{b_2}{b_1})} \\
&= e^{a_2 t} \sum_{i\geq 0} \binom{\frac{b_2}{b_1}+i}{i} \left[b_1(q+\lambda)t\right]^i \\
&= e^{a_2 t} \sum_{i\geq 0} \prod_{j=1}^{i} (b_2 + b_1 j) \times (q+\lambda)^i \frac{t^i}{i!},
\end{aligned}$$

which implies

$$T_{n,k} = \sum_{i\geq k} \prod_{j=1}^{i} (b_2 + b_1 j) \times \binom{n}{i} \binom{i}{k} \lambda^{i-k} a_2^{n-i}.$$

This completes the proof. □

5 Stieltjes moment property and continued fractions

In this section, we will present a continued fraction expansion of $\sum_{n\geq 0} T_n(q) t^n$ and Stieltjes moment property of $T_n(q)$. Continued fraction is an important tool in combinatorics, which is closely related to many aspects, *e.g.*, combinatorial lattice paths and combinatorial interpretations, combinatorial identities, combinatorial positivity, determinants of sequences, and so on. We refer the reader to Flajolet [18] for more information of continued fraction expansions related to many important combinatorial objects.

Continued fraction plays an important role in Hankel-total positivity and Stieltjes moment sequences. Given a sequence $\alpha = (a_k)_{k\geq 0}$, define its *Hankel matrix* $H(\alpha)$ by

$$H(\alpha) = [a_{i+j}]_{i,j\geq 0} = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ a_1 & a_2 & a_3 & a_4 & \\ a_2 & a_3 & a_4 & a_5 & \\ a_3 & a_4 & a_5 & a_6 & \\ \vdots & & & & \ddots \end{bmatrix}.$$

We say that α is a *Stieltjes moment* (*SM* for short) sequence if it has the form

$$a_k = \int_0^{+\infty} x^k d\mu(x), \tag{5.1}$$

where μ is a non-negative measure on $[0, +\infty)$ (see [29, Theorem 4.4] for instance). It is well-known that the following are equivalent:

(i) α is a Stieltjes moment sequence;

(ii) Its Hankel matrix $H(\alpha)$ is TP;

(iii) Its generating function has the Stieltjes continued fraction expansion

$$\sum_{n\geq 0} a_n z^n = \cfrac{1}{1 - \cfrac{\beta_0 z}{1 - \cfrac{\beta_1 z}{1 - \cfrac{\beta_2 z}{1 - \cdots}}}}$$

with $\beta_i \geq 0$ for $i \geq 0$.

(iv) Positivity characterization: $\sum_{n=0}^{N} c_n a_n \geq 0$ for every polynomial $\sum_{n=0}^{N} c_n q^n \geq 0$ on $[0, \infty)$.

It is well known that Stieltjes moment problem is one of classical moment problems and arises naturally in many branches of mathematics [33, 45]. Let $\mathbf{x} = (x_i)_{i\in I}$ is a set of indeterminates. A polynomial sequence $(\alpha_n(\mathbf{x}))_{n\geq 0}$ in $\mathbb{R}[\mathbf{x}]$ is called a $\mathbf{x}$-Stieltjes moment ($\mathbf{x}$-SM for short) sequence if its associated infinite Hankel matrix is $\mathbf{x}$-TP, see Zhu [51, 54] for instance. When $(\alpha_n(\mathbf{x}))_{n\geq 0}$ is a sequence of real numbers, $\mathbf{x}$-SM sequence reduces to the classical Stieltjes moment sequence. For $\mathbf{x}$-SM sequences, the following criterion was proved in [51, 54].

Lemma 5.1. [54] *Let* $\{s_n(\boldsymbol{x}), r_n(\boldsymbol{x}), H_n(\boldsymbol{x})\} \subseteq \mathbb{R}^{\geq 0}[\boldsymbol{x}]$ *for* $n \in \mathbb{N}$ *and*

$$\sum_{n\geq 0} H_n(\boldsymbol{x}) t^n = \cfrac{1}{1 - s_0(\boldsymbol{x})t - \cfrac{r_1(\boldsymbol{x})t^2}{1 - s_1(\boldsymbol{x})t - \cfrac{r_2(\boldsymbol{x})t^2}{1 - \cdots}}}.$$

If there exists $\{\lambda_n(\boldsymbol{x}), u_n(\boldsymbol{x}), v_n(\boldsymbol{x})\} \subseteq \mathbb{R}^{\geq 0}[\boldsymbol{x}]$ *such that* $s_n = \lambda_n + u_n + v_n$ *and* $r_{n+1} = u_{n+1}v_n$ *for* $n \geq 0$*, then polynomials* $H_n(\boldsymbol{x})$ *form a* $\boldsymbol{x}$*-Stieltjes moment sequence for* $n \geq 0$.

If a polynomial sequence $(A_n(q))_{n\geq 0}$ in a value q is q-Stieltjes moment, then its triangular convolution preserves the Stieltjes moment property in terms of the next result.

Lemma 5.2. [44] *For $n \in \mathbb{N}$, let $A_n(q) = \sum_{k=0}^{n} A_{n,k} q^k$ be the nth row generating function of a matrix $[A_{n,k}]_{n,k\geq 0}$. Assume that $(A_n(q))_{n\geq 0}$ is a Stieltjes moment sequence for any fixed $q \geq 0$. If both $(x_n)_{n\geq 0}$ and $(y_n)_{n\geq 0}$ are Stieltjes moment sequences, then so is $(z_n)_{n\geq 0}$ defined by*

$$z_n = \sum_{k=0}^{n} A_{n,k} x_k y_{n-k}. \tag{5.2}$$

In recent years there has been a growing interest in the $\mathbf{x}$-log-convexity of combinatorial polynomials. It is necessary for $\mathbf{x}$-Stieltjes moment property. For a polynomial sequence $(f_n(\mathbf{x}))_{n\geq 0}$, it is ***x**-log-convex* if

$$f_{n+1}(\mathbf{x}) f_{n-1}(\mathbf{x}) - f_n(\mathbf{x})^2$$

is a polynomial with nonnegative coefficients for $n \geq 1$. Define the operator $\mathcal{L}$ which maps a polynomial sequence $(f_n(\mathbf{x}))_{n\geq 0}$ to a polynomial sequence $(g_i(\mathbf{x}))_{i\geq 1}$ given by

$$g_i(\mathbf{x}) := f_{i-1}(\mathbf{x}) f_{i+1}(\mathbf{x}) - f_i(\mathbf{x})^2.$$

Then the $\mathbf{x}$-log-convexity of $(f_n(\mathbf{x}))_{n\geq 0}$ is equivalent to the $\mathbf{x}$-positivity of $\mathcal{L}\{f_i(\mathbf{x})\}$, i.e., the coefficients of $g_i(\mathbf{x})$ are nonnegative for all $i \geq 1$. Generally, we say that $(f_i(\mathbf{x}))_{i\geq 0}$ is *k-**x**-log-convex* if the coefficients of $\mathcal{L}^m\{f_i(\mathbf{x})\}$ are nonnegative for all $m \leq k$, where $\mathcal{L}^m = \mathcal{L}(\mathcal{L}^{m-1})$.

Lemma 5.3. [54] *If all minors of order ≤ 4 of the Hankel matrix $[T_{i+j}(\boldsymbol{x})]_{i,j\geq 0}$ are polynomials with nonnegative coefficients, then the sequence $(T_n(\boldsymbol{x}))_{n\geq 0}$ is 3-**x**-log-convex.*

Obviously, if $(A_n(\mathbf{x}))_{n\geq 0}$ is a $\mathbf{x}$-Stieltjes moment sequence, then $[A_{i+j}(\mathbf{x})]_{i,j}$ is $\mathbf{x}$-Hankel-TP$_4$. Thus it is 3-$\mathbf{x}$-log-convex by Lemma 5.3.

If $\mathbf{x}$ is a valuable q, then it has been proved that many famous polynomials have the q-log-convexity, e.g., the Bell polynomials, the classical Eulerian polynomials, the Narayana polynomials of type A and B, Jacobi-Stirling polynomials, and so on, see Liu and Wang [25], Chen *et al.* [11], Zhu [46, 47, 48, 49] for instance. These polynomials also have 3-q-log-convexity, see Zhu [50, 52]. In addition, the next is an important criterion for 3-q-log-convexity.

In order to compute continued fraction, we need the following addition formulae of the Stieltjes-Rogers type.

Lemma 5.4. *[32, 37] For a sequence $(\alpha_n)_{n\geq 0}$, define the function*

$$h(x) \;=\; \sum_{n\geq 0} \alpha_n \frac{x^n}{n!}.$$

If there exists two sequences $(e_n)_{n\geq 0}$ and $(w_n)_{n\geq 0}$ such that the generating function

$$h(x+y) = \sum_{n\geq 0} w_k f_k(x) f_k(y),$$

where

$$f_k(x) = \frac{x^k}{k!} + e_{k+1}\frac{x^{k+1}}{(k+1)!} + O(x^{k+2}),$$

then we have

$$\sum_{n\geq 0} \alpha_n t^n = \cfrac{1}{1 - s_0 t - \cfrac{r_1 t^2}{1 - s_1 t - \cfrac{r_2 t^2}{1 - s_2 t - \cfrac{r_3 t^2}{1 - s_3 t - \cdots}}}},$$

where $s_n = e_{n+1} - e_n$ *and* $r_{n+1} = w_{n+1}/w_n$ *for* $n \geq 0$.

We present a continued fraction expansion of $\sum_{n\geq 0} T_n(q)t^n$ as follows.

Theorem 5.5. *Let* $T_n(q)$ *be the row-generating function of the Stirling-Whitney-Riordan triangle* $[T_{n,k}]_{n,k\geq 0}$. *Then we have the next results.*

(i) *The ordinary generating function of* $T_n(q)$ *has a Jacobi continued fraction expression*

$$\sum_{n\geq 0} T_n(q) t^n = \cfrac{1}{1 - s_0 t - \cfrac{r_1 t^2}{1 - s_1 t - \cfrac{r_2 t^2}{1 - s_2 t - \cdots}}},$$

where $s_n = a_2 + a_1 n + [b_1(2n+1) + b_2](q+\lambda)$ *and* $r_{n+1} = [b_1(n+1) + b_2](q+\lambda)[b_1(q+\lambda) + a_1](n+1)$ *for* $n \geq 0$.

(ii) *The sequence* $(T_n(q))_{n\geq 0}$ *is* $\boldsymbol{x}$*-SM and* 3-$\boldsymbol{x}$*-LCX with* $\boldsymbol{x} = (a_1, a_2, b_1, b_2, \lambda, q)$.

(iii) *The convolution* $z_n = \sum_{k\geq 0} T_{n,k} x_k y_{n-k}$ *preserves SM property if* $\{\lambda, a_1, a_2, b_1, b_2\} \subseteq \mathbb{R}^{\geq 0}$.

(iv) *We have Hankel-determinants as follows*

$$\det_{0\leq i,j\leq n-1}(T_{i+j}(q)) \quad = \quad r_1^{n-1} r_2^{n-2} \ldots r_{n-2}^2 r_{n-1}$$

and

$$\det_{0\leq i,j\leq n-1}(T_{i+j+1}(q)) \quad = \quad r_1^{n-1} r_2^{n-2} \cdots r_{n-2}^2 r_{n-1} Q_n,$$

where $(Q_n)_{n\geq 0}$ *is defined by* $Q_{n+1} = s_n Q_n - r_n Q_{n-1}$ *with* $Q_0 = 1$ *and* $Q_1 = s_0$.

Proof. Let the exponential generating function

$$G(q,t) = \sum_{n\geq 0} T_n(q)\frac{t^n}{n!}.$$

Then

$$G(q,t)=e^{a_2t}\left[1+\frac{b_1(q+\lambda)(1-e^{a_1t})}{a_1}\right]^{-(1+\frac{b_2}{b_1})}.$$

In the following, we only need to consider the case $a_1b_1\neq 0$ (for the case $a_1b_1=0$, it is the corresponding limits in terms of continuity). Let $\gamma=\frac{b_1}{a_1}$, $\beta=\frac{b_2}{b_1}$ and $p=q+\lambda$. Assume that

$$h(x+y)=e^{(x+y)a_2}\left[1-\gamma p(e^{a_1(x+y)}-1)\right]^{-(1+\beta)}.$$

Then

$$\begin{aligned}h(x+y)&=e^{(x+y)a_2}\left\{[1-\gamma p(e^{a_1x}-1)][1-\gamma p(e^{a_1y}-1)]-\gamma p(1+\gamma p)(e^{a_1y}-1)(e^{a_1x}-1)\right\}^{-(1+\beta)}\\&=\sum_{k\geq 0}k!a_1^{2k}\langle 1+\beta\rangle_k(\gamma p)^k(1+\gamma p)^kf_k(x)f_k(y),\end{aligned}$$

where $\langle 1+\beta\rangle_k=(1+\beta)(2+\beta)\cdots(k+\beta)$ and

$$\begin{aligned}f_k(x)&=\frac{1}{k!a_1^k}e^{xa_2}(e^{a_1x}-1)^k\left[1-\gamma p(e^{a_1x}-1)\right]^{-(1+\beta+k)}\\&=\frac{1}{k!a_1^k}\left(1+a_2x+\frac{(a_2x)^2}{2}+\cdots\right)\left(a_1x+\frac{a_1^2x^2}{2}+\cdots\right)^k[1+\gamma p(1+\beta)a_1x+\cdots]\\&=\frac{x^k}{k!}+\frac{x^{k+1}}{(k+1)!}\left[a_2+\frac{a_1k}{2}+(1+\beta+k)\gamma pa_1\right](k+1)+O(x^{k+2}).\end{aligned}$$

By Lemma 5.4, we have

$$w_k=k!a_1^{2k}\langle 1+\beta\rangle_k(\gamma p)^k(1+\gamma p)^k,\quad e_{k+1}=\left[a_2+\frac{a_1k}{2}+(1+\beta+k)\gamma pa_1\right](k+1).$$

Thus, we get

$$s_k=e_{k+1}-e_k=a_2+a_1k+(2k+1+\beta)\gamma pa_1,\quad r_{k+1}=\frac{w_{k+1}}{w_k}=a_1^2(k+1+\beta)\gamma p(\gamma p+1)(k+1)$$

for $k\geq 0$. So

$$\sum_{n\geq 0}T_n(q)t^n=\cfrac{1}{1-s_0t-\cfrac{r_1t^2}{1-s_1t-\cfrac{r_2t^2}{1-s_2t-\cdots}}},$$

where $s_n=a_2+a_1n+[b_1(2n+1)+b_2](q+\lambda)$ and $r_{n+1}=[b_1(n+1)+b_2](q+\lambda)[b_1(q+\lambda)+a_1](n+1)$ for $n\geq 0$.

Let $v_n=(nb_1+b_2+b_1)(q+\lambda)$ and $u_n=n[a_1+b_1(q+\lambda)]$ for $n\geq 0$. It is obvious that $s_n=a_2+u_n+v_n$ and $t_n=v_nu_{n+1}$ for $n\geq 0$. It follows from Lemma 5.1 that $(T_n(q))_{n\geq 0}$ is a $\mathbf{x}$-SM sequence with $\mathbf{x}=(a_1,a_2,b_1,b_2,\lambda,q)$. Then $[T_{i+j}(q)]_{i,j}$ is $\mathbf{x}$-TP$_4$. Thus by Lemma 5.3, we immediately have $(T_n(q))_{n\geq 0}$ is 3-$\mathbf{x}$-log-convex. In addition, it follows from Lemma 5.2 that the Stirling-Whitney-Riordan-triangle-convolution

$$z_n=\sum_{k=0}^{n}T_{n,k}x_ky_{n-k},\quad n=0,1,2,\ldots$$

preserves the SM property. Finally, for (iv), it follows the next general criterion (see [28] for instance). If the generating function of $(u_i)_{i\geq 0}$ can be expressed by

$$\sum_{i=0}^{\infty} u_i x^i = \cfrac{u_0}{1 - s_0 x - \cfrac{t_1 x^2}{1 - s_1 x - \cfrac{t_2 x^2}{1 - s_2 x - \ldots}}},$$

then

$$\det_{0\leq i,j\leq n-1}(u_{i+j}) \;=\; u_0^n t_1^{n-1} t_2^{n-2} \ldots t_{n-2}^2 t_{n-1}$$

and

$$\det_{0\leq i,j\leq n-1}(u_{i+j+1}) \;=\; u_0^n t_1^{n-1} t_2^{n-2} \cdots t_{n-2}^2 t_{n-1} q_n,$$

where $(q_n)_{n\geq 0}$ is defined by $q_{n+1} = s_n q_n - t_n q_{n-1}$ with $q_0 = 1$ and $q_1 = s_0$.

□

Remark 5.6. Note that for a Jacobi continued fraction expansion, it has a general combinatorial interpretation of weighted Motzkin paths due to Flajolet [18]. For the Jacobi continued fraction expansion

$$\sum_{n\geq 0} T_n(q)\, t^n = \cfrac{1}{1 - s_0 t - \cfrac{r_1 t^2}{1 - s_1 t - \cfrac{r_2 t^2}{1 - s_2 t - \cfrac{r_3 t^2}{1 - s_3 t - \cdots}}}}$$

with $s_n = a_2 + a_1 n + [b_1(2n+1) + b_2](q+\lambda)$ and $r_{n+1} = (n+1)[b_1(n+1) + b_2](q+\lambda)[b_1(q+\lambda) + a_1]$ for $n \geq 0$ for $n \geq 0$, we have weighted Motzkin paths starting from the origin $(0,0)$ never falling below the x-axis and ending at $(n,0)$ with up diagonal steps $(1,1)$ weighted 1, down diagonal steps $(1,-1)$ weighted r_{i+1} and horizontal steps $(1,0)$ weighted s_i on the line $y = i$. Then $T_n(q)$ counts the number of these weighted paths ending at $(n,0)$. Thus if let $\mathscr{M}_n$ denote the set of the weighted Motzkin paths of length n and $w(\beta) = (w(\beta_1), w(\beta_2), \ldots, w(\beta_n))$ be a weighted Motzkin path of length n with $w(\beta_i) \in \{1, s_n, r_{n+1}\}_{n\geq 0}$, then we have

$$T_n(q) = \sum_{\beta\in\mathscr{M}_n} \prod_{i=1}^{n} w(\beta_i).$$

6 Properties of the first column

The first column $(T_{n,0})_{n\geq 0}$ of the Stirling-Whitney-Riordan triangle $[T_{n,k}]_{n,k}$ has similar properties to those of $(T_n(q))_{n\geq 0}$. In this section, we will further derive some properties of $(T_{n,0})_{n\geq 0}$ as follows.

Theorem 6.1. *Let $(T_{n,0})_{n\geq 0}$ be the first column of the Stirling-Whitney-Riordan triangle $[T_{n,k}]_{n,k}$.*

(i) *The ordinary generating function of $T_{n,0}$ has a Jacobi continued fraction expression*

$$\sum_{n\geq 0} T_{n,0}t^n = \cfrac{1}{1-s_0t-\cfrac{r_1t^2}{1-s_1t-\cfrac{r_2t^2}{1-s_2t-\cdots}}},$$

where $s_n = a_2+a_1n+[b_1(2n+1)+b_2]q$ and $r_{n+1} = [b_1(n+1)+b_2]q(b_1q+a_1)(n+1)$ for $n \geq 0$.

(ii) *The sequence $(T_{n,0})_{n\geq 0}$ are $\boldsymbol{x}$-SM and 3-$\boldsymbol{x}$-LCX with $\boldsymbol{x} = (a_1, a_2, b_1, b_2, \lambda)$.*

(iii) *The exponential generating function of $T_{n,0}$ is*

$$\sum_{n\geq 0} T_{n,0}\frac{t^n}{n!} \quad = \quad e^{a_2t}\left[1+\frac{b_1\lambda(1-e^{a_1t})}{a_1}\right]^{-(1+\frac{b_2}{b_1})}.$$

(iv) *$T_{n,0}$ is a polynomial in λ having only real zeros.*

(v) *The Turán-type polynomial $T_{n+1,0}T_{n-1,0} - T_{n,0}^2$ is a weakly stable polynomial in λ for $n \geq 1$.*

Proof. (i) Assume that $s_n = a_2 + a_1n + [b_1(2n + 1) + b_2]\lambda$, $r_n = b_1(n + 1) + b_2$ and $t_n = \lambda(b_1\lambda + a_1)n$ for $n \geq 0$. Let $h_k(z) = \sum_{n\geq k} T_{n,k}z^n$ for $k \geq 0$. It follows from the recurrence relation:

$$T_{n,k} = r_{k-1}T_{n-1,k-1} + s_kT_{n-1,k} + t_{k+1}T_{n-1,k+1}$$

that we have

$$\begin{aligned} h_0(z) &= 1 + s_0zh_0(z) + t_1zh_1(z), \\ h_k(z) &= r_{k-1}zh_{k-1}(z) + s_kzh_k(z) + t_{k+1}zh_{k+1}(z) \end{aligned}$$

for $k \geq 1$, which imply

$$\begin{aligned} \frac{h_0(z)}{1} &= \frac{1}{1-s_0z-t_1z\frac{h_1(z)}{h_0(z)}}, \\ \frac{h_1(z)}{h_0(z)} &= \frac{r_0z}{1-s_1z-t_2z\frac{h_2(z)}{h_1(z)}}, \\ &\vdots \\ \frac{h_k(z)}{h_{k-1}(z)} &= \frac{r_{k-1}z}{1-s_kz-t_{k+1}z\frac{h_{k+1}(z)}{h_k(z)}}. \end{aligned}$$

Thus we get

$$\sum_{n=0}^{\infty} T_{n,0} z^n = h_0(z) = \cfrac{1}{1 - s_0 z - \cfrac{r_0 t_1 z^2}{1 - s_1 z - \cfrac{r_1 t_2 z^2}{1 - s_2 z - \dots}}}.$$

If let $\mathscr{T}_n(\lambda) := T_{n,0}$, then by (i) and Theorem 5.5 (i), we immediately get for $n \geq 0$ that

$$\mathscr{T}_n(\lambda + q) = T_n(q).$$

So we immediately get (ii) and (iii) by Theorem 5.5 (ii) and Theorem 4.1 (i), respectively. We also have (iv) and (v) by Theorem 3.2. This completes the proof. □

References

[1] A. Baricz, S. Koumandos, Turán-type inequalities for some Lommel functions of the first kind, Proc. Edinb. Math. Soc. (2) 59 (2016) no. 3, 569–579.

[2] M. Benoumhani, On some numbers related to Whitney numbers of Dowling lattices, Adv. Appl. Math. 19 (1997) 106–116.

[3] M. Benoumhani, Log-concavity of Whitney numbers of Dowling lattices, Adv. Appl. Math. 22 (1999) 181–189.

[4] P. Brändén, On linear transformations preserving the Pólya frequency property, Trans. Amer. Math. Soc. 358 (2006) 3697–3716.

[5] P. Brändén, Unimodality, log-concavity, real-rootedness and beyond, in Handbook of Combinatorics (M. Bona, ed.), CRC Press, 2015, pp. 437–483.

[6] F. Brenti, Unimodal, log-concave, and Pólya frequency sequences in combinatorics, Mem. Amer. Math. Soc. 413 (1989).

[7] F. Brenti, Log-concave and unimodal sequences in algebra, combinatorics, and geometry: an update, Contemp. Math. 178 (1994) 71–89.

[8] F. Brenti, Combinatorics and total positivity, J. Combin. Theory Ser. A 71 (1995) 175–218.

[9] X. Chen, H. Liang, Y. Wang, Total positivity of recursive matrices, Linear Algebra Appl. 471 (2015) 383–393.

[10] X. Chen, H. Liang, Y. Wang, Total positivity of Riordan arrays, European J. Combin. 46 (2015) 68–74.

[11] W.Y.C. Chen, R.L. Tang, L.X.W. Wang, A.L.B. Yang, The q-log-convexity of the Narayana polynomials of type B, Adv. in Appl. Math. 44(2) (2010) 85–110.

[12] W.Y.C. Chen, L.X.W. Wang, A.L.B. Yang, Recurrence relations for strongly q-log-convex polynomials, Canad. Math. Bull. 54 (2011) 217–229.

[13] G.-S. Cheon and J.-H. Jung, r-Whitney numbers of Dowling lattices, Discrete Math. 312 (2012) 2337–2348.

[14] L. Comtet, Advanced combinatorics, Revised and enlarged edition, Reidel, Dordrecht, 1974.

[15] T.A. Dowling, A class of geometric lattices based on finite groups, J. Combin. Theory Ser. B 14 (1973) 61–86.

[16] J. Engbers, D. Galvin, J. Hilyard, Combinatorially interpreting generalized Stirling numbers, European J. Combin. 43 (2015) 32–54.

[17] S. Fisk, Polynomials, roots, and interlacing, arXiv:math/0612833.

[18] P. Flajolet, Combinatorial aspects of continued fractions, Discrete Math. 32 (1980) 125–161.

[19] I. Gessel and G. Viennot, Binomial determinants, path, and hook length formulae, Adv. Math. 58 (1985) 300–321.

[20] I. Gessel and R.P. Stanley, Stirling polynomials, J. Combin. Theory Ser. A 24 (1978), 25–33.

[21] G.H. Hardy, J.E. Littlewood, G. Pólya, Inequalities, Cambridge University Press, Cambridge, 1952.

[22] L. Hsu, P.-S. Shiue, A unified approach to generalized Stirling numbers, Adv. Appl. Math. 20 (1998) 366–384.

[23] S. Karlin, Total Positivity, Vol. I, Stanford University Press, Stanford, 1968.

[24] L.L. Liu, Y. Wang, A unified approach to polynomial sequences with only real zeros, Adv. in Appl. Math. 38(4) (2007) 542–560.

[25] L.L. Liu, Y. Wang, On the log-convexity of combinatorial sequences, Adv. in Appl. Math. 39 (2007) 453–476.

[26] M. Marden, Geometry of Polynomials, Math. Surveys Monogr., vol. 3, Amer. Math. Soc., Providence, RI, 1966.

[27] P. Mongelli, Total positivity properties of Jacobi–Stirling numbers, Adv. in. Appl. Math. 48 (2012) 354–364.

[28] L. Mu, Y. Wang, Y.-N. Yeh, Hankel determinants of linear combinations of consecutive Catalan-like numbers, Discrete Math. 340 (2017) 3097–3103.

[29] A. Pinkus. Totally Positive Matrices. Cambridge University Press, Cambridge, 2010.

[30] G. Pólya and G. Szegö, Problems and Theorems in Analysis, Vol. II, Springer-Verlag, Berlin, 1976.

[31] J. Riordan, Letter, Oct 31 1977, http://oeis.org/A001861/a001861$_1$.pdf.

[32] L.J. Rogers, On the representation of certain asymptotic series as convergent continued fractions, Proc. Lond. Math. Soc. (Ser. 2) 4 (1907) 72–89.

[33] J.A. Shohat and J.D. Tamarkin, The Problem of Moments, Amer. Math. Soc., New York, 1943.

[34] N.J.A. Sloane, The On-Line Encyclopedia of Integer Sequences.

[35] R.P. Stanley, Log-concave and unimodal sequences in algebra, combinatorics, and geometry, Ann. New York Acad. Sci. 576 (1989) 500–534.

[36] J. Stembridge, Nonintersecting paths, pfaffians, and plane partitions, Adv. Math. 83 (1990) 96–131.

[37] T. Stieltjes, Sur la réduction en fraction continue d'une série procédant selon les puissances descendantes d'une variable, Ann. Fac. Sci. Toulouse 4 (1889) 1–17.

[38] G. Szegö, On an inequality of P. Turán concerning Legendre polynomials, Bull. Am. Math. Soc. 54 (1948) 401–405.

[39] S. Tanny, On some numbers related to the Bell numbers, Canad. Math. Bull. 17 (1975) 733–738.

[40] P. Turán, On the zeros of the polynomials of Legendre, Časopis Pest. Mat. Fys. 75 (1950), 113–122.

[41] D.G. Wagner, Total positivity of Hadamard products, J. Math. Anal. Appl. 163 (1992) 459–483.

[42] Y. Wang and Y.-N. Yeh, Polynomials with real zeros and Pólya frequency sequences, J. Combin. Theory Ser. A 109 (2005) 63–74.

[43] Y. Wang, Y.-N. Yeh, Log-concavity and LC-positivity, J. Combin. Theory Ser. A 114 (2007) 195–210.

[44] Y. Wang, B.-X. Zhu, Log-convex and Stieltjes moment sequences, Adv. in Appl. Math. 81 (2016) 115–127.

[45] D.V. Widder, The Laplace Transform, Princeton University Press, Princeton, 1946.

[46] B.-X. Zhu, Log-convexity and strong q-log-convexity for some triangular arrays, Adv. in Appl. Math. 50 (2013) 595–606.

[47] B.-X. Zhu, Some positivities in certain triangular array, Proc. Amer. Math. Soc. 142(9) (2014) 2943–2952.

[48] B.-X. Zhu, Log-concavity and strong q-log-convexity for Riordan arrays and recursive matrices, Proc. Roy. Soc. Edinburgh Sect. A 147 (2017) 1297–1310.

[49] B.-X. Zhu, q-log-convexity from linear transformations and polynomials with real zeros, European J. Combin. 73 (2018) 231–246.

[50] B.-X. Zhu, Positivity of iterated sequences of polynomials, SIAM J. Discrete Math. 32 (2018) 1993-2010.

[51] B.-X. Zhu, Total positivity, continued fractions and Stieltjes moment sequences, preprint, 2018.

[52] B.-X. Zhu, Positivity and continued fractions from the binomial transformation, Proc. Roy. Soc. Edinburgh A 149 (2019) 831–847.

[53] B.-X. Zhu, A generalized Eulerian triangle from staircase tableaux and tree-like tableaux, J. Combin. Theory Ser. A 172 (2020) 105206, 32pp.

[54] B.-X. Zhu, Stieltjes moment properties and continued fractions from combinatorial triangles, preprint, 2020, arXiv:2007.14924.

[55] B.-X. Zhu, Total positivity from a generalized cycle index polynomial, preprint, 2020, arXiv:2006.14485.