GENERALISATIONS OF HECKE ALGEBRAS FROM LOOP BRAID GROUPS

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Abstract. We introduce a generalisation LH_n of the ordinary Hecke algebras informed by the loop braid group LB_n and the extension of the Burau representation thereto. The ordinary Hecke algebra has many remarkable arithmetic and representation theoretic properties, and many applications. We show that LH_n has analogues of several of these properties. In particular we introduce a class of local (tensor space/functor) representations of the braid group derived from a meld of the (non-functor) Burau representation [Bur35] and the (functor) Rittenberg representations [MR92], here thus called Burau-Rittenberg representations. In its most supersymmetric case somewhat mystical cancellations of anomalies occur so that the Burau-Rittenberg representation extends to a loop Burau-Rittenberg representation. And this factors through LH_n. Let SP_n denote the corresponding (not necessarily proper) quotient algebra, k the ground ring, and $t \in k$ the loop-Hecke parameter. We prove the following:

1. LH_n is finite dimensional over a field.
2. The natural inclusion $LB_n \hookrightarrow LB_{n+1}$ passes to an inclusion $SP_n \hookrightarrow SP_{n+1}$.
3. Over $k = \mathbb{C}$, SP_n/rad is generically the sum of simple matrix algebras of dimension (and Bratteli diagram) given by Pascal's triangle. (Specifically $SP_n/rad \cong \mathbb{C}S_n/\mathbf{e}^1_{(2,2)}$ where S_n is the symmetric group and $\mathbf{e}^1_{(2,2)}$ is a $\lambda = (2, 2)$ primitive idempotent.)
4. We determine the other fundamental invariants of SP_n representation theory: the Cartan decomposition matrix; and the quiver, which is of type-A.
5. the structure of SP_n is independent of the parameter t, except for $t = 1$.
6. For $t^2 \neq 1$ then $LH_n \cong SP_n$ at least up to rank $n = 7$ (for $t = -1$ they are not isomorphic for $n > 2$; for $t = 1$ they are not isomorphic for $n > 1$).

Finally we discuss a number of other intriguing points arising from this construction in topology, representation theory and combinatorics.

1. Introduction

Until the 1980s, methods to construct linear representations of the braid group B_n were relatively scarce. We have those factoring through the symmetric group and the Burau representation [Bur35], and those factoring through the Hecke algebra [Hoe74] and the Temperley–Lieb algebra [TL71]; and, as for every group, the closure in the monoidal category $Rep(B_n)$. These proceed essentially through 'combinatorial' devices such as Artin's presentation. Then there are some more intrinsically 'topological' constructions such as Artin's representation [Art47] (and Burau can be recast in this light [LP93]).

In the 80s there were notable steps forward. Algebraic formulations of the Yang–Baxter equation began to yield representations (see e.g. [Bax82]). And Jones' discovery [Jon86] of link invariants from finite dimensional quotients of the group algebra

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Date: August 12, 2020.

$\mathbb{K}[B_n]$ inspired a revolution in braid group representations and topological invariants [Kau90, BW89, Mur87, FYH$^+$85, Wen88]. Work of Drinfeld, Reshetikhin, Turaev, Jimbo, [Dri87, RT91, Jim86] and others on quantum groups yielded yet further representations. Enriched through modern category theory [Tur10, KT08, BK01, DMM19], constructions are now relatively abundant.

The connections among B_n representations, $2+1$-dimension topological quantum field theory (see e.g. [Wit89]) and statistical mechanics (see e.g. [Bax82, Mar88]) were already well established in the 1980s. Even more recently, the importance of such representations in topological phases of matter [FKLW03, RW18] in two spacial dimensions has led to an invigoration of interest, typically focused on unitary representations associated with the 2-dimensional part of a $(2+1)$-TQFT. In this context the braid group is envisioned as the group of motions of point-like quasi-particles in a disk, with the trajectories of these *anyons* forming the braids in 3-dimensions. Here the braid group generators σ_i correspond to exchanging the positions of the i and $i+1$st anyons. The density of such braid group representations in the group of (special) unitary matrices is intimately related to the universality of quantum computational models built on these topological phases of matter [FKW02, FLW02], as well as the (classical) computational complexity of the associated link invariants [Row09]. Indeed, there is a circle of conjectures relating *finite* braid group images [NR11, RSW09], *classical* link invariants, *non-universal* topological quantum computers and *localizable* unitary braid group representations [RW12, GHR13]. The other side of this conjectured coin relates the holy grail of universal topological quantum computation with powerful 3-manifold invariants through surgery on links in the three sphere.

What is a non-trivial generalization of the braid group to 3-dimensions? Natural candidates are groups of motions: heuristically, the elements are classes of trajectories of a compact submanifold N inside an ambient manifold M for which the initial and final positions of N are set-wise the same. The group of motions of points in a 3-manifold in effect simply permutes the points, but the motion of circles or more general links in a 3-manifold is highly non-trivial. This motivates the study of these 3-dimensional motion groups, as defined in the mid-20th century by Dahm [Dah62] and expounded upon by Goldsmith [Gol81, Gol82].

More formally, a motion of N inside M is an ambient isotopy $f_t(x)$ of N in M so that $f_0 = id_M$ and $f_1(N) = N$. Such a motion is *stationary* if $f_t(N) = N$ for all t; and given any motion f, we have the usual notion of the reverse $\bar{f}$. We say two motions f, g are equivalent if the composition of f with $\bar{g}$ (via concatenation) gives a motion endpoint-fixed homotopic to a stationary motion as isotopies $M \times [0,1] \to M$. The *motion group* $\mathcal{M}o(M, N)$ is the group of motions modulo this equivalence. When M and N are both oriented we will consider only motions f so that $f_1(N) = N$ as an oriented submanifold, although one may consider the larger groups allowing for orientation reversing motions.

The motion groups of links inside $\mathbb{R}^3$, S^3 or D^3 and their representations are very rich, and only recently explored in the literature [BB16, DK19, KMRW16, BKMR20, BWC07]. Further enticement is provided by the prospect of applications to 3-dimensional topological phases of matter with loop-like excitations (i.e. vortices) [WL14]. The fruitful symbiosis between braid group representations and 2-dimensional condensed matter systems give us hope that 3-dimensional systems could enjoy a similar relationship with motion group representations, $3+1$-TQFTs, and invariants of surfaces embedded in 4-manifolds (see e.g. [Kam07, CKS04]).

There are a few hints in the literature that the $3+1$-dimensional story has some key differences from the $2+1$-dimensional situation. Reutter [Reu20] has shown that semisimple $3+1$-TQFTs cannot detect smooth structures on 4-manifolds. Wang and Qiu [QW20] provided evidence that the mapping class group and motion group representations associated with $3+1$-dimensional Dijkgraaf-Witten TQFTs are determined via dimension reduction by the corresponding $2+1$-dimensional DW theory. As the representation theory of motion groups has been largely neglected until very recently, it is hard to speculate on precise statements analogous to the 2-dimensional conjectures and theorems.

In this article we take hints from the the classical works [Bur35, Hoe74], from the braid group revolution [Jon87], and more directly from statistical mechanics [MR92], to study representations of the motion group of free unlinked circles in 3-dimensional space, the *loop braid group* LB_n. Presentations of LB_n are known (see [FRR97] and [Dam17] and references therein). As LB_n contains the braid group B_n as an abstract subgroup, a natural approach to finding linear representations is to extend known B_n representations to LB_n. This has been considered by various authors, see e.g. [BCH$^+$15, Bar05, KMRW16]. Another idea is to look for finite dimensional quotients of the group algebra, mimicking the techniques of [Jon87, BW89]. As non-trivial finite-dimensional quotients of the braid group are not so easy to find, we take a hybrid approach: we combine the extension of the Burau representation to LB_n [Bur35,Bar05] with the Hecke algebras $\mathcal{H}_n$ obtained from $\mathbb{Q}(t)[B_n]$ as the quotient by the ideal generated by $(\sigma_i+1)(\sigma_i-t)$. While the naive quotient of $\mathbb{Q}(t)[LB_n]$ by this ideal does not provide a finite dimensional algebra, certain additional quadratic relations (satisfied by the extended Burau representation) are sufficient for finite dimensionality, with quotient denoted LH_n. We find a local representation of LH_n that aids in the analysis of its structure — the Loop Burau-Rittenberg representation. One important feature of the algebras LH_n is that they are not semisimple; in fact, the image of the loop Burau-Rittenberg representation has a 1-dimensional center, but is far from simple. Its semisimple quotient by the Jacobson radical gives an interesting tower of algebras with Bratteli diagram exactly Pascal's triangle.

Our results suggest new lines of investigation into motion group representations. What other finite dimensional quotients of motion group algebras can we find (confer e.g. [Ban13])? What is the role of (non)-semisimplicity in such quotients? Can useful topological invariants be derived from these quotients? What do these results say about $3+1$-dimensional TQFTs?

Outline of the paper: In Sec.2 we recall the Burau representation and corresponding knot invariants. In Sec.3 we introduce loop Hecke algebras and prove they are finite dimensional. In Sec.4 we develop arithmetic tools (calculus) that we will need. In Sec.5 we construct our local representations and hence prove our main structure Theorems. In Sec.6 we apply the results from §5 to LH_n, and make several conjectures on the open cases with $t^2=1$. We conclude with a discussion of new directions opened up by this work.

Acknowledgements
CD and PM would like to thank Leverhulme for support under grant RPG-2018-029: “Emergent Physics From Lattice Models of Higher Gauge Theory”; and João Faria Martins (PI) for encouragement. CD would like to thank Paolo Bellingeri and Emmanuel Wagner for their guidance during the time this idea was being born. PM would like to thank Raphael Bennett-Tennenhaus and Paula Martin for useful conversations. We all thank Alex Bullivant (and hence EPSRC), and the University of Leeds, for partial support of ER's visit to Leeds. ER is partially supported by US NSF grant DMS-1664359, a Presidential Impact Fellowship from Texas A&M University and a Simons Foundation Fellowship. This research was carried out while ER was participating in a semester-long program on Quantum Symmetries at MSRI, a research institute partially supported by US NSF grant DMS-1440140.

2. Burau representation, Hecke algebra and invariants of knots

Let $\underline{n} := \{1, 2, ..., n\}$. Then the braid group B_n may be identified with the motion group $\mathcal{M}o(\mathbb{R}^2, \underline{n} \times \{0\})$. Artin showed that, for $n \geq 1$, B_n admits the presentation

$$\left\langle \sigma_1, \dots, \sigma_{n-1} \,\middle|\, \begin{array}{cr} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for } |i-j| > 1 \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & \text{for } i = 1, \dots, n-2 \end{array} \right\rangle \tag{2.1}$$

We will write $\mathfrak{A}_n(\sigma)$ for the set of relations here.

We will also need the symmetric group S_n. In a ‘motion group spirit’ this can be identified with $\mathcal{M}o(\mathbb{R}^3, \underline{n} \times \{0\} \times \{0\})$. It can be presented as a quotient of B_n by the relation ${\sigma_1}^2 = 1$ (however since we will often want to have both groups together we will soon rename the S_n generators).

2.1. Burau representation.

We define Burau representation $\varrho\colon B_n \to GL_n(\mathbb{Z}[t, t^{-1}])$ as follows:

$$\sigma_i \mapsto I_{i-1} \oplus \begin{pmatrix} 1-t & t \\ 1 & 0 \end{pmatrix} \oplus I_{n-i-1}. \tag{2.2}$$

The Burau representation has Jordan–Holder decomposition into a 1-dimensional representation (the vector $(1, \dots, 1)^T$ remains fixed) and an $(n-1)$-dimensional irreducible representation known as reduced Burau representation $\bar{\varrho}\colon B_n \to GL_{n-1}(\mathbb{Z}[t, t^{-1}])$. The decomposition is not split over $\mathbb{Z}[t, t^{-1}]$ — an inverse of $t+1$ is needed (see later).

Remark 2.1. One can also use the transpose matrix of (2.2) (depending on orientation choices while building the “carpark cover” of the punctured disc in the homological definition of Burau). The transpose fixes $(1, \dots, 1, t, t^2, 1, \dots, 1)^T$.

2.2. Facts about the Burau representation.

1. Burau is unfaithful for $n \geq 5$ (Moody [Moo91] proved unfaithfulness for $n \geq 9$, Long and Paton [LP93] for $n \geq 6$, Bigelow [Big99] for $n = 5$).
2. The case $n = 4$ is open, Beridze and Traczyk [BT18] recently published some advances toward closing the problem.
3. It is faithful for $n = 2, 3$ (Magnus and Peluso [MP69]).

(4) If we consider the braid group in its mapping class group formulation, it has a homological meaning (attached *a posteriori* to it, since Burau used only combinatorial aspects of matrices [Bur35]). The Burau representation describes the action of braids on the first homology group of the (covering of) the punctured disk. On the other hand the Alexander polynomial is extracted from the presentation matrix of the first homology group of the knot complement (the Alexander matrix). When we close up a braid, each element of homology of the punctured disk on the bottom becomes identified with its image in the punctured disk at the top. At this point the Alexander matrix of the closed braid is (roughly) the Burau matrix of the braid with the modification of identifying the endpoints.

More specifically, let K be a knot, and b a braid such that $\hat{b}$ is equivalent to K. Then the Alexander polynomial $\Delta_K(t)$ can be obtained by computing:

$$\Delta_K(t) = \frac{\det(\bar{\varrho}(b) - I_{n-1})}{1 + t + \ldots + t^{n-1}}.$$

So one can think of the Alexander polynomial of $K \sim \hat{b}$ as a rescaling of the characteristic polynomial of the image of b in the reduced representation.

Representations of B_n are partially characterised by the eigenvalue spectrum of the image of σ_i. Observe that

$$\varrho({\sigma_i}^2) = (1-t)\varrho(\sigma_i) + tI_n, \tag{2.3}$$

i.e., the eigenvalue spectrum is $Spec(\varrho(\sigma_i)) = \{1, -t\}$. Recall also that Kronecker products obey $Spec(A \otimes B) = Spec(A).Spec(B)$, so $Spec(\varrho(\sigma_i) \otimes \varrho(\sigma_i) = \varrho \otimes \varrho(\sigma_i)) = \{1, -t, t^2\}$. From this we see that the spectrum is fixed under tensor product only if $t = \pm 1$.

2.3. **Hecke algebras.** Let R be an integral domain and q_1, q_2 elements of R with q_2 invertible. We define the Hecke algebra $H_n^R(q_1, q_2)$ to be the algebra with generators $\{1, T_1, \ldots, T_{n-1}\}$ and the following defining relations:

$$T_i T_j = T_j T_i \qquad \text{for } |i-j| > 1 \tag{2.4}$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \qquad \text{for } i = 1, \ldots, n-2 \tag{2.5}$$

$$T_i^2 = (q_1 + q_2) T_i - q_1 q_2 \qquad \text{for } i = 1, \ldots, n-1. \tag{2.6}$$

Remark 2.2.
(1) Relation (2.6) coincides with the characteristic equation of the images of the generators under the Burau representation (2.1) when $(q_1, q_2) = (1, -t)$. We denote the resulting 1-parameter Iwahori-Hecke algebra by $H_n^R(t)$.
(2) If $t = 1$ then $H_n^R(t)$ is the group algebra $R[S_n]$ (the free R-module RS_n made an R-algebra in the usual way).
(3) There is a map from B_n to $H_n^R(t)$ sending σ_i to T_i. Thus representations of $H_n^R(t)$ are equivalent to representations of B_n for which the generators satisfy relation (2.6). This is described in [Big06, Section 3], [Jon87, Section 4] in [Mar91, §5.7] and many other places.
(4) Fixing $R = \mathbb{C}$, point (3) allows us to think of $H_n^R(t)$ as being isomorphic to the quotient $H_n(t) := {}^{\mathbb{C}[B_n]}\!\big/_{\sigma_i^2 = (1-t)\sigma_i + t}$.

(5) Using the map in (3) we can represent any element of $H_n(t)$ as a linear combination of braid diagrams. The quadratic relation can be seen as a *skein relation* on elementary crossings. Knowing a basis for $H_n(t)$ makes this fact usable.

Question 2.3. Why these parameters and this quadratic relation?
As noted, Hecke algebras can be defined with two units of R as parameters. We chose to fix these parameters to $(1, -t)$ because from this quotient one should recover the Alexander polynomial. Choosing $(-1, t)$ one should get the quotient on which Ocneanu traces are defined (see [KT08, Chapter 4.2]). With the Ocneanu trace being a 1-parameter family over a 1-parameter algebra, we end up with polynomials in two variables. These polynomials are attached to the braid diagrams that we can see representing elements of $H_n(t)$. Moreover they are defined in such a way to respect Markov moves, so they are invariants for the closures of said braids. Hence, they are knot invariants. The quadratic relation from 2.2(3) translates the trace in a *skein relation.* Through the Ocneanu trace (normalised) the invariant that is obtained is the HOMFLY-PT polynomial, which specialises in both Alexander and Jones. Each specialisation corresponds to factoring through a further quotient of the Hecke algebra (in the case of Jones, this is a quotient of the Temperley-Lieb algebra). Below we "reverse engineer" this process.

3. Generalising Burau and Hecke to loop braid groups

3.1. **The loop braid group.** Here S^1 denotes the unit circle. We now consider the loop braid group

$$LB_n = \mathcal{M}o(\mathbb{R}^3, \underline{n} \times S^1)$$

(see e.g. [Gol81, Sav96, FRR97, BH13, Dam17, KMRW16, BCH$^+$15]).

Consider the set $\Xi_n = \{\sigma_i, \rho_i, i = 1, 2, ..., n-1\}$ and group $\langle \Xi_n | \mathfrak{Q}_n \rangle$ presented by generators σ_i and ρ_i, and relations $\mathfrak{Q}_n$ as follows. The σ_is obey the braid relations as in (2.1); the ρ_is obey the braid relations and also

$$\rho_i \rho_i = 1 \tag{3.1}$$

and then there are mixed braid relations

$$\rho_i \rho_{i+1} \sigma_i = \sigma_{i+1} \rho_i \rho_{i+1}, \tag{3.2}$$

$$\rho_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \rho_{i+1}, \tag{3.3}$$

$$\sigma_i \rho_{i \pm j} = \rho_{i \pm j} \sigma_i \qquad (j > 1) \qquad\qquad (\textit{all distant commutators}). \tag{3.4}$$

Remark 3.1. The first mixed relation (3.2) implies its reversed order counterpart:

$$\sigma_i \rho_{i+1} \rho_i = \rho_{i+1} \rho_i \sigma_{i+1} \tag{3.5}$$

whereas the reversed order second mixed relation does not hold.
The relations also imply

$$\rho_2 \sigma_1 \rho_2 = \rho_1 \sigma_2 \rho_1 \tag{3.6}$$

We have (see e.g. [FRR97]) that

$$LB_n \;\cong\; \langle \Xi_n | \mathfrak{Q}_n \rangle. \tag{3.7}$$

It will be convenient to give an *algebra presentation* for the group algebra. Recall that in an algebra presentation inverses are not present automatically by freeness, so we may put them in by hand as formal symbols and then impose the inverse relations. Thus as a presented algebra we have

$$k\langle \Xi_n | \mathfrak{Q}_n \rangle \;=\; \langle \Xi_n \cup \Xi_n^- \mid \mathfrak{Q}_n, \mathfrak{I}_n \rangle_k$$

— here kG means the group k-algebra of group G; and $\langle - | - \rangle_k$ means a k-algebra presentation; and $\mathfrak{I}_n$ is the set of inverse relations $\sigma_i \sigma_i^- = 1$.

3.2. **The loop–Hecke algebra LH_n.** With §2 in mind, there *is* a suitable generalisation of the Burau representation to LB_n.

Proposition 3.2 ([Ver01])**.** *The map on generators of LB_n given by*

$$\sigma_i \mapsto I_{i-1} \oplus \begin{pmatrix} 1-t & t \\ 1 & 0 \end{pmatrix} \oplus I_{n-i-1}. \tag{3.8}$$

$$\rho_i \mapsto I_{i-1} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus I_{n-i-1}. \tag{3.9}$$

extends to a representation $\varrho_{GB} : LB_n \to GL_n(\mathbb{Z}[t, t^{-1}])$.

Proof. Direct calculation. □

This group representation is not faithful for $n \geq 3$ [Bar05], and corresponds to an Alexander polynomial for welded knots.

We consider a quotient algebra of the group algebra (over a suitable commutative ring) of the group $\langle \Xi_n | \mathfrak{Q}_n \rangle$. The quotient algebra is

$$LH_n^{\mathbb{Z}} := \mathbb{Z}[t, t^{-1}]\langle \Xi_n | \mathfrak{Q}_n \rangle \big/ \mathfrak{R}_n \;=\; \langle \Xi_n \cup \Xi_n^- \mid \mathfrak{Q}_n, \mathfrak{I}_n, \mathfrak{R}_n \rangle_{\mathbb{Z}[t,t^{-1}]} \tag{3.10}$$

where $\mathfrak{R}_n$ is the set of (algebra) relations:

$$\sigma_i^2 = (1-t)\sigma_i + t \qquad \text{(i.e. } (\sigma_i - 1)(\sigma_i + t) = 0) \tag{3.11}$$

$$\rho_i \sigma_i = -t\rho_i + \sigma_i + t \qquad \text{(i.e. } (\rho_i - 1)(\sigma_i + t) = 0) \tag{3.12}$$

$$\sigma_i \rho_i = -\sigma_i + \rho_i + 1. \qquad \text{(i.e. } (\sigma_i - 1)(\rho_i + 1) = 0) \tag{3.13}$$

(NB we already have $(\rho_i - 1)(\rho_i + 1) = 0$).

Observe that (3.11) yields an inverse for σ_i (the inverse to t is specifically needed), so we have the following:

$$LH_n^{\mathbb{Z}} = \;\langle \Xi_n \mid \mathfrak{Q}_n, \mathfrak{R}_n \rangle_{\mathbb{Z}[t,t^{-1}]} \tag{3.14}$$

Observe then that the relations as such do not require an inverse to t, so we could consider the variant algebra over $\mathbb{Z}[t]$.

For any field K that is a $\mathbb{Z}[t, t^{-1}]$ algebra we then define the base change $LH_n^K = K \otimes_{\mathbb{Z}[t,t^{-1}]} LH_n^{\mathbb{Z}}$ and, for given $t_c \in \mathbb{C}$,

$$LH_n(t_c) = LH_n = LH_n^{\mathbb{C}}$$

where $\mathbb{C}$ is a $\mathbb{Z}[t]$-algebra by evaluating t at t_c (the choice of which we notationally suppress). Note that there is no reason to suppose that this gives a flat deformation (i.e. the same dimension) in all cases. (It will turn out that it does, at least in low rank, if we can localise at $t^2 - 1$. In particular, perhaps surprisingly, in the variant $t = 0$ is isomorphic to the generic case.)

Remark. The relations (3.11) et seq are suggested by (2.3) and the following calculations (on σ_1 and ρ_1 in LB_3, noting that blocks work the same way for all generators):

$$\varrho_{GB}(\sigma_1\rho_1) = \begin{pmatrix} t & 1-t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = -\begin{pmatrix} 1-t & t & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + I_3$$

$$\varrho_{GB}(\rho_1\sigma_1) = \begin{pmatrix} 1 & 0 & 0 \\ 1-t & t & 0 \\ 0 & 0 & 1 \end{pmatrix} = -t\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1-t & t & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + tI_3.$$

3.3. **Notable direct consequences of the relations: Finiteness.** Given a word in the generators, of form $\sigma_3\sigma_4\rho_2$ say, by a *translate* of it we mean the word obtained by shifting the indices thus: $\sigma_{3+i}\sigma_{4+i}\rho_{2+i}$.

With the $\mathfrak{Q}$ and $\mathfrak{R}$ relations we can derive the following ones, together with the natural translates thereof (here $\overset{*}{=}$ uses (3.1); $\overset{\rho\rho\sigma}{=}$ uses (3.2); $\overset{\sigma\rho}{=}$ uses (3.13), and so on):

$$\begin{aligned}
\text{(M1)}\quad \boldsymbol{\sigma_2\rho_1\sigma_2} &\overset{*}{=} \sigma_2\rho_2\rho_2\rho_1\sigma_2 \overset{\rho\rho\sigma}{=} \sigma_2\rho_2\sigma_1\rho_2\rho_1 \overset{\sigma\rho}{=} -\sigma_2\sigma_1\rho_2\rho_1 + \rho_2\sigma_1\rho_2\rho_1 + \sigma_1\rho_2\rho_1 \\
&\overset{\rho\sigma\sigma,\rho\rho\sigma}{=} -\rho_1\sigma_2\sigma_1\rho_1 + \rho_2\rho_2\rho_1\sigma_2 + \sigma_1\rho_2\rho_1 \overset{\sigma\rho}{=} \boldsymbol{\sigma_1\rho_2\rho_1} + \boldsymbol{\rho_1\sigma_2\sigma_1} - \boldsymbol{\rho_1\sigma_2\rho_1}, \\
\text{(M2)}\quad \boldsymbol{\rho_2\sigma_1\sigma_2} &\overset{*}{=} \rho_2\sigma_1\rho_2\rho_2\sigma_2 \overset{\rho\sigma}{=} -t\rho_2\sigma_1\rho_2\rho_2 + \rho_2\sigma_1\rho_2\sigma_2 + t\rho_2\sigma_1\rho_2 \overset{*,\rho\sigma\rho}{=} -t\rho_2\sigma_1 + \rho_1\sigma_2\rho_1\sigma_2 + t\rho_1\sigma_2\rho_1 \\
&\overset{M1}{=} -t\rho_2\sigma_1 + \rho_1(\rho_1\sigma_2\sigma_1 - \rho_1\sigma_2\rho_1 + \sigma_1\rho_2\rho_1) + t\rho_1\sigma_2\rho_1 \\
&\overset{*,\rho\sigma}{=} -t\rho_2\sigma_1 + \sigma_2\sigma_1 - \sigma_2\rho_1 + (-t\rho_1 + \sigma_1 + t)\rho_2\rho_1 + t\rho_1\sigma_2\rho_1 \\
&= \sigma_1\rho_2\rho_1 + t\rho_1\sigma_2\rho_1 - t\rho_1\rho_2\rho_1 + \sigma_2\sigma_1 - \sigma_2\rho_1 - t\rho_2\sigma_1 + t\rho_2\rho_1.
\end{aligned}$$

Definition 3.3. For given n and $m \leq n$ let $LH^{\langle}_m$ denote the subalgebra of LH_{n+1} generated by Ξ_m (it is a quotient of LH_m, as per the Ψ map formalism in §4.2).

Lemma 3.4. *For any n let X_i be the vector subspace of LH_n spanned by $\{1, \sigma_i, \rho_i\}$. Then $LH_{n+1} = LH^{\langle}_n X_n LH^{\langle}_n$.*

Proof. It is enough to show that $X_n LH^{\langle}_n X_n$ lies in $LH^{\langle}_n X_n LH^{\langle}_n$. We work by induction on n. The case $n = 1$ is clear, since $LH_1 = \mathbb{C}$. Assume true in case $n - 1$ and consider case n. We have $X_n LH^{\langle}_n X_n = X_n LH^{\langle}_{n-1} X_{n-1} LH^{\langle}_{n-1} X_n$ by assumption. But $LH^{\langle}_{n-1}$ and X_n commute so we have $LH^{\langle}_{n-1} X_n X_{n-1} X_n LH^{\langle}_{n-1}$. The inductive step follows from the relations $\mathfrak{Q}$ and $\mathfrak{R}$ and the relations (M1,2) above. ☐

Corollary 3.5. *LH_n is finite dimensional.* ☐

Remark 3.6. We may also treat certain other quotients of $\mathbb{C}LB_n$. For example, eliminating either relations (3.12) or (3.13) we still obtain finite dimensional quotients. In particular, if we only include (3.13) and not (3.12) then the analogous proof with X_n replaced by $\{1, \rho_n, \sigma_n, \rho_n\sigma_n\}$ proves finite dimensionality.

3.4. **Refining the spanning set.** Can we express elements of LH_3 as sums of length-2 words (and hence eventually solve word problem)? We have, for example,

$$\rho_1\rho_2\rho_1 = -1 + \rho_2 + \frac{(-t-1)}{(t-1)}(-\rho_1 + \rho_2\rho_1 - \rho_1\rho_2) + \frac{2}{(t-1)}(-\sigma_1 + \sigma_2\rho_1 - \rho_1\sigma_2) \tag{3.15}$$

But in general this is not easy. And another problem is that we do not have immediately manifest relationships between different ranks (such as inclusion) that would be useful. With this (and several related points) in mind it would be useful to have a tensor space representation. In what follows we address the construction of such a representation.

4. BASIC ARITHMETIC WITH LH_n

Here we briefly report some basic arithmetic in LH_n that gives the clues we need for our local representation constructions below.

4.1. **Fundamental tools, locality.** In what follows, B denotes the *braid category*: a strict monoidal category with object monoid $(\mathbb{N}_0, +)$ generated by 1, and $\mathsf{B}(n,n) = B_n$, $\mathsf{B}(n,m) = \emptyset$ otherwise, and monoidal composition is via side-by-side concatenation of suitable braid representatives (see e.g. [Mac98, XI.4]). Similarly S is the permutation category (of symmetric groups). Let H denote the ordinary Hecke category — again monoidal, but less obviously so [Hum94]. (We have not yet shown that LH, the loop-Hecke category, is monoidal.)

Let LB denote the loop-braid category — this is the strict monoidal category analogous to the braid category where the object monoid is $(\mathbb{N}, +)$, $\mathsf{LB}(n,n) = LB_n$, $\mathsf{LB}(n,m) = \emptyset$ otherwise, and monoidal composition $\otimes$ is side-by-side concatenation of loop-braids.

Suppose C is a strict monoidal category with object monoid $(\mathbb{N}_0, +)$ generated by 1 (for example, LB). Write 1_1 for the unique element of $\mathsf{C}(1,1)$ and for $x \in \mathsf{C}(n,n)$ define the *translate*

$$x^{(t)} = 1_1^{\otimes t} \otimes x \in \mathsf{C}(n+t, n+t) \tag{4.1}$$

For k a commutative ring, define translates of elements of kLB_n (i.e. $k\mathsf{LB}(n,n)$), and kS_n and so on, by linear extension.

Caveat: Note that it is a property of the geometric topological construction of loop braids that the composition $\otimes$ in LB makes manifest sense. It requires that side-by-side concatenation of rank n with rank m passes to $n+m$. This is clear by construction. But in groups/algebras defined by generators and relations it would not be intrinsically clear. For example, how do we know that the subalgebra of LH_n generated *in* LH_n by the elements p_i, s_i, $i = 1, 2, ..., n-2$ is isomorphic to LH_{n-1}? (Some of our notation requires care at this point since it may lead us to take isomorphism for granted!)

4.2. **The Ψ maps.** Let $A = \langle X|R\rangle_k$ be an algebra presented with generators X and relations R. Then there is a homomorphism from the free algebra generated by any subset X_1 of X to A, taking $s \in X_1$ to its image in A. This factors through the quotient by any relations, R_1 say, expressed only in X_1. We may consider it as a homomorphism from this quotient. But of course the kernel may be bigger — relations induced indirectly by the relations in R. A Ψ map is such a homomorphism:

$$\langle X_1|R_1\rangle_k \overset{\Psi}{\twoheadrightarrow} \langle X_1|R\rangle_k \hookrightarrow \langle X|R\rangle_k$$

Note that arithmetic properties such as idempotency, orthogonality and vanishing are preserved under Ψ maps. Thus for example a decomposition of 1 into orthogonal idempotents in kS_n passes to such a decomposition in LH_n (see (4.3)). However conditions such as primitivity, inequality and even non-zero-ness are not preserved in general.

Note that there is a natural (not generally isomorphic) image of

$$kS_n \;\cong\; k\langle p_1, ..., p_i, ..., p_{n-1} \mid \mathfrak{A}_n(p), p_i p_i = 1\rangle \tag{4.2}$$

in LH_n obtained by the map of generators $p_i \mapsto \rho_i$. Let us call it LH_n^ρ. Thus

$$kS_n \overset{\Psi}{\twoheadrightarrow} LH_n^\rho \hookrightarrow LH_n \tag{4.3}$$

Similarly $H_n = \langle T_1, ..., T_i, ..., T_{n-1}| \,\mathfrak{A}_n(T), ...\rangle_k$ has image LH_n^σ under $T_i \mapsto \sigma_i$:

$$H_n \twoheadrightarrow LH_n^\sigma \hookrightarrow LH_n \tag{4.4}$$

Let us consider the image of a primitive idempotent decomposition in kS_n:

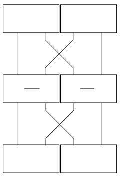
$$1 = \sum_{\lambda\in\Lambda_n} \sum_{i=1}^{d_\lambda} \mathbf{e}^i_\lambda$$

under $\Psi : kS_n \to LH_n$. Here Λ_n denotes the set of integer partitions of n, and d_λ is the dimension of the S_n irrep. See §A.1 for explicit constructions. We will also write $(\Lambda, \subseteq)$ for the poset of all integer partitions ordered by the usual inclusion as a Young diagram.

Proposition 4.1. *Let k be the field of fractions of $\mathbb{Z}[t, t^{-1}]$.*
(I) The image $\Psi(\mathbf{e}^i_\lambda)$ in LH_n^k of every idempotent with $(2, 2) \subseteq \lambda \in \Lambda_n$ is zero.
(II) On the other hand all other $\lambda \in \Lambda_n$, i.e. all hook shapes, give non-zero image.

Proof. (I) Note that $\mathbf{e}^1_\mu$ with $\mu \in \Lambda_m$ is defined in kS_n for $n \geq m$ by $S_m \hookrightarrow S_n$. It is shown for example in [MR92] that if the relation $\mathbf{e}^1_\mu = 0$ is imposed in a quotient of kS_n then $\mathbf{e}^i_\nu = 0$ holds for $\mu \subseteq \nu \in \Lambda_n$ (a proof uses $S_{n-1} \hookrightarrow S_n$ restriction rules, from which we see that $\mathbf{e}^1_\mu$ is expressible as a sum of orthogonal such idempotents). Consider $\mathbf{e}^1_{(2,2)}$ (i.e. with $(2, 2) \in \Lambda_4$) which may be expressed as

$$\mathbf{e}^1_{(2,2)} \;=$$



$$\propto\; (p_1 + 1)(p_3 + 1)p_2(p_1 - 1)(p_3 - 1)p_2(p_1 + 1)(p_3 + 1)$$

(using notation and a choice from (A.5)). By a direct calculation in LH_4

$$\Psi((p_1 + 1)(p_3 + 1)p_2(p_1 - 1)(p_3 - 1)) = 0 \tag{4.5}$$

(N.B. we know no elegant way to do this calculation; the result holds also for generic t, but *not* for $t = 1$).

(II) This can be verified by evaluation as non-zero in a suitable representation. (For simplicity it is sufficient to work in the 'SP quotient' that we give in Prop.5.2 below, working with Kronecker products. We will omit the explicit calculation.) □

With identity (4.5) in mind, recall that in [MR92] local representations of ordinary Hecke (and hence S_n) with this property were constructed from spin chains. In §5 we will combine this with Burau and thus find the representations of loop-Hecke that we need here.

By Prop.4.1 we have a decomposition of 1 in LH_n according to hook partitions:

$$1 \;=\; \sum_{i=0}^{n-1} \sum_{j=1}^{d_{(n-i,1^i)}} \Psi(\mathbf{e}^j_{(n-i,1^i)}) \tag{4.6}$$

(NB j varies over idempotents that are equivalent in the sense that they induce isomorphic modules — it will be sufficient to focus on $j = 1$).

(Left) multiplying by $A = LH_n$ we thus have a decomposition of the algebra

$$A \;\cong\; \oplus_{i=0}^{n-1} \oplus_j A\Psi(\mathbf{e}^j_{(n-i,1^i)})$$

as a left-module for itself, into projective summands.

We have not yet shown that these summands are indecomposable. But consider for a moment the action of LH_n on the image under Ψ of

$$Y^n_\pm = \sum_{g\in S_n} (\pm 1)^{len(g)} g$$

in LH_n (we write Y^n_+ for unnormalised $\mathbf{e}^1_{(n)}$ and Y^n_- for $\mathbf{e}^1_{(1^n)}$ — again see §A.1 for a review). By abuse of notation we will write $Y^n_\pm$ also for the image. By (3.13) and the classical identities $Y^{a(1)}_\pm Y^n_\pm = a! Y^n_\pm$ (recall $Y^{a(1)}_\pm$ means $Y^a_\pm$ with indices shifted by $+1$, see (A.2) *et seq*) we have

$$\sigma_i Y^n_+ = Y^n_+, \qquad\qquad Y^n_-\,\sigma_i \;=\; -tY^n_- \tag{4.7}$$

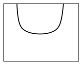
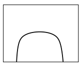
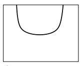
It follows that Y^n_+ spans a 1-d left ideal in LH_n. If we work over a field containing the rationals then it is normalisable as an idempotent, and so we have an indecomposable projective left module

$$P_{(n)} \;=\; LH_n Y^n_+ \;=\; LH_n \mathbf{e}^1_{(n)} \;=\; k\mathbf{e}^1_{(n)}.$$

5. ON LOCAL REPRESENTATIONS

Here Mat is the monoidal category of matrices over a given commutative ring (and Mat_k the case over commutative ring k), with object monoid $(\mathbb{N}, \times)$ and tensor product on morphisms given by a Kronecker product (NB there is a convention choice in defining the Kronecker product). We often focus on the monoidal subcategory Mat^m generated by a single object $m \in \mathbb{N}$ — usually $m = 2$. Then the object monoid $(2^{\mathbb{N}}, \times)$ becomes $(\mathbb{N}, +)$ in the natural way.

In the study of ordinary Hecke algebras (and particularly quantum-group-controlled quotients like Temperley–Lieb) a very useful tool is the beautiful set of local tensor space representations generalising those arising from XXZ spin chains and Schur–Weyl duality. For example we have the following.

Consider the TL diagram category T with object monoid $(\mathbb{N},+)$ k-linear-monoidally generated by the morphisms represented by diagrams $\mathsf{u} =$  $\in \mathsf{T}(2,0)$ and $\mathsf{u}^* =$ . This has a TQFT F_2 given by  $\mapsto (0,\tau,\tau^{-1},0)$ (the target category is Mat) and taking $*$ to transpose. Of course for $1_1 \in \mathsf{T}(1,1)$ we have $\mathsf{F}_2(1_1) = I_2$.

To pass to our present topic we note that $1_1 \otimes 1_1 = 1_2$ and that the Yang–Baxter construction $\sigma_1 \mapsto 1_2 - \tau^2\mathsf{u}^*\mathsf{u}$ gives

$$\sigma_1 \;\mapsto\; \mathsf{F}_2(1_2) - \tau^2 \begin{pmatrix} 0 & & & \\ & \tau^2 & 1 & \\ & 1 & \tau^{-2} & \\ & & & 0 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & 1-\tau^4 & -\tau^2 & \\ & -\tau^2 & 0 & \\ & & & 1 \end{pmatrix} \tag{5.1}$$

thus a representation of the braid category B (note that eigenvalues are 1 and $-\tau^4$ so τ^4 here passes to t in our parameterisation for loop-Hecke). But also note that u,u^* can be used for a Markov trace. And also for idempotent localisation functors: let $\mathsf{U} = \mathsf{u}^*\mathsf{u}$, $\mathsf{U}_1 = \mathsf{U} \otimes 1_{n-2}$, and $T_n = \mathsf{T}(n,n)$ regarded as a k-algebra; then we have the algebra isomorphism $\mathsf{U}_1 T_n \mathsf{U}_1 \cong T_{n-2}$. This naturally gives a category embedding G_{U} of T_{n-2}-mod in T_n-mod. Recall that irreps are naturally indexed by partitions of n into at most two parts: $\lambda = (n-m,m)$, or equivalently (for given n) by 'charge' $\lambda_1 - \lambda_2 = n-2m$, thus by $\Upsilon_n = \{n, n-2, n-4, ..., 0/1\}$ (depending on n is odd or even). This latter labelling scheme is stable under the embedding. That is, indecomposable projective modules are mapped by G_{U} according to $\Upsilon_{n-2} \hookrightarrow \Upsilon_n$.

5.1. **Charge conservation.**

Another useful property of F_2 is 'charge conservation'. We may label the row/colum index for object 2 in Mat by $\{\varepsilon_1,\varepsilon_2\}$ or $\{+,-\}$. Then $2\otimes 2$ has index set $\{\varepsilon_1\otimes\varepsilon_1, \varepsilon_2\otimes\varepsilon_1, \varepsilon_1\otimes\varepsilon_2, \varepsilon_2\otimes\varepsilon_2\}$ (which we may abbreviate to $\{11,21,12,22\}$) and so on. The 'charge' ch of an index is $ch = \#1 - \#2$. Note from (5.1) that F_2 does not mix between different charges (hence charge conservation).

For a functor with the charge conservation property the representation of B_n (say) obtained has a direct sum decomposition according to charge, with 'Young blocks' β_i of charge $i = n, n-2, ..., -n$. The dimensions of the blocks are given by Pascal's triangle. It will be convenient to express this with the semiinfinite Toeplitz matrices $\mathcal{U}$ and $\mathcal{T}$:

$$\mathcal{U} = \begin{pmatrix} 1 & 1 & & & \\ & 1 & 1 & & \\ & & 1 & 1 & \\ & & & 1 & 1 \\ & & & & \ddots \end{pmatrix}, \quad \mathcal{U}^2 = \begin{pmatrix} 1 & 2 & 1 & & & \\ & 1 & 2 & 1 & & \\ & & 1 & 2 & 1 & \\ & & & 1 & 2 & 1 \\ & & & & & \ddots \end{pmatrix}, \quad \mathcal{T} = \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & 1 & & \\ & 1 & 0 & 1 & \\ & & 1 & 0 & 1 \\ & & & & \ddots \end{pmatrix}$$

and semiinfinite vectors $v_1 = (1,0,0,0,...)$, $v_2 = (0,1,0,0,...)$, ... Thus $v_1\mathcal{U}^n$ (respectively $v_{n+1}\mathcal{T}^n$) gives the numbers in the $n{+}1$-th row of Pascal (followed by a tail of zeros). (The two different formulations correspond to two different thermodynamic limits — $\mathcal{T}$ corresponds to the $\Upsilon_{n-2} \hookrightarrow \Upsilon_n$ limit — see later.) Then

$$dim(\beta_i) \;=\; (v_1\mathcal{U}^n)_{(n-i+2)/2} \;=\; (v_{n+1}\mathcal{T}^n)_{n-i+1} \tag{5.2}$$

In the case of F_2 these blocks are not linearly irreducible in general (the generic irreducible dimensions are given by $v_1\mathcal{T}^n$). But they still provide a useful framework. We return to this later.

With this construction and Prop.3.2 in mind, it is natural to ask if we can make a local version of generalised Burau. (Folklore is that this cannot work, and directly speaking it does not. But we now have some more clues at our disposal.)

5.2. **'Burau–Rittenberg' representations of B_n.** Now we have in mind Prop.4.1; and brute force calculations in low rank showing (see §6) that LH_n is non-semisimple but has irreps with dimensions given by Pascal's triangle. This is all reminiscent of work of Rittenberg on quantum spin chains over Lie superalgebras [MR92] (it is also reminiscent of work of Saleur on 'type-B' braids — but for this see later). Inspired by this and the Burau representation (and cf. [DF18] and references therein) we proceed as follows.

Proposition 5.1. *Fix a commutative ring k, $\tau \in k^\times$, and $t = \tau^4$. There is a monoidal functor F_M from the Braid category B to Vect (or at least Mat) given by object 1 mapping to $V = \mathbb{C}\{e_1, e_2\}$ (i.e. to 2 in $\mathsf{Mat}_{\mathbb{Z}[t]}$) and the positive braid σ in $\mathsf{B}(2,2)$ mapping to*

$$M_t(\sigma) = \begin{pmatrix} 1 & & & \\ & 1-t & t & \\ & 1 & 0 & \\ & & & 1 \end{pmatrix} \tag{5.3}$$

The conjugation of this matrix to $\mathsf{F}_2(\sigma)$ lifts to a natural isomorphism of functors. There is one other natural isomorphism class of charge conserving functors that factors through Hecke. A representative functor is $\mathsf{F}_{M'}$ given by

$$M'_t(\sigma) = \begin{pmatrix} 1 & & & \\ & 1-t & t & \\ & 1 & 0 & \\ & & & -t \end{pmatrix} \tag{5.4}$$

(According to Rittenberg's scheme this is the (1,1)-super class, cf. for example [MR92]. But note that in extending to LB below, isomorphism will not be preserved, so we are focussing on the specific representative.)

Proof. There is a proof in each case by direct calculations and properties of the construction. Let 1_n denote the identity morphism in $\mathsf{B}(n,n)$. Then

$$M_t((\sigma \otimes 1_1)(1_1 \otimes \sigma)(\sigma \otimes 1_1)) = M_t((1_1 \otimes \sigma)(\sigma \otimes 1_1)(1_1 \otimes \sigma))$$

with other relations then holding by construction. Invertibility of the image of σ is clear. The M' case is similar. The proof of completeness is elementary analysis. □

Indeed

$$M'_t(\sigma \otimes 1_1)M'_t(1_1 \otimes \sigma)M'_s(\sigma \otimes 1_1) = M'_s(1_1 \otimes \sigma)M'_t(\sigma \otimes 1_1)M'_t(1_1 \otimes \sigma) \tag{5.5}$$

while the tss version of this identity does *not* hold (unless we force $s = 1$) (NB care must be taken with conventions here).

Note that the M_t case is a smooth (not necessarily flat) deformation of the $t = 1$ case, which is the classical $N = 2$ tensor space rep of S — a special case of our F_2 functor in (5.1) above. For $n = 3$ the M_t case has 2 copies of the trivial rep and 2 copies of the Burau rep (thus dim: $2^3 = 1 + 3 + 3 + 1$). Thus at $t = 1$ it is a direct sum of four copies of the trivial and two copies of the 2d S_n irrep. For $n = 4$ we have 16=1+4+6+4+1, so as well as two trivial and two Burau (4=3+1), now also a 6 (with $6 = 2 + 3 + 1$ when $t = 1$) — see e.g. [Mar91, §9.6]. The M'_t case has the same block decomposition but different irreducible decomposition. See §6.

5.3. **Extending to LB.** Recall we introduced the loop-braid category LB. We write $\sigma \in \mathsf{LB}(2, 2)$ for the positive braid exchange and $\rho \in \mathsf{LB}(2, 2)$ for the symmetric exchange.

Naively extending with elementary transpositions (cf. ϱ_{GB}), the F_M construction fails to satisfy the mixed braid relation (3.3). However the $(1, 1)$-Rittenberg functor $\mathsf{F}_{M'}$ fairs better:

Theorem 5.2. *(i) The $\sigma \mapsto M'_t(\sigma)$ construction extended using the super transposition $\rho \mapsto M'_1(\sigma)$ gives a monoidal functor $F^e_{M'}$ from the loop Braid category LB to Mat.*
(ii) $F^e_{M'}$ factors through LH.

Proof. The proof is a linear algebra calculation similar to the B case above, using Kronecker product identities. □

Definition 5.3. Fix a field k and $t \in k$. Then the k-algebra $SP_n = kLB_n/Ann\, F^e_{M'}$.

We conjecture that the extended super representation, which we call Burau–Rittenberg, or 'SP' rep for short, is faithful on LH unless $t^2 = 1$ (see later).

Proposition 5.4. *Fix a field k and $t \in k$, $t \neq 1$. Let $\chi_i = \frac{\sigma_i - \rho_i}{1-t}$. Then*
(a) χ_i and ρ_i $(i = 1, 2, ..., n - 1)$ are alternative generators of SP_n; and
(b) The k-algebra isomorphism class of SP_n is independent of t.

Proof. (a) Elementary. (b) The images of the alternative generators in the defining representation are independent of t. □

5.4. **Towards linear structure of SP.** Let us work out the linear structure of SP. (I.e. its Artin–Wedderburn linear representation theory over $\mathbb{C}$: simple modules, projective modules and so on. See §5.5 for a review.)

Proposition 5.5. *Suppose $t \neq 1 \in k$. Let $\chi = \frac{\sigma-\rho}{1-t}$ and $\chi_1 = \frac{\sigma_1-\rho_1}{1-t} \in SP_n$. Then*

$$\chi_1 SP_n \chi_1 \cong SP_{n-1} \tag{5.6}$$

and

$$SP_n \;/\; SP_n\, \chi_1\, SP_n \;\cong\; k. \tag{5.7}$$

(II) In particular the map $f_\chi : SP_{n-1} \to \chi_1 SP_n \chi_1$ given by $w \mapsto \chi_1 w^{(1)} \chi_1$ (recall the translation notation from (4.1)) is an algebra isomorphism.

Proof. Let us write simply $F = F_n$ for the defining representation $F^e_{M'}$ of SP_n. We write $\{1, 2\}^n$ for the basis (i.e. we write simply symbols $1, 2$ for e_1, e_2; and the word 112 for

$e_1 \otimes e_1 \otimes e_2$ and so on). Our convention for ordering the basis is given by 11,21,12,22. First observe that the image in F is (here with $n = 3$)

$$(\chi \otimes 1_2) \mapsto \begin{pmatrix} 0 &&& \\ & \boxed{1} & -1 & \\ && 0 & \\ &&& \boxed{1} \end{pmatrix} \otimes 1_2 = \begin{pmatrix} 0 &&&&&&& \\ & \boxed{1} & -1 &&&&& \\ && 0 &&&&& \\ &&& \boxed{1} &&&& \\ &&&& 0 &&& \\ &&&&& \boxed{1} & -1 & \\ &&&&&& 0 & \\ &&&&&&& \boxed{1} \end{pmatrix} \tag{5.8}$$

Note that the basis change conjugating by

$$\begin{pmatrix} 1 &&& \\ & 1 & 1 & \\ && 1 & \\ &&& 1 \end{pmatrix} \otimes 1_2 \tag{5.9}$$

(again this is the example with $n = 3$) brings this into diagonal form, projecting onto the $2\{1,2\}^{n-1}$ subspace (the subspace of $V^n = \mathbb{C}\{1,2\}^n$ spanned by basis elements of form $2w$ with $w \in \{1,2\}^{n-1}$, i.e. of form $e_2 \otimes ...$). That is: $\chi \mapsto \begin{pmatrix} 0 & \\ & 1 \end{pmatrix} \otimes 1_2$. Furthermore,

$$(\chi \otimes 1_2)(1_2 \otimes \sigma)(\chi \otimes 1_2) \mapsto \begin{pmatrix} 0 &&&&&&& \\ & \boxed{1} & -1 &&&&& \\ && 0 &&&&& \\ &&& \boxed{\text{1-t}} & 0 & \boxed{\text{t}} & -t & \\ &&&& 0 &&& \\ &&& \boxed{1} & 0 & \boxed{0} && \\ &&&&&& 0 & \\ &&&&&&& \boxed{\text{-t}} \end{pmatrix} \tag{5.10}$$

Note that after the (5.9) basis change this decomposes as a sum of several copies of the 0 module together with the submodule N with basis $2\{1,2\}^{n-1}$. Then the map from N to $\{1,2\}^{n-1}$ given by $2w \mapsto w$ gives

$$\chi_1\sigma_1\chi_1 = -t.\chi_1 \mapsto F_{n-1}(-t.1)$$

and $\chi_1\sigma_2\chi_1 \mapsto F_{n-1}(\sigma_1)$ and $\chi_1\rho_2\chi_1 \mapsto F_{n-1}(\rho_1)$. Also note that χ_1 commutes with σ_i for $i > 2$ so we have

$$\chi_1\sigma_i\chi_1 = \chi_1\sigma_i \mapsto F_{n-1}(\sigma_{i-1}), \qquad \chi_1\rho_i\chi_1 \mapsto F_{n-1}(\rho_{i-1}) \qquad i > 2. \tag{5.11}$$

Thus the images of the generators under $w \mapsto \chi_1 w \chi_1$ are the generators of SP_{n-1}, establishing (5.6) on generators. To show that the images of the generators span we proceed as follows. From Lemma 3.4 and the (sufficient) symmetry of the relations under $i \mapsto n-i$ on indices, writing $LH_n = L_n$ for short, we have

$$\begin{aligned} L_{n+1} = L_n X_n L_n = L_n^{(1)} X_1 L_n^{(1)} &= L_{n-1}^{(2)} X_2 L_{n-1}^{(2)} X_1 L_{n-1}^{(2)} X_2 L_{n-1}^{(2)} \\ &= L_{n-1}^{(2)} X_2 L_{n-2}^{(3)} X_3 L_{n-2}^{(3)} X_1 X_2 L_{n-1}^{(2)} = L_{n-1}^{(2)} X_2 X_3 X_1 X_2 L_{n-1}^{(2)} \end{aligned}$$

Thus

$$\chi_1 L_{n+1} \chi_1 = \chi_1 L^{(2)}_{n-1} X_2 X_3 X_1 X_2 L^{(2)}_{n-1} \chi_1 = L^{(2)}_{n-1} \chi_1 X_2 X_3 X_1 X_2 \chi_1 L^{(2)}_{n-1}$$

We can show by direct calculations that $\chi_1 X_2 X_3 X_1 X_2 \chi_1$ lies in the algebra generated by the images of the generators. (We can do this even in LH_4. The result then holds in SP_4 since it is a quotient; and then in SP_n by construction. — Note however that we have not shown that it holds in LH_n.) Also $L^{(2)}_{n-1} \chi_1$ evidently lies in the algebra generated by the images of the generators, by commutation, so we are done.

Finally (5.7) follows on noting that the quotient corresponds to imposing $\chi_1 = 0$, i.e. $\sigma_1 = \rho_1$. Noting that $t \neq 1$, this gives $\sigma_i = 1$.
(II) Note that f_χ inverses the map from (5.11) above. □

5.5. **Aside on linear/Artinian representation theory.** Since this paper bridges between topology and linear representation theory it is perhaps appropriate to say a few words on the bridge. While topology focuses on topological invariants, linear rep theory is concerned with invariants such as the spectrum of linear operators (and the generalised 'spectrum' of algebras of linear operators). The former is thus of interest for topological quantum field theories, and the latter for usual quantum field theories (where notions such as mass are defined). In this section we recall a few key points of linear/Artinian rep theory that are useful for us. (So of course it can be skipped if you are not interested in this aspect, or are already familiar.)

Recall that every finite dimensional algebra over an algebraically closed field is Morita equivalent to a basic algebra (see e.g. [NS43, Jac89, Ben95]). This allows us to track separately the combinatorial and homological data of an algebra.

Let A be a finite dimensional algebra over an algebraically closed field k (cf. e.g. [Ben95]). Let $J(A)$ denote the radical. Let $L = \{L_1, ... L_r\}$ be an ordered set of the isomorphism classes of simple A-modules, with projective covers $P_i = Ae_i$ (i.e. the e_is are a set of primitive idempotents). Given an A-module M let $Rad(M)$ denote the intersection of the maximal proper submodules. Now suppose A is basic. Recall that $Ext^1_A(L_i, L_j)$ codifies the non-split extensions between these modules — i.e. the 'atomic' components of non-semisimplicity. The corresponding 'Ext-matrix' $E_L(A)$ is given by

$$(E_L(A))_{ij} = \ dim_k Ext^1_A(L_i, L_j)$$

or equivalently

$$dim_k Ext^1_A(L_i, L_j) = \ dim_k(Hom_A(P_j, Rad(P_i))/Hom_A(P_j, Rad^2(P_i)))$$

$$= \ dim_k(e_j J(A) e_i / e_j J^2(A) e_i)$$

This perhaps looks technical, but note that $e_j J(A) e_i = e_j A e_i$ when $i \neq j$ and so then is essentially what we study in §4.2 et seq (and in our case the quotient factor is even conjecturally zero, so in fact we are already studying the Ext-matrix!). Note that the Ext-matrix defines a quiver and hence a path algebra $kE_L(A)$. For any finite dimensional algebra A, basic or otherwise, the Cartan decomposition matrix $C_L(A)$ is given by

$$(C_L(A))_{ij} \ = \ dim_k Hom_A(P_j, P_i) \tag{5.12}$$

that is, the i-th row gives the number of times each simple module occurs in P_i.

To pass back from the basic-algebra/homology to the full algebra we need the dimensions of the irreducibles. For an algebra A with Cartan matrix $C_L(A)$ and a vector $v_L(A)$ giving the dimensions of the irreducible heads of the projectives we have

$$dim(A) = v_L(A) C_L(A) v_L(A)^T \tag{5.13}$$

Definition 5.6. Let the $n \times n$ matrix $\bar{\mathcal{M}}_n$ be

$$\bar{\mathcal{M}}_n = \begin{pmatrix} 1 &&&&& \\ 1 & 1 &&&& \\ & 1 & 1 &&& \\ && 1 & 1 && \\ &&& \ddots & \ddots & \\ &&&& 1 & 1 \end{pmatrix}$$

— we label columns left to right (and rows top to bottom) by the ordered set h_n of hook integer partitions of n:

$$h_n = ((n), (n-1, 1), (n-2, 1^2), ..., (1^n)).$$

We will see in 5.8 that $\bar{\mathcal{M}}_n$ is the left Cartan decomposition matrix of SP_n (it follows that the Ext-matrix is the same except without the main diagonal entries).

5.6. **Linear structure of SP_n.**

A corollary of Prop.5.5 is that we have an embedding of module categories $G_\chi : SP_{n-1} - \text{mod} \to SP_n - \text{mod}$. In fact we can use this (together with our earlier calculations) to determine the structure of these algebras. Before giving the structure theorem let us recall the relevant general theory.

Lemma 5.7. (see e.g. Green [Gre80, §6.2]) *Let A be an algebra and $e \in A$ an idempotent. Then*
(i) the functor $Ae \otimes_{eAe} - : eAe - \text{mod} \to A - \text{mod}$ takes a complete set of inequivalent indecomposable projective left eAe-modules to a set of inequivalent indecomposable projective A-modules that is complete except for the projective covers of simple modules L in which $eL = 0$. (There is a corresponding right-module version.)
(ii) This functor and the functor $\bar{G}_e : A - \text{mod} \to eAe - \text{mod}$ given by $M \mapsto eM$ form a left-right adjoint pair.
(iii) The Cartan decomposition matrix of eAe embeds in that of A according to the labelling of modules in (i). □

Theorem 5.8. *(i) Isomorphism classes of irreps of SP_n are naturally indexed by h_n. (Indeed $SP_n/rad \cong \mathbb{Q}S_n/\mathbf{e}^1_{2,2}$ so the dimensions are given by the n-th row of Pascal's triangle — see Fig.1.)*
(ii) The left Cartan decomposition matrix is $\bar{\mathcal{M}}_n$. (Note that this determines the structure of SP_n. It gives the dimension as dim=$(2^{n-1} + choose)/2$.)
(iii) The image of the decomposition (4.6) is complete in SP_n.

Proof. (i) Consider Lemma 5.7(i). In our case, putting $A = SP_n$ and $e = \chi_1$, then by (5.7) there is exactly one module L such that $eL = 0$ (at each n) — the trivial module. Thus by Prop.5.5 SP_n has one more class of projectives and hence irreps than SP_{n-1}.

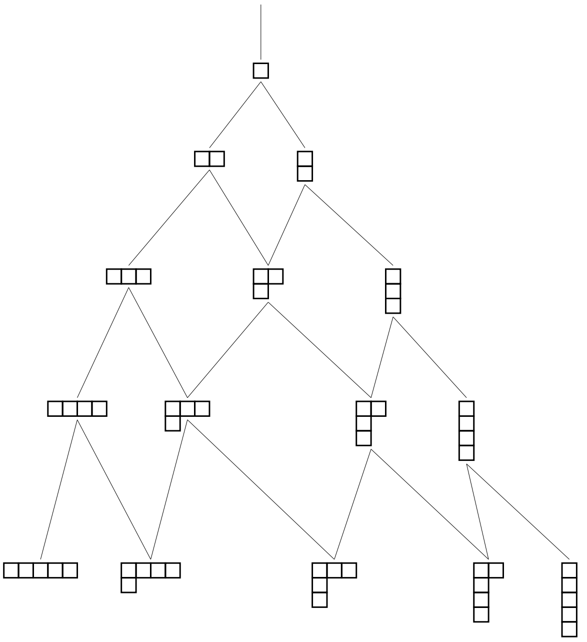


FIGURE 1. Young graph up to rank 5 with 22-diagrams removed.

In particular write $G_\chi : SP_{n-1} - \text{mod} \;\to\; SP_n - \text{mod}$ for the functor in our case obtained using (5.6) from Prop.5.5, that is: $G_\chi(M) = SP_n\chi \otimes_{\chi SP_n \chi} f_\chi M$, suppressing the index n, where f_χ is as described above. Then a complete set of indecomposable projectives is

$$P^n_n = SP_n e^1_{(n)},$$

$$P^n_{n-1} = G_\chi(P^{n-1}_{n-1}) = G_\chi(SP_{n-1}e^1_{(n-1)}),$$

$$P^n_{n-2} = G_\chi(G_\chi(SP_{n-2}e^1_{(n-2)}), ..., P^n_{n-j} = G^{\circ j}_\chi(SP_{n-j}e^1_{(n-j)}), ..., P^n_1 = G^{\circ n-1}_\chi(k)$$

It follows that the Cartan decomposition matrix $C(n)$ contains $C(n-1)$ as a submatrix, with one new row and column with the label n. The new row gives the simple content of P^n_n. But by (4.7) (noting Prop.5.2(ii)) this projective is simple. Iterating, we deduce that $C(n)$ is lower-unitriangular.

Working by induction, suppose $C(n)$ is of the claimed form in (ii) at level $n-1$. Then at level n we have:

$$C(n) = \left(\begin{array}{c|cccccc} 1 & & & & & & \\ \hline * & 1 & & & & & \\ * & 1 & 1 & & & & \\ * & & 1 & 1 & & & \\ * & & & 1 & 1 & & \\ \vdots & & & & \ddots & \ddots & \\ * & & & & & 1 & 1 \end{array}\right) \tag{5.14}$$

(omitted entries 0). To complete the inductive step we need to compute the $e^1_{(n)}P_{n-j}$ for each j. Write G^m_χ for G_χ and f^m_χ for f_χ at level $m<n$, and note that

$$\begin{aligned} G_\chi(SP_{n-1}e^1_\lambda) &= SP_n\chi_1 \otimes_{\chi SP_n\chi} f_\chi(SP_{n-1}e^1_\lambda) = SP_n\chi_1 \otimes_{\chi SP_n\chi} \chi_1 SP^{(1)}_{n-1}e^{(1)}_\lambda\chi_1 \\ &= SP_n\chi_1 SP^{(1)}_{n-1}e^{(1)}_\lambda\chi_1 \otimes_{\chi SP_n\chi} \chi_1 \cong SP_n\chi_1 SP^{(1)}_{n-1}e^{(1)}_\lambda\chi_1 \end{aligned}$$

where we have used that these modules are idempotently generated ideals to apply the tensor product up to isomorphism (and where again we use the notation from (4.1), so $SP^{(1)}_{n-1}$ is the 1-step translated copy of SP_{n-1} in SP_n). So in particular

$$e^1_{(n)}SP_nG_\chi(SP_{n-1}e^1_{(n-1)}) \cong e^1_{(n)}SP_n\chi_1SP^{(1)}_{n-1}e^{(1)}_{(n-1)}\chi_1 \subseteq e^1_{(n)}SP_n\chi_1$$

It follows from the form of the image of $e^1_{(n)}$ in the SP representation (cf. [Ham62], [Mar92, Append.B] and [MR92]) that the dimension of $e^1_{(n)}SP_n\chi_1$ is 1, so the first $*$ is 1. Specifically we have for example

$$e_2 = \frac{1+p_1}{2} \mapsto \left(\begin{array}{c|cc|c} 1 & & & \\ \hline & 1/2 & 1/2 & \\ & 1/2 & 1/2 & \\ \hline & & & 0 \end{array}\right), \qquad \chi \mapsto \left(\begin{array}{c|cc|c} 0 & & & \\ \hline & 1 & -1 & \\ & 0 & 0 & \\ \hline & & & 1 \end{array}\right)$$

and

$$e_3 \mapsto \frac{1}{3}\left(\begin{array}{c|ccc|ccc|c} 3 & & & & & & & \\ \hline & 1 & 1 & 1 & & & & \\ & 1 & 1 & 1 & & & & \\ & 1 & 1 & 1 & & & & \\ \hline & & & & 0 & 0 & 0 & \\ & & & & 0 & 0 & 0 & \\ & & & & 0 & 0 & 0 & \\ \hline & & & & & & & 0 \end{array}\right), \qquad \chi_1 \mapsto \left(\begin{array}{c|ccc|ccc|c} 0 & & & & & & & \\ \hline & 0 & & & & & & \\ & & 1 & -1 & & & & \\ & & 0 & 0 & & & & \\ \hline & & & & 1 & -1 & 0 & \\ & & & & 0 & 0 & 0 & \\ & & & & 0 & 0 & 1 & \\ \hline & & & & & & & 1 \end{array}\right)$$

where we have reordered the basis into fixed charge sectors, i.e. as 111, 112, 121, 211, 122, 212, 221, 222 (the charge of a basis element is $\#(1)-\#(2)$, where $\#(1)$ is the number of 1's [Bax82, Mar92]). Note from the construction that charge is conserved in SP, so each charge sector is a submodule. We see that in each charge sector except $(n-1,1)$ we have that either the image of $e^1_{(n)}$ is zero or the image of χ_1 is zero. Finally in the $(n-1,1)$ sector both of these have rank 1. We deduce that $e^1_nA\chi_1$ is 1-dimensional as required.

Similarly we have to consider

$$\begin{aligned} &G_\chi G^{n-1}_\chi(SP_{n-2}e^1_{(n-2)}) \cong SP_n\chi_1f_\chi f^{n-1}_\chi(SP_{n-2}e^1_{(n-2)}) \\ &\cong SP_n\chi_1f_\chi(\chi_1SP^{(1)}_{n-2}e^{(1)}_{(n-2)}\chi_1) = SP_n\chi_1\chi_1\chi^{(1)}_1SP^{(2)}_{n-2}e^{(2)}_{(n-2)}\chi^{(1)}_1\chi_1 \end{aligned}$$

(NB $\chi^{(1)}_1=\chi_2$) giving

$$e^1_{(n)}SP_nG_\chi G_\chi(SP_{n-2}e^1_{(n-2)}) \cong e^1_{(n)}SP_nf_\chi f_\chi(SP_{n-2}e^1_{(n-2)}) = e^1_{(n)}SP_n\chi_1\chi_2 ...$$

We have, in the charge block basis,

$$\chi_2 \mapsto \left(\begin{array}{c|ccc|ccc|c} 0 &&&&&&& \\ \hline & 1 & -1 & &&&& \\ && 0 & 0 &&&& \\ && 0 & 0 &&&& \\ \hline &&&& 1 & 0 & 0 & \\ &&&& 0 & 1 & -1 & \\ &&&& 0 & 0 & 0 & \\ \hline &&&&&&& 1 \end{array}\right), \qquad \chi_1\chi_2 \mapsto \left(\begin{array}{c|ccc|ccc|c} 0 &&&&&&& \\ \hline & 0 & 0 & &&&& \\ && 0 & 0 &&&& \\ && 0 & 0 &&&& \\ \hline &&&& 1 & -1 & 1 & \\ &&&& 0 & 0 & 0 & \\ &&&& 0 & 0 & 0 & \\ \hline &&&&&&& 1 \end{array}\right)$$

(in general for a nonzero entry in $\chi_1\chi_2$ we need basis elements with 2 in the first and second position) so

$$e^1_{(n)} SP_n\ \chi_1\chi_2 \ = 0$$

(remark: indeed we can verify that $e^1_{(n)}\ \chi_2\chi_1 \ = 0$ holds in LH_n) so the second $*$ and indeed the other $*$s in (5.14) are all zero. We have verified the inductive step for (ii).

Statement (iii) may be deduced from (i,ii) as follows. Note that we have n isomorphism classes in the decomposition, and their multiplicities are the dimensions of the hook irreps of S_n in the natural order. On the other hand the $n+1$ charge blocks of the SP representation are each either an irrep or contains two irreps, since each contains one or two irreps upon restricting to S_n. The first is an irrep (since dimension 1). By the proof of (ii) the second contains the first irrep, so two irreps in total, and the other again has the same dimension as the corresponding S_n hook representation. Furthermore no other block contains the first irrep so this block must be indecomposable (else the SP representation could not be faithful, which it is by definition). Proceeding through the blocks then by (ii) the first n of them are a complete set of projective modules, so each one except the first and last contains two simple modules ('adjacent' in the hook order). But then by the construction of the Pascal triangle and (ii) these simple modules have the same dimension as the corresponding S_n irreps, and (iii) follows. □

6. ON REPRESENTATION THEORY OF LH_n

Combining (5.2) with (5.13) and Th.5.8 we have

$$dim(SP_n(t \neq 1)) = v_1 \mathcal{U}^{n-1} \begin{pmatrix} 1 &&&& \\ 1 & 1 &&& \\ & 1 & 1 && \\ && 1 & 1 & \\ &&& \ddots & \ddots \end{pmatrix} \left(v_1 \mathcal{U}^{n-1}\right)^T = \binom{2n-1}{n-1} = \frac{1}{2}\binom{2n}{n}$$

— NB we have used the obvious 'global' limit of all the Cartan matrices (it is a coincidence that this and the $\mathcal{U}$ matrix are similar).

Given a vector v we write $Diag(v)$ for the diagonal matrix with v down the diagonal. Let p^n be the vector with the n-th row of Pascal's triangle as the entries, thus for example $p^4 = (1,3,3,1)$. We have

$$\mathcal{M}^p_n \ := \ Diag(p^n)\ \bar{\mathcal{M}}_n\ Diag(p^n)$$

(examples are given in (6.2) below) and the dimension is the sum of all the entries. The closed form follows readily from this. Also from Th.5.8 we have:

Corollary 6.1. *For* $t \neq 1$ *the Morita class of* SP_n *is of the path algebra with* A_n *quiver (directed* $1 \to 2 \to \cdots \to n$*) and relations given by vanishing of all proper paths of length 2. In particular the radical-squared vanishes.*

6.1. **Properties determined from Th.5.8 and direct calculation in low rank.**

Our results for LH_n may be neatly given as follows. Firstly,

Proposition 6.2. *For* $t^2 \neq 1$ *and* $n < 8$, $LH_n \cong SP_n$.

Proof. Here we can compute dimensions directly, which saturates the bound on the kernel. ☐

Conjecture 6.3. *For* $t^2 \neq 1$, $LH_n \cong SP_n$.

A summary of what we learn for the algebra dimensions, and irreducible reps, of LH_n is given by the following tables.

(6.1)

	$t=1$	$t=-1$	$t^2 \neq 1$	$t \neq 1$	*irreps* : */dims*				*labels*								
n	*dim*	*dim*	*dim*	*ssdim*	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
1	1	1	1	1							1						
2	3	3	3	2						1		1					
3	15	11	10	6					1		2		1				
4	114	42	35	20				1		3		3		1			
5	1170	163	126	70			1		4		6		4		1		
6	15570	638	462	252		1		5		10		10		5		1	
7		2510	1716	924	1		6		15		20		15		6		1

The irrep labels here are given by $(n-i, 1^i) \mapsto n - 2i - 1$.

Combining (5.2) with (5.13), Th.5.8 and 6.3 we have the conjecture

$$dim(LH_n(t^2 \neq 1)) = v_1 \mathcal{U}^{n-1} \begin{pmatrix} 1 & & & & \\ 1 & 1 & & & \\ & 1 & 1 & & \\ & & 1 & 1 & \\ & & & \ddots & \ddots \end{pmatrix} \left(v_1 \mathcal{U}^{n-1} \right)^T = \binom{2n-1}{n-1} = \frac{1}{2} \binom{2n}{n}$$

For $t = -1$ we note that SP_n is generally a proper quotient of LH_n, and that LH_n has larger radical (the square does not vanish). We define the semiinfinite matrix:

$$C(LH(t=-1)) = \begin{pmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 1 & 1 & & \\ 1 & 1 & 1 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and conjecture that the Cartan matrix $C(\mathit{LH}_n(t=-1))$ is this truncated at $n \times n$ (i.e. the quiver is the same as the generic case, but without quotient relations); and thus we conjecture

$$dim(\mathit{LH}_n(t=-1)) \;=\; v_1\mathcal{U}^{n-1}\begin{pmatrix} 1 &&&& \\ 1 & 1 &&& \\ 1 & 1 & 1 && \\ 1 & 1 & 1 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}\left(v_1\mathcal{U}^{n-1}\right)^T = \frac{n^2+\binom{2n-2}{n-1}}{2}$$

(cf. OEIS A032443). Note that our calculations verify this for $n \leq 7$.

For $t=1$ we see that $LH_n(t=1)$ has semisimple quotient at least as big as $\mathbb{C}S_n$, which is of dimension $n!$. Indeed, in this case the quotient by the relation $\sigma_i = \rho_i$ is precisely $\mathbb{C}S_n$, since in this case $\sigma_i^2 = 1$. For $n \leq 4$ we have computationally verified that the semisimple subalgebra of $LH_n(t=1)$ is precisely $\mathbb{C}S_n$, and we conjecture that this is the case for all n. The Jacobson radical grows quite quickly however, and we do not have a conjecture on the general structure.

Observe that the numbers in (6.1) follow the conjectured patterns. Since the vector v_1 has finite support the nominally infinite sums above are all finite. To inspect the supported part, in the generic case consider matrices $\mathcal{M}_n^p$ $(n=2,3,4,5)$:

$$\begin{pmatrix} 1 & \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 && \\ 2 & 2^2 & \\ & 2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 &&& \\ 3 & 3^3 && \\ & 9 & 3^3 & \\ && 3 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 &&&& \\ 4 & 4^2 &&& \\ & 24 & 6^2 && \\ && 24 & 4^2 & \\ &&& 4 & 1 \end{pmatrix} \tag{6.2}$$

Here the semisimple dimension is given by the sum down the diagonal and the radical dimension is given by the sum in the off-diagonal.

For $t=-1$:

$$\begin{pmatrix} 1 & \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 && \\ 2 & 2^2 & \\ 1 & 2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 &&& \\ 3 & 3^3 && \\ 3 & 9 & 3^3 & \\ 1 & 3 & 3 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 &&&& \\ 4 & 4^2 &&& \\ 6 & 24 & 6^2 && \\ 4 & 16 & 24 & 4^2 & \\ 1 & 4 & 6 & 4 & 1 \end{pmatrix}$$

6.2. **On χ elements.** Let us define

$$\chi^{(m+1)} = (\sigma_1 - \rho_1)(\sigma_2 - \rho_2)...(\sigma_m - \rho_m), \tag{6.3}$$

understood as an element in LH_n with $n > m$. Thus in particular $\chi^{(2)} = \chi_1$. Similarly for sequence $X = (x_1, x_2, ..., x_k)$ define

$$\chi^{(X)} = (\sigma_{x_1} - \rho_{x_1})(\sigma_{x_2} - \rho_{x_2})...(\sigma_{x_k} - \rho_{x_k}), \tag{6.4}$$

and

$$\chi_-^{(m+1)} = (\sigma_m - \rho_m)(\sigma_{m-1} - \rho_{m-1})...(\sigma_2 - \rho_2)(\sigma_1 - \rho_1), \tag{6.5}$$

It is easy to verify that if X is non-increasing then $\chi^{(X)}\chi^{(X)} = (1-t)^k\chi^{(X)}$. Thus (for $t \neq 1$) the non-increasing cases can all be normalised as idempotents. However it is also easy to check that no increasing case can. (A nice illustration of the 'chirality' present in the defining relations.)

Observe that imposing the relation $\sigma_1 = \rho_1$ in LH_n forces $\sigma_1 = 1$, unless $t = 1$. Thus the quotient algebra

$$LH_n/\chi^{(2)} \cong k \qquad t \neq 1 \tag{6.6}$$

i.e. only the trivial, or label $\lambda = +n$, irrep survives. And the same holds for SP_n. The following has been checked up to rank 5.

Conjecture 6.4. *The structure of the quotient $LH_n/\chi^{(j+1)}$ is given by the $j \times j$ truncation of $\mathcal{M}_n^p$.*

7. Discussion and avenues for future work

Above we give answers to the main structural questions for SP_n and LH_n. But exploration of generalisations is also well-motivated, since these algebras (even taken together with the constructions discussed in [KMRW16]) cover a relatively small quotient inside $Rep(LB_n)$. With this in mind, there are a number of other questions worth addressing around SP_n and LH_n, offering clues on generalisation, and hence towards understanding more of the structure of the group algebra. Remark 3.6 suggests that for most values of t we obtain larger finite dimensional quotients by eliminating one of the local relations (3.12) or (3.13). Computational experiments suggest that for $t = 0$ eliminating (3.13) yields infinite dimensional algebras. This parameter-dependence should be further explored.

In light of the results of [Reu20] the non-semisimplicity of LH_n is an important feature, rather than a shortcoming. Extracting topological information from the non-semisimple part requires some further work, as Markov traces typically 'see' the semisimple part. Another aspect of our work is the (conjectural) localisation of the regular representation of LH_n. It is worth pointing out that localisations of *unitary* sequences of B_n representations are relatively rare, conjecturally corresponding to representations with finite braid group image [RW12, GHR13]. Since LH_n is non-semisimple and hence non-unitary this does not contradict this conjectural relationship, but gives us some hope that localisations are possible for other parameter choices and other quotients.

The quotient of LB_n by the relation ${\sigma_i}^2 = 1$ is a potentially interesting infinite group, which we call the mixed double symmetric group MDS_n. The reason for this nomenclature is that MDS_n is a quotient of the free product of two copies of the symmetric group. In particular, MDS_n surjects onto S_n by $\sigma_i \to \rho_i$. It is of special interest here as $LH_n(1)$ is a quotient of $\mathbb{Z}[MDS_n]$. We expect it could be of quite general interest.

In [KMRW16] constructions are developed based on BMW algebras, but still starting from 'classical' precepts. It would be very interesting to meld the super-Burau-Rittenberg construction to the KMRW construction. For example, one might try to use cubic local (eigenvalue) relations among the generators ρ_i, σ_i to obtain finite dimensional quotients, possibly inspired by the relations satisfied by a subsequence of LB_n lifts of BMW algebra representations.

Appendix A. Preparatory arithmetic and notation for left ideals

A.1. **Symmetric group and Hecke algebra arithmetic.** Recall Young's (anti-)symmetrizers in kS_n. Unnormalised in $\mathbb{Z}S_n$ they are:

$$Y^n_{\pm} = \sum_{g \in S_n} (\pm 1)^{len(g)} g \tag{A.1}$$

where $len(g)$ is the usual Coxeter length function. If k has characteristic 0 then kS_n is semisimple and these elements are simply the (unnormalised) idempotents corresponding to the trivial and alternating representations respectively. Note that exactly the same classical construction works for the Hecke algebra over any field where it is semisimple. (The corresponding idempotents are sometimes called Jones–Wenzl projectors.) Specifically (see e.g. [CR81, §9B])

$$X^n_{\pm} = \sum_{g \in S_n} (-\lambda_{\mp})^{-len(g)} T_g \qquad \text{i.e. } X^2_- = 1 - \sigma_1, \;\; X^2_+ = 1 + t^{-1}\sigma_1, ...$$

where for us $\lambda_- = -t$ and $\lambda_+ = 1$ (the apparent flip of labels is just because we use non-Lusztig scaling), and T_g is the product of generators obtained by writing g in reduced form then applying $\rho_i \mapsto T_i$.

Working in kS_{n+m} we understand $Y^n_{\pm}$ and translates such as $Y^{n(1)}_{+}$ in the obvious way. Note then that we have many identities like

$$Y^2_+ Y^n_+ = 2Y^n_+, \qquad Y^{a(1)} Y^n_+ = a! Y^n_+ \quad (a < n) \tag{A.2}$$

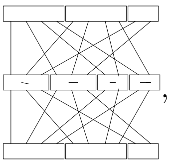
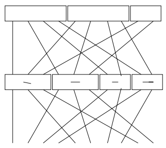
Recall Λ_n denotes the set of integer partitions of n. Over the rational field we have a decomposition of $1 \in kS_n$ into primitive central idempotents

$$1 = \sum_{\lambda \in \Lambda_n} \epsilon_\lambda \tag{A.3}$$

where each ϵ_λ is a known unique element (see e.g. Cohn [Coh82, §7.6] or Curtis–Reiner [CR81] for gentle expositions). There is a further (not generally unique) decomposition of each ϵ_λ into primitive orthogonal idempotents

$$\epsilon_\lambda = \sum_{i=1}^{dim_\lambda} \mathbf{e}^i_\lambda \tag{A.4}$$

where dim_λ is the number of walks from the root to λ on the directed Young graph. The elements $\mathbf{e}^i_\lambda$ are conjugate to each other. The elements $\mathbf{e}^i_\lambda$ are not uniquely defined in general. Two possible constructions of one for each λ are exemplified pictorially by (case $\lambda = 442$)

(A.5) $e^1_\lambda = \; c_\lambda$ , $\hat{e}^1_\lambda = \; c_\lambda$ 

where an undecorated box is a symmetrizer and a −-decorated box an antisymmetrizer, and the factor c_λ is just a scalar. (NB For the moment we write e^1_λ instead of $\mathbf{e}^1_\lambda$ for this specific choice.) In particular though, $\mathbf{e}^1_{(n)}$ *is* unique: $\mathbf{e}^1_{(n)} = \frac{1}{n!} Y^n_+$. (The whole story lifts to the Hecke case — see e.g. [Mar91] for a gentle exposition.)

An idempotent decomposition of 1 in a subalgebra B of an algebra A is of course a decomposition in A. Thus in particular we can take an idempotent in kS_n and consider it as an idempotent in kS_{n+1} by the inclusion that is natural from the presentation ($p_i \mapsto p_i$). Understanding $\mathbf{e}^1_\lambda$ with $\lambda \vdash n$ in kS_{n+1} in this way, a useful property in our $k = \mathbb{C}$ case will be

$$\mathbf{e}^j_\lambda = \sum_{\mu \in \lambda +} \mathbf{e}'_\mu \tag{A.6}$$

where $\lambda+$ denotes the set of partitions obtained from λ by adding a box, and the prime indicates that we identify this idempotent only up to equivalence. (Various proofs exist. For example note that the existence of such a decomposition follows from the induction rules for $S_n \hookrightarrow S_{n+1}$.) For example

$$\mathbf{e}^1_{(2,2)} = \mathbf{e}'_{(3,2)} + \mathbf{e}'_{(2,2,1)}$$

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