A p-adic analogue of Chan and Verrill's formula for $1/\pi$

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Abstract. We prove three supercongruences for sums of Almkvist–Zudilin numbers, which confirm some conjectures of Zudilin and Z.-H. Sun. A typical example is the Ramanujan-type supercongruence:

$$\sum_{k=0}^{p-1}\frac{4k+1}{81^k}\gamma_k\equiv\left(\frac{-3}{p}\right)p\pmod{p^3},$$

which is corresponding to Chan and Verrill's formula for $1/\pi$:

$$\sum_{k=0}^{\infty}\frac{4k+1}{81^k}\gamma_k=\frac{3\sqrt{3}}{2\pi}.$$

Here γ_n are the Almkvist–Zudilin numbers.

Keywords: Supercongruences; Almkvist–Zudilin numbers; Harmonic numbers

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1 Introduction

For $n\ge 0$, the following sequence:

$$\gamma_n=\sum_{j=0}^{n}(-1)^{n-j}\frac{3^{n-3j}(3j)!}{(j!)^3}\binom{n}{3j}\binom{n+j}{j}$$

are known as Almkvist–Zudilin numbers (see [1] and A125143 in [20]). This sequence appears to be first recorded by Zagier [29] as integral solutions to Apéry-like recurrence equations.

These numbers also appear as coefficients of modular forms. Let $q=e^{2\pi i\tau}$ and

$$\eta(\tau)=q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$$

be the Dedekind eta function. Chan and Verrill [6] showed that if

$$t_3(\tau)=\left(\frac{\eta(3\tau)\eta(6\tau)}{\eta(\tau)\eta(2\tau)}\right)^4\quad\text{and}\quad F_3(\tau)=\frac{(\eta(\tau)\eta(2\tau))^3}{\eta(3\tau)\eta(6\tau)},$$

and $|t_3(\tau)|$ is sufficiently small, then

$$F_3(\tau) = \sum_{n=0}^{\infty} \gamma_n t_3^n(\tau).$$

They also constructed some new series for $1/\pi$ in terms of the numbers γ_n, one of the typical examples is the following formula [6, Theorem 3.14]:

$$\sum_{k=0}^{\infty} \frac{4k+1}{81^k}\gamma_k = \frac{3\sqrt{3}}{2\pi}. \tag{1.1}$$

The above interesting example motivates us to prove the following supercongruence, which was originally conjectured by Zudilin [30, (33)].

Theorem 1.1 *For any prime $p \geq 5$, we have*

$$\sum_{k=0}^{p-1} \frac{4k+1}{81^k}\gamma_k \equiv \left(\frac{-3}{p}\right) p \pmod{p^3}, \tag{1.2}$$

where $\left(\frac{\cdot}{p}\right)$ denotes the Legendre symbol.

The supercongruence (1.2) may be regarded as a p-adic analogue of (1.1). In the past two decades, Ramanujan-type series for $1/\pi$ as well as related supercongruences and q-supercongruences have attracted many experts' attention (see, for instance, [3, 5, 6, 8–11, 14–16, 24, 26, 28, 30]).

The second result of this paper consists of the following two related supercongruences involving the numbers γ_n, which were originally conjectured by Z.-H. Sun [21, Conjecture 6.8].

Theorem 1.2 *For any prime $p \geq 5$, we have*

$$\sum_{k=0}^{p-1} (4k+3)\gamma_k \equiv 3\left(\frac{-3}{p}\right) p \pmod{p^3}. \tag{1.3}$$

Theorem 1.3 *For any prime $p \geq 5$, we have*

$$\sum_{k=0}^{p-1} \frac{2k+1}{(-9)^k}\gamma_k \equiv \left(\frac{-3}{p}\right) p \pmod{p^3}. \tag{1.4}$$

We remark that congruence properties for the Almkvist–Zudilin numbers have been widely investigated by Amdeberhan and Tauraso [2], Chan, Cooper and Sica [4], and Z.-H. Sun [21–23].

The rest of the paper is organized as follows. In Section 2, we recall some necessary combinatorial identities involving harmonic numbers and prove a preliminary congruence. The proofs of Theorems 1.1–1.3 are presented in Sections 3–5, respectively.

2 Preliminary results

Let

$$H_n = \sum_{j=1}^{n} \frac{1}{j}$$

denote the nth harmonic number. The Fermat quotient of an integer a with respect to an odd prime p is given by $q_p(a) = (a^{p-1} - 1)/p$.

In order to prove Theorems 1.1 and 1.2, we need the following two lemmas.

Lemma 2.1 *For any non-negative integer n, we have*

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} \binom{n+i}{i} = (-1)^n, \tag{2.1}$$

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} \binom{n+i}{i} H_i = 2(-1)^n H_n, \tag{2.2}$$

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} \binom{n+i}{i} H_{n+i} = 2(-1)^n H_n. \tag{2.3}$$

In fact, such identities can be discovered and proved by the symbolic summation package `Sigma` developed by Schneider [19]. One can also refer to [14] for the same approach to finding and proving identities of this type. For human proofs of (2.1)–(2.3), one refers to [18].

Lemma 2.2 *For any prime $p \geq 5$, we have*

$$\sum_{k=0}^{p-1} \frac{(3k)!}{3^{3k} k!^3} (H_{3k} - H_k) \equiv \left(\frac{-3}{p}\right) q_p(3) \pmod{p}. \tag{2.4}$$

Proof. Note that

$$\sum_{k=0}^{p-1} \frac{(3k)!}{3^{3k} k!^3} (3H_{3k} - H_k) = \sum_{k=0}^{p-1} \frac{(1/3)_k (2/3)_k}{(1)_k^2} \sum_{j=0}^{k-1} \left(\frac{1}{1/3 + j} + \frac{1}{2/3 + j} \right). \tag{2.5}$$

Recall the following identity due to Tauraso [27, Theorem 1]:

$$\frac{(1/3)_k (2/3)_k}{(1)_k^2} \sum_{j=0}^{k-1} \left(\frac{1}{1/3 + j} + \frac{1}{2/3 + j} \right) = \sum_{j=0}^{k-1} \frac{(1/3)_j (2/3)_j}{(1)_j^2} \cdot \frac{1}{k - j}. \tag{2.6}$$

Substituting (2.6) into (2.5) and exchanging the summation order gives

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{(3k)!}{3^{3k}k!^3}(3H_{3k}-H_k)&=\sum_{k=0}^{p-1}\sum_{j=0}^{k-1}\frac{(1/3)_j(2/3)_j}{(1)_j^2}\cdot\frac{1}{k-j}\\&=\sum_{j=0}^{p-2}\frac{(1/3)_j(2/3)_j}{(1)_j^2}H_{p-1-j}\\&\equiv\sum_{j=0}^{p-2}\frac{(3j)!}{3^{3j}j!^3}H_j \pmod p,\end{aligned}\tag{2.7}$$

where we have utilized the fact that $H_{p-1-j}\equiv H_j \pmod p$. By (2.7), we obtain

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{(3k)!}{3^{3k}k!^3}(H_{3k}-H_k)&\equiv\frac{1}{3}\left(\sum_{k=0}^{p-2}\frac{(3k)!}{3^{3k}k!^3}H_k-2\sum_{k=0}^{p-1}\frac{(3k)!}{3^{3k}k!^3}H_k\right)\\&\equiv-\frac{1}{3}\sum_{k=0}^{\lfloor p/3\rfloor}\frac{(3k)!}{3^{3k}k!^3}H_k \pmod p,\end{aligned}\tag{2.8}$$

because $(3k)!\equiv 0 \pmod p$ for $k>\lfloor p/3\rfloor$.

Let $m=\lfloor p/3\rfloor$. From [2, Lemma 2.3], we see that for $0\le k\le m$,

$$\frac{(3k)!}{3^{3k}k!^3}\equiv(-1)^k\binom{m}{k}\binom{m+k}{k} \pmod p.\tag{2.9}$$

It follows from (2.2), (2.8) and (2.9) that

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{(3k)!}{3^{3k}k!^3}(H_{3k}-H_k)&\equiv-\frac{1}{3}\sum_{k=0}^{m}(-1)^k\binom{m}{k}\binom{m+k}{k}H_k \pmod p\\&=-\frac{2(-1)^m}{3}H_m.\end{aligned}$$

Finally, noting

$$(-1)^{\lfloor p/3\rfloor}=\left(\frac{-3}{p}\right),\tag{2.10}$$

and the following congruence [13, page 359]:

$$H_{\lfloor p/3\rfloor}\equiv-\frac{3}{2}q_p(3) \pmod{p^2},\tag{2.11}$$

we complete the proof of (2.4). □

3 Proof of Theorem 1.1

We begin with the transformation formula due to Chan and Zudilin [7, Corollary 4.3]:

$$\gamma_n=\sum_{i=0}^{n}\binom{2i}{i}^2\binom{4i}{2i}\binom{n+3i}{4i}(-3)^{3(n-i)}. \tag{3.1}$$

Using (3.1) and exchanging the summation order, we obtain

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{4k+1}{81^k}\gamma_k&=\sum_{k=0}^{p-1}\frac{4k+1}{81^k}\sum_{i=0}^{k}\binom{2i}{i}^2\binom{4i}{2i}\binom{k+3i}{4i}(-3)^{3(k-i)}\\&=\sum_{i=0}^{p-1}\frac{1}{(-3)^{3i}}\binom{2i}{i}^2\binom{4i}{2i}\sum_{k=i}^{p-1}\frac{4k+1}{(-3)^k}\binom{k+3i}{4i}.\end{aligned} \tag{3.2}$$

Note that

$$\sum_{k=i}^{n-1}\frac{4k+1}{(-3)^k}\binom{k+3i}{4i}=(n-i)\binom{n+3i}{4i}(-3)^{1-n}, \tag{3.3}$$

which can be easily proved by induction on n. Combining (3.2) and (3.3) gives

$$\sum_{k=0}^{p-1}\frac{4k+1}{81^k}\gamma_k=3^{1-p}\sum_{i=0}^{p-1}\frac{p-i}{(-3)^{3i}}\binom{2i}{i}^2\binom{4i}{2i}\binom{p+3i}{4i}. \tag{3.4}$$

Furthermore, we have

$$\begin{aligned}&(-1)^i(p-i)\binom{2i}{i}^2\binom{4i}{2i}\binom{p+3i}{4i}\\&=\frac{(-1)^ip(p+3i)\cdots(p+1)(p-1)\cdots(p-i)}{i!^4}\\&\equiv\frac{p(3i)!}{i!^3}\left(1+p\left(H_{3i}-H_i\right)\right)\pmod{p^3}.\end{aligned}$$

Thus,

$$\sum_{k=0}^{p-1}\frac{4k+1}{81^k}\gamma_k\equiv 3^{1-p}p\sum_{i=0}^{p-1}\frac{(3i)!}{3^{3i}i!^3}\left(1+p\left(H_{3i}-H_i\right)\right)\pmod{p^3}.$$

Finally, noting (2.4) and Mortenson's supercongruence [17, (1.2)]:

$$\sum_{i=0}^{p-1}\frac{(3i)!}{3^{3i}i!^3}\equiv\left(\frac{-3}{p}\right)\pmod{p^2},$$

we arrive at

$$\sum_{k=0}^{p-1}\frac{4k+1}{81^k}\gamma_k \equiv p\left(\frac{-3}{p}\right)\left(3^{1-p}+3^{1-p}pq_p(3)\right) \pmod{p^3}$$

$$= p\left(\frac{-3}{p}\right),$$

as desired.

4 Proof of Theorem 1.2

Recall the following transformation formula [21, (5.1)]:

$$\gamma_n = \sum_{i=0}^{\lfloor n/3\rfloor}\binom{2i}{i}^2\binom{4i}{2i}\binom{n+i}{4i}(-3)^{n-3i}. \tag{4.1}$$

By (4.1), we have

$$\sum_{k=0}^{p-1}(4k+3)\gamma_k = \sum_{k=0}^{p-1}(4k+3)\sum_{i=0}^{\lfloor k/3\rfloor}\binom{2i}{i}^2\binom{4i}{2i}\binom{k+i}{4i}(-3)^{k-3i}$$

$$= \sum_{i=0}^{p-1}\frac{1}{(-3)^{3i}}\binom{2i}{i}^2\binom{4i}{2i}\sum_{k=i}^{p-1}(-3)^k(4k+3)\binom{k+i}{4i}. \tag{4.2}$$

It can be easily proved by induction on n that

$$\sum_{k=i}^{n-1}(-3)^k(4k+3)\binom{k+i}{4i} = 3(n-3i)\binom{n+i}{4i}(-3)^{n-1}. \tag{4.3}$$

It follows from (4.2) and (4.3) that

$$\sum_{k=0}^{p-1}(4k+3)\gamma_k = 3^p\sum_{i=0}^{\lfloor p/3\rfloor}\frac{p-3i}{(-3)^{3i}}\binom{2i}{i}^2\binom{4i}{2i}\binom{p+i}{4i}.$$

Note that

$$(-1)^i(p-3i)\binom{2i}{i}^2\binom{4i}{2i}\binom{p+i}{4i}$$

$$= \frac{(-1)^ip(p+i)\cdots(p+1)(p-1)\cdots(p-3i)}{i!^4}$$

$$\equiv \frac{p(3i)!}{i!^3}\left(1-p\left(H_{3i}-H_i\right)\right) \pmod{p^3}.$$

Thus,

$$\sum_{k=0}^{p-1}(4k+3)\gamma_k \equiv 3^p p \sum_{i=0}^{\lfloor p/3 \rfloor} \frac{(3i)!}{3^{3i} i!^3}\left(1 - p\left(H_{3i} - H_i\right)\right) \pmod{p^3}. \tag{4.4}$$

Let $m = \lfloor p/3 \rfloor$. Since

$$\frac{(3i)!}{3^{3i} i!^3} = (-1)^i \binom{-1/3}{i}\binom{-1/3+i}{i} = (-1)^i \binom{-2/3}{i}\binom{-2/3+i}{i},$$

we have

$$\begin{aligned}\frac{(3i)!}{3^{3i} i!^3} &= (-1)^i \binom{m-p/3}{i}\binom{m-p/3+i}{i} \\ &= \frac{(-1)^i (m+i-p/3)\cdots(m-i+1-p/3)}{i!^2} \\ &\equiv (-1)^i \binom{m}{i}\binom{m+i}{i}\left(1 - \frac{p}{3}\left(H_{m+i} - H_{m-i}\right)\right) \pmod{p^2}. \end{aligned} \tag{4.5}$$

Substituting (4.5) into the right-hand side of (4.4) gives

$$\begin{aligned} &\sum_{k=0}^{p-1}(4k+3)\gamma_k \\ &\equiv 3^p p \sum_{i=0}^{m} (-1)^i \binom{m}{i}\binom{m+i}{i} \\ &\times \left(1 - \frac{p}{3}\left(H_{m+i} - H_{m-i} + 3H_{3i} - 3H_i\right)\right) \pmod{p^3}. \end{aligned} \tag{4.6}$$

Furthermore, we have

$$\begin{aligned} H_{3i} &= \frac{1}{3}\left(H_i + \sum_{j=1}^{i}\frac{1}{j-1/3} + \sum_{j=1}^{i}\frac{1}{j-2/3}\right) \\ &\equiv \frac{1}{3}\left(H_i + \sum_{j=1}^{i}\frac{1}{m+j} - \sum_{j=1}^{i}\frac{1}{m+1-j}\right) \pmod{p} \\ &= \frac{1}{3}\left(H_i + H_{m+i} + H_{m-i} - 2H_m\right). \end{aligned} \tag{4.7}$$

It follows from (2.1)–(2.3), (4.6) and (4.7) that

$$\begin{aligned}
&\sum_{k=0}^{p-1}(4k+3)\gamma_k \\
&\equiv 3^p p\sum_{i=0}^{m}(-1)^i\binom{m}{i}\binom{m+i}{i}\left(1-\frac{2p}{3}\left(H_{m+i}-H_m-H_i\right)\right) \pmod{p^3} \\
&= 3^p p(-1)^m\left(1+\frac{2p}{3}H_m\right).
\end{aligned}$$

Finally, using (2.10) and (2.11), we obtain

$$\begin{aligned}
\sum_{k=0}^{p-1}(4k+3)\gamma_k &\equiv 3p\left(\frac{-3}{p}\right)3^{p-1}\left(2-3^{p-1}\right) \\
&= 3p\left(\frac{-3}{p}\right)\left(1-\left(3^{p-1}-1\right)^2\right) \\
&\equiv 3p\left(\frac{-3}{p}\right) \pmod{p^3},
\end{aligned}$$

where we have used the Fermat's little theorem in the last step.

5 Proof of Theorem 1.3

Recall the following transformation formula [21, Lemma 4.1]:

$$\gamma_n=\sum_{i=0}^{n}(-9)^{n-i}\binom{2i}{i}\binom{n+i}{2i}\sum_{j=0}^{i}\binom{i}{j}^2\binom{2j}{j}. \tag{5.1}$$

Let

$$g_n=\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}.$$

By (5.1), we have

$$\begin{aligned}
\sum_{k=0}^{p-1}\frac{2k+1}{(-9)^k}\gamma_k &= \sum_{k=0}^{p-1}\frac{2k+1}{(-9)^k}\sum_{i=0}^{k}(-9)^{k-i}\binom{2i}{i}\binom{k+i}{2i}g_i \\
&= \sum_{i=0}^{p-1}\frac{g_i}{(-9)^i}\sum_{k=i}^{p-1}(2k+1)\binom{k+i}{2i}\binom{2i}{i}.
\end{aligned} \tag{5.2}$$

Note that

$$\sum_{k=i}^{n-1}(2k+1)\binom{k+i}{2i}\binom{2i}{i}=\frac{n^2}{i+1}\binom{n-1}{i}\binom{n+i}{i}, \tag{5.3}$$

which can be proved by induction on n. It follows from (5.2) and (5.3) that

$$\sum_{k=0}^{p-1}\frac{2k+1}{(-9)^k}\gamma_k=p^2\sum_{i=0}^{p-1}\frac{g_i}{(-9)^i(i+1)}\binom{p-1}{i}\binom{p+i}{i}.$$

Since

$$\binom{p-1}{i}\binom{p+i}{i}\equiv(-1)^i \pmod{p^2},$$

we have

$$\sum_{k=0}^{p-1}\frac{2k+1}{(-9)^k}\gamma_k\equiv p^2\sum_{i=0}^{p-1}\frac{g_i}{9^i(i+1)} \pmod{p^3}. \tag{5.4}$$

From [12, Lemma 2.7], we see that for $0\le i\le p-1$,

$$\frac{g_i}{9^i}\equiv\left(\frac{-3}{p}\right)g_{p-1-i} \pmod{p},$$

and so

$$\begin{aligned}\sum_{i=0}^{p-2}\frac{g_i}{9^i(i+1)}&\equiv\left(\frac{-3}{p}\right)\sum_{i=0}^{p-2}\frac{g_{p-1-i}}{i+1}\\&=\left(\frac{-3}{p}\right)\sum_{i=1}^{p-1}\frac{g_i}{p-i}\\&\equiv-\left(\frac{-3}{p}\right)\sum_{i=1}^{p-1}\frac{g_i}{i} \pmod{p}.\end{aligned}$$

Using the congruence [25, (1.8)]:

$$\sum_{i=1}^{p-1}\frac{g_i}{i}\equiv 0 \pmod{p},$$

we obtain

$$\sum_{i=0}^{p-2}\frac{g_i}{9^i(i+1)}\equiv 0 \pmod{p}. \tag{5.5}$$

Furthermore, combining (5.4) and (5.5) gives

$$\sum_{k=0}^{p-1} \frac{2k+1}{(-9)^k} \gamma_k \equiv \frac{p g_{p-1}}{9^{p-1}} \pmod{p^3}.$$

By [25, Lemma 3.2], we have

$$g_{p-1} \equiv \left(\frac{-3}{p}\right) \left(2 \cdot 3^{p-1} - 1\right) \pmod{p^2},$$

and so

$$\begin{aligned} \sum_{k=0}^{p-1} \frac{2k+1}{(-9)^k} \gamma_k &\equiv p \left(\frac{-3}{p}\right) \left(1 - \frac{(3^{p-1}-1)^2}{9^{p-1}}\right) \\ &\equiv p \left(\frac{-3}{p}\right) \pmod{p^3}, \end{aligned}$$

where we have utilized the Fermat's little theorem.
Remark. Z.-H. Sun [21, Conjecture 6.8] also conjectured a companion supercongruence of (1.4):

$$\sum_{k=0}^{p-1} \frac{2k+1}{9^k} \gamma_k \equiv \left(\frac{-3}{p}\right) p \pmod{p^3}. \tag{5.6}$$

In a similar way, by using (5.1) and the following identity:

$$\sum_{k=i}^{n-1} (-1)^k (2k+1) \binom{k+i}{2i} \binom{2i}{i} = (-1)^{n-1} n \binom{n-1}{i} \binom{n+i}{i},$$

we can show that

$$\sum_{k=0}^{p-1} \frac{2k+1}{9^k} \gamma_k \equiv p \sum_{i=0}^{p-1} \frac{g_i}{9^i} \pmod{p^3}.$$

Thus, the conjectural supercongruence (5.6) is equivalent to

$$\sum_{i=0}^{p-1} \frac{g_i}{9^i} \equiv \left(\frac{-3}{p}\right) \pmod{p^2},$$

which was originally conjectured by Z.-W. Sun [25, Remark 1.1].

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