

On Colored Factorizations

Jacob Sprittulla

Germany

sprittulla@alice-dsl.de

Abstract

We study the number of factorizations of a positive integer, where the parts of the factorization are of l different colors (or kinds). Recursive or explicit formulas are derived for the case of unordered and ordered, distinct and non-distinct factorizations with at most and exactly l colors.

1 Introduction

For $l \geq 1$ and $n \geq 2$ we denote the following factorization counting functions, where the parts of the factorization are always ≥ 2 :

- $A_l(n)$ denotes the number of *unordered* factorizations into parts of *at most* l different colors (or into parts of at most l different kinds),
- $B_l(n)$ denotes the number of *unordered* factorizations into *distinct* parts of *at most* l colors (more precisely, parts are considered to be distinct, if they have the same color but a different value or if they have a different color),
- $\tilde{A}_l(n)$ denotes the number of *ordered* factorizations into parts of *at most* l colors.
- $\tilde{B}_l(n)$ denotes the number of *ordered* factorizations into *distinct* parts of *at most* l colors.

We will denote the corresponding factorization counting functions into parts with *exactly* l colors by lowercases. For example $a_l(n)$ denotes the number of *unordered* factorizations into parts of *exactly* l colors, and analogously for $b_l(n)$, $\tilde{a}_l(n)$ and $\tilde{b}_l(n)$.

We will sometimes also briefly refer to $A_l(n)$ as *the number of l -colored factorizations* and analogously for the other factorization counting functions.

We give an example. Let $n = 12 = (2^2)(3)$ and $l = 2$. We list all factorizations and the values of the factorization counting functions below.

$$\begin{aligned}
A_2(12) &= 16 = \#\{(12), (\bar{1}\bar{2}), (6, 2), (6, \bar{2}), (\bar{6}, 2), (\bar{6}, \bar{2}), (4, 3), (4, \bar{3}), (\bar{4}, 3), (\bar{4}, \bar{3}), \\
&\quad (3, 2, 2), (\bar{3}, \bar{2}, \bar{2}), (3, 2, \bar{2}), (\bar{3}, 2, \bar{2}), (3, \bar{2}, \bar{2}), (\bar{3}, 2, 2)\}, \\
B_2(12) &= 12 = \#\{(12), (\bar{1}\bar{2}), (6, 2), (6, \bar{2}), (\bar{6}, 2), (\bar{6}, \bar{2}), (4, 3), (4, \bar{3}), (\bar{4}, 3), (\bar{4}, \bar{3}), \\
&\quad (3, 2, \bar{2}), (\bar{3}, 2, \bar{2})\}, \\
a_2(12) &= 8 = \#\{(6, \bar{2}), (\bar{6}, 2), (4, \bar{3}), (\bar{4}, 3), (3, 2, \bar{2}), (\bar{3}, 2, \bar{2}), (3, \bar{2}, \bar{2}), (\bar{3}, 2, 2)\}, \\
b_2(12) &= 6 = \#\{(6, \bar{2}), (\bar{6}, 2), (4, \bar{3}), (\bar{4}, 3), (3, 2, \bar{2}), (\bar{3}, 2, \bar{2})\}, \\
\tilde{a}_2(12) &= 26 = \#\{(6, \bar{2}), (\bar{2}, 6), (\bar{6}, 2), (2, \bar{6}), (4, \bar{3}), (\bar{3}, 4), (\bar{4}, 3), (3, \bar{4}), \\
&\quad (3, 2, \bar{2}), (2, 3, \bar{2}), (2, \bar{2}, 3), (3, \bar{2}, 2), (\bar{2}, 3, 2), (\bar{2}, 2, 3), \\
&\quad (\bar{3}, 2, \bar{2}), (2, \bar{3}, \bar{2}), (2, \bar{2}, \bar{3}), (\bar{3}, \bar{2}, 2), (\bar{2}, \bar{3}, 2), (\bar{2}, 2, \bar{3}), \\
&\quad (\bar{3}, 2, 2), (2, \bar{3}, 2), (2, 2, \bar{3}), (\bar{3}, \bar{2}, 2), (\bar{3}, 2, \bar{2}), (2, \bar{3}, \bar{3})\}, \\
\tilde{b}_2(12) &= 20 = \#\{(6, \bar{2}), (\bar{2}, 6), (\bar{6}, 2), (2, \bar{6}), (4, \bar{3}), (\bar{3}, 4), (\bar{4}, 3), (3, \bar{4}), \\
&\quad (3, 2, \bar{2}), (2, 3, \bar{2}), (2, \bar{2}, 3), (3, \bar{2}, 2), (\bar{2}, 3, 2), (\bar{2}, 2, 3), \\
&\quad (\bar{3}, 2, \bar{2}), (2, \bar{3}, \bar{2}), (2, \bar{2}, \bar{3}), (\bar{3}, \bar{2}, 2), (\bar{2}, \bar{3}, 2), (\bar{2}, 2, \bar{3})\}.
\end{aligned}$$

Here $\#$ denotes the number of elements of a set and (d, \bar{d}) denote the two colors of an integer d . For completeness, we also note that $\tilde{A}_2(12) = 42$ and $\tilde{B}_2(12) = 30$. For small values of l , some of these sequences can be found in the *On-Line Encyclopedia of Integer Sequences* (OEIS) [7], for example [A301830](#) $((A_2(n))_{n \geq 1})$ and [A328706](#) $((B_2(n))_{n \geq 1})$.

For $l = 1$, we get the standard (uncolored) factorization counting functions, which we will denote by $f(n) := A_1(n)$, $g(n) := B_1(n)$, $\tilde{f}(n) := \tilde{A}_1(n)$ and $\tilde{g}(n) := \tilde{B}_1(n)$.

It is easy to see that all the above functions are *prime independent*, meaning that their value is completely determined by the prime signature of n . For example, we have $A_2(12) = A_2(75) = 16$, since $12 = (2^2)(3)$ and $75 = (3)(5^2)$ share the same prime signature $(2, 1)$.

We will denote primes by π and π_i ($i = 1, 2, \dots$). Evaluated at prime powers π^n , we get the corresponding partition or composition functions. For example $A_l(\pi^n)$ is the number of partitions of n into parts of at most l kinds, $B_l(\pi^n)$ the number of compositions of n into parts of at most l kinds. For small values of l , these sequences can be found in the OEIS, for example [A000712](#) $((A_2(\pi^n))_{n \geq 1})$, [A022567](#) $((B_2(\pi^n))_{n \geq 1})$, [A025192](#) $((\tilde{A}_2(\pi^n))_{n \geq 1})$ and [A032005](#) $((\tilde{B}_2(\pi^n))_{n \geq 1})$.

The number of l -colored factorizations doesn't seem to have attracted much attention in the literature yet. The only reference we found is the paper of Subbarao [9], where the Dirichlet generating function (dgf) of $A_l(n)$ is given. The aim of this paper is to study the recursive structure and the interdependence of these functions. We will give recursive or explicit formulas and the dgf's for all variants mentioned. Further, we derive an average order for the number of l -colored ordered factorizations (Corollary 10).

2 Unordered colored factorizations

In the case of l -colored unordered factorizations, the recursive structure of the factorization counting functions can be deduced with standard methods from their dgf's. Subbarao [9, Equation 2.4] noted that the dgf of $A_l(n)$ is given by the l -th power of the dgf of $f(n)$. We will motivate and generalize this result in the next lemma.

For an arithmetic function $\phi(n)$, we denote by $\phi^{*l}(n)$ the l -fold *Dirichlet convolution* of ϕ with itself, i.e. $\phi^{*l} = \phi^{*(l-1)} * \phi$, for $l \geq 2$ and $\phi^{*1} = \phi$, where $*$ denotes Dirichlet convolution. We will also say that ϕ^{*l} is the l -th *Dirichlet power* of ϕ .

Further, we denote by $f_{k,l}(n)$ ($g_{k,l}(n)$) the number of (distinct) unordered l -colored factorizations with exactly k parts. These auxiliary functions are needed to deduce formulae for $\tilde{B}_l(n)$ (Theorem 11 below) and $\tilde{b}_l(n)$. For $l = 1$, $f_k(n) := f_{k,1}(n)$ ($g_k(n) := g_{k,1}(n)$) denotes the number of unordered (distinct) factorizations of n with exactly k parts. The dgfs of these sequences are given by

$$1 + \sum_{n=2}^{\infty} \sum_{k=1}^{\infty} f_k(n) n^{-s} z^k = \prod_{n=2}^{\infty} (1 - zn^{-s})^{-1} \quad (2.1)$$

$$1 + \sum_{n=2}^{\infty} \sum_{k=1}^{\infty} g_k(n) n^{-s} z^k = \prod_{n=2}^{\infty} (1 + zn^{-s}), \quad (2.2)$$

see Hensley [3, Equation 1.4] and Subbarao [9, Equation 2.4]. We denote complex numbers by s and z .

Lemma 1. *For an integer $l \geq 1$ and $\operatorname{Re}(s) > 1$, we have $A_l = f^{*l}$ and $B_l = g^{*l}$ and therefore*

$$\mathcal{A}_l(s) := 1 + \sum_{n=2}^{\infty} A_l(n) n^{-s} = \prod_{n=2}^{\infty} (1 - n^{-s})^{-l} \quad (2.3)$$

$$\mathcal{B}_l(s) := 1 + \sum_{n=2}^{\infty} B_l(n) n^{-s} = \prod_{n=2}^{\infty} (1 + n^{-s})^l. \quad (2.4)$$

Further, for $k, l \geq 1$ and $\operatorname{Re}(s), \operatorname{Re}(z) > 1$ we have

$$\mathcal{F}_l(s) := 1 + \sum_{n=2}^{\infty} \sum_{k=1}^{\infty} f_{k,l}(n) n^{-s} z^k = \prod_{n=2}^{\infty} (1 - zn^{-s})^{-l} \quad (2.5)$$

$$\mathcal{G}_l(s) := 1 + \sum_{n=2}^{\infty} \sum_{k=1}^{\infty} g_{k,l}(n) n^{-s} z^k = \prod_{n=2}^{\infty} (1 + zn^{-s})^l. \quad (2.6)$$

Proof. Let $n \geq 2$ be given. Any unordered l -colored factorization of n can be constructed of all pairs of integers n_1 and n_2 with $n_1 n_2 = n$, by considering factorizations of the first $(l-1)$ colors of n_1 and factorizations of the l -th color of n_2 ; hence $A_l(n) = \sum_{n_1 n_2 = n} A_{l-1}(n_1) f(n_2) = (A_{l-1} * f)(n)$. This shows (2.3) by induction on l ; the proof of (2.4) is similar.

Analogously, all factorizations with exactly k parts of at most l colors can be constructed of all pairs of integers $n_1 n_2 = n$ and $k_1 + k_2 = k$, where n_1 is made of k_1 parts of the first $l - 1$ colors and n_2 is made of k_2 parts of the last l -th color. Therefore

$$\begin{aligned} \sum_{n=2}^{\infty} \sum_{k=1}^{\infty} f_{k,l}(n) n^{-s} z^k &= \sum_{n=2}^{\infty} \sum_{k=1}^{\infty} \left(\sum_{n_1 n_2 = n} \sum_{k_1 + k_2 = k} f_{k_1, l-1}(n_1) f_{k_2, 1}(n_2) \right) n^{-s} z^k \\ &= \sum_{n_1, n_2=2}^{\infty} \sum_{k_1, k_2=1}^{\infty} f_{k_1, l-1}(n_1) f_{k_2, 1}(n_2) n_1^{-s} z^{k_1} n_2^{-s} z^{k_2} \\ &= \left(\sum_{n=2}^{\infty} \sum_{k=1}^{\infty} f_{k, l-1}(n) n^{-s} z^k \right) \left(\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} f_{k, 1}(n) n^{-s} z^k \right). \end{aligned}$$

It follows by induction on l that $\mathcal{F}_{k,l}(s)$ is the l -th power of the the dgf of $f_k(n)$. Together with (2.1), this shows (2.5). The proof of (2.6) is similar, using (2.2). \square

Remark 2. We can change the set admissible parts $\mathcal{S} = \{2, 3, 4, 5, \dots\}$ in Lemma 1 to any other nonempty subset of \mathcal{S} by changing the range of the products of the dgfs accordingly. For example, if we put $\mathcal{S} = \{2, 3, 5, 7, \dots\}$, the set of all primes, we get the dgfs of the number of l -colored *prime factorizations* of n . By the Euler product of the Riemann zeta function, we then get that this dgf equals $\zeta(s)^l$. But $\zeta(s)^l$ is also the dgf of the number of ordered factorizations into l parts ≥ 1 (the l -th divisor function, denoted by $d_l(n)$). Therefore, we can conclude that, for every $n \geq 2$, the number of l -colored unordered prime-factorizations equals the number of ordered factorizations with l parts ≥ 1 , since both sequences share the same dgf $\zeta(s)^l$. A similar argument shows that the number of ordered factorizations into l parts ≥ 2 (denoted by $f_l(n)$) equals the number of exactly l -colored unordered prime factorizations, since both sequences share the same dgf $(\zeta(s) - 1)^l$ (see Remark 13 below). For example, for $n = 12$ and $l = 2$, we have

$$\begin{aligned} d_2(12) &= 6 = \#\{(12, 1), (1, 12), (6, 2), (2, 6), (4, 3), (3, 4)\} \\ &= \#\{(3, 2, 2), (\bar{3}, 2, 2), (3, \bar{2}, 2), (\bar{3}, \bar{2}, 2), (3, \bar{2}, \bar{2}), (\bar{3}, \bar{2}, \bar{2})\}, \\ f_2(12) &= 4 = \#\{(6, 2), (2, 6), (4, 3), (3, 4)\} \\ &= \#\{(\bar{3}, 2, 2), (3, \bar{2}, 2), (\bar{3}, \bar{2}, 2), (3, \bar{2}, \bar{2})\}. \end{aligned}$$

Remark 3. An effective algorithm to produce a list of all unordered factorizations for a given $n \geq 2$ can be found in Knopfmacher and Mays [6]. As explained in the proof of Lemma 1, Equation (2.3), all l -colored factorizations of an integer n can be constructed from $(l - 1)$ -colored factorizations of the divisors d of n and the factorizations of n/d . This can be used to produce recursively a list of all unordered l -colored factorizations based on a list of all unordered factorizations. This algorithm can be extended to produce lists of factorizations for the other factorization counting functions analyzed in this paper.

We can now derive a recursive equations for $A_l(n)$, $B_l(n)$, $f_{k,l}(n)$ and $g_{k,l}(n)$, by making use of the dgf's.

Theorem 4. For $n \geq 2$, $k \geq 1$ and an integer $l \neq 0$, we have

$$A_l(n) \log n = l \sum_{\substack{d^i | n \\ d \geq 2}} A_l(n/d^i) \log d \quad (2.7)$$

$$B_l(n) \log n = l \sum_{\substack{d^i | n \\ d \geq 2}} (-1)^{i+1} B_l(n/d^i) \log d \quad (2.8)$$

$$f_{k,l}(n) \log n = l \sum_{\substack{d^i | n \\ d \geq 2, k \geq i}} f_{k-i,l}(n/d^i) \log d \quad (2.9)$$

$$g_{k,l}(n) \log n = l \sum_{\substack{d^i | n \\ d \geq 2, k \geq i}} (-1)^{i+1} g_{k-i,l}(n/d^i) \log d, \quad (2.10)$$

with boundary conditions $A_l(1) = B_l(1) = 1$ and

$$f_{k,l}(1) = g_{k,l}(1) = \begin{cases} 1, & \text{if } k = 0; \\ 0, & \text{if } k \geq 1. \end{cases}$$

Proof. Fix an integer $l \neq 0$. We proof (2.8) by using logarithmic differentiation of the dgf and the identity $\log(1+x) = \sum_{i=1}^{\infty} \frac{1}{i} (-1)^{i+1} x^i$ (for $|x| < 1$). For positive integers i and n , we denote by $\lambda_i(n)$ the indicator function of the i -th power, i.e.

$$\lambda_i(n) := \begin{cases} 1, & \text{if } n = d^i, \text{ for some integer } d; \\ 0, & \text{else.} \end{cases}$$

We have

$$\begin{aligned} \frac{\partial}{\partial s} \log \mathcal{B}_l(s) &= \frac{\partial}{\partial s} \left(l \sum_{n=2}^{\infty} \log(1+n^{-s}) \right) \\ &= \frac{\partial}{\partial s} \left(l \sum_{n=2}^{\infty} \sum_{i=1}^{\infty} \frac{1}{i} (-1)^{i+1} n^{-is} \right) \\ &= -l \sum_{n=2}^{\infty} \sum_{i=1}^{\infty} (-1)^{i+1} n^{-is} \log n \\ &= -l \sum_{n=2}^{\infty} \sum_{i=1}^{\infty} (-1)^{i+1} \lambda_i(n) n^{-s} \log(n^{1/i}). \end{aligned}$$

By the definition of $\mathcal{B}_l(s)$, we also have

$$\begin{aligned} \frac{\partial}{\partial s} \log \mathcal{B}_l(s) &= \frac{\frac{\partial}{\partial s} \mathcal{B}_l(s)}{\mathcal{B}_l(s)} \\ &= \frac{-1}{\mathcal{B}_l(s)} \sum_{n=2}^{\infty} B_l(n) n^{-s} \log n. \end{aligned}$$

Equating both equations for the logarithmic derivative, we get

$$\begin{aligned}
\sum_{n=2}^{\infty} B_l(n)n^{-s} \log n &= l \left(\sum_{n=2}^{\infty} B_l(n)n^{-s} \right) \left(\sum_{n=2}^{\infty} \sum_{i=1}^{\infty} (-1)^{i+1} \lambda_i(n)n^{-s} \log(n^{1/i}) \right) \\
&= l \sum_{n=2}^{\infty} n^{-s} \sum_{\substack{d|n \\ d \geq 2}} (-1)^{i+1} B_l(n/d) \lambda_i(d) \log(d^{1/i}) \\
&= l \sum_{n=2}^{\infty} n^{-s} \sum_{\substack{d^i | n \\ d \geq 2}} (-1)^{i+1} B_l(n/d^i) \log d.
\end{aligned}$$

Extracting coefficients on both sides of this equation, we get (2.8). Analogously, by using the identity $\log(1-x) = -\sum_{i=1}^{\infty} \frac{1}{i} x^i$ (for $|x| < 1$), we get (2.7).

Next, we prove (2.10). As in the proof of (2.8), we get

$$\begin{aligned}
\frac{\partial}{\partial s} \log \mathcal{G}_l(s) &= \frac{\partial}{\partial s} \left(l \sum_{n=2}^{\infty} \log(1 + zn^{-s}) \right) \\
&= -l \sum_{n=2}^{\infty} \sum_{i=1}^{\infty} (-1)^{i+1} z^i \lambda_i(n) n^{-s} \log(n^{1/i})
\end{aligned}$$

and

$$\frac{\partial}{\partial s} \log \mathcal{G}_l(s) = \frac{-1}{\mathcal{G}_l(s)} \sum_{k=1}^{\infty} z^k \sum_{n=2}^{\infty} g_{k,l}(n) n^{-s} \log n.$$

Equating these two expressions, we get (by absolute convergence)

$$\begin{aligned}
\sum_{n=2}^{\infty} \sum_{k=1}^{\infty} g_{k,l}(n) n^{-s} z^k \log n &= \left(\sum_{n=2}^{\infty} \sum_{k=1}^{\infty} g_{k,l}(n) n^{-s} z^k \right) \left(l \sum_{n=2}^{\infty} \sum_{i=1}^{\infty} (-1)^{i+1} z^i \lambda_i(n) n^{-s} \log(n^{1/i}) \right) \\
&= l \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} z^k z^i \left(\sum_{n=2}^{\infty} n^{-s} \sum_{\substack{d|n \\ d \geq 2}} (-1)^{i+1} g_{k,l}(n/d) \lambda_i(d) \log(d^{1/i}) \right) \\
&= l \sum_{k=1}^{\infty} \sum_{n=2}^{\infty} n^{-s} z^k \sum_{\substack{d|n \\ d \geq 2}} \sum_{i=1}^k (-1)^{i+1} g_{k-i,l}(n/d) \lambda_i(d) \log(d^{1/i}).
\end{aligned}$$

This shows (2.10) by extracting coefficients. The proof of (2.9) is similar. \square

Remark 5. The parameter l in Theorem 4 is not restricted to positive integers. For $l = -1$, we get recursive equations for the *generalized Möbius functions* $\mu_f(n)$ and $\mu_g(n)$ defined by

$$\mathcal{B}_{-1}(s) = 1 + \sum_{n=2}^{\infty} \mu_f(n)n^{-s} = \prod_{n=2}^{\infty} (1 + n^{-s})^{-1} \quad (2.11)$$

$$\mathcal{A}_{-1}(s) = 1 + \sum_{n=2}^{\infty} \mu_g(n)n^{-s} = \prod_{n=2}^{\infty} (1 - n^{-s}). \quad (2.12)$$

The function $\mu_f(n)$ ($\mu_g(n)$) counts the surplus of unordered (distinct) factorizations with an even number of parts over the unordered (distinct) factorizations with an odd number of parts, see Chamberland et al. [1, Chapter 5] and Subbarao [9, Section 6.3]. The term *generalized Möbius functions* is motivated by the observation that the Möbius function

$$\mu(n) = \begin{cases} (-1)^{\omega(n)}, & \text{if } n \text{ is squarefree;} \\ 0, & \text{otherwise,} \end{cases}$$

counts the surplus of the number of ordered factorizations with an even number of parts over the number of ordered factorizations with an odd number of parts, see Friedlander and Iwaniec [2, Equation 17.2]. We denote by $\omega(n)$ the number of distinct prime factors of n .

If n is square free, we can derive an explicit formula for the number of l -colored unordered (distinct) factorizations of n . We denote by $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$ the Stirling numbers of the second kind.

Theorem 6. *Let $n = \prod_{i=1}^m \pi_i$ be a product of distinct primes and $l \geq 1$. Then*

$$A_l(n) = B_l(n) = \sum_{k=1}^m l^k \left\{ \begin{smallmatrix} m \\ k \end{smallmatrix} \right\}.$$

Proof. Let an integer $m \geq 1$ be given. For an integer $1 \leq k \leq m$, the number of unordered (distinct) factorizations of n with exactly k parts is given by $\left\{ \begin{smallmatrix} m \\ k \end{smallmatrix} \right\}$. Denote the parts of such a factorization by d_1, \dots, d_k . By the structure of n , we can assume that $d_1 > \dots > d_k$. Every l -colored factorization of n with parts d_1, \dots, d_k can be constructed by assigning α_i of the parts to the i -th color ($i = 1, \dots, l$), with $\alpha_1 + \dots + \alpha_l = k$ and $\alpha_i \geq 0$. This shows that for every such partition $\alpha_1, \dots, \alpha_l$ there are $\binom{m}{\alpha_1 \dots \alpha_l}$ l -colored factorizations. By the multinomial theorem, we have

$$\sum_{\alpha_1 + \dots + \alpha_l = k} \binom{m}{\alpha_1 \dots \alpha_l} = l^k.$$

This shows that the number of l -colored unordered (distinct) factorizations with k parts is given by $l^k \left\{ \begin{smallmatrix} m \\ k \end{smallmatrix} \right\}$. The claim now follows by summing up over k . \square

Let Q_m denote the m -th primorial. For small values of l , the sequences $(A_l(Q_m))_{m \geq 1}$ can be found in the OEIS, e.g. [A001861](#) ($l = 2$) and [A027710](#) ($l = 3$).

3 Ordered colored factorizations

For l -colored ordered factorizations, a simple combinatorial argument shows that the value of $\tilde{A}_l(n)$ can be derived from the number of factorizations with exactly k different parts, denoted by $\tilde{f}_k(n)$. Recall that for $n = \prod_{j=1}^{\omega} \pi_j^{e_j}$ and $k \geq 1$, we have

$$\tilde{\mathcal{F}}_k(s) := \sum_{n=1}^{\infty} \tilde{f}_k(n) n^{-s} = (\zeta(s) - 1)^k \quad (3.1)$$

and

$$\tilde{f}_k(n) = \sum_{i=0}^{k-1} (-1)^i \binom{k}{i} \prod_{j=1}^{\omega} \binom{e_j + k - i - 1}{e_j}, \quad (3.2)$$

see Knopfmacher and Mays [5, Chapter 2.1]. We will denote by $\Omega = \Omega(n)$ the number of prime factors of n , counted with multiplicity.

Theorem 7. *For $n \geq 2$ and $l \geq 1$, we have*

$$\tilde{\mathcal{A}}_l(s) := \sum_{n=1}^{\infty} \tilde{A}_l(n) n^{-s} = \frac{1}{(l+1) - l\zeta(s)} \quad (3.3)$$

and

$$\tilde{A}_l(n) = \sum_{k=1}^{\Omega} l^k \tilde{f}_k(n). \quad (3.4)$$

Proof. Fix $n \geq 2$ and $l \geq 1$. Let $d_1 \cdots d_k$ be an ordered factorization of n . Every ordered factorization of n with k parts of l kinds has a representation as $d_{1,i_1} \cdots d_{k,i_k}$ ($i_j \in \{1, 2, \dots, l\}$ for $j = 1, \dots, k$), where the second subscript indicates the kind of the part. This shows (3.4), by summing up over k .

From (3.1), (3.4) and the formula for the geometric series we get

$$\tilde{\mathcal{A}}_l(s) = \sum_{k=1}^{\infty} l^k (\zeta(s) - 1)^k = \frac{1}{1 - l(\zeta(s) - 1)}.$$

This shows (3.3). □

Remark 8. It follows from (3.3) that the sequence $(\tilde{A}_l(n))_{n \geq 1}$ is the Dirichlet inverse of the sequence $(c_l(n))_{n \geq 1}$ with

$$c_l(n) := \begin{cases} 1, & \text{if } n = 1; \\ -l, & \text{for } n \geq 2. \end{cases}$$

Remark 9. Let Q_m denote the m -th primorial ($m \geq 1$). Then, we have $\tilde{f}_k(Q_m) = k! \left\{ \begin{smallmatrix} m \\ k \end{smallmatrix} \right\}$ and therefore

$$\tilde{A}_l(Q_m) = \tilde{B}_l(Q_m) = \sum_{k=1}^m l^k k! \left\{ \begin{smallmatrix} m \\ k \end{smallmatrix} \right\}.$$

The same holds for any other square free number composed of m distinct primes. For small values of l , the sequences $(\tilde{A}_l(Q_m))_{m \geq 1}$ can be found in the OEIS, e.g. [A004123](#) ($l = 2$) and [A032033](#) ($l = 3$).

We can now derive an average order of $\tilde{A}_l(n)$ from its dgf (3.3) as a generalization of the formula for the average order of $\tilde{f}(n)$ of Kalmár [4]. For real $x > 1$, we denote by $\zeta^{-1}(x)$ the unique real solution of the equation $\zeta(s) = x$.

Corollary 10. *For an integer $l \geq 1$ we have*

$$\frac{1}{x} \sum_{n \leq x} \tilde{A}_l(n) \sim \alpha_l x^{\beta_l - 1} \quad (x \rightarrow \infty) \quad \text{with} \quad \beta_l = \zeta^{-1}\left(\frac{l+1}{l}\right) \quad \text{and} \quad \alpha_l = -\frac{1}{l\beta_l \zeta'(\beta_l)}.$$

Proof. For $\operatorname{Re}(s) > 1$ and $l \geq 1$, we get

$$\begin{aligned} \mathcal{F}_l(s) &:= (l+1) - l\zeta(s) \\ &= (l+1) - l \sum_{i=0}^{\infty} \frac{1}{i!} \zeta^{(i)}(\beta_l) (s - \beta_l)^i \\ &= -l \sum_{i=1}^{\infty} \frac{1}{i!} \zeta^{(i)}(\beta_l) (s - \beta_l)^i \\ &= -l(s - \beta_l) \sum_{i=1}^{\infty} \frac{1}{i!} \zeta^{(i)}(\beta_l) (s - \beta_l)^{i-1}. \end{aligned}$$

This shows that $\mathcal{F}_l(s)$ has a simple zero at $s = \beta_l$. Therefore, by (3.3), the dgf $\tilde{\mathcal{A}}_l(s) = 1/\mathcal{F}_l(s)$ of $\tilde{A}_l(n)$ has a simple pole at β_l with

$$\operatorname{Res}_{s=\beta_l} \tilde{\mathcal{A}}_l(s) = \frac{1}{\mathcal{F}_l'(\beta_l)} = -\frac{1}{l\zeta'(\beta_l)}.$$

The claim now follows from the Wiener-Ikehara Theorem [10]. □

In the next theorem, we derive a formula for the number of l -colored ordered distinct factorizations.

Theorem 11. *For integers $n \geq 2$ and $l \geq 1$, we have*

$$\tilde{B}_l(n) = \sum_{k=1}^{\Omega} k! g_{k,l}(n), \tag{3.5}$$

where $g_{k,l}(n)$ is defined recursively by (2.10).

Proof. For $1 \leq k \leq \Omega$, let $d_{1,i_1} \cdots d_{k,i_k}$ be an l -colored unordered distinct factorization of n , where $i_j \in \{1, \dots, l\}$ ($j = 1, \dots, k$), are indicating the l -colors. Since the order of the parts is irrelevant, we can assume that $d_{1,i_1} > \cdots > d_{k,i_k}$. By changing the order of the parts, we can assign $k!$ ordered l -colored factorizations to each such factorization. Summing up over k gives (3.5). \square

4 The number of exactly l -colored factorizations

In the next theorem, we show that the number of factorizations of *exactly* l colors can easily be derived from the number of factorizations of *at most* l colors.

We say that the sequence $(\Phi_l)_{l \geq 0}$ is the *binomial transform* of the sequence $(\phi_l)_{l \geq 0}$, if $\Phi_l = \sum_{i=0}^l \binom{l}{i} \phi_i$ ($l \geq 0$) holds. It then follows that $\phi_l = \sum_{i=0}^l (-1)^{l-i} \binom{l}{i} \Phi_i$ ($l \geq 0$), see Sloane and Plouffe [8, Chapter 2.7]. We set $A_0(n) = a_0(n) = B_0(n) = b_0(n) = \tilde{A}_0(n) = \tilde{a}_0(n) = \tilde{B}_0(n) = \tilde{b}_0(n) = 0$ for $n \geq 1$.

Theorem 12. $A_l(n)$ is the binomial transform of $a_l(n)$, $B_l(n)$ is the binomial transform of $b_l(n)$, $\tilde{A}_l(n)$ is the binomial transform of $\tilde{a}_l(n)$ and $\tilde{B}_l(n)$ is the binomial transform of $\tilde{b}_l(n)$.

Proof. We proof the assertion for (A_l, a_l) . Let $n \geq 2$ and $l \geq 1$ be given. Among the $A_l(n)$ unordered factorizations of at most l colors, there are exactly $\binom{l}{i} a_i(n)$ unordered factorizations of exactly i of the l colors, for $i = 1, \dots, l-1$. Therefore, there are $a_l(n) = A_l(n) - \sum_{i=1}^{l-1} \binom{l}{i} a_i(n)$ unordered factorizations of exactly l colors, so that

$$A_l(n) = \sum_{i=0}^l \binom{l}{i} a_i(n).$$

The same argument applies for unordered distinct factorizations, ordered factorizations and ordered distinct factorizations of parts with l colors. \square

Remark 13. By Theorem 12, (2.3) and the binomial theorem, we have

$$\begin{aligned} \sum_{n=1}^{\infty} a_l(n) n^{-s} &= \sum_{n=1}^{\infty} n^{-s} \sum_{i=0}^l (-1)^{l-i} \binom{l}{i} A_i(n) \\ &= \sum_{i=0}^l (-1)^{l-i} \binom{l}{i} \prod_{n=2}^{\infty} (1 - n^{-s})^{-i} \\ &= \left(\prod_{n=2}^{\infty} (1 - n^{-s})^{-1} - 1 \right)^l. \end{aligned}$$

This last expression is the l -th power of the dgf of the series $(\bar{f}(n))_{n \geq 1}$, where $\bar{f}(1) = 0$ and $\bar{f}(n) = f(n)$ for $n \geq 2$. Therefore, we find that a_l is the l -th Dirichlet power of \bar{f} , i.e. $a_l = \bar{f}^{*l}$. An analogous argument gives that $b_l = \bar{g}^{*l}$, where $\bar{g}(1) = 0$ and $\bar{g}(n) = g(n)$ for $n \geq 2$.

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