# Frierson's 1907 Parameterization of Compound Magic Squares Extended to Orders $3^{l}, l=1,2,3, .$. , with Information Entropy 

P.D.Loly and I.D.Cameron Department of Physics and Astronomy, University of Manitoba, Winnipeg, Manitoba, R3T 2N2, Canada


#### Abstract

Frierson used a powerful parameterization of the pattern of the order 3 associative magic square to construct a family of six related order $3^{2}=9$ compound (or composite) magic squares, several of them ancient. Stimulated by Bellew's 1997 extension to order $27=3^{3}$, we extend these ideas to all orders $3^{l}, l=1,2,3, .$. , and in addition find simple formulae for the matrix spectra and entropic measures for all those orders. This construction is fractal and we give numerical results to order $243=3^{6}$ which show an information entropy measure converging to a constant value of about 1.168.. for the lowest entropy members.

We also briefly consider compounding of an order 4 magic square with the lowest entropy, for which we find a similar trend to constant entropy.


## 1 Introduction

Magic squares (MSs) have the same line sum for all Rows, Columns, and their two main Diagonals (RCDs), with most interest in the full cover of sequential integers $1,2,3, . . n^{2}$ with RCD sums of $S(n)=n\left(n^{2}+1\right) / 2$. From many sources, e.g. the cover of Swetz[49], and with its vertical invert $M_{3}$, to which we include the order 3 addition table, $A T_{3}$, of the same elements in which successive rows are augmented by 3 , all in matrix notation:

$$
\text { Luoshu }=\left[\begin{array}{lll}
4 & 9 & 2  \tag{1}\\
3 & 5 & 7 \\
8 & 1 & 6
\end{array}\right], M_{3}=\left[\begin{array}{lll}
8 & 1 & 6 \\
3 & 5 & 7 \\
4 & 9 & 2
\end{array}\right], A T_{3}=\left[\begin{array}{lll}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9
\end{array}\right], E_{3}=\left[\begin{array}{lll}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{array}\right] .
$$

The first and smallest Luoshu magic square is the sole 3 -by- 3 magic square (see Andrews[1][MSC1,2], Swetz[48],[49]) and dates before the

Warring States period in China 403-221 BCE, and possibly even two millenia earlier. We also added $E_{3}$, a constant order 3 matrix of all 1's that will soon prove useful.

Both Luoshu (sometimes called Lo Shu) and $M_{3}$ have RCDs of 15 , but $A T_{3}$ does not for its outer rows and columns, and so is not magic, but nevertheless affords an example of a pandiagonal[56] square in which all the continued broken diagonals have the same sum as the main diagonals. The pandiagonal property is easily seen by placing a copy of $A T_{3}$ to its right:

$$
\begin{align*}
& 123123 \\
& 456456  \tag{2}\\
& 789789
\end{align*}
$$

and noting that the parallels of the diagonals of $A T_{3}$, i.e. $\mathbf{2 + 6}+\mathbf{7}$, $3+4+8,1+\mathbf{6}+8$, and $2+4+9$ of this rectangle all have the same sum of 15 , the RCD of $M_{3}$, which is not pandiagonal. In the Appendix we discuss an order 4 magic square which is pandiagonal and exhibits some of the same trends under compounding that we are able to demonstrate with our main theme.
$M_{3}$ (and Luoshu) are unique aside from their 8 variants under rotations and reflections. These are not counted as distinct in most literature on magic squares and are consistent with most modern literature in running $1,2, . ., 3^{2}=9$.

### 1.1 Movement in $M_{3}$ (and Luoshu)

Here one notes the Knight's move sequence from 1 to 2 , followed by another from 2 to 3 , then a jump to 4 before sliding along the diagonal $4,5,6$ followed by another jump to 7 , followed by two more Knight's moves, 7 to 8 and finally 8 to 9 . The RCDs are all 15 . Swetz[49] has described the movement from the first cell to the last as steps in the 'Yubu' dance.

### 1.2 Associative (or Regular) Magic Squares (AMSs)

Luoshu and $M_{3}$ are called associative[55] (or regular) magic squares (AMS) as their antipodal pairs all sum to a common value, so that here $10=1+9=2+8=3+7=4+6$.

## 2 Frierson's Associative Compound Magic Squares [CMSs] begin at order $3^{2}=9$

The smallest and most famous CMS is taken from Frierson's chapter $5[15]$ in Andrews MSC1[1], which is also used as Andrews' Figure 96 on page 44 from his introductory chapter[1]. We label it $T_{9 A}$, where $T$
stands for tessellated and our terminology reflects the Aggregation of successive groups of nine integers in a tiled fashion using the pattern of $M_{3}$, augmented by successive increments of nine times $E_{3}$ in the same pattern, so it is clear that it can be compacted to a 3 -by- 3 "compact matrix" using multiples of $9 E_{3}$ as:

$$
\begin{equation*}
T_{9 A}= \tag{3}
\end{equation*}
$$

which is one of six related CMS's in three pairs found by Frierson, which pairs exhibit a different information entropy - see later discussion under "Spectral Measures".

It is also an associative magic square, with RCDs of 369 , since all RCDs sum the linesums of: $3 M_{3}+108 E_{3}=369$. This associative property is preserved in all the larger Frierson Compound Magic Squares (CMSs) studied here that follow for orders of the powers of 3, i.e. $9,27,81, \ldots$ of which Frierson found 6 at order $3 * 3=9$, and we find 90 at order $3 * 9=27$, then 2520 at order $81, \ldots$

## $2.1 T_{9 A}$ in its explicit 9-by-9 form is quite ancient

$T_{9 A}=$| 71 | 64 | 69 | $\mathbf{8}$ | $\mathbf{1}$ | $\mathbf{6}$ | 53 | 46 | 51 |
| :--- | :--- | :--- | ---: | :--- | :--- | :--- | :--- | :--- |
| 66 | 68 | 70 | $\mathbf{3}$ | $\mathbf{5}$ | $\mathbf{7}$ | 48 | 50 | 52 |
| 67 | 72 | 65 | $\mathbf{4}$ | $\mathbf{9}$ | $\mathbf{2}$ | 49 | 54 | 47 |
| 26 | 19 | 24 | 44 | 37 | 42 | 62 | 55 | 60 |
| 21 | 23 | 25 | 39 | 41 | 43 | 57 | 59 | 61 |
| 22 | 27 | 20 | 40 | 45 | 38 | 58 | 63 | 56 |
| 35 | 28 | 33 | 80 | 73 | 78 | 17 | 10 | 15 |
| 30 | 32 | 34 | 75 | 77 | 79 | 12 | 14 | 16 |
| 31 | 36 | 29 | 76 | 81 | 74 | 13 | 18 | 11 |

CMSs of multiplicative order $m n$, whose tiled subsquares of orders $m$ and $n$ are also magic squares within each subsquare, are found back to at least the 10th century CE in Persia for the smallest order 9 case $m=n=3$, see Swetz[49], and a top-bottom reflection of $T_{9 A}$ was recorded in Arabia by Abul-Wafa al-Buzani (940-997/8 CE), and is found in Descombes[12] (p.253/4) and Sesiano [43]. See also Lam[23] and Li Yen[24].

A partner CMS to $T_{9 A}$ appears in Frierson's 1907 paper, $T_{9 D}$, for which it helps to consider the order 9 addition table of the first 81 integers, $A T_{9}$, an obvious generalization of $A T_{3}$ with a first row of $1,2, . .9$,
since in 1960 Cammann[9] suggested that this first CMS may also have been constructed from the rows of such a table.

Judging from the lack of citations to prior work, the first type has been rediscovered, apparently independently, by many authors, particularly over the past two centuries, with the partner CMS rarely mentioned. The original method of construction may have been done by incrementing the upper middle subsquare by 9's and placing them in the same pattern as $M_{3}$. In 2002 Chan and Loly [CL] [10] realized that this construction of $T_{9 A}$ is likely why it has been rediscovered many times. The row and column line sums are clearly magic, as also are the diagonal line sums. CL also suggested that this amounted to a fractal construction.

Frierson further established a partner square, $T_{9 D}$, as well as two other pairs that we will discuss shortly for a sextet $T_{9 A, D, B, E, C, F}$, while for this order 9 several estimates indicate an astronomically large number of $7.8448(38) \times 10^{79}$ magic squares at this order for which we note Walter Trump's website[53].

We note that in 1908 Andrews[1] stated: "The writer believes that these highly ingenious combinations were first devised by Prof. Hermann Schubert[41]", whose publication dates to 1898, a sentence which was deleted in the 1917 edition[1], but we now know that they were at least some 900 years earlier! We also found that W.H.Thompson[51] constructed the vertical invert of $T_{9 A}$ in 1869. Also Pickover[36] gave this CMS in 2002, the same year as CL's[10] first CMS report - see Pickover's Chapter 2 (page $81=9^{*} 9$ !) but without attribution to Frierson, who is only mentioned later on his pages 222,3 for an order 8 MS! Pickover also used the section title "Composite (or Compound) Squares", and on the same page says "This reminds me of a fractal, ...".

Cammann[9] pointed out that the sums of the subsquares in $T_{9 A}$, themselves magic squares, also form a magic square, and staring us in the face are in fact many other CMSs due to rotation and reflection of each 3 -by -3 subsquare.

### 2.2 Frierson's Sextet

Frierson[15] arranged his six CMSs in vertical pairs on a single diagram (his page 134):

| $T_{9 A}$ | $T_{9 B}$ | $T_{9 C}$ |
| :--- | :--- | :--- |
| $T_{9 D}$ | $T_{9 E}$ | $T_{9 F}$ |

Table 1 - Schematic of Frierson's display of the sextet of $F_{9}$ 's.
This style will be useful later for order 27 CMSs . It is easy to see how the top centre $M_{3}$ subsquare of (4) incremented by 9's and a Knight's
move to the lower right subsquare, etc. in the same pattern as $M_{3}$ itself, followed by another increment of 9 in another Knight's move to the middle of the LHS. Then down to the bottom LHS with another increment of 9 , before incremented diagonal moves up to the to RHS, followed by drop to the middle RHS, followed by two more Knight's moves to the upper RHS and finally the bottom centre.

For order $9, T_{9 A}$ above is the most obvious of CMSs given the uniqueness of a single order 3 , versus the existence of 880 distinct order 4 magic squares, and thus many more CMSs.

### 2.3 Our extensions to larger CMSs

We discuss here an infinite family of orders $n$ of the powers of three, $n=3^{l}, l=1,2, .$. , as an extension of the $1907 / 8$ study by Frierson, and a 1997 sequel by Bellew [2], who cited Andrews[1], but not Frierson explicitly!

We are able to give unprecedented insight into a remarkable family of MS's that began a full millenium ago by extending the spectral properties of magic squares treated as matrices in LAA[25], and extended by our later studies [6] [CRL] using the more powerful singular value decomposition (SVD) [6], which demonstrates clearly the SV clan structure of this family by extending the algebraic formulation begun by Frierson[15].

## 3 Frierson's parameterization of the order 3 in Andrews Magic Squares and Cubes

Our starting point is a paper in The Monist journal (editor Paul Carus) in 1907 by Lorraine Screven Frierson[15] of Shreveport[42], Louisiana, which extends a parameterization of the smallest magic square of order 3 in order to generate a sextet of related magic squares of order $3 * 3=9$, called Compound (or Composite) Magic Squares [CMS]. This article was reprinted in the classic Magic Squares and Cubes by W.S.Andrews[1][MSC1] 1908, with chapters and sections by W.S.Andrews, L.S.Frierson and others, which are essentially edited versions of papers originally published in The Monist from 1905 for most of a decade. MSC1 is a critical reference despite shortcomings in referencing even to The Monist, whose included papers share the same lack of references to earlier sources. A second edition in 1917 will be denoted MSC2, and was reprinted more recently.

For $l=1, n=3$ we use Frierson's notation [15] but omit a common constant $c$ in all cells by starting with 0 at the top centre, followed by a Knight step to the bottom right corner, placing $y$, and a second Knight step $y$ increment to the middle of the left column for $2 y$. Different
increments, $v$ and $2 v$ are made on the opposite sides by another pair of Knight moves from top centre. Then the centre cell is made the average of those to its left and right, i.e. $v+y$, so that the linesum of this middle row and column is then $3(v+y)$. The remaining cells are completed so that all the RCDs have this same linesum. This is shown in matrix notation to facilitate later spectral function operations:

$$
M(v, y)=\left[\begin{array}{ccc}
2 v+y & 0 & v+2 y  \tag{5}\\
2 y & v+y & 2 v \\
v & 2 v+2 y & y
\end{array}\right]
$$

where we will call the pair $v, y$ a "couple". It is clear that swapping $v, y$ exchanges the first and third columns, i.e. flips the square from left to right. As already noted, these are not regarded as distinct.

Frierson's parameterization of 3rd order can now be written as:

$$
\begin{equation*}
F_{3}(k, v, y)=k E_{3}+M(v, y) \tag{6}
\end{equation*}
$$

and since we use $v, y>0, k$ gives the smallest entry, which is usually chosen as either 1 here (or 0 by some authors).
[We note that Bellew[2] in 1997 used capital $V, Y$ variables instead of our lowercase $v, y$.]

We have used the constant term $k$ in place of Frierson's $x$ since $x$ has a standard use in matrix calculus which is needed later, and $c$ was used in an alternate parameterization by Lucas[28], described in the Appendix.

Then $F_{3}(1,3,1)$ is $M_{3}$, while $F_{3}(1,1,3)$ is the Luoshu. Note that swapping $v$ and $y$ swaps the left and right columns of $F_{3}$.

Then $F_{3}$ has the following properties:

- it is associative - and this property is maintained in the iterative compounding of this paper,
- rotating or flipping $F_{3}$ about the centre still describes all 8 possible variants of third order magic squares under rotation and reflection,
- the RCDs all sum to $3 k+3 v+3 y=3(k+v+y)$.

The present study extends the powerful parameterized construction of these by Frierson[15] for order $3^{2}=9$, to higher orders that are powers of 3 , and first reported by us [LC] at a 2009 conference[26] with an emphasis on counting the number of such magic squares for orders $n=$ $3^{l}, l=1,2,3, \ldots$, where $l$ will now be called "level".

### 3.1 Frierson's order 9 parameterization for the smallest (level $l=2$ ) CMSs

CMS's of order $n=3^{l}, l=2,3$.. are constructed in an iterative manner from the fundamental parameterized order 3 pattern, as done in Frierson's [15] algebraic study of the smallest CMS's of order $l=2, n=3^{2}=$ 9, which consist of six distinct CMS's.

Now we are able to give a complete account of those of the powers of 3 opened up by Frierson[15] and Bellew[2], now including their spectra.

Frierson [15] repeated the same associative pattern with two more parameters, $s, t$, by replacing $v, y$ in (6) for the couple of level $l=2$ :

$$
M(s, t)=\left[\begin{array}{ccc}
2 s+t & 0 & s+2 t  \tag{7}\\
2 t & s+t & 2 s \\
s & 2 s+2 t & t
\end{array}\right]
$$

to help in describing $n=9$ associative compound magic squares (CMS9's) whose elements are then used to provide $s, t$ increments to copies of $F_{3}(k, v, y)$ placed in the nine order $m=3$ submatrices tiled to fill a larger $n=9$ matrix, producing a general ninth order associative compound magic square, $F_{9}(c, v, y, s, t)$ which has a magic linesum of $9(k+v+y+s+t)$. We denote this process by:

$$
\begin{equation*}
F_{9}(k, v, y, s, t)=k E_{9}+M(s, t) \otimes M(v, y) \tag{8}
\end{equation*}
$$

in which $\otimes$ is suggested by the Kronecker product formulation for CMS's described by Rogers and us [39][RCL].
N.B. Bellew[2] used capital $V, Y, S, T$ in place of Frierson's lowercase variables. Another useful reference for a broader context of larger component magic squares is given by Derksen, Eggermont and van den Essen[11].

Then a 9-by-9 matrix is constructed from the elements that are the sum of the components stacked vertically in each cell of Frierson's Figure 228 , which includes a common term, here $k$, for his $x$, added to each.

### 3.2 Frierson's Order 9 Sextet

Frierson generated 6 distinct numerical $F_{9}$ CMSs by adding a second couple, $s=27, t=9$, to his first couple, $v=3, y=1$, to guarantee full cover (without gaps or overlap). The explicit algebraic form of $F_{9}$ is identical with that of Frierson, aside from a common $x$ in all his elements which we have replaced when needed by the constant $k$ given earlier.

For $n=9$ at level $l=2$ Frierson simply stated that: 'Only six forms may be made, because, excluding our $k$ [his $x$ ] whose value is fixed,
only six different couples may be made from the four remaining symbols $v, y, s, t$. '.

These 6 couples are: three for $v$ with $y, s, t$; then two for $y$ with $s, t$; and finally $s$ with $t$. Note that this is still at the algebraic level before specific parameters are used to produce natural CMSs.

Later we show that the entropy $H$ which decreases from the right column to the left, is the same for vertical pairs.

- They are associative by construction, as are the individual tiled subsquares, i.e. all antipodal pair cells sum to $2(v+y+s+t)$, which is twice the centre cell.
- The $T_{9}$ 's are $T_{9 A}=F_{9}(1,3,1,27,9)$, where the first 1 is the constant $k$, then $T_{9 D}=F_{9}(1,9,27,1,3)$, then $T_{9 B}, T_{9 E}$ and $T_{9 C}, T_{9 F}$ - see our later Table 3.
- Since $F_{9}$ is associative by construction, the sextet are also, as are all the tiled 3 -by- 3 subsquares.
- Moreover Table 1 contains all the basic ninth order compound magic squares, aside from variants due to rotations and reflections of subsquares.
- The centre cell of $F_{9}$ is the sum of the 4 variables, $k+v+y+s+t$, and is the average of antipodal pairs, while the bottom centre cell is always twice that expression, less a $k$.
- The RCD linesum of $9(k+v+y+s+t)$ summing the values $1,3,9,27$ and adding $k=1$ is $9 \times 40+9=369$.


### 3.3 Coding using Mathematica[31]

First the $v y$ pair, then the st pair, and finally their Kronecker product:

$$
v y\left[v_{-}, y_{-}\right]:=\{\{2 v+y, 0, v+2 y\},\{2 y, v+y, 2 v\},\{v, 2 v+2 y, y\}\}
$$

$$
s t\left[s_{-}, t_{-}\right]:=\{\{2 s+t, 0, s+2 t\},\{2 t, s+t, 2 s\},\{s, 2 s+2 t, t\}\}
$$

f9algebra $=$ KroneckerProduct $[s t[s, t], v y[v, y]]$
to which one adds the constant matrix $k E_{9}$, where $E_{9}$ is the order 9 matrix of all 1's.

### 3.4 Counting the six order 9 CMS's

Citing only Andrews' book, but not Frierson's article, Bellew [2] nevertheless uses Frierson's algebra before giving an argument expressed in integer values of the parameters that since $k$ is fixed (usually at 0 or 1 ), distinct values of the 2 pairs (couples) of parameters $v, y$ and $s, t$ in
which both $v$ and $y$, as well as $s$ and $t$, are interchangeable mean that there are only:

$$
\begin{equation*}
[(4 \times 3) / 2][(2 \times 1) / 2]=6 \tag{9}
\end{equation*}
$$

unique ways to assign the variables, shown later in Table 3.

## 4 Beyond Frierson's $n=9$ sextet to order 27 (level $l=3$ )

Compounding in a similar fashion to Frierson to order 27 was suggested briefly by Bellew[2] in 1997, even though such a large square is rather unwieldy. In fact we chose $p, q$ variables above after Bellew, but later noted that he used those for Frierson's $s, t$, so we have followed Frierson here at order 9 , and then we use $p, q$ for order 27 . Clearly more parameter pairs can be used for orders $81,243,729, .$. which have much larger CMSs.

Bellew[2] actually considered the counting the magic squares for two themes, the first reviewing Frierson's parameterization for order 9 CMSs and briefly suggesting its extension to order 27 , which is developed fully here, but also a second theme for pandiagonal or Nasiq MSs for orders $\geq 5$ which included an order 9 .

With this background, and including spectra not included in most earlier compounding, we can now proceed to our main theme - to give a complete account of the generalization of Frierson's scheme to the next order of $n=27$, and later we extend (generalize) this logic for all levels $l$, before using this powerful formulation to give an algebraic account of the main spectral function, specifically the singular values for entropic measures.

### 4.1 Order 27

The extension to $n=27$ follows similarly with the addition of another pair of parameters in $M(p, q)$ which has a magic linesum of $3(p+q)$. When this is compounded with $F_{9}$ it produces $F_{27}$, which is again associative, and aside from an overall constant term, describes all possible compound magic squares with tiled subsquares of orders 3 and 9 .
N.B. Our use of $p, q$ here for order 27 CMS's is not the same as Bellew's[2] use of $P, Q$ for his discussion of ninth order.

Now extend (16) to the next compound order of $n=27$ :

$$
\begin{equation*}
F_{27}(k, v, y, s, t, p, q)=k E_{27}+M(p, q) \otimes[M(s, t) \otimes M(v, y)] \tag{10}
\end{equation*}
$$

Since $F_{27}$ is rather large to display explicitly here we continue with our compact representation for order 27.

### 4.2 The lowest entropy case for order 27

For order 27 the obvious generalization of the lowest entropy order 9 pair adds a pair $p, q$ with $p=243, q=81$, which has $T_{A}=F_{9 A}$ in the top middle order 9 subsquare with versions incremented by multiples 81 of an order 9 with all its elements unity, $E_{9}$, placed in the corresponding cells in the pattern of $M_{3}$ for a compact representation of an order 27 matrix (which otherwise are a challenge to exhibit explicitly):

$$
F_{27 A}=\begin{array}{|c|c|c|}
\hline T_{9 A}+7 \times 81 E_{9} & T_{9 A} & T_{9 A}+5 \times 81 E_{9}  \tag{11}\\
\hline T_{9 A}+2 \times 81 E_{9} & T_{9 A}+4 \times 81 E_{9} & T_{9 A}+6 \times 81 E_{9} \\
\hline T_{9 A}+3 \times 81 E_{9} & T_{9 A}+8 \times 81 E_{9} & T_{9 A}+81 E_{9} \\
\hline
\end{array}
$$

where now the multiples of $9 E_{3}$ in $T_{9 A}$ are now multiples of $81 E_{9}$. Clearly $F_{27}$ 's are both 3- and 9 -partitioned.

### 4.3 Counting the 90 order 27 CMS's at level $l=3$

From the 6 parameters $v, y, s, t, p, q$ there are $6!=720$ ways of doing this, of which some are to be counted as 'basic', while others not. We interpret the logic of Frierson [15] and Bellew [2] as an extension of (9) to give $90 F_{27}$ 's:

$$
\begin{equation*}
[(6 \times 5) / 2]([(4 \times 3) / 2][(2 \times 1) / 2]=6)=90 \tag{12}
\end{equation*}
$$

Here there are six first couples, then four second couples, and finally two third couples. There are now 15 distinct 'first' couples now multiplied by 6 'second' couples, the number found in $F_{9}$. These are counted as follows: 5 for $y=1,4$ for $y=3,3$ for $y=9,2$ for $y=27,1$ for $y=81$, for a total of 15 , all multiplied by 6 from the second couples.

Having extended Frierson style parameterization for the construction of order 27 CMSs , we now turn to spectral measures that give deeper insight into their properties. To proceed further we need the SVs, $\sigma_{i}$, for orders 9 and 27 which we obtained from Mathematica[31] and Maple[30] symbolic calculations next.

## 5 Matrix Properties - Singular Values (SVs, $\sigma_{i}$ ) versus Eigenvalues (EVs, $\lambda_{i}$ ) for Magic Squares

Our first foray into the spectra of Frierson's CMSs was presented at a 2009 conference only used EVs, but all our subsequent MS,CMS studies now use the always positive and declining SVs, whose number of nonzero SVs gives the rank of the matrix. Few of the sources that we find in the literature examine matrix spectra, except notably Kirkland and

Neumann[21] in 1995, drawn to our attention by Adam Rogers c. 2005 in connection with MATLAB[32], which has a magic $[n]$ function that delivers a single magic square of odd, even and doubly-even orders. Since then we have progressively shown how SVs lead to powerful measures for comparing magic squares, first in 2007 by Loly, Cameron, Schindel and Trump [LCTS][25], then a big leap was made by us in extending Shannon information entropy measures that Newton and DeSalvo[33] used for Sudoku matrices[33] to magic square issues in 2012-13 [6] [CRL], and most recently detailed by Rogers, Cameron and Loly[39] [RCL] in 2017. See a standard text such as Horn and Johnson[19] for SV background.

### 5.1 Eigenvalues - EVs, $\lambda_{i}$

First we set the determinant of $M_{3}$ less $x$ times the column vector of three ones equal to zero:

$$
\operatorname{Det}\left[\begin{array}{lll}
8 & 1 & 6  \tag{13}\\
3 & 5 & 7 \\
4 & 9 & 2
\end{array}\right]-x\left[\begin{array}{l}
1 \\
1 \\
1
\end{array}\right]=0
$$

for the characteristic polynomial:

$$
\begin{equation*}
x^{3}-15 x^{2}-24 x+360=(x-15)\left(x^{2}-24\right)=0 \tag{14}
\end{equation*}
$$

for eigenvalues $\lambda_{i}=15, \pm 2 \sqrt{6}$, noting that the effect of rotation and reflection on $M_{3}$ is to change the imaginary eigenvalues to real ones in an alternating fashion[25].

Since some larger magic squares have just one non-zero eigenvalue, $\lambda_{1}$, the RCD linesum, in 2017 Loly, Cameron and Rogers[27][LCR] concluded that the Singular Values, SVs, $\sigma_{i}$, presented as positive values declining from the linesum, provided a more useful tool for assessing magic squares than the eigenvalues, as introduced next.

An invitation to give the lead keynote talk at IWMS2007 allowed Loly and Cameron to show that matrix eigenvalue analysis of highly singular magic squares needed to be replaced by the more powerful Singular Value [SV] analysis. Here the squares of the SVs are the EVs of the product of a matrix and its transpose, for which we refer to Horn and Johnson[19], to understand 1EV MSs of order 4 and 5, as reported in 2009 by Loly, Cameron, Trump and Schindel in LAA[25], and fully by Rogers, Cameron and Loly[39][RCL] in 2017.

### 5.2 Singular Values - SVs, $\sigma_{i}$

Now the critical Singular Values (SVs, $\sigma_{i}$ ) which are always positive or zero, never complex nor imaginary, and will be the same for both

F3 $(1,3,1)$ and $F 3(1,1,3)$. We began to use the SVs in LCTS[25] 2007/9 when encountering magic squares of orders greater than three with some vanishing EVs, for which the number of non-zero SVs gives the matrix rank, $r$.

As an example take the matrix product of $M_{3}$ with its transpose:

$$
\left[\begin{array}{lll}
8 & 1 & 6  \tag{15}\\
3 & 5 & 7 \\
4 & 9 & 2
\end{array}\right]\left[\begin{array}{lll}
8 & 3 & 4 \\
1 & 5 & 9 \\
6 & 7 & 2
\end{array}\right]=\left[\begin{array}{ccc}
101 & 71 & 53 \\
71 & 83 & 71 \\
53 & 71 & 71
\end{array}\right]
$$

and using $X=\sigma^{2}$ for the characteristic polynomial, $X^{3}-255 X^{2}+$ $8556 X-29340$, which is a cubic equation with the factorization:

$$
\begin{equation*}
\left(X-15^{2}\right)(X-48)(X-12)=0 \tag{16}
\end{equation*}
$$

so that the squares of the SVs, $\sigma_{i}^{2}=15^{2}, 48=3 \times 4^{2}, 12=3 \times 2^{2}$, where their positive square roots are the $\mathrm{SVs}, \sigma_{i}$ always presented in declining positive values:

$$
\begin{equation*}
\sigma_{i}=15,4 \sqrt{3}, 2 \sqrt{3}, \text { with numerical values } 15,6.9282 . ., 3.464 \ldots \tag{17}
\end{equation*}
$$

See LCTS[25] and CRL[6] for more on SVs, the latter having decreasing positive values from the leading $\mathrm{SV}, \sigma_{1}=15$, which is the same as the RCD linesum EV. Also the reverse product is different, but has the same spectra - a useful feature of SVDs! The SVs are also invariant to rotations and reflections of these (square) matrices.

With the SVs established, we note that our 2013 study[6] showed that the 880 of order 4 have 63 different singular value "clans" in 2013 (CRL) [6]. After Loly gave a talk at McGill later in Summer 2009 noting that we had not found any magic squares with rank less than 3, Sam Drury[13] proved that MSs have a minimum rank of 3 .

### 5.3 The couple $v, y$ for level $l=1$

In preparation for level $l=2$ for $n=9$ it will be useful to examine this simplest case as follows. Here the linesum SV: $\sigma_{1}=3(k+v+y)$, and the pair:

$$
\begin{equation*}
\sigma_{2,3}^{2}=3(v \pm y)^{2} \tag{18}
\end{equation*}
$$

These are included in our later Table 4.
So for $M_{3}$ and Luoshu when $v, y$ are 3,1 , or vice versa, the (positive) SVs are $\sigma_{1}=15$, and $\sigma_{2,3}=4 \sqrt{3}, 2 \sqrt{3}$, as already noted above, for full rank 3 .

By contrast, $A T_{3}$ has singular values: [16.848.., 1.0684.., 0], and rank 2 - see Table 2 later.

The present authors and colleagues have extended earlier studies of singular values spectra of magic squares to the complete set of the 880 , as well as to selected higher order magic squares at a 2007 conference [LAA[25] 2009], and in greater detail at with a virtual presentation at another in 2012 [DMPS2013].

### 5.4 Factorization of the SV characteristic polynomial for $T_{9 A}$

Using $X=\sigma^{2}$ in the characteristic polynomial which factors as:

$$
\begin{equation*}
\left(X-369^{2}\right)\left(X-3 \times 108^{2}\right)\left(X-3 \times 54^{2}\right)\left(X-3 \times 12^{2}\right)\left(X-3 \times 6^{2}\right)=0 \tag{19}
\end{equation*}
$$

for five non-zero SVs shown later as part of Table 6.

### 5.5 Numerical calculations for the SVs

We have used Mathematica[31] and Maple[30], including a subset of the latter in the ScientificWorkplace[50] [SWP]TeX system used for the preparation of this manuscript. The Python[37] libraries Numpy and Sympy were also used, and we note that other online tools for calculating SVs include Keisan[20] and "bluebit"[5].

Now we are able to provide other measures related to Shannon information entropy, which measures the degree of order in a system, and can now show an asymptotic behaviour for increasingly large order CMSs in the present study of Frierson's partner CMSs.

### 5.6 Spectral Measures - Entropy $H$ and Compression $C$

With the SVs, $\sigma_{i}$, we can proceed further we introduce some measures introduced for us in 2010 by Newton and DeSalvo[33] [NDS] who considered Sudoku matrices, which are special order 9 Latin squares of elements 1..9 in every row and column arranged so that each occurs in every tiled 3-by-3 subsquare. In 2013 we extended NDS to magic squares of orders $3,4, . .9$ as well as Latin squares from orders $2,3,4,5,8,9$ in CRL[6].

These powerful measures for assessing different magical squares, notably the Shannon information entropy $H$ and a very useful percentage Compression $C$, which we found in 2010 in NDS for completed Sudoku puzzles which may be regarded as compounded order three Latin squares (Sudoku appeared in newspapers c.2004). Useful measures of these matrices are now shown in a tabular report, whose components will now be defined.

First the $\mathrm{SVs}, \sigma_{i}$, are normalized by their sum:

$$
\begin{equation*}
\hat{\sigma}_{i}=\frac{\sigma_{i}}{\sum_{i}^{n} \sigma_{i}} \tag{20}
\end{equation*}
$$

then the Shannon information entropy, $H$, is calculated:

$$
\begin{equation*}
H=-\sum_{i}^{n} \hat{\sigma}_{i} \ln \left(\hat{\sigma}_{i}\right) \tag{21}
\end{equation*}
$$

named after Boltzmann's $H$-theorem, and finally a very useful percentage compression measure:

$$
\begin{equation*}
C=\left(1-\frac{H}{\ln (n)}\right) \times 100 \% \tag{22}
\end{equation*}
$$

which being bounded between $0 \%$ and $100 \%$ is very useful for comparisons between different magic squares.

See the Appendix for a sample numerical calculation for $M_{3}$.

### 5.7 Additional measures $R, L$

In CRL[6] we introduced some integer measures for integer square matrices based on the sums of the even powers of the SVs, ,

$$
\begin{equation*}
L=\Sigma_{i}^{n} \sigma_{i}^{4} \tag{23}
\end{equation*}
$$

and especially its shorter version:

$$
\begin{equation*}
R=\sum_{i=2}^{n} \sigma_{i}^{4}=L-\sigma_{1}^{4}, \tag{24}
\end{equation*}
$$

which is also integer for MSs. These are included in Table 2 below. CRL[6] called the distinct sets of SVs "clans" [6], which usually have a distinct value of $R$, except so far only for one pair at order 4 .

### 5.8 Matrix rank of CMSs

Drury[13] showed that magic squares have a minimum rank of 3, and therefore if less than their order $n$, are singular with one or more zero eigenvalues. In 2017 a theorem was given by Adam Rogers and the present authors [39] [RCL], for understanding the matrix rank of CMSs of combinations of all orders which gives their rank as the sum of their component ranks, here for $n=9$ each 3 less 1 for rank $r=3+3-1=5$.

Our 2009 conference report on Frierson's compound squares[26] occured before we encountered the Shannon entropy measures later in 2010, so did not include these powerful measures for the entropy and compression, which we later encountered later from 2010 paper by Newton and DeSalvo[33]. These were then used in a conference in 2012 with Adam Rogers in 2013 [RCL][6], which included a table for order 9 magic squares including both $T_{9 A}$ and $T_{9 D}$, but without further elaboration.

## 6 Matrix Properties for $n=3$

Our first tabular presentation of the matrix spectra:

| matrix | $5 E_{3}$ | $M_{3}$, Luoshu | $A T_{3}$ |
| :--- | :--- | :--- | :--- |
| $\lambda_{i}$ | $15,0,0$ | $15, \pm 2 i \sqrt{6}$ | $\frac{3}{2}(5 \pm \sqrt{33})$ |
| $\sigma_{i}^{2}$ | $225,0,0$ | $225,48,12$ | $\frac{3}{2}(95 \pm \sqrt{8881})$ |
| $\sigma_{1}$ | 15 | 15 | $16.8481 .$. |
| $\sigma_{2}$ | 0 | $4 \sqrt{3}=6.928 .$. | $1.06837 .$. |
| $\sigma_{3}$ | 0 | $2 \sqrt{3}=3.464 .$. | 0 |
| $H$ | 0.0 | $0.937098 .$. | $0.22595 .$. |
| $C$ | $100 \%$ | $14.7017 . . \%$ | $79.4332 . . \%$ |
| rank, $r$ | 1 | 3 | 2 |
| $R$ | 0 | 2448 | $1.30282 .$. |
| $L$ | 50,625 | 53,073 | 80,577 |

Table 2 - Matrix properties for $5 E_{3}, M_{3}$ and $A T_{3}$.
N.B. For $A T_{3}$, since $\sigma_{1}$ is not integer then nor is $R$.

For $M_{3}$ the pair $\sigma_{2}, \sigma_{3}$ differ by a factor of 2 , a feature found in later pairs in Table 6. The $14.7017 . . \%$ compression for $M_{3}$, Luoshu is one of the smallest that CRL[6] found in a wide ranging study of magic squares and Latin squares, while we will see that the larger CMSs here trend to much higher $C \%$ 's than we found for the smallest CMSs of order 9 of $48.57 . . \%$ that we showed earlier[6]. Extended in the present work to orders $27,81,243, \ldots$, we find systematically larger values that tend towards the uniformity of $100 \%$. Since any uniform square matrix of all 1's has full compression of $100 \%$, a low compression reflects a more "lumpy" matrix! Most other (larger) magic squares have a much higher compression, especially the compound magic squares studied here.
N.B. After this table we drop further discusson of the EVs $\left(\lambda_{i}\right)$ since the SVs $\left(\sigma_{i}\right)$ give us all the information needed for the entropy and compression.

Also all versions of $E_{n}$ have just the linesum $\operatorname{EV}\left(\lambda_{1}\right)$ and $\operatorname{SV}\left(\sigma_{1}\right)$, both $n$.

It is worth noting that in DMPS we did find lower compressions than for $M_{3}$ 's $14.7017 . . \%$ for some order 5 and 9 MSs , and that $A T_{3}$ 's high compression shows the high ordering of its elements, only surpassed by the completely ordered matrices of identical elements, e.g. the $100 \%$ of $E_{3}$.

### 6.1 Zero-based MSs

If the elements of a MS are chosen to run $0,1,2,3, . .\left(n^{2}-1\right)$ instead of the $1,2,3, . . n^{2}$ used here, then the $\hat{\sigma}_{i}$ will be smaller since the RCD's are
smaller, so that the entropy will be larger and the compression smaller, e.g. for $M_{3}^{\prime}$ these change to: $H^{\prime}=0.985975, C^{\prime}=10.2527 \%$.

## 7 Frierson's partner CMSs

Now fill a new $D_{3}$ in place of $M_{3}$ with the elements of $A T_{9}$ 's first column, $1,10,19,28,37,46,55,64,72$, in the $M_{3}$ pattern for $D_{3}$, enhanced by simple multiples of $E_{3}$ to obtain $T_{9 D}$, a spectral partner MS to $T_{9 A}$ :

$$
D_{3}=\left[\begin{array}{ll|c|l|}
\hline 64 & 1 & 46  \tag{25}\\
1 & 3 & 3 & 55 \\
28 & 73 & 10
\end{array}\right], T_{9 D}=\begin{array}{|c|c|c|}
\hline D_{3}+7 E_{3} & D_{3} & D_{3}+5 E_{3} \\
\hline D_{3}+2 E_{3} & D_{3}+4 E_{3} & D_{3}+6 E_{3} \\
\hline D_{3}+3 E_{3} & D_{3}+8 E_{3} & D_{3}+E_{3} \\
\hline
\end{array}
$$

and is magic, having the same SVs as $T_{9 A}$ - see Table 3 below.
Here we used $D$ to indicate that the elements of the subsquares of $T_{A}$ have been $D$ ispersed to other subsquares in a systematic way.
$T_{9 D}$ also has an early date before 1000 CE - Cammann[9] noted that this magic square was found in China by the 13th CE by Yang Hui 1275 CE, and suggested that $T_{9 A}$ and $T_{9 D}$ were originally derived from the order 9 addition table, $A T_{9}$. See also Table 3 below for its spectra.

### 7.1 Frierson's second pair $T_{9 B}=F_{9}(1,27,9,3)$ and $T_{9 E}$

$T_{9 B}$ uses the first rows of the left hand subsquares of $A T_{9}, 1,2,3$ with $28,29,30$ and $55,56,57$, to fill a $B_{3}$ with the $M_{3}$ pattern:

$$
B_{3}=\left[\begin{array}{ccc}
56 & 1 & 3  \tag{26}\\
3 & 29 & 55 \\
28 & 5 & 2
\end{array}\right], T_{9 B}=\begin{array}{|l|l|l|}
\hline B_{3}+21 E_{3} & B_{3} & B_{3}+15 E_{3} \\
\hline B_{3}+6 E_{3} & B_{3}+12 E_{3} & B_{3}+18 E_{3} \\
\hline B_{3}+9 E_{3} & B_{3}+24 E_{3} & B_{3}+3 E_{3} \\
\hline
\end{array} .
$$

with SVs: 369, 145.49.., 135.10.., 62.354..,31.177.., and four zeros, as does its partner $T_{9 E}$, not shown.

### 7.2 Frierson's third pair $T_{9 C}=F_{9}(1,9,27,3)$ and $T_{9 F}$

$T_{9 C}$ then uses $1,2,3$ with $10,11,12$ and $19,20,21$ from the top left order 3 subsquare of $A T_{9}$ arranged in the $M_{3}$ pattern:

$$
C_{3}=\left[\begin{array}{ccc}
20 & 1 & 12  \tag{27}\\
3 & 11 & 19 \\
10 & 21 & 2
\end{array}\right], T_{9 C}=\begin{array}{|l|l|l|}
\hline C_{3}+57 E_{3} & C_{3} & C_{3}+33 E_{3} \\
\hline C_{3}+6 E_{3} & C_{3}+30 E_{3} & C_{3}+54 E_{3} \\
\hline C_{3}+27 E_{3} & C_{3}+60 E_{3} & C_{3}+3 E_{3} \\
\hline
\end{array} .
$$

Now $T_{9 C}=F_{9}(1,9,1,27,3)$, with a "partner" $T_{9 F}$.. see also Table 3 , with singular values: $369.0,155.88,124.71,51.962,41.5969$, and four zeros, as does its partner $T_{9 F}$, not shown.

### 7.3 Matrix Properties for Frierson's 6 natural 9th order 'basic' $F_{9}$ 's $l=2, n=9$

The next table gives the properties for Frierson's six squares (ordered $v>y$ from $M(v, y)$ where $v, y$ in the second row, and $s>t$ from $M(s, t)$ where $s$ is written to the left of $t$ ), showing pairs of isentropic variants:

| $F_{9}$ | $T_{9 A}$ | $T_{9 D}$ | $T_{9 B}$ | $T_{9 E}$ | $T_{9 C}$ | $T_{9 F}$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $v, y$ | 3,1 | 27,9 | 27,1 | 9,3 | 9,1 | 27,3 |
| $s, t$ | 27,9 | 3,1 | 9,3 | 27,1 | 27,3 | 9,1 |
| $C$ | $48.572 . . \%$ | $<-$ | $40.0241 . . \%$ | $<-$ | $39.8296 . . \%$ | $<-$ |
| $H$ | $1.12999 .$. | $<-$ | $1.31781 .$. | $<-$ | $1.32208 .$. | $<-$ |
| $R$ | $1,301,165,856$ | $<-$ | $797,281,056$ | $<-$ | $842,630,688$ | $<-$ |

Table 3 - Matrix Properties for Frierson's $F_{9}$ sextet with RCDs,

$$
\lambda_{1}, \sigma_{1}=369
$$

These $R$ values for $T_{9 A, D}$ agree with our 2017 RCL [39], but since $R$ becomes much larger for $n=27,81, \ldots$ it will be dropped henceforth, with an emphasis on the \% Compression which is always bounded between $0 \%$ and $100 \%$.

Here there are 3 sets of SVs, each with different entropies and compressions. We interpret the reduced compression and higher entropy values to show that the order decreases from $T_{9 A, D}$, through $T_{9 B, E}$, to $T_{9 C, F}$ are not quite as ordered as $T_{9 A, D}$, but are closer to each other. Clearly the spectral properties are not changed by swapping the parameters values of the pairs $y, v$ and $s, y$.

This gives a deeper insight into Frierson's construction than possible without the spectra.

Our spectral measures for $T_{9 A, B, C, D, E, F}$ differ from $M_{3}$, with $C \%$ of $14.7 \%$, having a much greater Compression, almost halfway to the $100 \%$ of a uniform matrix, e.g. $E_{9}$, a trend that increases as we explore order $27,81,243$, .. compounding later by continuation of the fractal pattern underlying this particular system, and apparently becomes asymptotic at about 1.1677038.. in our later Table 6.

Next we extend Frierson's ideas to the next level, $l=3$ for $n=27$.
8 Comparing Spectral Algebras for $l=1,2,3$ (or $n=3,9,27)$

We followed Frierson in the use of $v, y$ and then his $s, t$, for order 9, whereas Bellew[2] used $p, q$ instead of Frierson's $s, t$, so we now use $p, q$ for the step to order 27 .

It is clear that this process could be continued for orders $81,243, \ldots$ but already a clear pattern has emerged which renders that unnecessary as the next Table will show!

On the basis of Maple and Mathematica calculations we can now state the formulae for the singular values $\left(n=3^{l}\right)$ of all orders of Frierson compound squares which consists of the linesum eigenvalue, and $l$ signed pairs and rank:

| $l$ | 1 | 2 | 3 |
| :--- | :--- | :--- | :--- |
| $n=3^{l}$ | 3 | 9 | 27 |
| $r=2 l+1$ | 3 | 5 | 7 |
| $S(n)$ | 15 | 369 | 9855 |
| $\sigma_{1}-n k$ | $3(v+y)$ | $9(v+y+s+t)$ | $27(v+y+s+t+p+q)$ |
| $\sigma_{2,3}^{2}$ | $3(v \pm y)^{2}$ | $27(v \pm y)^{2}$ | $243(v \pm y)^{2}$ |
| $\sigma_{4,5}^{2}$ |  | $27(s \pm t)^{2}$ | $243(s \pm t)^{2}$ |
| $\sigma_{6,7}^{2}$ |  |  | $243(p \pm q)^{2}$ |

Table 4 -Singular Values for $n=3,9,27$.
In this table the $\sigma_{i}$ for $i>1$ increase by a factor of 3 , so that their squares increase by factors of 9 . It is clear how this table can be extended by adding extra pairs, e.g. $a, b ; c, d$, etc. for $l=4,5, .$. Considering orders 3, 9,27 in Table 6 above where it does not matter for the SVs if $v$ is greater or less than $y$ (because of the squares in the formulae for $\left.\sigma_{2,3}^{2}=3(v \pm y)^{2}\right)$, nor similarly their positive numerical magnitudes.
N.B. While numerical data for the SVs are usually listed in descending magnitude the magnitudes of $p, q, s, t, v, y$ vary in the next table the magnitudes of $\lambda_{6,7}, \sigma_{6,7}, \lambda_{4,5,}, \sigma_{4,5}, \lambda_{2,3}, \sigma_{2,3}$ will rarely be sequential!

### 8.1 Numerical $F_{27}$ spectra

Calculations were done with SV formulae in Table 4 above. All have rank $7=(3+3-1=5)+3-1$ in agreement with RCL[39].

Case A has the lowest entropy and its counterparts for different orders will be our main focus. Note that this case has two sets of parameters, $v, s, p$ and $y, t, q$, increasing monotonically.

The integer index $R$ devised by Loly in CRL[6] as the sum of the 4th powers of the SVs (less the one for the linesum) is rather long and we note just the two extremes:
$R(\mathrm{~A})=691,492,899,739,824$ with $\ln [R(\mathrm{~A})]=34.169874 \ldots$,
and $R(\mathrm{O})=420,327,995,019,696$ with $\ln [R(\mathrm{O})]=33.672056 \ldots$,
from which we conclude that $H$ and especially $C \%$ are more useful in comparing large MSs than the huge integer $R$ 's!

The 90 order 27 's would need a 6 -by- 15 table of 6 rows for the isentropic squares and 15 columns of the different entropies which are now listed:

|  | $v$ | $y$ | $s$ | $t$ | $p$ | $q$ | $H$ | $C \%$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| A | 1 | 3 | 9 | 27 | 81 | 243 | 1.16247 | 64.7291 |
| B | 1 | 27 | 3 | 9 | 81 | 243 | 1.20646 | 63.3944 |
| C | 1 | 9 | 3 | 27 | 81 | 243 | 1.20697 | 63.3788 |
| D | 1 | 3 | 81 | 27 | 243 | 9 | 1.34763 | 59.1110 |
| E | 1 | 3 | 243 | 27 | 81 | 9 | 1.35191 | 58.9813 |
| F | 1 | 243 | 9 | 3 | 81 | 27 | 1.38498 | 57.9778 |
| G | 1 | 9 | 243 | 3 | 81 | 27 | 1.38566 | 57.9573 |
| H | 1 | 81 | 3 | 9 | 27 | 243 | 1.38973 | 57.8338 |
| I | 1 | 9 | 81 | 3 | 243 | 27 | 1.39035 | 57.8149 |
| J | 1 | 243 | 81 | 3 | 9 | 27 | 1.46991 | 55.4010 |
| K | 1 | 81 | 27 | 9 | 243 | 3 | 1.46996 | 55.3995 |
| L | 1 | 243 | 3 | 27 | 9 | 81 | 1.47129 | 55.3593 |
| M | 1 | 27 | 81 | 9 | 243 | 3 | 1.47148 | 55.3533 |
| N | 1 | 81 | 3 | 27 | 9 | 243 | 1.47178 | 55.3443 |
| O | 1 | 27 | 81 | 3 | 243 | 9 | 1.47193 | 55.3398 |

Table 5- $F_{27}$ Matrix spectral measures for 15 clans with the lowest entropy at top and highest at bottom.

In an Appendix Browne's[4] order 27 is shown to have $v=27, y=$ $1 ; s=3, t=81 ; p=9, q=243$, so it is a variant of case " O " with the highest entropy, one of $90 / 6=15$ variants - see the next section.

### 8.2 Collecting the lowest entropy sets for higher values of $l, n$

Late in 2019 we realized that the SVs of higher order versions of the lowest entropy members, e.g. $T_{9 A}, T_{27 A}, .$. , could be used directly to obtain the Compression and entropy values so now the SVs for each pair differ by the same factor of 2 found in Table 2 for $l=1$, and these SVs increase by a factor of 27 as $l$ increases, while the SV's of each higher pair increase by a factor of 9 for every increase in $l$. Here we see these lowest entropies slowly increasing with order $n$ from 0.937 .. to 1.168.. and clearly becoming asymptotic - a feature that we now see was probably present in our earlier CRL study[6] for magic squares obtained form the MATLAB's magic[ $n$ ] function[32], where its "Figure 1" showed a slowing increase of the entropies of odd order to $n=99$ from $H=0.937$.. to ~ 3.5 (which only included the sole order $M_{3}$ in the present study since those for $n=9,27, .$. lie well above our lowest entropy members: for $n=9 \mathrm{c}$. 1.8.., for $n=27$ c. 2.7.. and for $n=81 . c$. 3.22..).

| $l$ | 1 | 2 | 3 | 4 | 5 |
| :--- | :--- | :--- | :--- | :--- | :--- |
|  | $M_{3}$ | $F_{9 A}$ | $F_{27 A}$ | $F_{81 A}$ | $F_{243 A}$ |
| $n=3^{l}$ | 3 | 9 | 27 | 81 | 243 |
| RCD $\sigma_{1}$ | 15 | 369 | 9855 | 265,761 | 7174575 |
| $\sigma_{2} / \sqrt{3}$ | 4 | 108 | 2916 | 78732 | 2125764 |
| $\sigma_{3} / \sqrt{3}$ | 2 | 54 | 1458 | 39366 | 1062882 |
| $\sigma_{4} / \sqrt{3}$ |  | 12 | 324 | 8748 | 236196 |
| $\sigma_{5} / \sqrt{3}$ |  | 6 | 162 | 4374 | 118098 |
| $\sigma_{6} / \sqrt{3}$ |  |  | 36 | 972 | 26244 |
| $\sigma_{7} / \sqrt{3}$ |  |  | 18 | 486 | 13122 |
| $\sigma_{8} / \sqrt{3}$ |  |  |  | 108 | 2916 |
| $\sigma_{9} / \sqrt{3}$ |  |  |  | 54 | 1458 |
| $\sigma_{10} / \sqrt{3}$ |  |  |  |  | 324 |
| $\sigma_{11} / \sqrt{3}$ |  |  |  |  | 162 |
| $\sigma_{\text {total }}$ | $26.3923 .$. | $680.76 .$. | $18366.3 .$. | $4.9847 . .10^{4}$ | $1.3387 . .10^{7}$ |
| $C \%$ | $14.7017 .$. | $48.572 .$. | $64.7291 .$. | $73.4364 .$. | $78.7368 .$. |
| $H$ | $0.93709 .$. | $1.1299 .$. | $1.16247 .$. | $1.16732 .$. | $1.1677038 .$. |
| $r=2 l+1$ | 3 | 5 | 7 | 9 | 11 |

Table 6 - The lowest entropy members of Frierson-type CMSs.
Our main goal in going beyond Frierson's order 9 CMSs to a full account of order 27 is now complete, but we can now make a further extension for the lowest entropy (highest order) cases.

Since this now completes $n=27$, we will now extrapolate to higher orders - see later for Sloane's A000680[45] and counting the isentropic variants illustrated here for $n=9,27$..

### 8.3 Asymptotic behaviour

For $F_{729 A}$ with $l=6, n=729$ we find $C \%=82.2829 . ., H=1.167856 \ldots$ The entropy $H$ is clearly flattening out to about 1.168 ..., while the Compression $C \%$ continues to increase more slowly towards $100 \%$.

Other CMSs using $T_{B, C, E, F}$, which begin with higher values of entropy, compounded with or without $T_{A, D}$, are expected to generate larger entropies than found above and are not pursued here.

## 9 Counting for $n=3^{l}$

The number of $F_{n}$ 's at level $l$ is the product of the number of first couples at level $l$, column 3 in the table below, the number of $F_{n}$ 's at the previous level $(l-1)$, for $l(2 l-1)$ first couples, as shown in column 4 , and the number of distinct SV sets in column 5:

| $n$ | $l$ | 1st couples | number of $F_{n}$ 's | no. of SV sets |
| :--- | :--- | :--- | :--- | :--- |
|  |  | $l(2 l-1)$ | $(2 l)!/ 2^{l}$ | $(2 l-1)!!$ |
| 3 | 1 | 1 | 1 | 1 |
| 9 | 2 | 6 | 6 | 3 |
| 27 | 3 | 15 | $15 \times 6=90$ | $5 \times 3=15$ |
| 81 | 4 | 28 | $28 \times 90=2520$ | $7 \times 15=105$ |
| 243 | 5 | 45 | $45 \times 2520=113400$ | $9 \times 105=945$ |

Table 7 - Counting couples, $F_{n}$ 's and SV sets.
Note that 15 in columns 3 and 5 is a coincidence.
Also only the $n=3,9,27$ results in columns 4,5 have been verified, and those prompted the formulae above and "OEIS" described next.

### 9.1 Integer Sequences - we use order $n=3^{l}$ in this paper

'The On-Line Encyclopedia of Integer Sequences', "OEIS", gives the following information on the three integer sequences used here:

### 9.1.1 Counting 1st couples

Sloane's [44] A000384: Hexagonal numbers: $n(2 n-1)$ :
$0,1,6,15,28,45, . .$,
in our notation: $l(2 l-1)$, and ignoring the zero!

### 9.1.2 Counting the number of $F_{n}$ 's

Sloane's [45] A000680: (2n)!/2 $2^{n}$ :
$1,1,6,90,2520,113400,7484400,681080400,81729648000, . .$,
in our notation: $(2 l)!/ 2^{l}$, again ignoring the first ' 1 '.

### 9.1.3 Number of SV sets

Sloane's[46] A001147: Double factorial of odd numbers:
$a(n)=(2 n-1)!!=1 \times 3 \times 5 \times \ldots \times(2 * n-1):$
$1,1,3,15,105,945,10395,135135, .$. ,
and again ignoring the first ' 1 '.

### 9.2 Factors of 8 for $F_{9}$ 's, $F_{27}$ 's ...

Bellew's factors of 8 drew our attention[26] to the significance of this aspect of compounding.

Now we note the effect of rotations and reflections of each subsquare, $m=3$ for $F_{9}$ 's for $8^{9}$ variations in $F_{9}$ 's, and both $m=3,9$ subsquares for $F_{27}$ 's which now give a factor of $8^{81+9}=8^{90}$ variants of each basic $F_{27}$ due to a factor of 8 for each of the $9 m=9$ subsquares multiplying the factor from $81 m=3$ subsquares.

Here for $n=27$ we have resolved disparate counts of $8^{18}$ of Trigg [52] (1980) and Bellew of $8^{81}$ to a new result of $8^{81+9}=8^{90}$ by taking account of all orders of tiled subsquares, before generalizing this for all $l$.

Then for $F_{81}$ 's we predict an additional factor of $8^{729}$ for a total $8^{729+81}=8^{819}$. We observe that the exponents $9,81,819, .$. may be found in Sloane's[47]

### 9.2.1 Number of variants due to subsquare rotations and reflections

A0523386: Number of integers from 1 to $10^{n}-1$ that lack 0 as a digit: $0,9,90,819,7380,66429,597870, .$. (ignoring the initial zero).

We also expect that these rotations and reflections of the magic subsquares in $F_{9}$ will increase the rank of the resultant CMS variants.

## 10 CMSs and Fractal patterns c. 2000

Earlier Chan and Loly[10] [CL] revived the compounding idea by using a pandiagonal order 4 and Euler's 1779 pandiagonal order 7 to produce an aggregated CMS of order $12,544=4^{4} * 7^{2}$, suggesting that this process is fractal[29], i.e. self-similar on all scales, in order to break records for large magic squares. CL also gave an argument for the preservation of pandiagonality on compounding that parallels our present observation of the preservation of associativity on compounding, and while referencing the important 1997 work of Bellew[2], focussed on his treatment of pandiagonal magic squares (PMSs), defined later, rather than Frierson's associative squares of concern here. They referenced a then recent paper 1997 paper by Bellew[2] as well as Andrews[1], neither of which explicitly referenced Frierson's parametric compounding of the order 3 to order 9.

## THIS COMPLETES OUR EXTENSIONS OF FRIERSON'S

 and BELLEW's IDEAS.
## 11 CONCLUSION

Frierson's parameterization set the stage for our generalization here. Extending his algebraic formulation from order 9 to highet powers of 3 has enabled us to project asymptotic behaviour for the lowest entropy members of this infinite family of CMSs of orders $3^{l}$, giving the first full account of order 27 ..

Our present achievement may be considered somewhat parallel to Ollerenshaw and Brée's[34] comprehensive study of Most-Perfect Pandiagonal [MPPD] MSs of orders all multiples of 4, but enhanced here with an account of the spectral properties. A preliminary study of compounding of one of those at order 4 in our Appendix indicates similar
asymptotic behaviour, which suggests a new look at their parameterization would be valuable, so the present work will be followed by a study of parameterizing order 4 MSs by Ian Cameron[8] using the 1910 scheme of Bergholt[3].

## 12 Acknowledgements

We thank Adam Rogers[39] [RCL] and Wayne Chan[10] [CL] for earlier compound collaborations. PDL has seen a copy of the original notes of the present topic and other L.S.Frierson material at Shreveport, Louisiana, courtesy of Fermand M. Garlington II, Archives and Special Collections, Louisiana State University in Shreveport. "Frierson" [42] is also an unincorporated community and Census-Designated Place (CDP) in DeSoto Parish, Louisiana, United States.

Loly also received early encouragement from John Hendricks[18], originally from our city of Winnipeg, and Harvey Heinz[16], who were early active members of a large online recreational mathematics community, whom we hope to encourage to include our spectral measures in their future investigations, including several extensive websites, e.g Harry White[57] and Walter Trump[53].

Email: loly@umanitoba.ca, Ian.Cameron@umr.umanitoba.ca.

## A An earlier parameterization by Édouard Lucas in 1894

Another parameterization for order 3 by Lucas[28] should be noted and was drawn to our attention in detail by Sallows[40], who used a parameter $c$ which plays the role of our $k$ and Frierson's $x$ :

$$
\operatorname{Lucas}(a, b, c)=\begin{array}{|c|c|c|}
\hline c+a & c-a-b & c+b  \tag{28}\\
\hline c-a+b & c & c+a-b \\
\hline c-b & c+a+b & c-a \\
\hline
\end{array} .
$$

On his page 3 Sallows uses $a=3, b=1, c=5$ to obtain the Luoshu in (1). However Sallows referenced neither Frierson, nor Andrews. See also Lachal[22].

We tested $\operatorname{Lucas}(3,1,5)$ finding numerical SVs 15, 6. 928 2.., 3. 464 1.. which agree with those of $M_{3}$, as expected.

## B Other Magic Squares

## B. 1 Order 4

At order 4 there are 880 distinct magic squares of $1,2, . ., 16$ which have been classified by the patterns of complementary number pairs within the square into 12 Groups by Dudeney[14]. Counted in 1693 by Frénicle de Bessy, amongst them 48 associative and another 48 of the pandiagonal variety defined soon. Since our 2013 study[6] the 880 are now known to have 63 different singular values (SV) clans, some of which have just one non-zero EV[27].

## B.1.1 Pandiagonal Magic Squares (PMSs)

Of the 880, the 48 in Dudeney[14] Group I are pandiagonal. These are characterised by having all parallel broken diagonals to the main ones with the same RCD linesum as noted earlier for $A T_{3}$, but we note that this is not the case for the present study of Frierson's associative compound squares which we are not pandiagonal[55]. There are also 16 ultramagic squares with both the associative and pandiagonal features.

However this does not rule out other magic squares of orders 9, 27, $81, \ldots$ from being pandiagonal, some are known and one noted below, and others we could construct by compounding.

## B.1.2 Ultramagic Squares

These have both the associative and pandiagonal properties and begin at order 5[25].

## C A low entropy order 4 Most-Perfect Pandiagonal [MPPD] Magic Square

These MPPDs are found at order 4 and multiples of order 4. Here we consider one of this variety from the classic study of Dame Kathleen Ollerenshaw and David Brée[35][34], from their cover but here using the classic elements $1,2, . . n^{2}$ instead of zero-based:

$$
M P P D_{4 \alpha}=\left[\begin{array}{cccc}
1 & 15 & 4 & 14 \\
8 & 10 & 5 & 11 \\
13 & 3 & 16 & 2 \\
12 & 6 & 9 & 7
\end{array}\right]
$$

This has the lowest entropy of the MSs in the 3 pandiagonal clans Dudeney[14] Groups 1, 2, 3, the Alpha clan[6], with $\lambda_{i}=34, \pm 8,0$, and $\sigma_{1}=34.0,17.889 . ., 4.4721 . ., 0$, rank 3.

Ollerenshaw \& Brée did not study any spectra, nor did Bellew, but the former did reference Bellew.

## C. 1 Compounding the lowest entropy $M P P D_{4 \alpha}$ for comparison with our $n=3^{l}$ CMSs

| $l$ | 1 | 2 | 3 | 4 |
| :--- | :--- | :--- | :--- | :--- |
| $n=4^{l}$ | 4 | 16 | 64 | 256 |
| $\sigma_{1}=\lambda_{1}$ | 34 | 2056 | 131,104 | $8,388,736$ |
| $\sigma_{2} / \sqrt{5}$ | $2^{3}$ | $2^{9}$ | $2^{15}$ | $2^{21}$ |
| $\sigma_{3} / \sqrt{5}$ | 2 | $2^{7}$ | $2^{13}$ | $2^{19}$ |
| $\sigma_{4} / \sqrt{5}$ |  | $2^{5}$ | $2^{11}$ | $2^{17}$ |
| $\sigma_{5} / \sqrt{5}$ |  | $2^{3}$ | $2^{9}$ | $2^{15}$ |
| $\sigma_{6} / \sqrt{5}$ |  |  | $2^{7}$ | $2^{13}$ |
| $\sigma_{7} / \sqrt{5}$ |  |  | $2^{5}$ | $2^{11}$ |
| $\sigma_{8} / \sqrt{5}$ |  |  |  | $2^{9}$ |
| $\sigma_{9} / \sqrt{5}$ |  |  |  | $2^{7}$ |
| $C \%$ | $37.2284 .$. | $64.3023 .$. | $75.9175 .$. | $81.9199 .$. |
| $H$ | $0.8702 .$. | $0.98975 .$. | $1.00156 .$. | $1.00257 .$. |
| $r$ | 3 | 5 | 7 | 9 |

Table 8 - A lowest entropy order 4 magic square compounded.
In Table 8 the $\sigma_{2,3,4,5, . .}$ increase by factors of 64 across columns from left to right as $l$ increases, and the $\sigma_{3,4,5, . .}$ decrease by factors of $1 / 4$ from their $\sigma_{2}$ 's. The trend to an asymptotic entropy mirrors that found in the main text for the lowest entropy members of the Frierson CMSs.

## C. 2 Higher orders $n=5,6, \ldots$

The populations of larger MSs continue to grow - see our colleague Walter Trump's table [53] which is regularly updated - so that the number of distinct order 9 MSs is astronomical, meaning that our Frierson-type CMS are rare, but possibly close to the lowest entropy member?

## D Numerical Compounding of Doubly Affine Matrices

From c. 2004 Rogers and Cameron explored the use of Kronecker products of MSs to generate larger ones of compound order - this was finally published in 2017 with Loly[39][RCL]. RCL gave a general study of CMSs which included these ancient pairs for arbitrary $m, n>2$ in terms of Kronecker products, including a full account of their spectral properties which showed that all CMSs are singular, a feature realized by them from earlier matrix eigenvalue studies c. 2004. This was first reported at IWMS-2007, but not included in the conference proceedings[25] in 2009. RCL used the entropy and compression measures from their 2013

CRL[6] for magic and Latin squares, including Frierson's first order 9 pair. Mixing orders 3 and 4 yields order 12 compound squares with either order 4 or 3 subsquares, and is extendable to high orders.

RCL contains much useful background to compounding that need not be repeated here as our focus is Frierson's different parameterized method, however one result useful in the present context is that the rank of a CMS is the sum of the ranks of its two components less 1 , e.g. the rank of a Frierson order 9 CMS is $3+3-1=5$, and for an order 27 is then $5+3-1=7$, in agreement with Table 5 .

Note that our terminology for CMSs of $T_{A, D}$ used here was changed in $\mathrm{RCL}[39]$ to $C_{A, D}$ which are vertical reflections using the sequence $0,1,2, . .\left(n^{2}-1\right)$.

## E Other tools for spectral calculations

The authors have used Mathematica[31] and Maple[30] software for both numerical and algebraic calculations as well as Numpy and Sympy from Python[37]. In previous studies with Adam Rogers[39], also MATLAB, which has a magic square generator for one of each odd, even and doublyeven orders. Earlier in "Online tools for calculating SVs" we noted Keisan and Bluebit. We add that Wolfram Alpha[58] enables access to some of Mathematica online, and well as via apps for iPhones and iPads.

Also this article has been edited with a version of LaTeX in Scientific Workplace[50], which also has a "Compute" section using Maple which has been used recently to check some of the matrices herein.

## E. 1 From the SVs to C\% - a sample entropy and Compression calculation for $M_{3}$

The (default) numerical precision in SWP's[50] "Evaluate Numerically" is used here, first the total sigmas:
$15+6.9282+3.4641=25.392$
then the contributions to the Shannon entropy, $H$, are calculated:
$-15 / 25.392 \times \ln [15 / 25.392]=0.31095$
$-6.9282 / 25.392 \times \ln [6.9282 / 25.392]=0.35439$
$-3.4641 / 25.392 \times \ln [3.4641 / 25.392]=0.27176$
For a total: $H=0.31095+0.35439+0.27176=0.9371$, and finally the \% Compression follows:
$C=(1-0.9371 / \ln [3]) \times 100=14.701$, which both agree with our CRL[6] calculations.

## E. 2 A Cautionary note

Since some computer software, e.g MATLAB[32] and "R"[38], offer just a single magic square for each order one must be careful to not draw strong conclusions from their single MSs as to the properties of others of the same order in view of the great diversity already apparent at order 4. Clearly our Frierson-type associative CMSs are going to be just a (small) fraction of the enormous number of order 9 magic squares, but perhaps of low entropy.

## F Browne's CMS27

An order 27 CMS by Browne[4], B27, with a commentary by Paul Carus, was shown in chapter VI of MSC1[1], Fig. 273 (Fig. 256 of MSC2[1]), but is not easy to read, in part because alternate cells are shaded.
$B 27$ may be a variant of our $\# 15$, " O " in Table $7[v=27, y=1 ; s=$ $81, t=3 ; p=243, q=9]$.

For compactness and accuracy we divide $F_{27}$ into 9 -by- 9 order 9 subsquares and those similarly to order 3 subsquares beginning with:
$B 3=\left[\begin{array}{ccc}28 & 57 & 2 \\ 3 & 29 & 55 \\ 56 & 1 & 30\end{array}\right]$, for which $v=27, y=1$,
and which has Knight path's $1 \rightarrow 2$ and $2 \rightarrow 3$, then a move up by 25 to begin a down diagonal $28 \rightarrow 29 \rightarrow 30$, and two more Knight path's $55 \rightarrow 56$ and $56 \rightarrow 57$.

Then using to construct the bottom middle order 9 subsquare:

$$
B 9=\left[\begin{array}{ccc}
B 3+81 E 3 & B 3+168 E 3 & B 3+3 E 3 \\
B 3+6 E 3 & B 3+84 E 3 & B 3+162 E 3 \\
B 3+165 E 3 & B 3 & B 3+87 E 3
\end{array}\right]
$$

and finally:

$$
B 27=\left[\begin{array}{ccc}
B 9+243 E 9 & B 9+504 E 9 & B 9+9 E 9 \\
B 9+18 E 9 & B 9+252 E 9 & B 9+486 E 9 \\
B 9+495 E 9 & B 9 & B 9+261 E 9
\end{array}\right],
$$

but not given explicitly as it takes a whole page - see the clarity issue in Browne's[4] example in MSC2[1], page 150, which is clearer in Swetz[49], page 136, with a duplicate 606 in row 7 , column 15 which should be 506 .

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