Arithmetics of Some Sequences via 2-determinants

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Abstract

We extend our investigations of 2-determinants, which we defined in a previous paper. For a linear homogenous recurrence of the second order, we define the notion of a fundamental sequence. We use fundamental sequences to investigate relations between different sequences satisfying the same recurrence of the second order. Also, we define the notion of a universal property and derive several results thereupon. At the end of the paper, we show that some standard identities, such as d'Ocagne's, Cassini's, and Catalan's, hold for every fundamental sequence. We illustrate our results with a number of examples.

1 Introduction

We continue our investigations of 2-determinants, defined in [1]. For a linear homogenous recurrence of the second order, we define the notion of a fundamental sequence. We use fundamental sequences to investigate relations between different sequences satisfying the same recurrence.

The main representatives of the fundamental sequences are: Fibonacci numbers and polynomials, bisection of Fibonacci numbers, positive integers, Pell numbers, Jacobhtal numbers, Mersenne numbers, and Chebyshev polynomials of the second kind.

We say that a property of a sequence which satisfies a linear recurrence of the second order is universal if it does not depend on the fundamental sequence of that recurrence. We list several such sequences.

Our method produces a number of identities. We restrict our attention to the so-called bilinear identities, such as d'Ocagne's, Cassini's, Vajda's, and Catalan's identities, and prove their analogs for some other fundamental sequences. We call universal those properties that depend only on their recurrences, and not on their fundamental sequences.

Let x and y be integer-valued variables. We consider the following recurrence of the second order:

$$a_{n+1}(x,y) = x \cdot a_n(x,y) + y \cdot a_{n-1}(x,y), \;\; n > 0, \tag{1}$$

where $a_0(x,y) = 0, \; a_1(x,y) = 1$.

Definition 1. We say that the sequence $(a) = (a_0(x,y), a_1(x,y), a_2(x,y), \ldots)$ defined by (1) is fundamental.

In two particular cases, we give combinatorial interpretations of fundamental sequences in terms of restricted words over a finite alphabet.

Proposition 2. *If $x > 0$ and $y > 0$, then $a_{n+1}(x,y)$ equals the number of words of length n over the alphabet $\{0, 1, \ldots, x+y-1\}$, such that the letters $\{0, 1, \ldots, y-1\}$ avoid runs of odd lengths.*

Proof. We denote by d_{n+1} the number of required words of length n. If $n = 0$, then $d_1 = 1$, since the empty word has no runs of zeros of an odd length. Also, $d_2 = x$, since a word of length 1 consists of one letter from $\{y, y+1, \ldots, x+y-1\}$. Obviously, there are x such words.

Assume that $n > 2$. If a word of length n begins with a letter from $\{y, y+1, \ldots, x+y-1\}$, then there are x choices for the first letter of such a word. If a word begins with a letter from $\{0, 1, \ldots, y-1\}$, then there are y choices for the first letter of such a word and the next letter must be the same since runs of length 1 are not allowed. The remaining part of such a word can be an arbitrary subword satisfying the given condition. We see that d_n satisfies the same recurrence as a_n, with the same initial conditions. This implies that $d_n = a_n$ for $n \geq 1$. □

In Proposition 12 in [2], when $x > 0$ and $y > 0$ we derived the following explicit formula for $a_n(x,y)$:

$$a_n(x,y) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-k}{k} \cdot x^{n-2k-1} \cdot y^k. \tag{2}$$

In fact, this formula is true in the general case, without any restrictions on x and y.

Proposition 3. *If $a_0(x,y) = 0, \; a_1(x,y) = 1$, and $a_{n+1}(x,y) = x \cdot a_n(x,y) + y \cdot a_{n-1}(x,y)$, $n > 1$, then*

$$a_n(x,y) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-k}{k} \cdot x^{n-2k-1} \cdot y^k.$$

Proof. We prove the formula by induction on n. It is obvious that the formula holds for $n = 0$ and $n = 1$. Assume that the formula holds for all integers less than $n+1$. Then from

$a_{n+1}(x,y) = x \cdot a_n(x,y) + y \cdot a_{n-1}(x,y)$ we have

$$\begin{aligned}
a_{n+1}(x,y) &= x \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-k}{k} \cdot x^{n-2k-1} \cdot y^k + y \sum_{k=0}^{\lfloor \frac{n-2}{2} \rfloor} \binom{n-2-k}{k} \cdot x^{n-2k-2} \cdot y^k \\
&= \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-k}{k} \cdot x^{n-2k} \cdot y^k + \sum_{k=0}^{\lfloor \frac{n-2}{2} \rfloor} \binom{n-2-k}{k} \cdot x^{n-2k-2} \cdot y^{k+1} \\
&= \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor - 1} \binom{n-2-k}{k+1} \cdot x^{n-2k-2} \cdot y^{k+1} + \sum_{k=0}^{\lfloor \frac{n-2}{2} \rfloor} \binom{n-2-k}{k} \cdot x^{n-2k-2} \cdot y^{k+1} + x^n.
\end{aligned}$$

If n is odd, then $\lfloor \frac{n-1}{2} \rfloor - 1 = \lfloor \frac{n-2}{2} \rfloor$, and we have

$$\begin{aligned}
a_{n+1}(x,y) &= \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor - 1} \left[\binom{n-2-k}{k+1} + \binom{n-2-k}{k} \right] \cdot x^{n-2k-2} \cdot y^{k+1} + x^n \\
&= \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor - 1} \binom{n-1-k}{k+1} \cdot x^{n-2k-2} \cdot y^{k+1} + x^n \\
&= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} \cdot x^{n-2k} \cdot y^k.
\end{aligned}$$

If n is even, then $\lfloor \frac{n-1}{2} \rfloor = \lfloor \frac{n-2}{2} \rfloor$, and we have

$$\begin{aligned}
a_{n+1}(x,y) &= \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor - 1} \left[\binom{n-2-k}{k+1} + \binom{n-2-k}{k} \right] \cdot x^{n-2k-2} \cdot y^{k+1} + x^n + y^{\lfloor \frac{n}{2} \rfloor} \\
&= \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor - 1} \binom{n-1-k}{k+1} \cdot x^{n-2k-2} \cdot y^{k+1} + x^n + y^{\lfloor \frac{n}{2} \rfloor} \\
&= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} \cdot x^{n-2k} \cdot y^k.
\end{aligned}$$

□

It would be nice to have a combinatorial description for $a_n(x,y)$ in the general case, not only when $x > 0$ and $y > 0$. In the following proposition, we give a combinatorial description of $a_n(x,y)$ in the case when $x > 0$, $y < 0$, and $-y < x$. This is the first situation where one of the variables x and y takes negative values.

Proposition 4. *Let $x > 0$, $y < 0$, and $-y < x$. If b_n is the number of words of length $n-1$ over $\{0, 1, \dots, x-1\}$ with no subword of the form $0i$, where $i \in \{1, 2, \dots, -y\}$, then*

$$b_n = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-k}{k} \cdot x^{n-2k-1} \cdot y^k. \tag{3}$$

Proof. Let d_n be the number of words of length $n-1$. It is obvious that $d_1 = 0$ and $d_2 = x$. Assume that $n > 2$. We have xd_{n-1} words beginning with an arbitrary letter. From this number, we must subtract the number of words which begin with subwords of the form $0i$, $1 \le i \le -y$. Hence, d_n satisfies the same recurrence as a_n does. □

2 Ten fundamental sequences

Some well-known integer sequences are fundamental, for instance, Fibonacci numbers F_n, Fibonacci polynomials, Jacobhstal numbers, and Pell numbers. They are obtained when $x > 0$, $y > 0$. The first example concerning the case $x > 0$, $y < 0$ shows that positive integers also make a fundamental sequence. Chebyshev polynomials of the second kind, bisection of Fibonacci numbers, and Mersenne numbers also belong to this class.

In the cases under consideration, x and y will always have fixed values, so that we can write a_n instead of $a_n(x, y)$, omitting x and y to simplify notation, as their values will always be clear from the context.

Example 5. For $x = 1$ and $y = 1$, we have $a_{n+1} = F_n$, $n \ge 1$. Also, (2) is the standard expression for the Fibonacci numbers in terms of the binomial coefficients:

$$F_n = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-k-1}{k}.$$

Combinatorially, the Fibonacci number F_n equals the number of binary words of length n avoiding a run of zeros of odd length.

Example 6. If $x > 0$ and $y = 1$, then $a_{n+1} = F_n(x)$, $n \ge 1$, where $F_n(x)$ is the nth Fibonacci polynomial. Also, (2) gives the explicit expression for $F_n(x)$:

$$F_n(x) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-k-1}{k} x^{n-2k-1}.$$

Combinatorially, if $x > 0$ is an integer, then $F_n(x)$ equals the number of words of length n over $\{0, 1, \dots, x-1\}$ in which 0 avoids runs of odd lengths.

Example 7. For $x = 2$ and $y = 1$, we have $a_{n+1} = P_n$, where P_n, $n \geq 0$, is the nth Pell number. From (2), we obtain

$$P_n = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} 2^{n-2k-1} \cdot \binom{n-k-1}{k}.$$

Also, P_n equals the number of ternary words of length n in which 0 avoids runs of odd lengths.

The Pell numbers are sometimes called "silver Fibonacci numbers".

Example 8. For $x = 1$ and $y = 2$, we have $a_{n+1} = J_n$, where J_n, $(n = 0, 1, 2, \ldots)$, are the Jacobsthal numbers. From (2), we obtain

$$J_n = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} 2^k \binom{n-k-1}{k-1}.$$

Also, the number J_n equals the number of ternary words of length n in which 0 and 1 avoid runs of odd lengths.

Example 9. Let $x = 3$, $y = 1$, and $n > 0$. If n is even, then $a_{n+1} = \lceil \Phi \cdot a_n \rceil$. If n is odd, then $a_{n+1} = \lfloor \Phi \cdot a_n \rfloor$. Here, Φ is the golden ratio. The numbers a_{n+1} are sometimes called "bronze Fibonacci numbers". Furthermore, we have

$$a_{n+1} = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} 3^{n-2k-1} \binom{n-k-1}{k}.$$

Also, a_{n+1} equals the number of quaternary words of length n in which 0 avoids runs of odd lengths.

Example 10. When $x = 2$ and $y = 2$, a_{n+1} is the number of ways to tile a board of length n using red and blue tiles of length 1 and 2. We also have

$$a_{n+1} = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} 2^{n-k-1} \binom{n-k-1}{k}.$$

The term a_{n+1} counts quaternary words of length n such that 0 and 1 avoid runs of length 1 and 2.

Example 11. If $x = 2$ and $y = -1$, then $a_0 = 0$, $a_1 = 1$, and $a_{n+1} = 2a_n - a_{n-1}$, which is the recurrence for non-negative integers. Thus, we obtain

$$n = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \cdot 2^{n-2k-1} \binom{n-k-1}{k}, \ n > 0.$$

This formula for n may seem rather complex, but its combinatorial meaning is very simple. Namely, n equals the number of binary words of length $n-1$ avoiding 01, which is obvious.

Example 12. When $x = 3$ and $y = -1$, we have $a_n = F_{2n}$. From (3), we obtain

$$F_{2n} = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \cdot 3^{n-2k-1} \cdot \binom{n-k-1}{k}.$$

Also, $F_{2n} + 1$ equals the number of ternary words of length n avoiding 01.

Example 13. When $x = 3$ and $y = -2$, we have $a_n = 2^n - 1$. These numbers are usually called the Mersenne numbers. We have

$$2^n - 1 = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-2)^k \cdot 3^{n-2k-1} \cdot \binom{n-k-1}{k}.$$

Also, $2^{n+1} - 1$ equals the number of ternary words of length n avoiding 01 and 02.

Example 14. If $x = 2z$ and $y = -1$, then $a_n = U_n(z)$, where $U_n(z)$ is the Chebyshev polynomial of the second kind. From (3), we obtain the following well-known formula:

$$U_n(z) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \cdot \binom{n-k-1}{k} (2z)^{n-2k-1}.$$

If $z > 0$ is an integer, then $U_{n+1}(z)$ equals the number of words of length n over the alphabet $\{0, 1, \ldots, 2z - 1\}$ avoiding the subword 01.

3 Generalized d'Ocagne's identity

Throughout this section, we assume that (a) is a fundamental sequence, and that $(b) = (b_0, b_1, \ldots)$ and $(c) = (c_0, c_1, \ldots)$ both satisfy (1).

The basic result that we use to investigate fundamental sequences is the one proved in Proposition 8 in [1]. The following identity is a direct corollary of Proposition 8 in [1].

Theorem 15 (Generalized d'Ocagne's identity)**.**

$$\begin{vmatrix} b_k & b_{k+m} \\ c_k & c_{k+m} \end{vmatrix} = (-y)^k \cdot a_m \cdot \begin{vmatrix} b_0 & b_1 \\ c_0 & c_1 \end{vmatrix}. \tag{4}$$

To clarify the name, we prove that the d'Ocagne's identity for Fibonacci numbers is a particular case of this identity. When $x = y = 1$, (2) becomes the recurrence for Fibonacci numbers. Hence, $a_m = F_m$, $m \geq 0$. For sequences (b) and (c), we again choose the Fibonacci numbers with the initial conditions such that the determinant on the right-hand side of (4) is equal to 1. For instance, we can choose $c_0 = 0, c_1 = 1$, and $b_0 = 1, b_1 = 1$, that is, $b_k = F_{k+1}$ and $c_k = F_k$. We thus obtain

$$\begin{vmatrix} F_{k+1} & F_{k+m+1} \\ F_k & F_{k+m} \end{vmatrix} = (-1)^k \cdot F_m,$$

which is d'Ocagne's identity.

In Section 4 of [1], we stated a number of d'Ocagne's identities for Fibonacci numbers and polynomials, and Chebyshev polynomials.

Definition 16. A property of sequences (b) and (c) is said to be universal if it does not depend on the fundamental sequence (a), that is, if it depends only on the recurrence, but not on the initial conditions.

We derive several universal properties. Note that, for $m = 1$, we have $a_1 = 1$, so that from (4) we obtain the following identity.

Identity 17 (Generalized Cassini's identity)**.**

$$\begin{vmatrix} b_k & b_{k+1} \\ c_k & c_{k+1} \end{vmatrix} = (-y)^k \cdot \begin{vmatrix} b_0 & b_1 \\ c_0 & c_1 \end{vmatrix}.$$

Assuming that $x = y = 1, b_0 = 1, b_1 = 1$, $c_0 = 0$, and $c_1 = 1$, we obtain the standard Cassini's identity for Fibonacci numbers.

As an immediate consequence of the Cassini's identity, we obtain the next basic property.

Corollary 18. *The determinant* $\begin{vmatrix} b_k & b_{k+1} \\ c_k & c_{k+1} \end{vmatrix}$ *is always divisible by* $(-y)^k$*. If* $\begin{vmatrix} b_0 & b_1 \\ c_0 & c_1 \end{vmatrix} = 1$*, then*

$$\begin{vmatrix} b_k & b_{k+1} \\ c_k & c_{k+1} \end{vmatrix} = (-y)^k.$$

Next, we assume that $k \geq p$. Replacing k by $k - p$ in (4) yields

$$\begin{vmatrix} b_{k-p} & b_{k+m-p} \\ c_{k-p} & c_{k+m-p} \end{vmatrix} = (-y)^{k-p} \cdot a_m \cdot \begin{vmatrix} b_0 & b_1 \\ c_0 & c_1 \end{vmatrix}.$$

If we apply (4) to the right-hand side of the previous equality, we obtain the following universal property.

Identity 19 (Index reduction formula)**.** *If* $k \geq p$*, then*

$$\begin{vmatrix} b_k & b_{k+m} \\ c_k & c_{k+m} \end{vmatrix} = (-y)^p \cdot \begin{vmatrix} b_{k-p} & b_{k-p+m} \\ c_{k-p} & c_{k-p+m} \end{vmatrix}.$$

In particular, if $k = p$, then by using the index reduction formula, we can write d'Ocagne's identity in a universal form:

$$\begin{vmatrix} b_k & b_{k+m} \\ c_k & c_{k+m} \end{vmatrix} = (-y)^k \cdot \begin{vmatrix} b_0 & b_m \\ c_0 & c_m \end{vmatrix}.$$

By comparing the last equality with (4), we obtain the following identity.

Identity 20 (Reduced d'Ocagne's identity)**.**

$$a_m \cdot \begin{vmatrix} b_0 & b_1 \\ c_0 & c_1 \end{vmatrix} = \begin{vmatrix} b_0 & b_m \\ c_0 & c_m \end{vmatrix}.$$

We illustrate this formula with two identities. The first identity concerns Fibonacci and Lucas numbers (L_n, $n \geq 0$).

Identity 21. *For arbitrary non-negative numbers m, p, and q, where $m > p$, the following holds*

$$F_m \cdot \begin{vmatrix} L_p & L_{p+1} \\ L_q & L_{q+1} \end{vmatrix} = \begin{vmatrix} L_p & L_m \\ L_q & L_m \end{vmatrix}.$$

The next identity concerns Chebyshev polynomials $U_n(x)$ of the second kind, and Chebyshev polynomials $T_n(x)$ of the first kind.

Identity 22.

$$U_m(x) \cdot \begin{vmatrix} T_p(x) & T_{p+1}(x) \\ T_q(x) & T_{q+1}(x) \end{vmatrix} = \begin{vmatrix} T_p(x) & T_m(x) \\ T_q(x) & T_m(x) \end{vmatrix}.$$

4 Some bilinear identities

We note that the left-hand side of (4) is bilinear, while the right-hand side is generally not bilinear. However, it is easy to make the right-hand side bilinear, and thus obtain a number of bilinear identities. For instance, we take $b_0 = b_p$, $b_1 = b_{p+1}$ and $c_0 = 0$, $c_1 = 1$. This implies that $(c) = (a)$, and subsequently, we obtain the following identity.

Identity 23 (Generalized Vajda's identity)**.**

$$\begin{vmatrix} b_{k+p} & b_{k+m+p} \\ a_k & a_{k+m} \end{vmatrix} = (-y)^k \cdot a_m \cdot b_p. \tag{5}$$

It is clear that, if $a_i = F_i, b_i = F_i$ $(i = 0, 1, \ldots)$, we obtain Vajda's identity for Fibonacci numbers.

We now give three examples of Vajda's identity for non-Fibonacci numbers.

Identity 24 (Vajda's identity for positive integers)**.** *If $a_i = i$, $i \geq 0$, then*

$$\begin{vmatrix} k+p & k+m+p \\ k & k+m \end{vmatrix} = m \cdot p.$$

In (5), if we set $m = p = r$, and $k = n - r$, we obtain the following identity.

Identity 25 (Generalized Catalan's identity)**.**

$$\begin{vmatrix} b_n & b_{n+r} \\ a_{n-r} & a_n \end{vmatrix} = (-y)^{n-r} \cdot a_r^2.$$

It is clear that this identity generalizes the standard Catalan's identity for Fibonacci numbers, which is obtained for $(a) = (b) = (F)$ and $y = 1$. We illustrate this case with several examples.

Identity 26 (Jacobhstal numbers)**.** *If $a_n = b_n = J_n$, then*

$$J_n^2 - J_{n-i} \cdot J_{n+i} = (-2)^{n-i} \cdot J_i^2.$$

Identity 27 (Pell numbers)**.** *If $a_n = b_n = P_n$, then*

$$P_n^2 - P_{n-i} \cdot P_{n+i} = (-1)^{n-i} \cdot P_i^2.$$

Identity 28 (Chebyshev polynomials of the second kind)**.** *Assume that $a_n = U_n(z)$. Then*

$$U_n^2(z) - U_{n-i}(z) \cdot U_{n+i}(z) = (-1)^{n-i} \cdot U_i^2(z).$$

Let p and q be arbitrary non-negative integers. By replacing b_0 by b_p and c_0 by c_q, we obtain

$$\begin{vmatrix} b_{k+p} & b_{k+m+p} \\ c_{k+q} & c_{k+m+q} \end{vmatrix} = (-y)^k \cdot a_m \cdot \begin{vmatrix} b_p & b_{p+1} \\ c_q & c_{q+1}. \end{vmatrix}.$$

We next assume that $p > q$. By using the index reduction theorem, we obtain

$$\begin{vmatrix} b_p & b_{p+1} \\ c_q & c_{q+1} \end{vmatrix} = (-y)^q \cdot \begin{vmatrix} b_{p-q} & b_{p-q+1} \\ c_0 & c_1. \end{vmatrix}.$$

Now, we again assume that $(c) = (a)$, so that we obtain

$$\begin{vmatrix} b_p & b_{p+1} \\ a_q & a_{q+1} \end{vmatrix} = (-y)^q \cdot b_{p-q}.$$

Additionally, by substituting $m - k$ for m, we finally obtain the following identity.

Identity 29 (Four parameter identity)**.**

$$\begin{vmatrix} b_{k+p} & b_{m+p} \\ a_{k+q} & a_{m+q} \end{vmatrix} = (-y)^{k+q} \cdot a_{m-k} \cdot b_{p-q}. \tag{6}$$

From this identity, we can easily obtain Vajda's generalized identity. But since there is one parameter more, we are able to derive some other identities too. For instance, if we set $p = m$ and $q = k$, we obtain the following identity.

Identity 30.

$$\begin{vmatrix} b_{k+m} & b_{2m} \\ a_{2k} & a_{k+m} \end{vmatrix} = (-y)^{2k} \cdot a_{m-k} \cdot b_{m-k}. \tag{7}$$

In the case when $(a) = (b) = (F)$ and $y = 1$, we obtain the following identity for Fibonacci numbers:

$$F_{k+m}^2 - F_{m-k}^2 = F_{2k} \cdot F_{2m}.$$

Also, if $p = m+1, q = k+1$, then the following identity holds.

Identity 31.

$$\begin{vmatrix} b_{k+m+1} & b_{2m+1} \\ a_{2k+1} & a_{k+m+1} \end{vmatrix} = (-y)^{2k+1} \cdot a_{m-k} \cdot b_{m-k}.$$

In particular, we have

$$F_{k+m+1}^2 + F_{m-k}^2 = F_{2k+1} \cdot F_{2m+1}.$$

References

[1] M. Janjić, Determinant and Recurrence Sequences, *J. Integer Sequences*, **15** (2012), Article 12.3.5

[2] M. Janjić, Words and Linear Recurrences, *J. Integer Sequences*, **21** (2018), Article 18.1.4

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