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## Maximal Sets of Equiangular Lines

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I introduce the problem of finding maximal sets of equiangular lines, in both its real and complex versions, attempting to write the treatment that I would have wanted when I first encountered the subject. Equiangular lines intersect in the overlap region of quantum information theory, the octonions and Hilbert's twelfth problem. The question of how many equiangular lines can fit into a space of a given dimension is easy to pose — a high-school student can grasp it — yet it is hard to answer, being as yet unresolved. This contrast of ease and difficulty gives the problem a classic charm.

To motivate the definition, we can start with the most elementary example: the diagonals of a regular hexagon. Any two of them cross and define what the schoolbooks call supplementary vertical angles. Without loss of information, we can take "the" angle defined by the pair of lines to be the smaller of these two values. Moreover, this value is the same for all three possible pairs of lines: For any two diagonals, their angle of intersection will be  $\pi/3$ . We can state this in a way amenable to generalization if we lay a unit vector along each of the three diagonals. Whichever way we choose to orient the vectors, their inner products will satisfy

$$|\langle v_j, v_k \rangle| = \begin{cases} 1, \ j = k; \\ \alpha, \ j \neq k. \end{cases}$$
(1)

When a set of vectors  $\{v_j : j = 1..., N\}$  enjoys this property, it yields a set of equiangular lines. This definition works equally well in  $\mathbb{R}^d$  and in  $\mathbb{C}^d$ . An orthonormal basis is equiangular, with  $\alpha = 0$ . The question becomes more intriguing when we push the size N of the set beyond the dimension d. It is not terribly difficult to prove the *Gerzon bound*: The size of a set of equiangular lines cannot exceed d(d+1)/2 in  $\mathbb{R}^d$  or  $d^2$  in  $\mathbb{C}^d$  [1]. When the Gerzon bound is met, the value of  $\alpha$  is fixed, to  $1/\sqrt{d+2}$  in  $\mathbb{R}^d$  and  $1/\sqrt{d+1}$  in  $\mathbb{C}^d$ . In the real case, we know that we cannot in general attain the Gerzon bound, and the question of how big N can be as a function of d ties in with some of the most remarkable structures in discrete mathematics. Meanwhile, in the complex case, it *appears* that we can attain the Gerzon bound in every dimension, but decades of work have not yet settled the matter one way or the other — and what we have seen so far has already led us into deep questions of number theory and quantum mechanics.

We take up the real case first. In general, the discrete choice of sign factors made when picking unit vectors to represent lines gives the study of maximal equiangular line-sets in  $\mathbb{R}^d$ a very combinatorial flavor, and the theory of finite simple groups plays an intriguing role. A mere sampling of the known solutions and the topics related to them will, regrettably, have to suffice. The three diagonals of a regular hexagon are a maximal set of equiangular lines in  $\mathbb{R}^2$ , saturating the Gerzon bound. Likewise, the six diagonals of a regular icosahedron attain the Gerzon bound in  $\mathbb{R}^3$ . We start to fall short at dimension d = 4, where it turns out we cannot exceed N = 6. The only two other cases where the Gerzon bound can be reached, as far as anyone knows, are in d = 7 and d = 23. While the regular icosahedron dates back to the semi-legendary centuries of ancient mathematics, thinking in terms of maximal sets of equiangular lines is nowadays credited to Haantjes, who solved the  $\mathbb{R}^2$  and  $\mathbb{R}^3$  cases in 1948. Van Lint and Seidel resolved  $\mathbb{R}^4$  through  $\mathbb{R}^7$  in 1966 [2]. Even today, uncertainty starts to creep in as soon as dimension d = 17 [3]. The Gerzon bound can only be attained above d = 3 if d+2 is the square of an odd integer, but not all odd integers qualify: The maximum is known to be strictly less than the Gerzon bound in dimensions 47 and 79 [4, 5].

But how remarkable those solutions in d = 7 and d = 23 are! These sets of equiangular lines can be extracted from celebrated structures in one dimension higher, the E<sub>8</sub> and Leech lattices respectively. The N = 28 lines in  $\mathbb{R}^7$  are the diagonals of the Gossett polytope  $3_{21}$ , and they correspond among other things to the 28 bitangents to a general plane quartic [6]. One way to obtain these lines — there are different constructions, but the results are all equivalent up to an overall rotation — stems from an observation by Van Lint and Seidel [2, 7] that if we want an interesting set that involves the number seven, we ought to turn to the Fano plane sconer or later. This geometry (the "combinatorialist's coat of arms") is a set of seven points grouped into seven lines such that each line contains three points and each point lies within three distinct lines, with each pair of lines intersecting at a single point. Consequently, if we take the *incidence matrix* of the Fano plane, writing a 1 in the *ij*-th entry if line *i* contains point *j*, then every two rows of the matrix have exactly the same overlap:

$$M = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}.$$
 (2)

The rows of the incidence matrix furnish us with seven equiangular lines in  $\mathbb{R}^7$ . To build this out into a full set of 28 lines, we can introduce sign factors [2, 7], and one way to do that is to add *orientations* to the Fano plane, exactly as one does when using it as a mnemonic for octonion multiplication. We can label the seven points with the imaginary octonions  $e_1$ through  $e_7$ . When drawn on the page, a useful presentation of the Fano plane has the point  $e_4$  in the middle and, reading clockwise, the points  $e_1$ ,  $e_7$ ,  $e_2$ ,  $e_5$ ,  $e_3$  and  $e_6$  around it in a regular triangle. The three sides and three altitudes of this triangle, along with the inscribed circle, provide the seven Fano lines:  $(e_1, e_2, e_3)$ ,  $(e_1, e_4, e_5)$ ,  $(e_1, e_7, e_6)$ ,  $(e_2, e_4, e_6)$ ,  $(e_2, e_5, e_7)$ ,  $(e_3, e_4, e_7)$ ,  $(e_3, e_6, e_5)$ . The sign of a product depends upon the order, for example,  $e_1e_2 = e_3$ but  $e_2e_1 = -e_3$ . The full multiplication table, due to Cayley and Graves, is

which we can express visually by carefully assigning arrows to the lines of the Fano plane.

d	<b>2</b>	3	4	5	6	7 - 14	15	16	17
$N_{\max}(d)$	3	6	6	10	16	28	36	40	48 - 49
d	18	19	20	21	22	<b>23</b> –41	42	43	
$N_{\max}(d)$	56 - 60	72 - 75	90 - 95	126	176	276	276 - 288	344	

TABLE I: Bounds on the largest possible size of a set of equiangular lines in  $\mathbb{R}^d$ . For some values of the dimension d, the bound is not known exactly. Dimensions where the Gerzon bound is saturated are shown in bold. For more details, see **OEIS:A002853** and references therein.

To build our set of 28 equiangular vectors, start by taking the first row of the incidence matrix M, which corresponds to the line  $(e_1, e_2, e_3)$ , and give it all possible choices of sign by multiplying by the elements not on that line. Multiplying by  $e_4$ ,  $e_5$ ,  $e_6$  and  $e_7$  respectively, we get

$$\begin{pmatrix} + & + & + & 0 & 0 & 0 \\ - & + & - & 0 & 0 & 0 \\ - & - & + & 0 & 0 & 0 \\ + & - & - & 0 & 0 & 0 \end{pmatrix} .$$
 (4)

Doing this with all seven lines of the Fano plane, we obtain a set of 28 vectors, each one given by a choice of a line and a point not on that line. For any two vectors derived from the same Fano line, two of the terms in the inner product will cancel, leaving an overlap of magnitude 1. And for any two vectors derived from different Fano lines, the overlap always has magnitude 1 because any two lines always meet at exactly one point.

As we go up from  $\mathbb{R}^7$  to  $\mathbb{R}^{23}$ , the properties tap into more of the esoteric. As mentioned above, we can obtain the N = 276 equiangular lines living in  $\mathbb{R}^{23}$  from the Leech lattice. This lattice is how a grocer would stack 24-dimensional oranges; the points of the lattice are the locations of the centers of the spheres in the densest possible packing thereof in 24 dimensions [8]. The vectors from the origin to the lattice points are classified by their "type", which is half their norm. To obtain a Gerzon-bound-saturating set of equiangular lines, start with a vector of type 3. For any such vector  $\vec{v}$ , there are 276 unordered pairs of other lattice vectors having minimal norm (type 2) that add to  $\vec{v}$ . Each pair specifies a line, and we obtain 276 lines in all. These lines have rich group-theoretic significance [1, 9]. First of all, their symmetry group is Conway's third sporadic group Co<sub>3</sub>. The stabilizer of a line is the subgroup comprising those symmetries of the set that leave that line fixed while permuting the others. For the N = 276 lines in  $\mathbb{R}^{23}$ , the stabilizer of any line is isomorphic to the McLaughlin sporadic simple group McL. Moreover, if we find the largest possible subset of the N = 276 lines that are all orthogonal to a common vector in  $\mathbb{R}^{23}$ , we obtain a set of 176 vectors that all squeeze into  $\mathbb{R}^{22}$ , and these furnish a maximal set of equiangular lines in that dimension. The symmetries of this set constitute the Higman–Sims sporadic simple group HS. By close consideration of the ways in which vector directions can be assigned to the N = 276 lines, it is also possible to distinguish special subsets of 23 lines and thence obtain the Mathieu group  $M_{23}$  [10].

We do not yet have a general theory of how the maximum N varies with the dimension d, and of course, we can only hazard guesses about what the textbooks of tomorrow might contain, but one conceptual connection that has proved quite important so far is to algebraic graph theory. If  $\{v_i : i = 1, ..., N\}$  is a set of unit vectors defining a set of equiangular lines, then by the definition of equiangularity, the Gram matrix of these unit vectors will

be 1 along the diagonal and  $\pm \alpha$  everywhere else. Note that specifying a choice of sign for each off-diagonal entry is exactly the same information needed to specify which edges are connected in a graph with N vertices. Changing the sign of any vector will flip the signs on some of the entries in the Gram matrix, which rewires the corresponding graph in a particular way. So, a set of N equiangular lines corresponds to an equivalence class of N-vertex graphs (a "switching class"), and there is a natural overlap of interest between equiangular lines and strongly regular graphs [5, 10–12].

Because the matrices one tries to construct when attempting to build sets of real equiangular lines will be filled with integers, conditions for their existence will be equations that algebraic number theory is suited to handle. In particular, the study of cyclotomic fields fields made by extending  $\mathbb{Q}$  with a primitive *n*th root of unity — becomes relevant [12].

One fruitful avenue of inquiry has been to fix an angle and ask how many lines in  $\mathbb{R}^d$  can be equiangular with that chosen angle [4, 5, 13]. Balla et al. have proved that for a fixed  $\alpha$ , when the dimension d becomes sufficiently large there can be at most 2(d-1) lines in  $\mathbb{R}^d$ that are equiangular with common overlap  $\alpha$  [14].

For students who wish to jump in and do calculations as quickly as possible, Tremain [15] provides a useful collection of constructions.

The complex case was first studied as a natural counterpart to the real one. Investigations of structures like complex polytopes [16] turned up maximal sets of complex equiangular lines in dimensions d = 2, 3 and 8. Then the 1999 PhD thesis of Zauner [17], followed by the independent work of Renes et al. [18], made complex equiangular lines into a physics problem. Now we have exact solutions for 102 different values of the dimension, including all dimensions from 2 through 40 inclusive, and some as large as d = 1299. Furthermore, numerical solutions are known to high precision for all dimensions  $d \leq 193$ , and some as large as d = 2208 [19, 20]. These lists have grown irregularly, since different simplifications have proved applicable in different dimensions. (We endeavor to keep the website [21] up to date.) Credit is due to M. Appleby, I. Bengtsson, T.-Y. Chien, S. T. Flammia, M. Grassl, G. S. Kopp, A. J. Scott and S. Waldron.

Physicists know sets of  $d^2$  equiangular lines in  $\mathbb{C}^d$  as "Symmetric Informationally Complete Positive-Operator-Valued Measures", a mouthful that is abbreviated to *SIC-POVM* and often further just to *SIC* (pronounced "seek"). The "Symmetric" refers to the equiangularity property, while the rest summarizes the role these structures play in quantum theory [22, 23]. A basic premise of quantum physics is that to each physical system of interest is associated a complex Hilbert space  $\mathcal{H}$ . The subdisciplines of quantum information and computation [24] often employ finite-dimensional Hilbert spaces,  $\mathcal{H}_d \simeq \mathbb{C}^d$ . The mathematical representation of a measurement process is a set of positive semidefinite operators  $\{E_j\}$ on  $\mathcal{H}_d$  that sum to the identity. Each operator in the set stands for a possible outcome of the measurement, and the set as a whole is known as a POVM. To represent the preparation of a quantum system, we ascribe to the system a density operator  $\rho$  that is also a positive semidefinite operator on  $\mathcal{H}_d$ , in this case normalized so that its trace is unity. The Born rule states that the probability for obtaining an outcome of a measurement is given by the Hilbert–Schmidt inner product of the density operator and the POVM element representing that outcome:

$$p(j) = \operatorname{tr} \rho E_j. \tag{5}$$

If the POVM  $\{E_j\}$  has at least  $d^2$  elements, then it is possible for it to span the space of Hermitian operators on  $\mathcal{H}_d$ . In this case, the POVM is *informationally complete* (IC), because any density operator can be expressed as its inner products with the POVM elements. Or, in more physical terms, the probabilities for the possible outcomes of an IC POVM completely specify the preparation of the quantum system. Given a set of unit vectors in  $\mathbb{C}^d$  defining  $d^2$  equiangular lines, the projectors  $\{\Pi_j\}$  onto these lines span the Hermitian operator space, and up to normalization they provide a resolution of the identity:

$$\sum_{j=1}^{d^2} \Pi_j = dI. \tag{6}$$

SICs satisfy a host of optimality conditions. By many standards, a SIC is as close as one can possibly get to having an orthonormal basis for Hermitian operator space while staying within the positive semidefinite cone. This is highly significant for quantum theory, because positivity is a crucial aspect of having operator manipulations yield well-defined probabilities.

Decades of work on the fundamentals of quantum physics have shown that quantum uncertainty cannot be explained away as ignorance about "hidden variables" intrinsic to physical systems but concealed from our view. Historically, this area has been home to deep theorems like the results of Bell, Kochen and Specker [25], while in the modern age, it is also a topic of increasingly practical relevance, since if we want our quantum computers to be worth the expense, we had better understand exactly which phenomena can be imitated classically and which cannot [26]. SICs provide a new window on these questions by furnishing a measure of the margin by which any attempt to model quantum phenomena with intrinsic hidden variables is guaranteed to fail [27].

Here we have a peculiar confluence of topics advanced by the late John Conway. It was his insight into the Leech lattice that gave us the maximal equiangular set in  $\mathbb{R}^{23}$ , and together with Simon Kochen he carried forward the study of how hidden-variable hypotheses break down [28]. Somewhere in the world's weight of loss is the fact that we will never know what he might have thought about these two problems coming together.

Above, we noted the group-theoretic properties of real equiangular lines. Group theory also manifests in the complex case. First, all known SICs are *group covariant*, meaning that they can be generated by taking a well-chosen initial vector and computing its orbit under a group action. Moreover, in all cases except a class of solutions in d = 8, the group in question is the Weyl-Heisenberg group for dimension d. Given an orthonormal basis  $\{e_n : i = 1, \ldots, d\}$ , we define a shift operator X that sends  $e_n$  to  $e_{n+1}$  modulo d, and a "clock" or "phase" operator Z that sends  $e_n$  to  $\exp(2\pi i n/d)e_n$ . The two unitary operators X and Z commute up to the phase factor  $\exp(2\pi i/d)$ , and together with phase factors their products define the Weyl-Heisenberg group. (The d = 2 special case is also known as the Pauli group. Weyl introduced X and Z in 1925, in order to define what the quantum mechanics of a discrete degree of freedom could mean [29]. The association of Heisenberg's name with this group is a convention that has less historical justification, since the "canonical commutation relation" of position and momentum that inspired Weyl was not due to Heisenberg [30, 31].) The Weyl–Heisenberg group is significant in multiple aspects of quantum theory, such as the study of when quantum computations can be efficiently emulated classically. Zhu has proved that when the dimension d is prime, group covariance implies Weyl–Heisenberg covariance [32]. However, it is not known whether SICs must be group covariant in general.

In dimension d = 8, there exist Weyl-Heisenberg SICs but also a class of SICs covariant under a different group (the tensor product of three copies of the Pauli group). These were first discovered by Hoggar and can be termed the *Hoggar-type* SICs [33–35]. All of them are equivalent to one another under unitary and anti-unitary conjugations. Zhu discovered that the stabilizer of an element in a Hoggar-type SIC is always isomorphic to PSU(3,3), the group of  $3 \times 3$  unitary matrices over the finite field of order 9 [36]. This is moreover the commutator subgroup of the automorphism group of the Cayley integral octonions [37], also known as the *octavians* [38]. In other words, the linear maps from the octavians to themselves that preserve the multiplication structure form a group, and an index-2 subgroup of that gives the ways to hold one vector in a Hoggar-type SIC in place and permute the other 63.

The octavians form a lattice, and up to an overall scaling, it is the same as the  $E_8$  lattice. So, a symmetric arrangement of *complex lines* is, under the surface, tied in with an optimal packing of *real hyperspheres* [39] — a development that was completely unforeseen.

Exploration of the Weyl-Heisenberg SICs leads into algebraic number theory, and at a more subtle and demanding level than that subject has so far manifested in the study of real equiangular lines. During the early years, SIC vectors were found by computer algebra, laborious calculations with Gröbner bases yielding coefficients that were, in technical terms, ghastly [40]. Pages upon pages of printouts were sacrificed. Yet, upon closer study, these solutions turned out to be not as bad as they seemingly could have been: In  $d \ge 4$ , they were always expressible in terms of nested radicals. Since the polynomial equations being solved were of degrees much higher than quintic, there was no reason to expect a solution by radicals, and so the Galois theory of the SIC problem became a topic of interest [41]. This has led to a series of surprises.

Recall that a group G is solvable if we can write a series starting with the trivial group,  $\langle 1 \rangle < H_1 < \cdots < H_m < G$ , where each of the  $H_i$  is a normal subgroup of the next and the quotients  $H_i/H_{i-1}$  are all abelian. The zeros of a polynomial f(x) over some field  $\mathbb{K}$  can be found by radicals exactly when the splitting field of f, the extension of  $\mathbb{K}$  in which ffalls apart into linear factors, can be made by stacking up abelian extensions of  $\mathbb{K}$ . So, if we want to understand solvability by radicals, getting a handle on abelian extensions of number fields is the thing to do. One obvious place to start is abelian extensions of the rationals  $\mathbb{Q}$ , and the *Kronecker–Weber theorem* tells us that every abelian extension of  $\mathbb{Q}$  is contained in a cyclotomic field.

Hilbert's twelfth problem asks for a broadening of the Kronecker–Weber theorem, or in other words, a classification of the abelian extensions of arbitrary number fields. This problem remains unresolved, although progress has been made. When we generalize beyond the case where the base field is  $\mathbb{Q}$ , the role of the cyclotomic fields is played by the *ray class fields*. The generalization of the *n* in a cyclotomic field's *n*th root of unity is a number called the *conductor* of the ray class field. In the original case covered by Kronecker–Weber, the fields in which the abelian extensions all live are generated by special values of a special function, i.e., the exponential function evaluated at certain points along the imaginary axis. What functions play the role of the exponential more generally, and at what points should they be evaluated? This is much more difficult to say.

Historically, the first to be understood beyond abelian extensions of the rationals themselves were abelian extensions of imaginary quadratic fields, that is,  $\mathbb{Q}(\sqrt{-n})$  where *n* is a positive integer. This is significantly more demanding than the case where the base field is  $\mathbb{Q}$ . The theory that does the job goes by the name *complex multiplication*, which to a mathematics student is deceptively simple. (Perhaps a term like "elliptic multiplication" would be better, but here as elsewhere, the jargon has solidified.) This theory is informally described as an order of magnitude harder than the Kronecker–Weber theorem, and the case of *real* quadratic fields is more difficult still, and only partially understood.

SICs know a lot about Hilbert's twelfth problem [42]. They have been found to generate ray class fields over real quadratic extensions of the rationals  $\mathbb{Q}(\sqrt{D})$ , where D is the squarefree part of the quantity (d-3)(d+1). The path to counterparts for roots of unity goes through the overlap phases, the phases that the absolute-value bars in Eq. (1) throw away. The overlap phases turn out to be *algebraic integer units* in ray class fields or extensions thereof. (Roots of monic polynomials over  $\mathbb{Z}$  are the algebraic integers, so called because their quotients yield the algebraic numbers just as quotients of ordinary integers  $\mathbb{Z}$  yield the rationals Q. The algebraic integers within a number field form a ring, and the units of this ring are those algebraic integers whose multiplicative inverses are also algebraic integers. In the real case, taking the absolute value discards a choice of  $\pm 1$ , while in the complex case, it discards a phase that is not arbitrary, but rather a generalized kind of " $\pm 1$ "!) Recently, Kopp has turned this connection around and, using conjectured properties of important numbers in ray class fields, constructed an exact SIC in d = 23 where none had been known before [43]. Kopp's SIC is constructed from overlap phases calculated as Galois conjugates of square roots of *Stark units*. These numbers figure largely in the Stark conjectures, which pertain to generating ray class fields explicitly. The conceptual waters here run deep, yet more remarkable still is the fact that these beguiling questions of number theory are, by way of almost schoolchildish geometry, intricated with quantum physics.

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