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# A CHAIN OF NORMALIZERS IN THE SYLOW 2-SUBGROUPS OF THE SYMMETRIC GROUP ON $2^n$ LETTERS

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ABSTRACT. On the basis of an initial interest in symmetric cryptography, in the present work we study a chain of subgroups. Starting from a Sylow 2-subgroup of AGL(2, n), each term of the chain is defined as the normalizer of the previous one in the symmetric group on  $2^n$ letters. Partial results and computational experiments lead us to conjecture that, for large values of n, the index of a normalizer in the consecutive one does not depend on n. Indeed, there is a strong evidence that the sequence of the logarithms of such indices is the one of the partial sums of the numbers of partitions into at least two distinct parts.

### 1. INTRODUCTION

Let n be a non-negative integer and let  $\operatorname{Sym}(2^n)$  denote the symmetric group on  $2^n$  letters. The study of the conjugacy class in  $\operatorname{Sym}(2^n)$  of the elementary abelian regular 2-subgroups has recently drawn attention for its application to block cipher cryptanalysis, and in particular to differential cryptanalysis [BS91]. The reader which is familiar with symmetric cryptography will not find hard to realize that the key-addition layer of a block cipher (see e.g. [DR13, BKL<sup>+</sup>07, NBoS77]) acts in general on the partially encrypted states as an elementary abelian regular 2subgroup of the message space. In a recent paper [CCS17], it has been shown that a cryptanalyst can derive from such subgroups new operations on the message space  $\mathbb{F}_2^n$  of the block cipher, which can be used to perform algebraic and statistical attacks. Indeed, although the encryption functions, in order to be secure, are designed to be far from being linear with respect to the classical bitwise addition modulo 2, it is possible to attack the encryption scheme by means of a variation of the classical differential attack, where instead a newly designed operation is used [CBS18]. Such operation is defined starting from a conjugate of the translation group T on the message space.

A study of regular subgroups of the affine group is carried out in [CDVS06, CR09] by means of radical algebras. We point out that there is an interesting connection between our study of the *position* of a regular subgroup in the symmetric group, in terms of the chain of normalizers defined below, and the rather recent theory of *braces*, introduced in [Rum07], since the above mentioned new operation can be used to construct a brace on T. Indeed, when + and  $\circ$  respectively denote the (additive) laws induced by T and by one of its affine conjugates, the structure  $(T, +, \circ)$  is a two-sided brace and  $(T, +, \cdot)$  is a radical ring, where  $a \cdot b$  is defined as  $a + b + a \circ b$ for each  $a, b \in T$ . For an extensive survey and detailed references on braces see e.g. [Ced18].

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In a recent paper [ACGS19], we considered the elements of the conjugacy class  $T^{\text{Sym}(2^n)}$  which are subgroups of the affine group AGL(T). We showed that, if  $T^g \cap T$  has index 4 in T, then there exists a Sylow 2-subgroup U < AGL(T) containing both  $T^g$  and T as normal subgroups. The normalizer  $N^1$  of U in Sym $(2^n)$  contains U as a subgroup of index 2 and interchanges T and  $T^{g}$  by conjugation. The 2-group  $N^{1}$  is therefore contained in a Sylow 2-subgroup  $\Sigma$  of Sym $(2^{n})$ . Motivated by a computational evidence, we prove here that this is the general behavior. We define a chain starting from U and where the k-th term  $N^k$  is the normalizer in  $Sym(2^n)$  of the previous  $N^{k-1}$ . We show that  $N^k$  is actually the normalizer of  $N^{k-1}$  in  $\Sigma$ , and thus the  $N^k$ s form a sequence of 2-groups ending at  $\Sigma$ . Philip Hall, indeed, proved that  $\Sigma$  is self-normalizing (see e.g. [CF64]). Using the software package GAP [GAP20], we computed the normalizer chain for  $n \leq 11$ . We experimentally noticed that the sequence defined by  $c_k = \log_2 |N^k : N^{k-1}|$  does not depend on n if  $k \leq n-2$  and, in such cases,  $\{c_k\}_{k\geq 1}$  represents the sequence of partial sum of the sequence  $\{b_{k+2}\}_{k\geq 1}$ , where  $b_k$  counts the number of partitions of k into at least two distinct parts, a well-known sequence of integers [OEI, https://oeis.org/A317910], also appearing in commutative algebra problems [ES14]. For larger values of n, the computational problem becomes intractable using the standard libraries, and so its investigation requires a theoretical approach. For small values of k, by way of an elementary but increasingly cumbersome analysis, we show that the previous claim is true. In the general case, the claim remains an open problem. We believe that more sophisticated combinatorial and group theoretical tools could prove that, for  $k \leq n-2$ , the integers  $c_k$  do not depend on n and are related to the sequence  $b_k$  as previously mentioned.

The paper is organized as follows: in Sec. 2 we introduce the notation and provide some preliminary results. The normalizer chain is defined in Sec. 3, which contains the main considerations that led us to formulate Conjecture 1. Some theoretical evidence in support of our conjecture, i.e. Theorem 4.7, is proved in Sec. 4, where we also discuss some open problems. To conclude, Sec. 5 is devoted to the computational aspects and contains the GAP code used for our computations.

### 2. NOTATION AND PRELIMINARIES

In this section, we recall some well known facts and a preliminary result on the imprimitivity action of subgroups of the symmetric group on a finite set.

**Definition 2.1.** Let  $\Omega \neq \emptyset$  and let  $G \leq \text{Sym}(\Omega)$  be a transitive permutation group. An imprimitivity system  $\mathcal{B}$  for G is a G-invariant partition of  $\Omega$ . The group G is primitive if G has only the trivial partitions  $\{\Omega\}$  and the set of the singletons of  $\Omega$  as imprimitivity systems. Otherwise, G is said to be imprimitive.

**Definition 2.2.** Let G act imprimitively on the set  $\Omega$ . An imprimitivity chain  $\mathcal{B}_0 \succ \cdots \succ \mathcal{B}_t$  of depht t is a sequence of imprimitivity systems for G acting on  $\Omega$ , where  $\mathcal{B}_0$  and  $\mathcal{B}_t$  are the trivial partitions. We also require that for each  $B \in \mathcal{B}_{m+1}$  there exists  $B' \in \mathcal{B}_m$  such that  $B \subset B'$  for  $0 \le m \le t-1$ .

Note that the imprimitivity chain  $\mathcal{B}_0 \succ \cdots \succ \mathcal{B}_t$  can be represented by its *imprimitivity tree* which is the rooted tree (V, E), where

- the set of vertices V is  $\bigcup_{m=0}^{t} \mathcal{B}_{m}$ , more precisely a vertex is a subset of  $\Omega$  belonging to some partition  $\mathcal{B}_{i}$  and the root vertex is  $\Omega$ ;
- two vertices X and Y in V are connected by an edge  $e \in E$  if and only if there exists m such that  $X \in \mathcal{B}_m, Y \in \mathcal{B}_{m+1}$  and  $Y \subset X$ .

In the remainder of this work, we will consider the special case of a subgroup G of the symmetric group Sym $(X_n)$ , where  $X_n \stackrel{\text{def}}{=} \{1, \ldots, 2^n\}$ .

For  $0 \le m \le n$  and  $0 \le k \le 2^m - 1$ , the following notation is used:

- $B_{m,k}^n \stackrel{\text{def}}{=} \{k2^{n-m} + 1, \dots, (k+1)2^{n-m}\}, \text{ and in particular } X_n = B_{0,0}^n;$
- $\mathcal{B}_m^n \stackrel{\text{def}}{=} \{B_{m,0}^n, \dots, B_{m,2^m-1}^n\};$  for  $1 \le i \le n$

$$s_i \stackrel{\text{def}}{=} \prod_{j=1}^{2^{i-1}} (j, j+2^{i-1});$$

• 
$$t_i^n \stackrel{\text{def}}{=} \begin{cases} s_i & \text{if } i = n \\ t_i^{n-1} \cdot (t_i^{n-1})^{s_n} & \text{if } 1 \le i < n. \end{cases}$$

The symmetric group  $\operatorname{Sym}(2^n)$  acts on the set of partitions of  $X_n$  and, with respect to this action, we define the subgroup

$$\Sigma_n \stackrel{\text{def}}{=} \bigcap_{m=1}^n \operatorname{Stab}_{\operatorname{Sym}(2^n)}(\mathcal{B}_m^n) = \langle s_1, \dots, s_n \rangle \cong \langle_{i=1}^n C_2,$$

which is the *n*-th iterated wreath product of copies of the cyclic group  $C_2$  of order 2, i.e. a Sylow 2-subgroup of Sym $(2^n)$ . Notice that  $\mathcal{B}_0^n \succ \cdots \succ \mathcal{B}_n^n$  is an imprimitivity chain  $\mathcal{C}_n$  of maximal depth for  $\Sigma_n$  and that  $\Sigma_n$  is the stabilizer of  $\mathcal{C}_n$  in  $\operatorname{Sym}(2^n)$ .

Let  $T_{n,0} \stackrel{\text{def}}{=} \{1\}$  and, for  $1 \leq i \leq n$ , let us define  $T_{n,i} \stackrel{\text{def}}{=} \langle t_1^n, \ldots, t_i^n \rangle$ . Clearly  $T_{n,i} \leq T_{n,i+1}$ , for  $0 \le i \le n-1$ . The group  $T_n \stackrel{\text{def}}{=} T_{n,n}$  is a regular elementary abelian subgroup of Sym $(2^n)$ of order  $2^n$  contained in  $\Sigma_n$ , whose normalizer in Sym $(2^n)$  is AGL $(T_n)$ , the affine general linear group. We also define

$$U_n \stackrel{\text{def}}{=} \operatorname{AGL}(T_n) \cap \Sigma_n = N_{\Sigma_n}(T_n).$$

The group  $T_n$  is a uniserial module for  $U_n$  whose maximal flag  $\mathcal{F}_n$  is defined as

$$\{1\} = T_{n,0} < \dots < T_{n,n} = T_n$$

Given a subgroup  $H \leq \Sigma_{n-1}$ , we define the *diagonal embedding* of H into  $\Sigma_n$  as

$$\Delta_n(H) \stackrel{\text{def}}{=} \{ (x, x^{s_n}) \mid x \in H \}.$$

Remark 1. It was already known to Dixon [Dix71] that the set of elementary abelian regular subgroups of  $Sym(2^n)$  form a unique conjugacy class. Moreover, a transitive abelian subgroup of Sym $(2^n)$  is regular and so is self-celtralizing. In particular,  $(T_n)^g$  is self-centralizing in  $\Sigma_n$ , for every  $q \in \text{Sym}(2^n)$ .

**Lemma 2.3.** Up to conjugation by elements of  $\Sigma_n$ , the group  $T_n$  is the unique elementary abelian regular subgroup of  $\operatorname{Sym}(2^n)$  having  $\mathcal{C}_n$  as imprimitivity chain.

*Proof.* First, recall that  $\Sigma_n$  stabilizes  $\mathcal{C}_n$  for every n. We argue by induction on n, the result being trivial when n = 1. Let T be an elementary abelian regular subgroup of  $\text{Sym}(2^n)$  having  $\mathcal{C}_n$  as imprimitivity chain and let M be the stabilizer in T of  $\{1,\ldots,2^{n-1}\}=B_{1,0}^n\in\mathcal{B}_1^n$ . In particular, M stabilizes also  $B_{1,1}^n = (B_{1,0}^n)^{s_n} = \{2^{n-1} + 1, \dots, 2^n\}$ . The group M acts on  $B_{1,0}^n$ as an elementary abelian regular subgroup  $M_1$  of  $S_{2^{n-1}}$  having  $\mathcal{C}_{n-1}$  as imprimitivity chain. By induction,  $M_1 = (T_{n-1})^{h_1}$  for some  $h_1 \in \Sigma_{n-1}$ . Similarly, the group M acts faithfully on  $B_{1,1}^n$ as an elementary abelian regular subgroup  $M_2$  of  $(S_{2^{n-1}})^{s_n}$  having  $(\mathcal{C}_{n-1})^{s_n}$  as imprimitivity

chain, and thus we find by induction  $M_2 = ((T_{n-1})^{h_2})^{s_n}$  for some  $h_2 \in \Sigma_{n-1}$ . Finally, we have that

$$M = \left\{ (m^{h_1}, m^{h_2 s_n}) \mid m \in T_{n-1} \right\} = \Delta_n (T_{n-1})^{(h_1, h_2^{s_n})}.$$

If  $t \in T \setminus M$  then t interchanges  $B_{1,0}^n$  and  $B_{1,1}^n$  and centralizes M. Let us write t in the form  $t = (a, b^{s_n})s_n$ , where  $a, b \in \Sigma_{n-1}$  and  $(m^{h_1}, m^{h_2 s_n}) = (m^{h_1}, m^{h_2 s_n})^t = (m^{h_2 b}, (m^{h_1 a})^{s_n})$ . Note that

- $1 = t^2 = (a, b^{s_n})s_n(a, b^{s_n})s_n = (ab, (ba)^{s_n})$ , and so  $a = b^{-1}$ ;  $m^{h_1} = m^{h_2 b}$  and  $m^{h_2 s_n} = m^{h_1 a s_n}$  for all  $m \in T_{n-1}$ , from which we derive

$$h_1ah_2^{-1}, h_2bh_1^{-1} \in C_{\Sigma_{n-1}}(T_{n-1}) = T_{n-1}$$

i.e.  $a = h_1^{-1}uh_2 = u^{h_1}h_1^{-1}h_2$  and  $b = a^{-1} = h_2^{-1}h_1u^{h_1} = u^{h_2}h_2^{-1}h_1$  for some  $u \in T_{n-1}$ .

Then we have

$$t = (a, b^{s_n})s_n = (u^{h_1}h_1^{-1}h_2, u^{h_2s_n}(h_2^{-1}h_1)^{s_n})s_n$$
$$\equiv (h_1^{-1}h_2, (h_2^{-1}h_1)^{s_n})s_n = s_n^{(h_1, h_2^{s_n})} \mod M.$$

Since  $T_n = \Delta_n(T_{n-1}) \rtimes \langle s_n \rangle$ , then  $T = T_n^{(h_1, h_2^{s_n})}$ , as required.

Remark 2. Notice that the chain  $C_n$  is a maximal imprimitivity chain for  $T_n$ , even though it is not the only one. It is known that every maximal imprimitivity chain for  $T_n$  determines and is determined by a maximal flag  $\{1\} = T_{n,0} < \cdots < T_{n,n} = T_n$ . Indeed, the partition  $\mathcal{B}_i$  is the set of the orbits of  $T_{n,n-i}$ , and conversely  $T_{n,n-i}$  is the pointwise stabilizer of the action of  $T_n$  over  $\mathcal{B}_i$ . Any Sylow 2-subgroup U of  $AGL(T_n)$  is the stabilizer by conjugation of a maximal flag of  $T_n$ , and therefore it stabilizes also the associated imprimitivity chain. In particular, the stabilizer of  $\mathcal{C}_n$  in  $\operatorname{AGL}(T_n)$  is  $U_n = \Sigma_n \cap \operatorname{AGL}(T_n)$ . More generally, any maximal flag  $\mathcal{F}$  of  $T_n$  determines a Sylow 2-subgroup  $U_{\mathcal{F}}$  of AGL $(T_n)$  and a Sylow 2-subgroup  $\Sigma_{\mathcal{F}}$  [Lei88, Theorem p. 226] of Sym $(2^n)$ such that  $U_{\mathcal{F}} = \Sigma_{\mathcal{F}} \cap \operatorname{AGL}(T_n)$ . The maps  $\mathcal{F} \mapsto U_{\mathcal{F}}$  and  $\mathcal{F} \mapsto \Sigma_{\mathcal{F}}$  are injective. Consequently, for every Sylow 2-subgroup U of  $AGL(T_n)$  there exists a unique Sylow 2-subgroup  $\Sigma$  of  $Sym(2^n)$ such that  $U = \Sigma \cap AGL(T_n)$ . In particular, the intersection  $AGL(T_n) \cap \Sigma_n = N_{\Sigma_n}(T_n) = U_n$  is a Sylow 2-subgroup of  $AGL(T_n)$ .

## 3. Experimental evidence on a normalizer chain

Let us start by defining the normalizer chain of  $T_n$ .

**Definition 3.1.** Using the same notation of the previous section, the normalizer chain of  $T_n$  is defined as the sequence  $\{N_n^k\}_{k>0}$ , where

$$N_n^0 \stackrel{\text{def}}{=} U_n = N_{\Sigma_n}(T_n), \quad N_n^1 \stackrel{\text{def}}{=} N_{\text{Sym}(2^n)}(U_n),$$

and recursively, for k > 1,

$$N_n^k \stackrel{\text{def}}{=} N_{\text{Sym}(2^n)}(N_n^{k-1}).$$

Considering  $\Sigma_n$  in place of Sym $(2^n)$  the resulting chain is the same, as proven in the next theorem.

**Theorem 3.2.** For every  $k \ge 1$ , we have  $N_n^k = N_{\Sigma_n}(N_n^{k-1})$ . In particular,  $N_n^k$  is a 2-group.

*Proof.* Suppose that  $\mathcal{B}$  is a system of imprimitivity for  $N_n^{k-1}$ . For each  $x \in N_n^k$ , the partition  $\mathcal{B}^x$  is a system of imprimitivity for  $(N_n^{k-1})^x$  and so for  $N_n^{k-1}$ , since  $(N_n^{k-1})^x = N_n^{k-1}$ . Thus, for a given  $x \in N_n^k$  and an imprimitivity chain  $\mathcal{C}$  for  $N_n^{k-1}$ , the set  $\mathcal{C}^x$  is also an imprimitivity chain for  $N_n^{k-1}$  and a fortiori for  $U_n$ . Since, by Remark 2, the imprimitivity chain  $\mathcal{C}_n$  is the unique

maximal one for  $U_n = N_n^0$ , we have  $C_n^x = C_n$ . Hence  $C_n$  is stabilized by  $N_n^k$  for every k, yielding  $N_n^k \leq \Sigma_n$ .

A direct consequence of the previous theorem is that there exists  $d(n) \in \mathbb{N}$  such that

$$N_n^k = N_n^{d(n)} = \Sigma_n$$

for every  $k \ge d(n)$ . We can interpret d(n)+1 as an upper bound for the defect  $\delta(n)$  of  $T_n$  in  $\Sigma_n$ , i.e. the length of the shortest subnormal series from  $T_n$  to  $\Sigma_n$ . Recalling that  $\Sigma_n$  is self-normalizing in Sym $(2^n)$  (see [CF64]), as already pointed out in the introduction, the fact previously stated represents a further argument showing that every Sylow 2-subgroup of AGL $(T_n)$  is contained in exactly one Sylow 2-subgroup of Sym $(2^n)$ .

We already recalled in Remark 2 that  $N_n^0 = U_n$  normalizes a maximal flag  $\mathcal{F}$  of  $T_n$ . Below we study the action by conjugation of  $N_n^1$  over  $\mathcal{F}$ .

# **Proposition 3.3.** The group $N_n^1$ normalizes each term of the flag $\{T_{n,0}, \ldots, T_{n,n-2}\}$ .

Proof. It is enough to prove that each element of  $N_n^1 \setminus U_n$  normalizes  $T_{n,i}$  for  $0 \le i \le n-2$ . For each  $g \in N_n^1 \setminus U_n$ , from [ACGS19, Corollary 3] we have that  $T_{n,n-2} = T_n \cap T_n^g$  is normal in  $N_n^1$ . Hence, for every subgroup  $H = T_{n,i}$  where i < n-2 and for every  $g \in N_n^1 \setminus U_n$ , we have  $H^g \le T_{n,n-2}$ . If  $x \in U_n$ , we clearly have  $(H^g)^x = (H^{gxg^{-1}})^g = H^g$ . Since  $T_n$  is a uniserial U-module, we conclude that  $H^g$  belongs to  $\mathcal{F}_n$ . Thus  $T_{n,i}^g = H^g = T_{n,i}$ .

We also used GAP to calculate  $N_n^k$  for  $n \leq 11$ . The computational results are summarized in Fig. 1, where the entry in position (k, n) denotes the logarithm in base 2 of the size of  $N_n^{k-1}$ . We observe that, in each column, consecutive values above the diagonal (bold values in the figure) have fixed differences. Such differences are listed in the (auxiliary) last column. For example, the number "+7" appearing in the last position of the fifth row denotes that the difference between  $\log_2 |N_j^4|$  and  $\log_2 |N_j^3|$  equals 7, where  $5 \leq j \leq 11$ , reading the table from left to right, starting from the position (5, 5) containing the bold number, i.e. the number 35.

n	2	3	4	5	6	7	8	9	10	11	
$\log_2  U_n $	3	6	10	15	21	28	36	45	55	66	
$\log_2  N_n^1 $	-	7	11	16	22	29	37	46	56	67	+1
$\log_2  N_n^2 $	-	-	13	18	24	31	39	48	58	69	+2
$\log_2  N_n^3 $	-	1	14	22	28	35	43	52	62	73	+4
$\log_2  N_n^4 $	-	-	15	23	35	42	50	59	69	80	+7
$\log_2  N_n^5 $	-	-	1	25	37	53	61	70	80	91	+11
$\log_2  N_n^6 $	-	-	1	27	41	57	77	86	96	107	+16
$\log_2  N_{,n}^7 $	-	-	-	28	45	64	84	109	119	130	+23
$\log_2  N_n^8 $	-	-	-	29	46	67	89	113	151	162	+32
$\log_2  N_n^9 $	-	-	-	30	47	71	95	122	155	205	+43

FIGURE 1. The logarithm of the size of the normalizers, when  $n \leq 11$ 

The table suggests that the values  $\log_2 |N_n^k : N_n^{k-1}|$ , reported in the last column of Fig. 1, do not depend on n, if  $n \ge k+2$ , and match with those of the sequence  $\{a_j\}_{j\ge 1}$  of the partial sums of the sequence  $\{b_j\}_{j\ge 1}$  counting the number of partitions of j into at least two distinct parts. The reader is referred to *The On-Line Encyclopedia of Integer Sequences* at [OEI, https://oeis.org/A317910] for a list of values and additional information. In the next section

j	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$b_j$	0	0	1	1	2	3	4	5	7	9	11	14	17	21
$a_j$	0	0	1	2	4	7	11	16	23	32	43	57	74	95

FIGURE 2. First values of the sequences  $a_j$  and  $b_j$ 

we show that for small values of k this is actually true. The above evidence is now summarized here as a conjecture.

**Conjecture 1.** For  $n \ge k+2 \ge 3$ , the number  $\log_2 |N_n^k : N_n^{k-1}|$  is independent of n and is equal to (k+2)-th term of the sequence  $\{a_j\}_{j\ge 1}$  of the partial sums of the sequence  $\{b_j\}_{j\ge 1}$  counting the number of partitions of j into at least two distinct parts.

The first values of the sequences  $a_j$  and  $b_j$  are listed in Fig. 2.

### 4. Theoretical evidence

In this section we prove Theorem 4.7 which solves Conjecture 1 in the cases  $1 \le k \le 4$ , by providing an explicit construction of  $N_n^k$ . We first need the following general lemma.

**Lemma 4.1.** Let  $G = A \rtimes B$  be a group and H a subgroup of G containing B. If  $[N_A(H \cap A), B] \leq H$ , then

$$N_G(H) = N_A(H \cap A) \rtimes B.$$

Proof. Clearly  $B \leq H \leq N_G(H)$ . Let  $x \in N_A(H \cap A) \leq A \leq G$ . Then  $[H, x] \subseteq A$ , since A is normal in G. Let  $h \in H$  and let us write h = bk where  $b \in B$  and  $k \in A \cap H$ . We have  $[h, x] = [b, x]^k [k, x] \in H$ , since  $[b, x] \in H$  as  $[N_A(H \cap A), B] \leq H$  by hypotheses. Thus  $[H, x] \subseteq H \cap A$  and thus  $N_A(H \cap A) \leq N_G(H)$ .

Let  $x \in N_A(H \cap A)$ ,  $k \in H \cap A$  and  $b \in B$ . Notice that

$$k^{x^{b}} = ((\underbrace{k^{b^{-1}}}_{\in H \cap A})^{x})^{b} \in H \cap A$$

This implies that  $N_A(H \cap A)$  is normalized by B. As a consequence, we have

$$N_G(H) \ge N_A(H \cap A) \rtimes B.$$

Conversely, let  $x \in N_G(H)$ . Since  $G = A \rtimes B$ , we can find  $b \in B \leq H \leq N_G(H)$  such that x = bu with  $u \in A$ . Clearly,  $u \in N_G(H) \cap A = N_A(H)$ . If  $h \in H \cap A$ , then  $[u, h] \in H \cap A$ , since A is normal in G. Thus  $x = bu \in N_A(H \cap A) \rtimes B$ .

Consider now the set-wise stabilizer  $Q_n$  in  $\Sigma_n$  of  $X_{n-1}$ . This group acts also on  $X_{n-1}^{s_n} = \{2^{n-1} + 1, \ldots, 2^n\}$  and so  $Q_n = \Sigma_{n-1} \times (\Sigma_{n-1})^{s_n}$  and  $\Sigma_n = Q_n \rtimes \langle s_n \rangle$ , where  $s_n$  interchanges the two direct factors of  $Q_n$ . We can give a general procedure for the construction of the normalizer  $N_{\Sigma_n}(Y)$  of a subgroup  $Y \leq \Sigma_n$  containing  $T_n$  such that  $[N_{Q_n}(Y \cap Q_n), s_n] \subseteq Y$ . Since  $t_n = s_n \in Y$ , we have  $Y^{s_n} = Y$  and  $N_{\Sigma_n}(Y) = N_{Q_n}(Y \cap Q_n) \rtimes \langle s_n \rangle$  by Lemma 4.1.

Let us apply the previous construction to obtain a description of  $U_n$  as the normalizer of  $T_n$ in  $\Sigma_n$ .

**Proposition 4.2.** We have that

$$U_n = \langle s_n \rangle \ltimes \left( \Delta_n(U_{n-1}) \cdot (T_{n-1} \times T_{n-1}^{s_n}) \right)$$
$$= \langle s_n \rangle \ltimes \left( \Delta_n(U_{n-1}) \cdot T_{n-1} \right).$$

*Proof.* Using the same notation as above, we notice that

$$T_n \cap Q_n = \Delta_n(T_{n-1}) = \{ (t, t^{s_n}) \mid t \in T_{n-1} \}.$$

We first claim that

$$N_{Q_n}(T_n \cap Q_n) = \Delta_n(U_{n-1}) \cdot T_{n-1}.$$

It is straightforward to check that  $\Delta_n(U_{n-1})$  normalizes  $T_n \cap Q_n = \Delta_n(T_{n-1})$  and that  $T_{n-1}$  centralizes  $T_n \cap Q_n$ , hence

$$N_{Q_n}(T_n \cap Q_n) \ge \Delta_n(U_{n-1}) \cdot T_{n-1}.$$

Now, let  $x = (a, b^{s_n}) \in N_{Q_n}(T_n \cap Q_n)$ , where  $a, b \in \Sigma_{n-1}$ , and  $y = (t, t^{s_n}) \in \Delta_n(T_{n-1}) = T_n \cap Q_n < T_{n-1} \times T_{n-1}^{s_n}$ . We have that

$$y^x = (t^a, (t^b)^{s_n}) = (\bar{t}, \bar{t}^{s_n}) \in \Delta_n(T_{n-1}) < T_{n-1} \times T_{n-1}^{s_n}$$

for some  $\overline{t} \in T_{n-1}$ , and so  $t^a = \overline{t} = t^b$ . It follows that  $a, b \in U_{n-1}$  and  $ab^{-1} \in C_{\Sigma_{n-1}}(T_{n-1}) = T_{n-1}$  by Remark 1. Therefore  $a = b\overline{t}$ , with  $\overline{t} \in T_{n-1}$  and  $x = (b, b^{s_n}) \cdot (\overline{t}, 1) \in \Delta_n(U_{n-1}) \cdot T_{n-1}$ , giving the opposite inclusion. In conclusion,

$$N_{Q_n}(T_n \cap Q_n) = \Delta_n(U_{n-1}) \cdot T_{n-1}.$$

In order to apply Lemma 4.1, it remains to be shown that  $[\Delta_n(T_{n-1}) \cdot T_{n-1}, s_n] \leq T_n$ . Notice that  $\Delta_n(T_{n-1}) \cdot T_{n-1} = T_{n-1} \times T_{n-1}^{s_n}$ , and thus  $[\Delta_n(T_{n-1}) \cdot T_{n-1}, s_n] \leq \Delta_n(T_{n-1}) \leq T_n$ , as claimed.

**Proposition 4.3.** Let  $H \leq K \leq \Sigma_{n-1}$  and  $U \stackrel{\text{def}}{=} \langle s_n \rangle \ltimes (\Delta_n(K) \cdot (H \times H^{s_n}))$ . If we define

• 
$$L \stackrel{\text{def}}{=} N_{\Sigma_{n-1}}(K) \cap N_{\Sigma_{n-1}}(H)$$
  
•  $M \stackrel{\text{def}}{=} C_K(K/H),$ 

then

$$N_{\Sigma_n}(U) = \langle s_n \rangle \ltimes \left( \Delta_n(L) \cdot (M \times M^{s_n}) \right).$$

Moreover,  $M \leq L \leq \Sigma_{n-1}$ .

*Proof.* The inclusion  $N_{\Sigma_n}(U) \ge \langle s_n \rangle \ltimes (\Delta_n(L) \cdot (M \times M^{s_n}))$  is straightforward since every factor of the second member is contained in the first one.

Note that  $U \cap Q_n = \Delta_n(K) \cdot (H \times H^{s_n})$ . Let us start considering the group

$$N \stackrel{\text{def}}{=} N_{Q_n}(U) = N_{\Sigma_n}(U) \cap Q_n$$

Let  $x = (a, b^{s_n}) \in N$ , where  $a, b \in \Sigma_{n-1}$ . First we note that  $[x, s_n] = (a^{-1}b, (b^{-1}a)^{s_n}) \in U \cap Q_n = \Delta_n(K) \cdot (H \times H^{s_n})$ . In particular,  $a^{-1}b \in K$ .

Let  $y = (h, 1^{s_n}) \in H \times H^{s_n} \leq U \cap Q_n$ , where  $h \in H$  and  $1 \in \Sigma_{n-1}$ . We have  $y^x = (h^a, 1^{s_n}) \in \Delta_n(K) \cdot (H \times H^{s_n}) = \Delta_n(K) \ltimes (H \times 1)$ . Since  $\Delta_n(K) \cap (H \times 1) = 1$ , then  $h^a \in H$  for all  $h \in H$  and so  $a \in N_{\Sigma_{n-1}}(H)$ . Similarly, we have that  $b \in N_{\Sigma_{n-1}}(H)$ . Now, letting  $u = (k, k^{s_n}) \in \Delta_n(K)$ , we have

$$u^{x} = (k^{a}, (k^{b})^{s_{n}}) = (k^{a}, (k^{a})^{s_{n}}) \cdot (1, ((k^{a})^{-1}k^{b})^{s_{n}})$$

$$\in \Delta_{n}(K) \cdot (H \times H^{s_{n}})$$

$$= \Delta_{n}(K) \ltimes (1 \times H^{s_{n}}),$$
(1)

and so  $a \in N_{\Sigma_{n-1}}(K) \cap N_{\Sigma_{n-1}}(H) = L$ . Similarly,  $b \in N_{\Sigma_{n-1}}(K) \cap N_{\Sigma_{n-1}}(H) = L$ . Again by Eq. (1) we have b = am with  $m = a^{-1}b \in C_L(K/H) \cap K = M$ . It follows that

$$x = (a, b^{s_n}) = (a, a^{s_n}) \cdot (1, m^{s_n}) \in \Delta_n(L) \cdot (M \times M^{s_n}).$$

Hence  $N \leq \Delta_n(L) \cdot (M \times M^{s_n})$ . As a consequence we have

$$N_{\Sigma_n}(U) = \langle s_n \rangle \ltimes N = \langle s_n \rangle \ltimes \left( \Delta_n(L) \cdot (M \times M^{s_n}) \right),$$

as required.

We also trivially have that  $M \leq L \leq \Sigma_{n-1}$ . If  $m \in M$ ,  $k \in K$  and  $l \in L$  then

$$[k,m^l] = l^{-1}\underbrace{[k^{l^{-1}},m]}_{\in H} l \in H,$$

and therefore  $m^l \in M$  and  $M \leq L$ .

The following technical definition is necessary to provide a recursive construction for the normalizer chain of  $T_n$  in  $\Sigma_n$ .

**Definition 4.4.** For a given natural number n we define the series  $\{C_n^k\}_{k\geq 0}$  and  $\{D_n^k\}_{k\geq 0}$  recursively as follows:

$$C_n^0 \stackrel{\text{def}}{=} T_n,$$

$$D_n^0 \stackrel{\text{def}}{=} N_{\Sigma_n}(C_n^0) = U_n,$$

$$C_n^k \stackrel{\text{def}}{=} C_{D_n^{k-1}}(D_n^{k-1}/C_n^{k-1}) \quad \text{for } k \ge 1,$$

$$D_n^k \stackrel{\text{def}}{=} N_{\Sigma_n}(C_n^{k-1}) \cap N_{\Sigma_n}(D_n^{k-1}) \quad \text{for } k \ge 1.$$

**Proposition 4.5.** For each  $k \ge 1$  we have that

$$\begin{split} N_n^k &= \langle s_n \rangle \ltimes \left( \Delta_n (D_{n-1}^k) \cdot \left( C_{n-1}^k \times (C_{n-1}^k)^{s_n} \right) \right) \\ &= \langle s_n \rangle \ltimes \left( \Delta_n (D_{n-1}^k) \ltimes \left( C_{n-1}^k \times \{1\} \right) \right). \end{split}$$

*Proof.* The result follows by a recursive application of Proposition 4.3, assuming  $H = C_{n-1}^{k-1}$ ,  $K = D_{n-1}^{k-1}$ ,  $L = D_{n-1}^k$  and  $M = C_{n-1}^k$ , beginning with  $C_{n-1}^0 = T_{n-1}$  which is normal in  $D_{n-1}^0 = U_{n-1}$ .

4.1. The case  $1 \le k \le 4$ . The main result of this work will be proved in this section. In order to do so, let us denote by  $\Theta_n$  the group of the upper unitriangular matrices and by  $Z_h(\Theta_n)$  the *h*-th term of its upper central series.

By Proposition 4.2

$$U_n = \langle s_n \rangle \ltimes \left( \Delta_n(U_{n-1}) \ltimes (T_{n-1} \times \{1\}) \right)$$

and

$$T_n = \langle s_n \rangle \cdot \Delta_n(T_{n-1}).$$

Hence

$$\Theta_n \cong U_n / T_n = \Delta_n (U_{n-1} / T_{n-1}) \ltimes (T_{n-1} \times \{1\}) = \Delta_n (\Theta_{n-1}) \ltimes (T_{n-1} \times \{1\}).$$

Moreover, notice that  $\Theta_n = U_{n-1}$ . It is easily checked that

$$Z_1(\Theta_n) = T_{n-1,1} \times \{1\}.$$

Proceeding by induction we obtain the following generalization.

Lemma 4.6. We have that

$$Z_h(\Theta_n) = \Delta_n(Z_{h-1}(\Theta_{n-1})) \ltimes (T_{n-1,h} \times \{1\}).$$

Proof. If  $G_h \stackrel{\text{def}}{=} \Delta_n(Z_{h-1}(\Theta_{n-1})) \ltimes (T_{n-1,h} \times \{1\})$ , then  $G_h/G_{h-1}$  is a central section of  $\Theta_n$ , hence  $G_h \leq Z_h(\Theta_n)$ . Notice that  $|G_h : G_{h-1}| = |Z_h(\Theta_n) : Z_{h-1}(\Theta_n)|$ , which is known to be  $2^h$ . Therefore  $Z_h(\Theta_n) = G_h$ .

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We are now ready to prove our main result.

**Theorem 4.7.** Let n be a non-negative integer. Then Conjecture 1 is true for  $1 \le k \le 4$ .

*Proof.* Let us prove each case separately. We will use Proposition 4.3 repeatedly without further mention. Since  $C_n^0 = T_n$  and  $D_n^0 = U_n$ , by Proposition 4.2 we have

$$N_n^0 = \langle s_n \rangle \ltimes \left( \Delta_n(U_{n-1}) \ltimes (T_{n-1} \times \{1\}) \right) = U_n.$$

 $[\mathbf{k} = \mathbf{1}]$  Since  $C_n^1 = C_{U_n}(U_n/T_n) = Z_1(\Theta_n) \ltimes T_n$  and  $D_n^1 = U_n \cap N_{\Sigma_n}(U_n) = U_n$ , we obtain

$$N_n^1 = \langle s_n \rangle \ltimes \left( \Delta_n(U_{n-1}) \ltimes \left( (Z_1(\Theta_{n-1}) \ltimes T_{n-1}) \times \{1\} \right) \right),$$

and so  $|N_n^1:N_n^0| = |N_n^1:U_n| = 2 = 2^1$ , since  $|Z_1(\Theta_{n-1})| = 2$ .  $[\mathbf{k} = \mathbf{2}]$  We have  $C_n^2 = C_{U_n}(U_n/(Z_1(\Theta_n) \ltimes T_n)) = Z_2(\Theta_n) \ltimes T_n$  and

$$D_n^2 = N_{\Sigma_n}(C_n^1) \cap N_{\Sigma_n}(U_n)$$
  
=  $N_{\Sigma_n}(Z_1(\Theta_n) \ltimes T_n) \cap N_n$   
=  $N_{N_n^1}(Z_1(\Theta_n) \ltimes T_n)$   
=  $U_n$ .

The last equality depends on the fact that

$$T_n \cdot T_n^g | = 2^{n+2},$$

where  $g \in N_n^1 \setminus U_n$  from [ACGS19, Corollary 3], and that  $|(Z_1(\Theta_n) \ltimes T_n)| = 2^{n+1}$ . We consequently obtain that

$$N_n^2 = \langle s_n \rangle \ltimes \left( \Delta_n(U_{n-1}) \ltimes ((Z_2(\Theta_{n-1}) \ltimes T_{n-1}) \times \{1\}) \right),$$

and so  $|N_n^2 : N_n^1| = 2^2 = 4$ , since  $|Z_2(\Theta_{n-1}) : Z_1(\Theta_{n-1})| = 2^2$ .  $[\mathbf{k} = \mathbf{3}]$  We have that  $C_n^3 = C_{U_n}(U_n/(Z_2(\Theta_n) \ltimes T_n)) = Z_3(\Theta_n) \ltimes T_n$  and

$$D_n^3 = N_{\Sigma_n}(Z_2(\Theta_n) \ltimes T_n) \cap N_{\Sigma_n}(U_n)$$
  
=  $N_{\Sigma_n}(Z_2(\Theta_n) \ltimes T_n) \cap N_n^1$   
=  $N_{N_n^1}(Z_2(\Theta_n) \ltimes T_n)$   
=  $N_n^1.$ 

In order to prove last equality, we first show that  $[N_n^1, T_n] \leq Z_2(\Theta_n) \ltimes T_n$ . For each  $g \in N_n^1 \setminus U_n$ , we have that  $[[T_n^g, U_n], U_n]$  is a normal subgroup of  $U_n$  of index at least 4 in  $T_n^g$ . Since  $T_n^g$  is uniserial for  $U_n$ , then  $[[T_n^g, U_n], U_n]$  is necessarily contained in the unique subgroup of index 4 in  $T_n^g$  and normal in  $U_n$ , which is  $T_n^g \cap T_n$  (see [ACGS19]). Hence  $(T_n \cdot T_n^g)/T_n$  lies in the second term of the upper central series of the quotient  $U_n/T_n = \Theta_n$ . Thus  $T_n^g \leq Z_2(\Theta_n) \ltimes T_n$ . We are left with proving that  $[N_n^1, Z_2(\Theta_n)] \leq Z_2(\Theta_n) \ltimes T_n$ , which is a direct consequence of the following straightforward properties: •  $Z_2(\Theta_n) = \Delta_n(Z_1(\Theta_{n-1})) \ltimes (T_{n-1,2} \times \{1\})$  (Lemma 4.6);

- $[s_n, \Delta_n(Z_1(\Theta_{n-1})) \ltimes (T_{n-1,2} \times \{1\})] \leq T_n;$
- $[3_n, \Delta_n(\Sigma_1(\bigcirc n-1)) \land (1_{n-1,2} \land (1_f))] \ge 1$
- $[\Delta_n(U_{n-1}), \Delta_n(Z_1(\Theta_{n-1}))] \leq T_n;$
- $[\Delta_n(U_{n-1}), T_{n-1,1} \times \{1\}] \le T_{n-1,2} \times \{1\} \le Z_2(\Theta_n);$
- $[Z_1(\Theta_{n-1}) \ltimes T_{n-1}, \Delta_n(Z_1(\Theta_{n-1}))] \le T_{n-1,1} \times \{1\} = Z_1(\Theta_n) \le Z_2(\Theta_n);$
- $[Z_1(\Theta_{n-1}) \ltimes T_{n-1}, T_{n-1,2} \times \{1\}] = \{1\}.$

In conclusion, we derive that

$$N_n^3 = \langle s_n \rangle \ltimes \left( \Delta_n(N_{n-1}^1) \ltimes \left( (Z_3(\Theta_{n-1}) \ltimes T_{n-1}) \times \{1\} \right) \right),$$

and so  $|N_n^3: N_n^2| = 2^4 = 16$ , as  $|N_{n-1}^1: U_{n-1}| = 2^1$  and  $|Z_3(\Theta_{n-1}): Z_2(\Theta_{n-1})| = 2^3$ . The same result can be also obtained as follows. Note that  $Z_2(\Theta_n) \ltimes T_n = Z_3(U_n) \cdot T_n$ . Since  $Z_3(U_n)$  is a characteristic subgroup of  $U_n$  we have  $(Z_2(\Theta_n) \ltimes T_n)^x = (Z_3(U_n) \cdot T_n)^x = Z_3(U_n) \cdot T_n \cdot T_n^x = Z_2(\Theta_n) \ltimes T_n$  for all  $x \in N_n^1$ . As a consequence  $[N_n^1, Z_2(\Theta_n)] \leq [N_n^1, Z_2(\Theta_n)T_n] \leq Z_2(\Theta_n) \ltimes T_n$ .

 $[\mathbf{k} = \mathbf{4}]$  We mimic the argument provided for the case k = 3. We start by computing

$$C_n^4 = C_{N_n^1} \left( N_n^1 / (Z_3(\Theta_n) \ltimes T_n) \right)$$
  
=  $C_M \left( \frac{\Delta_n(\Theta_{n-1}) \ltimes \left( (Z_1(\Theta_{n-1}) \ltimes T_{n-1}) \times \{1\} \right)}{\Delta_n(Z_2(\Theta_{n-1})) \ltimes \left( (Z_1(\Theta_{n-1}) \ltimes T_{n-1,3}) \times \{1\} \right)} \right) \ltimes T_n$   
=  $\left( \Delta_n(Z_3(\Theta_{n-1})) \ltimes \left( (Z_1(\Theta_{n-1}) \ltimes T_{n-1,4}) \times \{1\} \right) \right) \ltimes T_n,$ 

where  $M \stackrel{\text{def}}{=} \Delta_n(\Theta_{n-1}) \ltimes ((Z_1(\Theta_{n-1}) \ltimes T_{n-1}) \times \{1\})$ . Moreover

$$D_n^4 = N_{\Sigma_n}(Z_3(\Theta_n) \ltimes T_n) \cap N_{\Sigma_n}(N_n^1)$$
$$= N_{\Sigma_n}(Z_3(\Theta_n) \ltimes T_n) \cap N_n^2$$
$$= N_{N_n^2}(Z_3(\Theta_n) \ltimes T_n)$$
$$= N_n^2.$$

The last equality is derived proceeding as in the case k = 3, provided that n is sufficiently large (e.g.  $n \ge 5$ ). In conclusion, we derive that

$$N_n^4 = \langle s_n \rangle \ltimes \left( \Delta_n(N_{n-1}^2) \ltimes \left( C_{n-1}^4 \times \{1\} \right) \right),$$

and so  $|N_n^4:N_n^3| = 2^7$ . Indeed,  $|N_{n-1}^2:N_{n-1}^1| = 2^2$  and  $|C_{n-1}^4:C_{n-1}^3| = 2^5$ , since

$$C_{n-1}^{4} = \left(\Delta_{n-1}(Z_{3}(\Theta_{n-2})) \ltimes \left( (Z_{1}(\Theta_{n-2}) \ltimes T_{n-2,4}) \times \{1\} \right) \right) \ltimes T_{n-1}$$

and

$$C_{n-1}^3 = \left( \Delta_{n-1}(Z_2(\Theta_{n-2})) \ltimes (T_{n-2,3} \times \{1\}) \right) \ltimes T_{n-1},$$
  
where  $|Z_3(\Theta_{n-2}) : Z_2(\Theta_{n-2})| = 2^3, |T_{n-2,4}, T_{n-2,3}| = 2 \text{ and } |Z_1(\Theta_{n-2})| = 2$ 

Finally, notice that, if  $n \ge k+2$ , the construction of  $N_n^k$  described above does not depend on the dimension n and the integers corresponding to  $\log_2 |N_n^k : N_n^{k-1}|$  for  $1 \le k \le 4$  are respectively the 3-rd, the 4-th, the 5-th and the 6-th term of the sequence  $\{a_j\}$  of the partial sums of the sequence  $\{b_j\}$  counting the number of partitions of j into at least two distinct parts (see Fig. 2 and the auxiliary column of Fig. 1). Therefore, Conjecture 1 is true for  $1 \le k \le 4$ .

Although we have a recursive method, it appears that the construction of  $N_n^k$  requires *ad hoc* computations that become increasingly complex as k grows.

# **Open Problem 1.** Find a closed and concise formula for $N_n^k$ .

Moreover, even though the sequence of the indices  $\log_2 |N_n^k : N_n^{k-1}|$  seems to be predictable for  $k \leq n-2$ , as conjectured in this paper, it is hard to figure any conjecture on the values appearing under the bold diagonal of the table in Fig. 1.

**Open Problem 2.** Determine  $\log_2 |N_n^k : N_n^{k-1}|$  for all natural numbers k and n.

# 5. Some computational aspects

We can derive from Proposition 4.2 the following efficient construction of  $U_n$ , which has been useful to speed up the process of computing the normalizer chain.

The center  $Z(U_n)$  is the subgroup  $\langle t_1^n \rangle$ , which is actually the center of  $\Sigma_n$ . Let

$$u_{1,j}^{n} \stackrel{\text{def}}{=} t_{n-j+1}^{n} \text{ for } 1 \le j \le n, \quad u_{2,j}^{n} \stackrel{\text{def}}{=} u_{1,j-1}^{n-1} \text{ for } 2 \le j \le n,$$

and for  $3 \le i \le j \le n$ 

$$u_{i,j}^{n} \stackrel{\text{def}}{=} u_{i-1,j-1}^{n-1} (u_{i-1,j-1}^{n-1})^{s_n}$$

Using this notation, it easy to recognize that

$$U_{1} = T_{1},$$

$$U_{2} = \langle u_{1,1}^{2}, u_{1,2}^{2}, u_{2,2}^{2} \rangle,$$

$$U_{3} = \langle u_{1,1}^{3}, u_{1,2}^{3}, u_{1,3}^{3}, u_{2,2}^{3}, u_{2,3}^{3}, u_{3,3}^{3} \rangle$$

$$\vdots$$

$$U_{n} = \langle u_{i,j}^{n} \mid 1 \le i \le j \le n \rangle.$$

As an example of this construction, we conclude the paper by showing the GAP code which we used to build the normalizer chains displayed in Fig. 1. The orders of the normalizer are also provided. The code below is specialized to the case n = 8.

```
dim:=8;
```

```
gens:=[];
# will contain generators for T_n
ngens:=[];
# will contain generators for U_n
sgens:=[];
# will contain generators for Sigma_n
# construction of the previous list
for i in [1..dim] do
        x:=();
        for j in [1..2<sup>(i-1)</sup>] do
                 x:=x*(j,j+2^(i-1));
        od;
        newgens:=[];
        newngens:=[];
        for y in gens do
                 Add(newgens, y*y^x);
        od;
        for y in ngens do
                 Add(newngens, y*y^x);
```

```
Append(newngens,gens);
        od;
       newgens:=Set(newgens);
       newngens:=Set(newngens);
       gens:=newgens;
       ngens:=newngens;
        Add(gens, x);
        Add(ngens,x);
        Add(sgens,x);
       tmpsyl:=Group(ngens);
       ngens:=MinimalGeneratingSet(tmpsyl);
       tmpsyl:=false;
od;
t:=Group(gens); # the group T_n
sigma:=Group(sgens); # the group
Sigma_n u:= Group(ngens); # the group U_n
sym:=SymmetricGroup(2^dim);
n:=u;
# here the normalizer chain computation starts
sz:=Collected(Factors(Size(n)));
# the orders of the normalizer chain as power of 2
lst:=[[t,Collected(Factors(Size(t)))],[n,sz]];
# will contain the normalizers and their orders
flag:=true;
Print(Collected(Factors(Size(u))));
# construction of the normalizers and order display
while flag do
       m:=n;
       n:=Normalizer(sym,m);
       if n<>m then
                sz:=Collected(Factors(Size(n)));
               Print(sz,"\n"); Add(lst,[n,sz]);
        else
               flag:=false;
       fi;
od;
```

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