The Collatz tree is a Hilbert hotel: a proof of the $3x + 1$ conjecture

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Abstract

The yet unproven Collatz conjecture maintains that repeatedly connecting even numbers n to $n/2$, and odd n to $3n + 1$, connects all natural numbers to the *Collatz tree* with 1 as its root. The Collatz tree proves to be a *Hilbert hotel* for uniquely numbered birds. Numbers divisible by 2 or 3 fly off. An infinite binary tree remains with one "upward" and one "rightward" child per number. Next rightward descendants of upward numbers fly off, and thereafter generation after generation of their upward descendants. The Collatz tree is a Hilbert hotel because its bird population remains equally numerous. The *density of unique numbers* of birds flying off comes nevertheless arbitrarily close to 100% of the natural numbers. The latter proves the Collatz conjecture.

1 The Collatz tree

The Collatz conjuncture maintains that the Collatz function $C(n) = n/2$ if n is even, but $C(n) = 3n + 1$ if n is odd, reaches 1 for all natural numbers n, after a finite number of iterations.

$8 \to 4 \to 2 \to 1 \ (\to 4 \to 2 \to 1 \to \dots)$

$9 \to 28 \to 14 \to 7 \to 22 \to 11 \to 34 \to 17 \to 52 \to 26 \to 13 \to 40 \to 20 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1$

$10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1$

This one-sentence conjecture became famous after renowned mathematicians realized that its proof would not be as easy. In the words of Paul Erdős (1913-1996): "hopeless, absolutely hopeless" (*1*). Recently, numerical computations verified the conjecture for all numbers below 2^{68}(*2*), while Terence Tao proved the conjecture for "almost all" numbers. Nevertheless, Tao reckons that "the full resolution … remains well beyond the reach of current methods" (*3*).

If the Collatz conjecture would hold, then the node set $N(T_{\mathbb{C}})$ of one *Collatz tree* $T_{\mathbb{C}}$ with as its trivial cyclic root $\Omega(T_{\mathbb{C}}) = \{4, 2, 1\}$ would comprise all natural numbers n. No *isolated trajectory* would exist, neither a *divergent trajectory* from n to infinity, nor a *nontrivial cycle* from any $n > 4$ back to its origin n (*1*). To be proven is that $N(T_{\mathbb{C}})=\mathbb{N}$, with $\mathbb{N} = \{1, 2, 3, \ldots\}$. The originator of the conjecture, Lothar Collatz (1910-1990), hoped that picturing “number theoretic functions $f(n)$” with ”arrows from n to $f(n)$”, like in Figure 1, would elicit “relations between elementary number theory and elementary graph theory” to prove the conjecture (*4*).

Figure 1: The original Collatz tree

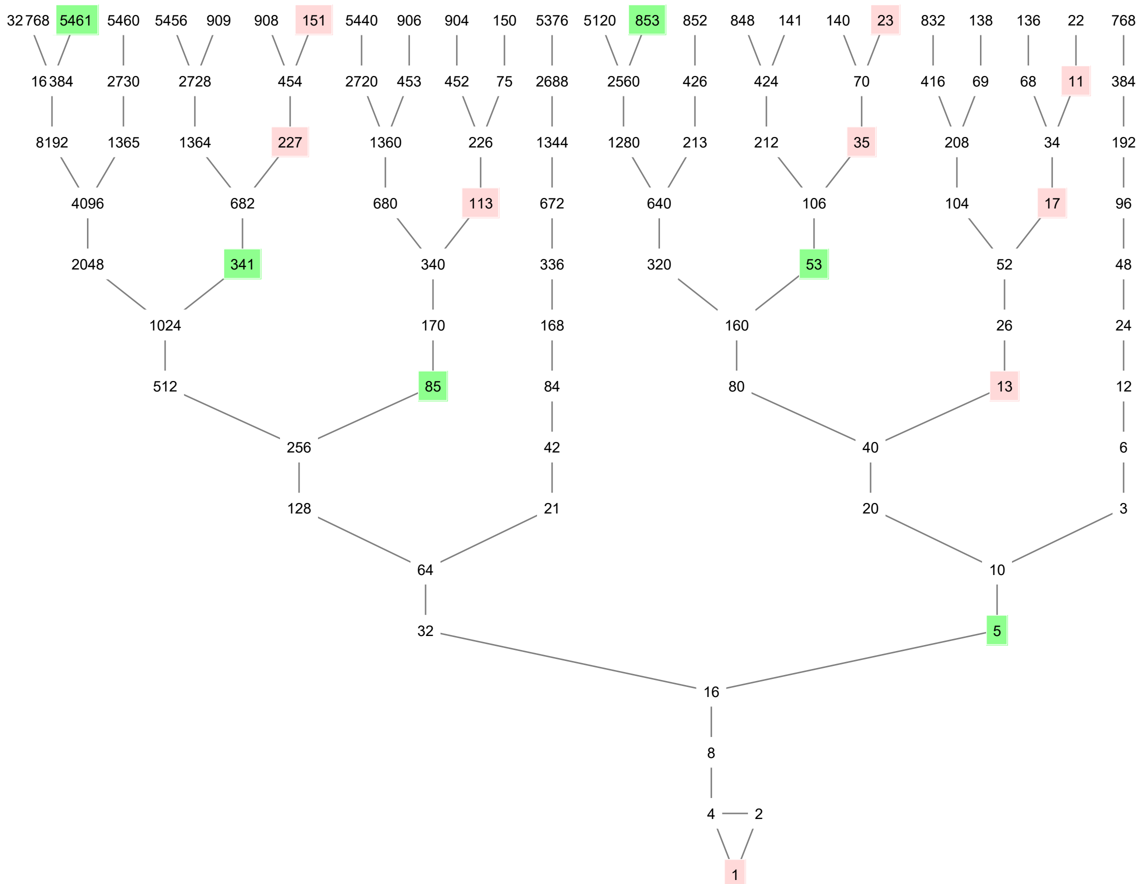

Figure 1 already distinguishes subset S_{-1} of uncolored numbers divisible by 2 and/or 3, which will fly off first, from green upward numbers and red rightward numbers to be defined in lemmas 1 and 2.

2 A Hilbert hotel

A *Hilbert hotel*, first introduced by David Hilbert (1862-1943) in a 1924 lecture, is an infinite hotel that never gets full upon the arrival of new guests because all hotel guests can move to higher numbered rooms (*5*). The key to proving the Collatz conjecture is to grasp from Figures 1, 2 and 3 that the Collatz tree is a Hilbert hotel from which an infinite number of swarms S, or subsets S, fly away, each with an infinite number of uniquely numbered birds. Their places are taken by infinite, *equally numerous* swarms of their children, separated by larger and larger numerical distances, indicating that their *density diminishes* ever further.

Our innovation is to erect a *binary tree* (Figure 2) instead of a Syracuse tree (explained below) after subset S_{-1} with numbers divisible by 2 and/or 3 has flown out. This binary tree empowers a hitherto deficient approach in the research literature to find ever less dense subsets $S \subset \mathbb{N}^+$, such that proving the conjecture on S implies it is true on $\mathbb{N}$ (*6*) [Suppl. C, Related literature]. Infinitely many successive subsets S_0, $S_1, S_2, S_3, \dots$ of *intermediary* numbers, which sit in between remaining numbers, fly off next from ever less dense binary trees $T_{\geq 0}$, $T_{\geq 1}, T_{\geq 2}, T_{\geq 3}, \dots$ (Figure 3). *Periodic* iterations of functions (with definitions 2 and 3) govern which numbers fly off in which subset. This enables *density calculations per period* mounting up to familiar sums of infinite geometric series. The proof of lemma 4 reveals that the density of all flown subsets $S_{-1}, S_0, S_1, S_2, S_3, \cdots, S_j, \cdots \subset \mathbb{N}$ comes arbitrarily close to 100% for $j \to \infty$. For $j \to \infty$, the ever lesser density of not yet flown swarms $S_{\geq j} = \biguplus_{i=j}^{\infty} S_i$ comes therefore arbitrarily close to 0, which is incompatible with the existence of isolated non-trivial cycles and divergent trajectories. Therefore $N(T_C) = \mathbb{N}$, the node set of the Collatz tree T_C comprises all natural numbers $\mathbb{N}$.

3 From the Syracuse tree to a binary tree

The uncolored numbers divisible by 2 and/or 3 in S_{-1} are *intermediaries* between descendants and ancestors in the remaining set of *admissible* numbers $S_{\geq 0}$ from congruence classes $\{1, 5\} \equiv n \bmod 6$. As Figure 1 shows, numbers divisible by both 2 and 3, which cannot be reached by $3n+1$ themselves, reduce through numbers that are divisible by 3 only (e.g. $12 = 2^2 \cdot 3 \to 2 \cdot 3 \to 3$) to even numbers n that are not divisible by 3 (e.g. $3 \to 10$). The latter numbers are intermediaries between $2^p n$ ancestors that can be reached from $S_{\geq 0}$ by the $3n+1$ operation, and an odd descendant $2^{-q} n$. For example, $10 = 2 \cdot 5 \in S_{-1}$ intermediates between $13 = \left(2^3 \cdot 5 - 1\right)/3 \in S_{\geq 0}$ and $5 \in S_{\geq 0}$.

It's not self-evident how to reconnect the remaining numbers in $S_{\geq 0}$ to a binary tree $T_{\geq 0}$. The *Syracuse tree* function $T(n)$ that is commonly used in the research literature connects any odd number n to the most nearby odd number. The *inverse Syracuse function* connects each odd node n to infinitely many children $T^{-1}(n) = (2^p n - 1)/3$, corresponding to the *infinitely many* powers $p \in \mathbb{N}^+$ for which $T^{-1}(n)$ is odd also [Suppl. Figures 3-4]. For example, $n = 1$ is connected *in parallel* to $1 = \left(2^2 1 - 1\right)/3$, thus to itself, and moreover to $5 = \left(2^4 1 - 1\right)/3$, $85 = \left(2^8 1 - 1\right)/3$, $341 = \left(2^{10} 1 - 1\right)/3$, $5461 = \left(2^{14} 1 - 1\right)/3$, and so on, e.g. to all green numbers to the left in Figure 1. Infinite amounts of children $T^{-1}(n)$ preclude easy density calculations *per finite period*. Recently, a measure of *logarithmic densit*y applied to the Syracuse tree enabled Tao nevertheless to prove the Collatz conjecture for "almost all" numbers (*3*).

The binary tree $T_{\geq 0}$ in Figure 2 [Suppl. Figures 5,7] connects $1 \to 5 \to 85 \to 341 \to 5461 \to \dots$ in a *serial* upward orbit, based on three adaptations of the inverse Syracuse function T^{-1}.

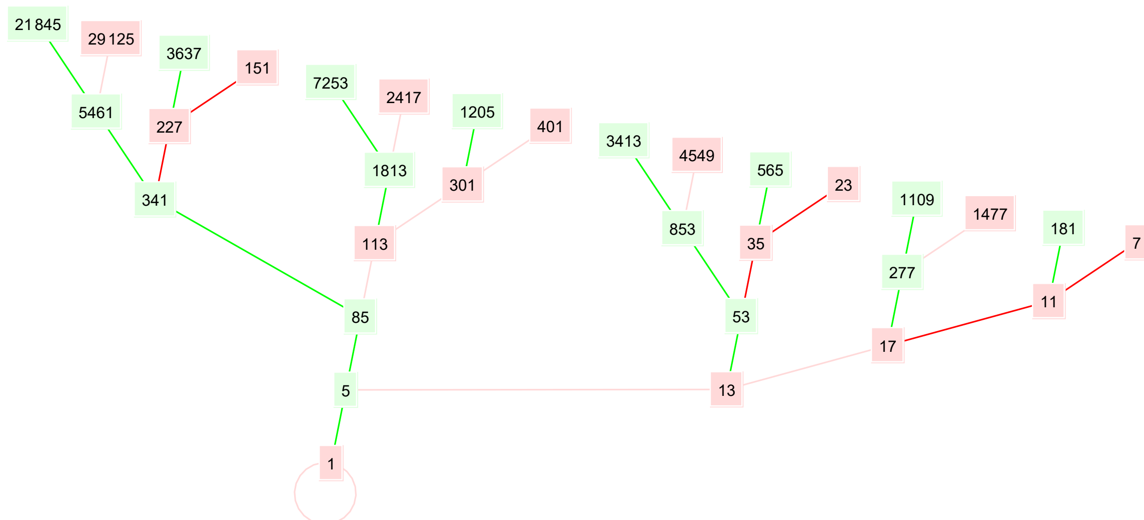

Figure 2: *Binary tree $T_{\geq 0}$ with root $\Omega(T_{\geq 0}) = 1$ of upward and rightward arcs,*
Legend: upward arcs and nodes in green, rightward arcs and nodes in red

First, odd arguments n and odd outputs $T^{-1}(n)$ are required to be non-divisible by 3 moreover. Next, n's infinite offspring $T^{-1}(n)$ is limited to the minimum value of p, thus to its first-born child, labeled as its *rightward*, red colored, child $R(n)$. Third, the first born $T^{-1}(3n+1)$ child to n's even, already flown, parent $3n+1$ becomes n's *upward*, green colored, child $U(n)$.

Definitions 1 and 2 below simplify the upward and rightward functions U and R to *periodic* functions without unknown minimum powers p. Lemmas 1 and 2 specify the disjoint *periodic* co-domains $\mathbb{N}^U \subset S_{\geq 0}$ and $\mathbb{N}^R \subset S_{\geq 0}$ to which U and R map their output. From the periodicities in lemmas 1 and 2 it's a small step to Lemma 3, which states their densities $\rho\left(\mathbb{N}^U\right) = 5/96$ and $\rho\left(\mathbb{N}^R\right) = 27/96$. Lemma 3 does still *not* imply that these densities are exclusively based on numbers included in the binary Collatz tree T_C, which has to wait until Lemma 4.

4 Periodicity proofs upward and rightward function

The proofs for the periodicities 6 (definition 1), 18 (definition 2) and 96 (lemmas 1 and 2) are variants on a similar recipe. First, rewrite each assumed congruence class $c \equiv n \bmod d$ as an equivalent arithmetic progression $n = d\ i + c$ for $i \in \mathbb{N}^0$ with $\mathbb{N}^0 = \{0, \mathbb{N}\}$. Next, verify that the definition/lemma holds for $i = 0$. Finally, prove its independence of the choice of $i \in \mathbb{N}^0$.

4.1 Argument periodicity upward function

Definition 1. Upward function $U: S_{\geq 0} \mapsto \mathbb{N}^U$;

$$U(n) = argmin_p \left(2^p(3n+1)-1\right)/3 = \begin{cases} 4n+1 & \text{if } 1 \equiv n \bmod 6 \\ 16n+5 & \text{if } 5 \equiv n \bmod 6 \end{cases}$$

Periodicity proof. The two admissible congruence classes ($c \equiv n \ mod \ 6) \subset S_{\geq 0}$ in definition 1 correspond to the arithmetic progressions $n = (6\ i + c) \subset S_{\geq 0}$ with $c \in \{1,5\} \subset S_{\geq 0}$. For $i = 0$ we obtain $n = c$ and $U(n) = U(c) = (2^p(3c+1)-1)/3$. For $c = 1$, $p = 1$ renders $7/3 \notin S_{\geq 0}$ and next a minimum $p = 2$ satisfying $U(c) = (16-1)/3 = 5 \in S_{\geq 0}$. Therefore $U(n) = \left(2^2(3 \cdot c+1)-1\right)/3 = 4c+1$ in case $1 \equiv c$ mod 6. For $c = 5$, p-values of $1,2$ and 3 render as inadmissible outcomes $31/3$, 21 and $127/3$ and as minimum $p = 4$ satisfying $U(c) = (256-1)/3 = 85 \in S_{\geq 0}$. Therefore $U(n) = \left(2^4(3 \cdot c+1)-1\right)/3 = 16c+5$ in case in case $5 \equiv c$ mod 6.

Let's prove next that $U(6i+c) \in S_{\geq 0}$ is independent of the choice of $i \in \mathbb{N}^+$. Collecting terms with and without i gives $U(n) = U(6i+c) = (2^p(3(6i+c)+1)-1)/3 = \left(2^{p+1}3i\right) + ((2^p(3c+1)-1)/3)$. Since the an i-dependent term $\left(2^{p+1}3i\right)$ is divisible by 2 and 3, divisibility of $U(n)$ by 2 and 3 only depends on the i-independent term $(2^p(3c+1)-1)/3$, which is the $U(n) = U(c) = (2^p(3c+1)-1)/3$ case solved above.

4.2 Argument periodicity rightward function

The rightward function $R(n)$ takes on a different power p depending on the congruence classes modulo $v^0 = 18$ of its argument n, where v^0 denotes the argument periodicity. The vector $p(a) \in \vec{\boldsymbol{p}}(a) = [2,3,4,1,2,1]$ corresponds to the *admissible* congruence classes $a \in \vec{\boldsymbol{a}} = [1,5,7,11,13,17]$. The upward function brings expansions only, but the rightward function brings contractions in case $p(a) = 1$, which is the case for the *contracting congruence classes* 11 and 17.

Definition 2. Rightward function $R: \ S_{\geq 0} \mapsto \mathbb{N}^R$;

$$R(n) = argmin_p\,(2^p n - 1)/3 = \begin{cases} \vec{\boldsymbol{p}}(a) & \qquad \vec{\boldsymbol{a}} \\ (2^2 n-1)/3 & \text{if} \quad 1 \equiv n \ mod\ 18 \\ (2^3 n-1)/3 & \text{if} \quad 5 \equiv n \ mod\ 18 \\ (2^4 n-1)/3 & \text{if} \quad 7 \equiv n \ mod\ 18 \\ (2^1 n-1)/3 & \text{if} \quad 11 \equiv n \ mod\ 18 \\ (2^2 n-1)/3 & \text{if} \quad 13 \equiv n \ mod\ 18 \\ (2^1 n-1)/3 & \text{if} \quad 17 \equiv n \ mod\ 18 \end{cases}$$

Periodicity proof. The six congruence classes $a \equiv n \ mod \ 18$ in definition 2 correspond to the arithmetic progressions $n = 18\ i + a \subset S_{\geq 0}$ with $a \in \vec{\boldsymbol{a}}$. For $i = 0$, we obtain $n = a$ and $R(n) = R(a) = (2^p a - 1)/3$, which enables the verification that $\vec{\boldsymbol{p}}(a) = [2,3,4,1,2,1]$ is the vector of minimum powers of p such that $R(a) \in S_{\geq 0}$. For example, for $a = 11$ we obtain $p = 1$, since $R(11) = (2^1 11 - 1)/3 = 7 \in S_{\geq 0}$.

Let's now split $R(n) = R(18i + a) = (2^p(18i+a)-1)/3$ in i-dependent and i-independent terms. This gives $\left(2^{p+1}3\ i\right) + (2^p a - 1\big)/3)$. Since the i-dependent term $\left(2^{p+1}3i\right)$ is divisible by 2 and 3, divisibility of $R(n)$ by 2 and 3 only depends on the i-independent term $(2^p a-1)/3$, which is the $R(n) = R(a) = (2^p a - 1)/3$ case solved above. ■

4.3 Output periodicity upward function

Lemmas 1 and 2 below delineate the periodic codomains $\mathbb{N}^U$ and $\mathbb{N}^R$ of the upward and rightward functions U and R. Lemma 1 states that the co-domain $\mathbb{N}^U$ of the upward function U consists of 5 out of the 32 admissible congruence classes modulo 96, also known as the multiplicative group $(\mathbb{N}/96\mathbb{N})^{\times}$.

Lemma 1. Co-domain upward numbers:

$$\mathbb{N}^U = \{\ n \mid \{5,\ 29,\ 53,\ 77,\ 85\} \equiv n \bmod 96\ \}$$

Proof: Let's consider the arithmetic progression $m = (24\,i + a) \in S_{\geq 0}$ for $i \in \mathbb{N}^0$ of arguments m for the rightward function $n = U(m)$. For $i = 0$, we obtain $m = a \in \{1,5,7,11,13,17,19,23\} \in S_{\geq 0}$ and $n = U(m) \in \{5,85,29,181,53,277,77,373\} \subset \mathbb{N}^U$. Reducing numbers to congruence classes gives $n \bmod 96 \in \{5,85,29,85,53,85,77,85\}$, thus $n \bmod 96 \in \{5,\ 29,\ 53,\ 77,\ 85\}$.

Let's prove next that this result for $i = 0$ is independent of the choice $i \in \mathbb{N}^0$ by an inductive proof that $U(24\,i + a) \bmod 96 = U(24\,(i+1) + a) \bmod 96$. By definition 1, proof is required for the congruence class $1 \equiv n \bmod 6$ that $(4(24\,i + a) + 1) \bmod 96 = (4(24\,(i+1) + a) + 1) \bmod 96$. Simplifying both terms further gives $(96i + 4a\ + 1) \bmod 96 = (96i + 96 + 4a\ + 1) \bmod 96$. Leaving out multiples of 96 on both sides gives $(4a\ + 1) \bmod 96$ at each side. Furthermore, proof is required for the congruence class $5 \equiv n\ mod\ 6$ that $(16(24i + a) + 5) \bmod 96 = (\ 16(24(i+1) + a) + 5) \bmod 96$. Simplifying both sides gives $(4 \cdot 96i + 16a + 5) \bmod 96 = (4 \cdot 96i + 4 \cdot 96 + 16a + 5) \bmod 96$. Both sides give $(16a + 5) \bmod 96$ after leaving out terms with 96. ■

4.4 Output periodicity rightward function

Lemma 2 states that the co-domain $\mathbb{N}^R$ of the rightward function R comprises 27 out of the 32 *admissible* congruence classes modulo 96. The *admissible* $\mathbb{N}^U$ congruence classes 5, 29, 53, 77 and 85 (Lemma 1) are excluded.

Lemma 2. Co-domain rightward numbers:

$$\mathbb{N}^R = \left\{ n \mid \left\{ \begin{array}{c} 1,7,11,13,17,19,23,25,31,35,37,41,43,47, \\ 49,55,59,61,65,67,71,73,79,83,89,91,95 \end{array} \right\} \equiv n \bmod 96 \right\}$$

Proof. The proof of Lemma 2 will be based on Table 1, with in its first two columns as givens the vector $\vec{a}$ with six admissible argument congruence classes $a \equiv n\ mod\ 18$ for arguments $n \in S_{\geq 0}$, followed by the corresponding vector $\vec{p}$ of powers of 2 (see definition 2). Its c_i column is for later use.

Table 1: From argument periodicity 18 in n to rightward output periodicity 96 in $R(n)$

$n \in S_{\geq 0}$		$R(n) \in \mathbb{N}^R$				
$a \equiv n$ mod 18	$p(a)$	$b(a) \equiv R(n)$ mod $m(a)$	$h(a) = 96/m(a)$	$\mathbb{r}(a) \equiv R(18i+a)$ mod 96	$c_i \equiv R(54j+18i+a)$ mod 18, for $i =$ 0, 1, 2	
$\vec{a}$	$\vec{p}$	$\vec{b}$, $\vec{m}$	$\vec{h}$	$\mathbb{N}^R$		$\vec{T}$
1	2	$1 \equiv R(n)$ mod 24	4	1,25,49,73	1,7,13	[[1,0,1,0,1,0],
5	3	$13 \equiv R(n)$ mod 48	2	13,61	13,7,1	[1,0,1,0,1,0],
7	4	$37 \equiv R(n)$ mod 96	1	37	1,7,13	[1,0,1,0,1,0],
11	1	$7 \equiv R(n)$ mod 12	8	7,19,31,43,55,67,79,91	7,1,13	[1,0,1,0,1,0],
13	2	$17 \equiv R(n)$ mod 24	4	17,41,65,89	17,5,11	[0,1,0,1,0,1],
17	1	$11 \equiv R(n)$ mod 12	8	11,23,35,47,59,71,83,95	11,5,17	[0,1,0,1,0,1]]

Let's rewrite the six argument congruence classes $a \equiv n$ mod 18 as arithmetic progressions $n = 18i + a$, with $i \in \mathbb{N}^0$. Splitting $R(n) = \left((18i+a)2^{p(a)} - 1\right)/3$ in an i-dependent part $m(a) = 2^{p(a)+1}3$, which represents the *intrinsic periodicity* of the output $R(n)$, and an i-independent part $b(a) = (a2^{p(a)} - 1)/3$, which represents the congruence class $b(a) \equiv R(n)$ *mod* $m(a)$. For example, the argument congruence class $a = 13$, with power $p(a) = 2$, gives as rightward output periodicity $m(a) = 2^{p(a)+1}3 = 24$ and as rightward output congruence class $b(a) = \left(13 \cdot 2^{p(a)} - 1\right)/3 = 17$.

Let's now use that a function with periodicity ν is also periodic in multiples of ν – the sine function, for example, has a periodicity of 2π, but also of 4π, 6π, $8\pi \ldots$. To obtain the same output periodicity for the rightward function as for the upward function (cf. Lemma 1) the intrinsic output periodicities $m(a) \in \vec{\boldsymbol{m}} = [24, 48, 96, 12, 24, 12]$ are transformed to their *Least Common Multiple rightward output periodicity* $v^1 = LCM(24, 48, 96, 12, 24, 12) = 96$. For each $a \in \vec{\boldsymbol{a}}$ the number of different arguments $n = 18i + a$, indexed by i, that map their rightward output $R(18i + a)$ onto a single rightward output period $(j-1)v^1 \ldots jv^1$, indexed by j, is given by the *heap vector* $\vec{\boldsymbol{h}} = v^1 / \vec{\boldsymbol{m}} = 96/[24, 48, 96, 12, 24, 12] = [4, 2, 1, 8, 4, 8]$ (cf. Table 1). For example, for $j = 2$ the co-domain range is $96 \ldots 192$, to which, for $a = 13$, $h(13) = 4$ successive arguments, indexed by $i = 4 \ldots 7$, map their rightward outputs $R(4 \cdot 18 + 13) = R(85) = 113$, $R(5 \cdot 18 + 13) = R(103) = 137$, $R(6 \cdot 18 + 13) = R(121) = 161$ and $R(7 \cdot 18 + 13) = R(139) = 185$.

Column $\mathbb{N}^R$ in Table 1 renders the rightward codomain $\mathbb{N}^R$ of 27 congruence classes with a common periodicity $v^1 = 96$ as the union of 6 sets of $h(a)$ different rightward congruence classes $R(18i + a)$ mod 96. ■

4.5 Density upward and rightward numbers - included in Collatz tree or not

Lemma 3 explicates the output densities of $\mathbb{N}^U$ and $\mathbb{N}^R$, without considering whether outputs are mapped to the Collatz tree or to isolated trajectories [Suppl. Figure 6].

Lemma 3. Output densities of U and R:

$$\rho\left(\mathbb{N}^U\right) = \frac{5}{96};\;\; \rho\left(\mathbb{N}^R\right) = \frac{27}{96}$$

Proof. By lemmas 1 and 2, $\mathbb{N}^U$ and $\mathbb{N}^R$ are disjoint subsets that cover 5, respectively 27, of the 32 admissible congruence classes modulo 96. ■

5 An infinite number of infinite departures from the binary tree

Let's first illustrate with Figure 2 our notation for inverses and iterates of the upward and rightward function that is required for density calculations of infinitely many departing swarms of infinitely many uniquely numbered birds. Let U^{-1} and R^{-1} denote the – downward, respectively leftward – inverses of U and R. Let $U^k(n)$ and $R^k(n)$ denote the k'th iterate of U, respectively R. Let $U^{-k}(n)$ and $R^{-k}(n)$ denote the k'th iterate of their inverses. For example: $U^{-1}(5) = 1$, $U^2(5) = 341$, $R^2(5) = 17$, $R^{-2}(17) = 5$, $U^{-2}(1109) = 17$. Each orbit has an inverse orbit, e.g. $m = U^j R^k U^{-l}(n)$ implies $n = U^l R^{-k} U^{-j}(m)$, thus $U^1 R^2 U^{-3}(5461) = 277$ implies $U^3 R^{-2} U^{-1}(277) = 5461$.

The fast way to grasp that the infinite binary tree $T_{\geq 0}$ in Figure 2 allows for *iterative departures* is to ask, while scrutinizing Figure 3 [Suppl. Notebook, Figures 7-10], what happens after successive departures of the subsets $S_0, S_1, S_2, \cdots$ whose numbers turn red just before flying off.

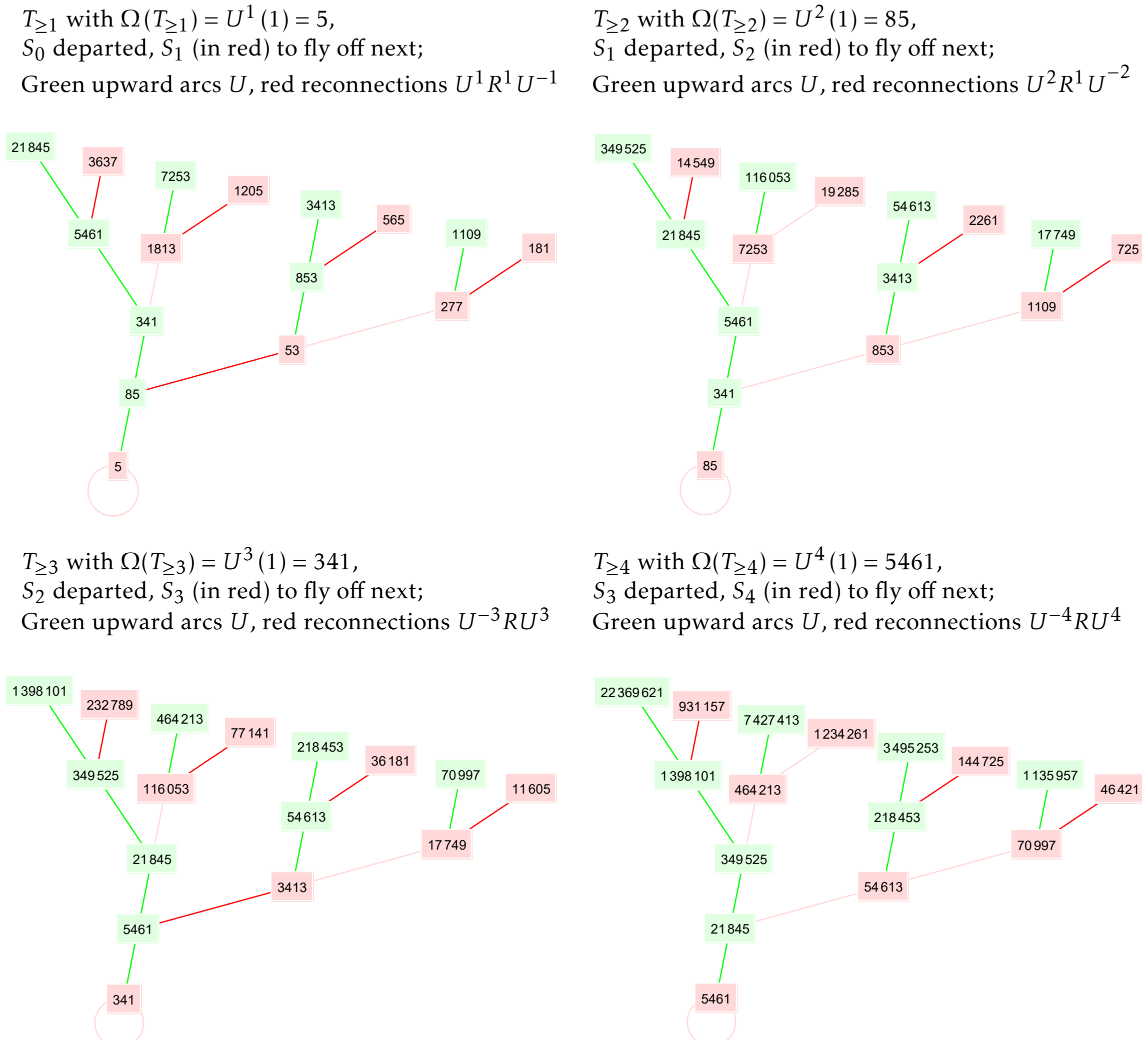

Figure 3: *Reconnected trees $T_{\geq 1}, T_{\geq 2}, T_{\geq 3}, T_{\geq 4}$ after the departures of subsets S_0, S_1, S_2, S_3*

The answer is that green-colored numbers that are thereby disconnected are reconnected to each other in red-colored orbits, indicating that they themselves are ready to fly off from the *less dense* binary trees $T_{\geq 1}$, $T_{\geq 2}$, $T_{\geq 3}, \dots$ The top-left pane of Figure 4 shows the less dense tree $T_{\geq 1}$ after the departure of subset S_0 with rightward orbits of upward descent.

In the top-left tree $T_{\geq 1}$ generation S_1 of reconnected (red) upward children of S_0 numbers (e.g. the orbit $53 \to 277 \to 181 \to \cdots \subset S_1$ with argument $85 \in S_{>1}$) replaces numbers on the nodes that were held in tree $T_{\geq 0}$ by subset S_0 (e.g. by the rightward orbit $13 \to 17 \to 11 \to \cdots \subset S_0$ with argument $5 \in S_{>0}$). In the top-right tree $T_{\geq 2}$ generation S_2 with reconnected (red) upward grandchildren of S_0 numbers (e.g. the orbit $853 \to 1109 \to \cdots \subset S_2$ with argument $341 \in S_{>2}$) takes over, and so on. In tree $T_{\geq j}$ upward generation S_j takes over, consisting of bridging, reconnected orbits $B_j^k(n) \subset S_j$ with arguments $n \in S_{>j}$ as specified in definition 3.

Definition 3: Reconnection orbit in node subset S_j of tree $T_{\geq j}$

$$B_j^k:\ S_{>j} \mapsto\ S_j:\ \forall k \in \mathbb{N}\colon B_j^k(n) = U^j R^k U^{-j}(n)$$

Definition 3 implies that subset S_j, which will fly off next from tree $T_{\geq j}$, includes only *intermediary* numbers in between remaining arguments $n \in S_{>j}$ and remaining upward children $U^{>j} R^k U^{-j}(n) \in S_{>j}$. Therefore numbers on hypothetical isolated trajectories are not included in any subset S_j, provided that they are not included in the remaining subsets $S_{>j}$. Definition 4 specifies the resulting tree $T_{\geq j}$.

Definition 4: Reconnected tree $T_{\geq j}$

node set: $n \in S_{\geq j}$

arc set: $\left\{\begin{array}{ll} \text{upward arcs} & n \to U(n), \\ \text{reconnection arcs} & n \to U^j R^1 U^{-j}(n) \end{array}\right\}$

cyclic root: $\Omega\left(T_{\geq j}\right) = U^j\left(\Omega(T_{\geq 0})\right) = U^j(1) = \left\{\begin{array}{ll} \left(2^{3j+1}-1\right)/3 & \text{if } j \text{ is odd} \\ \left(2^{3j+2}-1\right)/3 & \text{if } j \text{ is even} \end{array}\right.$

Figure 3 shows that the cyclic roots $U^1(1) \ldots U^8(1)$ of trees $T_{\geq 1} \ldots T_{\geq 8}$ respectively are 5, 85, 341, 5 461, 21 845, 349 525, 1 398 101 and 22 369 63. Cyclic root $\Omega(T_{\geq 23}) = U^{23}(1)$ exceeds already the computational record of 2^{68} by September 2020 for the highest number below which the Collatz conjecture is verified (*2*). Cyclic roots build the utmost left *upward trunk* $U^1(1) \to U^2(1) \to U^3(1) \to \ldots$ in tree $T_{\geq 0}$ (cf. Figure 2).

6 Density of departed subsets

Lemma 4 maintains that the density $\rho(T_C)$ of bird swarms leaving the Collatz tree approaches 1. Its proof outline below rests on the Supplementary Materials with elaborated proofs, plots, examples [Suppl. B11-13] and an underlying Mathematica notebook {Suppl., Notebook].

Lemma 4: density of departed infinite number subsets from the Collatz tree T_C

$$\rho(T_C) \ = \rho(S_{-1}) + \rho(S_0) + \rho(S_{1,3,5,\ldots}) + \rho(S_{2,4,6,\ldots}) = 4/6 + 27/96 + 1/21 + 1/224 = 1$$

The concepts used in the proofs are already defined and exemplified in the previous subsection. The condensed density proofs below are summarized in Table 15 with in its columns the four subsets S_{-1}, S_0, $S_{1,3,5,\ldots}$ and $S_{2,4,6,\ldots}$ to be considered, and in its rows their densities $\rho(S)$ and the periodicities ν – denoted by Greek nu, and cardinalities per period $\#(S|\nu)$ on which they rest.

Table 2: Densities, periodicities and cardinalities per period of arguments and outputs

Output node subsets	S_{-1}	S_0	$S_{i\in\{1,3,5,\ldots\}}$	$S_{j\in\{2,4,6,,\ldots\}}$
density $\rho(S)$	$\rho(S_{-1}) =$ $4/6$	$\rho(S_0) =$ $27/96$	$\rho(S_{1,3,5,\ldots}) =$ $1/21$	$\rho(S_{2,4,6,\ldots}) =$ $1/224$
periodicity ν	$\nu =$ $2 \cdot 3$	$\nu^k =$ $2^{4k+5}3$	$\nu^{ik} =$ $2^{4k+3i+4}3^2$	$\nu^{jk} =$ $2^{4k+3j+5}3^2$
cardinality per period $\#(S\|\nu)$	$\#(S_{-1}\|\nu) =$ 2^2	$\#(S_0\|\nu^k) =$ $3^4 \cdot 13^{k-1}$	$\#(S_{i=1,3,5,\cdots}\|\nu^{ik}) =$ $2 \cdot 3^4 \cdot 13^{k-1}$	$\#(S_{j=2,4,6,\cdots}\|\nu^{jk}) =$ $3^5 \cdot 13^{k-1}$
Argument subsets		$S_{>0}$	$S_{>i}$	$S_{>j}$
periodicity ν		$\nu_k^0 =$ $2^5 3^{k+1}$	$\nu_{ik}^0 =$ $2^{3i+4}3^{k+2}$	$\nu_{jk}^0 =$ $2^{3j+5}3^{k+2}$
cardinality per period $\#(S\|\nu)$		$\#(S_{>0}\|\nu_k^0) =$ $5 \cdot 3^k$	$\#(S_{>i}\|\nu_{ik}^0) =$ $6 \cdot 3^k$	$\#(S_{>j}\|\nu_{jk}^0) =$ $5 \cdot 3^{k+1}$
Collatz orbits generating function		$R^k(n \in S_{>0}) \in S_0$ for $k = 1,2,3,\ldots$	$B_j^k(n \in S_{>i}) \in S_i$ for odd $i = 1,3,5,\ldots$ and $k = 1,2,3,\ldots$	$B_j^k(n \in S_{>j}) \in S_j$ for even $j = 2,4,6,\ldots$ and $k = 1,2,3,\ldots$

Let's elucidate the structure of this summarizing table, before elaborating in the proofs below on the formulas in the cells of the table. The bottom of the table shows that for 3 out of the 4 output subsets first periodicities and cardinalities per period of their corresponding *argument subsets* have to be determined. Output subset S_0 is generated by iterations k of the rightward function $R^k(n) \in S_0$, starting from upward *arguments* $n \in S_{>0}$. The odd upward output subsets $S_{1,3,5,\ldots}$ are generated by iterations of the reconnection function $B_j^k(n)$, with iterator k for its implied rightward iterations and iterator i for its implied odd upward iterations, starting from *arguments* $n \in S_{>i}$. Similarly, the generation of even upward subsets $S_{2,4,6,\ldots}$ involve an iterator j for even upward iterations, starting from *arguments* $n \in S_{>j}$.

The table columns show that the Collatz orbit generating function split any node set $S_{\geq j}$ in arguments $S_{>j}$ that will remain as the node set $S_{\geq j+1}$ of tree $T_{\geq j+1}$, after having generated as their complement the departing output numbers S_j whose density $\rho(S_j)$ is assessed. The table rows reveal the periodicities and cardinalities per period that undergird these density assessments.

6.1 Density $\rho(S_{-1})$ of numbers divisible by 2 and/or 3

Let's illustrate for the first term, the density $\rho(S_{-1})$ of set S_{-1} of flown numbers divisible by 2 and/or 3, that the density calculations require a *periodicity* ν. The Least Common Multiple of divisors 2 and 3 gives as periodicity $\nu = LCM(2,3) = 6$. Counting remainders 0, 2, 3 and 4 per period ν gives as *cardinality per period* $\#(S_{-1}|\nu) = 4$. Division by ν gives as *density* $\rho(S_{-1}) = \#(S_{-1}|\nu)/\nu = 4/6$.

6.2 Density $\rho(S_0)$ of rightward orbits with upward $S_{>0}$ arguments

The next term is the density $\rho(S_0)$ of rightward function iterates $R^k(n)$ for $k = 1\ldots\infty$, given upward arguments $n \in S_{>0}$, with as a suitable argument periodicity $\upsilon_1^0 = LCM(18,96) = 288 = 2^5 3^2$. This periodicity incorporates the output periodicity 96 of upward arguments in $S_{>0}$ (lemma 1) and the input periodicity 18 of the rightward function (definition 2). The cardinality of upward arguments per argument period $\upsilon_1^0 = 288 = 3\cdot 96$ is $\#(S_{\geq 1}|\upsilon_1^0) = 3\cdot 5 = 15$ (cf. lemma 1), which can be split up in a vector of argument cardinalities $\overrightarrow{\#}\left(S_{\geq 1} \mid \upsilon_1^0\right) = [1,4,1,4,1,4]$, representing the upward argument congruence classes $[\{181\},\{5,77,149,221\},\{277\},\{29,101,173,245\},\{85\},\{53,125,197,269\}]$ mod 288. The last c_i-column of Table 1 shows that to account for all possible congruence classes modulo 18 at iteration k, an argument periodicity of 3.18 = 54 at iteration $k-1$ guarantees for successive values $i = 0,1,2$ three different output congruence classes $R(54j+18i+a)$ mod 18 as input for iteration k. Consequently, the argument periodicity to account for cardinalities at iteration k amounts to $\nu_k^0 = 3^{k-1}\nu_1^0 = 2^5 3^{k+1}$.

At the first rightward iteration starting from upward arguments, the corresponding output periodicity $\nu^1 = \upsilon_1^0\theta = 1536 = 2^{1\cdot 4+5}3$ results from multiplication with the conversion factor $\theta = 96/18 = 2^4 3^{-1}$ (cf. Table 1). Correspondingly, the output periodicity at iteration k amounts to $\nu^k = \nu_k^0\theta^k = 2^{4k+5}3$. The multiplier $2^4 = 16$ observed in ν^k for successive output periodicities will determine the *denominator* in the density calculation below.

Calculating the rightward output cardinality per period requires additionally the *heap vector* $\overrightarrow{\boldsymbol{h}} = [4,2,1,8,4,8]$ from Table 1 to handle intrinsic rightward output periodicities for different congruence classes modulo 18. This gives as output cardinality for the first rightward iteration given upward arguments $\overrightarrow{\#}\left(S_0|\nu^1\right) = \overrightarrow{\#}\left(S_{>0}|\nu_1^0\right) \circ \overrightarrow{\boldsymbol{h}} = [1,4,1,4,1,4]\circ[4,2,1,8,4,8] = [4,\ 8,\ 1,\ 32,\ 4,\ 32]$. For rightward iterations $k > 1$ it also requires the *transformation matrix* $\boldsymbol{T}$ from the last column in Table 1 to incorporate that each specific congruence class modulo 18 of arguments for the previous iteration $k-1$ sends its output to 3 different congruence classes modulo 18 of arguments for the current iteration k. Congruence classes 1, 5,7,11 at iteration $k-1$ produce congruence classes 1,7,13 at iteration k. Congruence classes 13,17 at iteration $k-1$ shift their output to congruence classes 5,11,17 at iteration k. Matrix pre-multiplication with $\boldsymbol{T}'$, thus with the transpose of $\boldsymbol{T}$, is required to calculate the cardinality per congruence class modulo 18 in iteration k from the cardinality per congruence class modulo 18 in iteration $k-1$. This gives as cardinalities per congruence class modulo 18 for the second rightward iterations $\overrightarrow{\#}\left(R^2(n)|\nu^2\right) = \overrightarrow{\boldsymbol{T}'}\cdot\overrightarrow{\#}\left(R^1(n)|\nu^1\right) \circ \overrightarrow{\boldsymbol{h}} = [180,72,45,288,180,288]$. Here $\cdot$ is used for matrix multiplication and $\circ$ for elementwise multiplication. Iterative application of $\overrightarrow{\#}\left(R^k(n)|\nu^k\right) = \overrightarrow{\boldsymbol{T}'}\cdot\overrightarrow{\#}\left(R^{k-1}(n)|\nu^{k-1}\right) \circ \overrightarrow{\boldsymbol{h}}$ gives as cardinalities per congruence class modulo 18 for the k'th iteration $\overrightarrow{\#}\left(R^k(n)|\nu^k\right) = 13^{k-2}\ [180,72,45,\ 288,\ 180,288]$. The multiplier 13 observed in $\overrightarrow{\#}\left(R^k(n)|\nu^k\right)$ for successive cardinalities per output period will determine the *nominator* in the density calculation below.

The density across iterations $k > 1$ happens to be an infinite geometric series sum with decay factor $r = 13/16$, with a growth rate of 13 in the cardinality $\#\left(R^k(n)|\nu^k\right)$ relative to a growth rate of 16 in the output periodicity ν^k. This gives $\rho(S_0) = \#\left(R^1(n)|\nu^1\right)/\nu^1 + \sum_{k=2}^{\infty}\left(R^k(n)|\nu^k\right)/\nu^k = 81/1536+$

351/1536 = 27/96. This is equal to the density $\rho(\mathbb{N}^R) = 27/96$ of all rightward numbers (lemma 3).

6.3 Densities $\rho(S_{1,3,5,...})$ and $\rho(S_{2,4,6,...})$ of upward descendants of S_0-ancestors

Let's start with a preview. Odd upward subsets $\rho(S_{1,3,5,...})$ and even upward subsets $\rho(S_{2,4,6,...})$ are treated separately because each second upward iteration invariably yields $U^2(n) = 64n + 21$. This can be checked by applying definition 1 twice. It can be seen from Figure 3 in which the odd trees $T_{\geq 1}$ and $T_{\geq 3}$ to the left are isomorphic with respect to contractions and expansions. The even trees $T_{\geq 0}$ (Figure 2), $T_{\geq 2}$ and $T_{\geq 4}$ (right side Figure 3) are also isomorphic with respect to contractions and expansions. We will use iterator i to iterate over odd subsets, and iterator j to iterate over even subsets.Within the bridging functions $B_i^k(n) = U^i R^k U^{-i}(n)$ and $B_j^k(n) = U^j R^k U^{-j}(n)$ The R^k iterations obey the S_0 logic above, with increasing powers of 13 in cardinalities at each further rightward iteration. The cardinalities of rightward S_0 ancestors apply also their upward children in S_1, upward grandchildren in S_2, and so on, since each S_0 ancestor has exactly one upward child in S_1, one upward grandchild S_2, and so on. Given $U^2(n) = 64n+21$, more distant descendants are spread over larger periodicities. With this preview in mind, let's now clarify the determination of the argument periodicities, output periodicities, cardinalities per output period, and densities.

Let's set argument periodicities for odd and even upward generations to $\nu_{ik}^0 = 2^{4\,+\,3\,i}3^{k+2}$ respectively $\nu_{jk}^0 = 2^{5+3j}3^{k+2}$. This choice guarantees that the reconnection output cardinality in one output period depends only on the argument cardinality within one argument period, and on the heap vector and the transformation matrix applied to each rightward iteration k in the reconnection function. The corresponding reconnection output periodicities amount to $v_i^k = v_{ik}^0\theta^k = 2^{4k+3i\,+\,4}3^2$ and $v^{jk} = v_{jk}^0\theta^k = 2^{4k+3j\,+\,5}3^2$. Note that the periodicity, which is the nominator in the density calculations below, increases at each further rightward iteration with a factor $2^4 = 16$, but with a factor $2^{3\cdot 2} = 64$ at each further odd iteration, and also at each further even iteration – in line with $U^2(n) = 64n + 21$. For odd generations S_i, the cardinality vector of upward arguments $n \in S_{>i}$ per argument period ν_{ik}^0 amounts to $\overrightarrow{\#}\left(S_{>i} \mid \nu_{ik}^0\right) = \left[3^k, 3^k, 3^k, 3^k, 3^k, 3^k\right]$. For odd generations, the cardinality vector of upward arguments $n \in S_{>i}$ per argument period ν_{jk}^0 amounts to $\overrightarrow{\#}\left(S_{>j} \mid \nu_{jk}^0\right) = \left[3^k, 12^k, 3^k, 12^k, 3^k, 12^k\right]$ (cf. Suppl.B12 for lists of argument and output congruence classes).

We obtain as cardinalities per period $\#\left(S_{i=1,3,5,\cdots}|\nu_i^k\right) = 2^1 3^4 13^{k-1}$ and $\#\left(S_{j=2,4,6,\cdots}|\nu_j^k\right) = 3^5 13^{k-1}$, with increasing powers of 13, just as for the underlying rightward iterations $R^k(n) \in S_0$ in the previous subsection – regardless which generation $S_1, S_2, \ldots.$ is considered, in line with the second observation above.

Infinite geometric sums result once again when calculating the required densities, with as final densities $\rho(S_{1,3,5,...}) = 1/21$ and $\rho(S_{2,4,6,...}) = 1/224$. The density of odd upward generations $\rho(S_{1,3,5,...}) = \sum_{i=1,3,5,\cdots}^{\infty}\sum_{k=1}^{\infty}\#\left(S_{i=1,3,5,\cdots}|v_i^k\right)/v_i^k$ is first summated over the inner rightward iterator k, giving $\sum_{i=1,3,5,\cdots}^{\infty}\sum_{k=1}^{\infty}(2^1 3^4 13^{k-1})/\left(2^{4k+3i\,+\,4}3^2\right)$. The resulting sum depends only on the odd upward iterator i, gving $\sum_{i=1,3,5,\cdots}^{\infty} 2^{-3i-5}3^2 = 1/21$. Similarly, the density of even upward generations $\rho(S_{2,4,6,...}) = \sum_{j=2,4,6,\cdots}^{\infty}\sum_{k=1}^{\infty}\#\left(S_{j=2,4,6,\cdots}|\nu_j^k\right)/\nu_j^k$ is also summated first over the inner rightward iterator k, giving $\sum_{j=2,4,6,\ldots}^{\infty}\sum_{k=1}^{\infty}\left(3^5 13^{k-1}\right)/(2^{4k+3j+5}3^2)$. The resulting sum depends only on the even upward iterator j, giving $\sum_{j=2,4,6,\ldots}^{\infty} 2^{-3j-5}3^2 = 1/224$. The sum of the odd upward densities $\rho(S_{1,3,5,\ldots}) = 1/21$ and the even upward densities $\rho(S_{2,4,6,\ldots}) = 1/224$ amounts to $1/21 + 1/224 = 5/96$, which is equal to the density of all upward numbers $\rho\left(\mathbb{N}^U\right) = 5/96$ (lemma 3).

7 Discussion

Proving the Collatz conjecture by picturing Collatz trees as a number theoretical Hilbert hotel reveals new links between graph theory and number theory, as anticipated by its originator (5). To our knowledge the Collatz tree is the first Hilbert hotel that is explicitly construed as a number tree rather than as a number line.

A momentous feature of the binary tree is its one-to-one mapping of unique numbers on its nodes to unique binary sequences of either upward or rightward jumps from its root to each of its nodes. Just as with prime factorization, natural numbers can be mapped one-to-one to sequences, e.g. $35 = 2^0 3^0 5^1 7^1 \leftrightarrow \langle 0,0,1,1\rangle$ if 35 is factorized, or $35 = 1 \xrightarrow{1} 5 \xrightarrow{0} 13 \xrightarrow{1} 53 \xrightarrow{0} 35 \leftrightarrow \langle 1,0,1,0\rangle$ if the binary Collatz path from the root to 35 is taken. The proof of the Collatz conjecture may enhance recently explored applications of the Collatz tree to random number generation, encryption, and watermarking.

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Ancillary Materials. *ElaborationProofsCollatzTree.pdf* gives an account of the notation used with further examples and elaborations, especially of various tree plots, and of the periodicities and cardinalities that underlie the density calculations for Lemma 4. *Math12NotebookCollatzTree.nb*, with additionally its static pdf, is the Mathematica 12 version of the Mathematica notebook underlying the paper.

Works 7-66 from the references below are cited in the ancillary materials. [1] [2] [3] [4] [5] [6] [7] [8] [9] [10] [11] [12] [13] [14] [15] [16] [17] [18] [19] [20] [21] [22] [23] [24] [25] [26] [27] [28] [29] [30] [31] [32] [33] [34] [35] [36] [37] [38] [39] [40] [41] [42] [43] [44] [45] [46] [47] [48] [49] [50] [51] [52] [53] [54] [55] [56] [57] [58] [59] [60] [61] [62] [63] [64] [65] [66]

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