Prime Values of Quadratic Polynomials

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***Abstract*:** This note investigates the prime values of the polynomial $f(t) = qt^2 + a$ for any fixed pair of relatively prime integers $a \geq 1$ and $q \geq 1$ of opposite parity. For a large number $x \geq 1$, an asymptotic result of the form $\sum_{n \leq x^{1/2},\, n \text{ odd}} \Lambda(qn^2 + a) \gg qx^{1/2}/2\varphi(q)$ is achieved for $q \ll (\log x)^b$, where $b \geq 0$ is a constant.

1 Introduction

The basic problem of prime values of linear polynomials $f(t) = qt + a \in \mathbb{Z}[t]$ is completely solved. Dirichlet theorem for primes in arithmetic progressions proves that any admissible linear polynomial has infinitely many prime values. The quantitative form of this theorem has the asymptotic formula

$$\sum_{\substack{n \leq x \\ n \equiv a \bmod q}} \Lambda(n) \sim \frac{1}{\varphi(q)} x, \tag{1}$$

where $\gcd(a, q) = 1$, as $x \to \infty$. The next basic problem of prime values of quadratic polynomials $f(t) = at^2 + bt + c \in \mathbb{Z}[t]$ has very precise heuristics and many partial results, but there is no qualitative nor quantitative results known. This note investigates the prime values of the admissible quadratic polynomials $f(t) = qt^2 + a \in \mathbb{Z}[t]$, and proposes the following result.

Theorem 1.1. *Let $x \geq 1$ be a large number. Let a and q be a pair of relatively prime integers, with opposite parity, and $q \ll (\log x)^b$, where $b \geq 0$ is a constant. Then,*

$$\sum_{\substack{n \leq x^{1/2} \\ n \text{ odd}}} \Lambda(qn^2 + a) \gg \frac{q}{2\varphi(q)} x^{1/2} + O\left(x^{1/2} e^{-c\sqrt{\log x}}\right), \tag{2}$$

where $c > 0$ is an absolute constant.

The core of the proof in Section 2 consists of the *quadratic to linear identity* in Section 6, and other results proved in Section 3 to Section 8. Theorem 1.1 proves the predicted asymptotic formula

$$\sum_{\substack{n \leq x^{1/2} \\ n \text{ odd}}} \Lambda(qn^2 + a) \sim \frac{c_f}{2} x^{1/2}, \tag{3}$$

but not the constant $c_f \geq 0$, see [3] for finer details. The conjectured constant, which depends on the polynomial $f(t) = qt^2 + a$, has the form

$$c_f = \epsilon \prod_{\substack{p \geq 3 \\ p \mid q}} \left(\frac{p}{p-1}\right) \prod_{\substack{p \geq 3 \\ p \nmid q}} \left(1 - \left(\frac{-aq}{p}\right) \frac{1}{p}\right), \tag{4}$$

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where

$$\epsilon = \begin{cases} 1/2 & \text{if } q \not\equiv 0 \bmod 2, \\ 1 & \text{if } q \equiv 0 \bmod 2. \end{cases} \tag{5}$$

The conjectured general formula for the constant $c_f(a,c) \geq 0$ attached to an admissible quadratic polynomial $f(t) = at^2 + bt + c \in \mathbb{Z}[t]$ appears in [16, p. 46], [3, p. 364], et alii. Discussions on the convergence of the product (4) appears in [3], [13, Section 5], et alii. Results on the average value $\overline{c_f(a,c)}$, and other properties appear in [5], [25], et cetera, optimization and numerical techniques appear in [18], and similar references.

The result in Theorem 1.1 is a special case of the Bateman-Horn Conjecture for polynomials over the integers, see [3], and [13] for a survey. Some references on the vast literature on the theory of prime values of polynomials are provided here. The general circle methods are introduced in [22], [28], and the heuristics for admissible quadratic polynomials was proposed in [16, p. 46]. More recent discussions are given in [26, p. 406], [23, p. 342], et cetera. Some partial results are proved in [15], [20], [5], [8], [17], [19], and the recent literature. The results for the associated least common multiple problem $\log \operatorname{lcm}[f(1)f(2)\cdots f(n)]$ appears in [6], et cetera. The related problem for almost primes appears in [17], [19], et alii. Topics on the Bateman-Horn Conjecture for polynomials over numbers fields and functions fields appear in [4], [7], [10], et alii, and for multivariable polynomials appears in [9], et cetera. A very recent proof for certain collection of quadratic polynomials $f(t) = a(u)t^2 + b(u)t + c(u) \in \mathbb{F}_q[u][t]$ over function fields of odd characteristic is proposed in [27, Theorem 1.2].

2 Prime Values of Quadratic Polynomials

For any pair of fixed integers $1 \leq a \leq q$ such that $\gcd(a,q) = 1$, the polynomial $f(t) = qt^2 + a$ is irreducible, and it has fixed divisor $\operatorname{div}(f) = \gcd(f(\mathbb{Z})) = 1$, see [14, p. 395] for more details.

For an integer $n \geq 1$, the vonMangoldt function $\Lambda : \mathbb{N} \longrightarrow \mathbb{R}$ is defined by

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k, \\ 0 & \text{if } n \neq p^k, \end{cases} \tag{6}$$

where $n = p^k$ is a prime power, and the Euler totient function $\varphi : \mathbb{N} \longrightarrow \mathbb{Q}$ is defined by $\varphi(n) = n \prod_{p \mid n} (1 - 1/p)$. A primes counting function, weighted by $\Lambda(n)$, is defined by

$$\psi_2(x,q,a) = \sum_{\substack{n \leq x^{1/2} \\ n \text{ odd}}} \Lambda(qn^2 + a). \tag{7}$$

Proof. (Theorem 1.1): Given a large number $x \geq 1$, let $p \equiv 1 \bmod 4$ be a large prime such that $x < p$, and let $N = 2p$. Further, assume that $[x^{1/2}] = 2k$ is an even integer. Now, in terms of the *quadratic to linear identity* in Lemma 6.1, the weighted primes counting function has the form

$$\begin{aligned} \sum_{\substack{n \leq x^{1/2} \\ n \text{ odd}}} \Lambda(qn^2 + a) &= \frac{1}{\varphi(N)} \sum_{\substack{n \leq x \\ n \text{ odd}}} \Lambda(qn + a) \sum_{s \leq x^{1/2},} \sum_{\substack{1 \leq u < N \\ \gcd(u,N)=1}} e^{i2\pi(s^2 - n)u/N} \\ &= \frac{1}{\varphi(N)} \sum_{\substack{n \leq x \\ n \text{ odd}}} \Lambda(qn + a) \sum_{\substack{1 \leq u < N \\ \gcd(u,N)=1}} e^{-i2\pi un/N} \sum_{1 \leq s \leq x^{1/2}} e^{i2\pi us^2/N}. \end{aligned} \tag{8}$$

This step removes any reference to nonlinear polynomial. Next step employs the quadratic symbol

$$\left(\frac{s}{N}\right)=\begin{cases}-1 & \text{if } s \text{ is a quadratic nonresidue } \bmod N,\\ 0 & \text{if } \gcd(s,N)\neq 1,\\ 1 & \text{if } s \text{ is a quadratic residue } \bmod N,\end{cases} \tag{9}$$

to remove the nonlinear exponential term in the finite inner sum in (8). This procedure yields

$$\begin{aligned}\psi_2(x,q,a) &= \frac{1}{\varphi(N)}\sum_{\substack{n\leq x\\ n \text{ odd}}}\Lambda(qn+a)\sum_{\substack{1\leq u<N\\ \gcd(u,N)=1}} e^{-i2\pi un/N}\sum_{1\leq s\leq x^{1/2}}\left(1+\left(\frac{s}{N}\right)\right)e^{i2\pi us/N}\\ &= \frac{1}{\varphi(N)}\sum_{\substack{n\leq x\\ n \text{ odd}}}\Lambda(qn+a)\sum_{\substack{1\leq u<N\\ \gcd(u,N)=1}}\sum_{1\leq s\leq x^{1/2}} e^{i2\pi u(s-n)/N} \qquad (10)\\ &\quad +\frac{1}{\varphi(N)}\sum_{\substack{n\leq x\\ n \text{ odd}}}\Lambda(qn+a)\sum_{\substack{1\leq u<N\\ \gcd(u,N)=1}} e^{-i2\pi un/N}\sum_{1\leq s\leq x^{1/2}}\left(\frac{s}{N}\right)e^{i2\pi us/N}\\ &= M(x)+E(x),\end{aligned}$$

Applying Lemma 3.1 to the main term $M(x)$ and Lemma 4.1 to the error term $E(x)$ yield

$$\begin{aligned}\sum_{\substack{n\leq x^{1/2}\\ n \text{ odd}}}\Lambda(qn^2+a) &= M(x)+E(x) \qquad (11)\\ &\gg \left[\frac{q}{2\varphi(q)}x+O\left(xe^{-c\sqrt{\log x}}\right)\right]+[0]\\ &\gg \frac{q}{2\varphi(q)}x+O\left(xe^{-c\sqrt{\log x}}\right),\end{aligned}$$

where $c>0$ is an absolute constant, as $x\to\infty$. ■

3 The Main Term

The partial main term is evaluated in this section.

Lemma 3.1. *Given a large number $x\geq 1$, let $p\equiv 1 \bmod 4$ be a large prime such that $x<p$, and let $N=2p$. Further, assume that $[x^{1/2}]=2k$ is an even integer. Let a and q be a pair of relatively prime integers, with opposite parity, and $q\ll(\log x)^b$, where $b\geq 0$ is a constant. Then,*

$$\frac{1}{\varphi(N)}\sum_{\substack{n\leq x\\ n \text{ odd}}}\Lambda(qn+a)\sum_{\substack{1\leq u<N\\ \gcd(u,N)=1}}\sum_{1\leq s\leq N^{1/2}} e^{i2\pi u(s-n)/N}\gg\frac{q}{2\varphi(q)}x^{1/2}+O\left(x^{1/2}e^{-c\sqrt{\log x}}\right), \tag{12}$$

where $c>0$ is an absolute constant.

Proof. Consider the dyadic partition

$$\begin{aligned} M(x) &= \frac{1}{\varphi(N)} \sum_{\substack{n \leq x \\ n \text{ odd}}} \Lambda(qn+a) \sum_{\substack{1 \leq u < N \\ \gcd(u,N)=1}} \sum_{1 \leq s \leq x^{1/2}} e^{i2\pi u(s-n)/N} \qquad (13) \\ &= \frac{1}{\varphi(N)} \sum_{\substack{n \leq x \\ n \text{ odd} \\ n=s}} \Lambda(qn+a) \sum_{1 \leq s \leq x^{1/2}} \sum_{\substack{1 \leq u < N \\ \gcd(u,N)=1}} e^{i2\pi u(s-n)/N} \\ &\quad + \frac{1}{\varphi(N)} \sum_{\substack{n \leq x \\ n \text{ odd} \\ n \neq s}} \Lambda(qn+a) \sum_{1 \leq s \leq x^{1/2}} \sum_{\substack{1 \leq u < N \\ \gcd(u,N)=1}} e^{i2\pi u(s-n)/N} \\ &= M_0(x) + M_1(x). \end{aligned}$$

The first term in (13) has the value

$$\begin{aligned} M_0(x) &= \frac{1}{\varphi(N)} \sum_{\substack{n \leq x \\ n \text{ odd} \\ n=s}} \Lambda(qn+a) \sum_{1 \leq s \leq x^{1/2}} \sum_{\substack{1 \leq u < N \\ \gcd(u,N)=1}} e^{i2\pi u(s-n)/N} \qquad (14) \\ &= \sum_{\substack{n \leq x^{1/2} \\ n \text{ odd}}} \Lambda(qn+a) \\ &\gg \frac{q}{2\varphi(q)} x^{1/2} + O\left(x^{1/2} e^{-c\sqrt{\log x}}\right), \end{aligned}$$

since $qn + a \leq qx^{1/2} + a$, as $x \to \infty$, see [11, Theorem 8.8], [21, Corollary 11.19], et cetera. The second term in (13) has the value

$$\begin{aligned} M_1(x) &= \frac{1}{\varphi(N)} \sum_{\substack{n \leq x \\ n \text{ odd} \\ n \neq s}} \Lambda(qn+a) \sum_{1 \leq s \leq x^{1/2}} \sum_{\substack{1 \leq u < N \\ \gcd(u,N)=1}} e^{i2\pi u(s-n)/N} \qquad (15) \\ &= \frac{1}{\varphi(N)} \sum_{\substack{n \leq x \\ n \text{ odd} \\ n \neq s}} \Lambda(qn+a) \sum_{1 \leq s \leq x^{1/2}} c_N(s-n) \\ &= 0, \end{aligned}$$

where $s \neq n$ implies that $n > x^{1/2}$, and $c_N(s-n)$ is a Ramanujan sum. The last equality in (15) follows from Lemma 7.2 since $n \leq x$ is odd, and $[x^{1/2}] = 2k$. ■

4 The Error Term

The error term is evaluated in this section.

Lemma 4.1. *Given a large number $x \geq 1$, let $p \equiv 1 \bmod 4$ be a large prime such that $x < p$, and let $N = 2p$. Further, assume that $[x^{1/2}] = 2k$ is an even integer. Then,*

$$\frac{1}{\varphi(N)} \sum_{\substack{n \leq x \\ n \text{ odd}}} \Lambda(qn+a) \sum_{\substack{1 \leq u < N \\ \gcd(u,N)=1}} e^{-i2\pi un/N} \sum_{1 \leq s \leq x^{1/2}} \left(\frac{s}{N}\right) e^{i2\pi us/N} = 0. \qquad (16)$$

Proof. By definition the integers u and N are relatively prime; $\gcd(u,N)=1$. Thus, the change of variable $z=us$ yields

$$\begin{aligned} E(x) &= \frac{1}{\varphi(N)} \sum_{\substack{n\le x \\ n \text{ odd}}} \Lambda(qn+a) \sum_{\substack{1\le u<N \\ \gcd(u,N)=1}} e^{-i2\pi un/N} \sum_{1\le s\le x^{1/2}} \left(\frac{s}{N}\right) e^{i2\pi us/N} \qquad (17) \\ &= \frac{1}{\varphi(N)} \sum_{\substack{n\le x \\ n \text{ odd}}} \Lambda(qn+a) \sum_{\substack{1\le u<N \\ \gcd(u,N)=1}} \left(\frac{u^{-1}}{N}\right) e^{-i2\pi un/N} \sum_{1\le z\le x^{1/2}} \left(\frac{z}{N}\right) e^{i2\pi z/N}. \end{aligned}$$

Since $n\le x<p$ is odd and $N=2p$, the integers n and N are relatively prime; $\gcd(n,N)=1$. Thus, the change of variable $w=un$ yields

$$\begin{aligned} E(x) &= \frac{1}{\varphi(N)} \sum_{\substack{n\le x \\ n \text{ odd}}} \Lambda(qn+a) \sum_{1\le z\le x^{1/2}} \left(\frac{z}{N}\right) e^{i2\pi z/N} \sum_{\substack{1\le u<N \\ \gcd(u,N)=1}} \left(\frac{u^{-1}}{N}\right) e^{-i2\pi un/N} \qquad (18) \\ &= \frac{1}{\varphi(N)} \sum_{\substack{n\le x \\ n \text{ odd}}} \left(\frac{n^{-1}}{N}\right) \Lambda(qn+a) \sum_{1\le z\le x^{1/2}} \left(\frac{z}{N}\right) e^{i2\pi z/N} \sum_{\substack{1\le w<N \\ \gcd(w,N)=1}} \left(\frac{w}{N}\right) e^{-i2\pi w/N}. \end{aligned}$$

The congruence relations $p\equiv 1 \bmod 4$, and $N=2p\equiv 2 \bmod 4$ imply that the Gauss sum

$$\sum_{\substack{1\le w<N \\ \gcd(w,N)=1}} \left(\frac{w}{N}\right) e^{-i2\pi w/N} = 0 \qquad (19)$$

vanish, see Theorem 8.2. Therefore, $E(x)=0$. ■

5 Characteristic Functions For Integer Powers

An explicit representation of the characteristic function $\mathcal{Q}:\mathbb{N}\longrightarrow\{0,1\}$ of square odd integers on an interval $[1,x]$ is introduced below. The parameters were chosen to fit the application within.

Lemma 5.1. *Given a large number $x\ge 1$, let $p\ge 2$ be a large prime such that $x<p$, and let $N=2p$. Further, assume that $[x^{1/2}]=2k$ is an even integer. If $n\le x$ is a fixed odd integer, then,*

$$\mathcal{Q}(n) = \frac{1}{\varphi(N)} \sum_{1\le s\le x^{1/2},} \sum_{\substack{0\le u\le N-1 \\ \gcd(u,N)=1}} e^{i2\pi(s^2-n)u/N} = \begin{cases} 1 & \text{if } n=s^2, \\ 0 & \text{if } n\ne s^2. \end{cases} \qquad (20)$$

Proof. Assume $n=s^2$ is a square. The hypothesis $n\le x$ implies that the equation $s^2-n=0$ has a unique integer solution $s\in[1,x^{1/2}]$ for each square integer $n\in[1,x]$. Thus, the double finite sum has the value

$$\frac{1}{\varphi(N)} \sum_{1\le s\le x^{1/2},} \sum_{\substack{0\le u\le N-1 \\ \gcd(u,N)=1}} e^{i2\pi(s^2-n)u/N} = 1. \qquad (21)$$

Assume $n\ne s^2$ is not a square. The hypothesis $N=2p$, with $p>2$ prime, and an odd integer $n\le x$ and $s\le x^{1/2}$, imply that the Ramanujan sum has the value $c_N(s^2-n)=$

$(-1)^s$ for $s^2 - n \neq 0$, see Lemma 7.1. Hence,

$$\begin{aligned} \frac{1}{\varphi(N)} \sum_{1 \leq s \leq x^{1/2},} \sum_{\substack{0 \leq u \leq N-1 \\ \gcd(u,N)=1 \\ n \neq s^2}} e^{i2\pi\left(s^2-n\right)u/N} &= \frac{1}{\varphi(N)} \sum_{\substack{1 \leq s \leq x^{1/2} \\ n \neq s^2}} c_N(s^2 - n) \qquad (22) \\ &= 0, \end{aligned}$$

the last equality follows from Lemma 7.2 since $[x^{1/2}] = 2k$ is an even integer. ■

This technique is very flexible, and has the advantages of being easily extended to other classes of integer powers as cubic integers, and quartic integers, et cetera.

6 Quadratic To Linear Identity

The quadratic to linear inequality trades off the evaluation of $\sum_{n \leq x^{1/2}, \text{odd } n} \Lambda(qn^2 + a)$ for the evaluation of a product of some exponential sums and $\sum_{n \leq x, \text{odd } n} \Lambda(qn + a)$.

Lemma 6.1. *Given a large number $x \geq 1$, let $p \geq 2$ be a large prime such that $x < p$, and let $N = 2p$. Further, assume that $[x^{1/2}] = 2k$ is an even integer. If a and q is a pair of relatively prime integers, and opposite parity, then,*

$$\sum_{\substack{n \leq x^{1/2} \\ odd\ n}} \Lambda(qn^2 + a) = \frac{1}{\varphi(N)} \sum_{\substack{n \leq x \\ n\ odd}} \Lambda(qn + a) \sum_{s \leq x^{1/2},} \sum_{\substack{0 \leq u < N \\ \gcd(u,N)}} e^{i2\pi(s^2-n)u/N}. \qquad (23)$$

Proof. Summing the product of $\Lambda(qn + a)$ and the characteristic function $\mathcal{Q}(n)$ of square odd integers $n \leq x$ return

$$\begin{aligned} \sum_{\substack{n \leq x \\ n \text{ odd}}} \Lambda(qn + a)\mathcal{Q}(n) &= \sum_{\substack{n \leq x \\ n \text{ odd}}} \Lambda(qn + a) \frac{1}{\varphi(N)} \sum_{s \leq x^{1/2},} \sum_{\substack{0 \leq u < N \\ \gcd(u,N)}} e^{i2\pi(s^2-n)u/N} \qquad (24) \\ &= \sum_{\substack{m \leq x^{1/2} \\ \text{odd } m}} \Lambda(qm^2 + a), \end{aligned}$$

where $m^2 = n \leq x$. The last line follows from the definition of $\mathcal{Q}(n)$, see Lemma 5.1. ■

This concept has a straight forward extension to the *cubic to linear identity*, the *quartic to linear identity*, et cetera.

7 Ramanujan Sums

Lemma 7.1. *Given a large number $x \geq 1$, let $p \geq 2$ be a large prime such that $x < p$, and let $N = 2p$. Further, assume that $[x^{1/2}] = 2k$ is an even integer. If $n \leq x$ is a fixed odd integer, and $s \leq x^{1/2}$, then,*

1. $c_N(s - n) = \sum_{\substack{1 \leq u < N \\ \gcd(u,N)=1 \\ s-n \neq 0}} e^{i2\pi u(s-n)/N} = (-1)^s.$

2. $c_N(s^2 - n) = \sum_{\substack{1 \leq u < N \\ \gcd(u,N)=1 \\ s^2-n \neq 0}} e^{i2\pi u(s^2-n)/N} = (-1)^s.$

Proof. (1) For fixed odd integer $n \leq x$, and $s \leq x^{1/2}$, where $s-n \neq 0$, the absolute value of the difference of these integers satisfies $0 < |s-n| < p$. Hence, for $N = 2p$,

$$\gcd(s-n,2p) = \begin{cases} 1 & \text{if } s \equiv 0 \bmod 2, \\ 2 & \text{if } s \equiv 1 \bmod 2. \end{cases} \tag{25}$$

The Ramanujan sum, see [1, Theorem 8.6], [21, Theorem 4.1], et cetera, has the value

$$c_N(s-n) = \sum_{d \mid \gcd(s-n,N)} d\mu(N/d) = \begin{cases} 1 & \text{if } \gcd(s-n),2p) = 1, \\ -1 & \text{if } \gcd(s-n),2p) = 2. \end{cases} \tag{26}$$

Combining (25) and (26) yield $c_N(s-n) = (-1)^s$. (2) The same proof applies to this case. ■

Lemma 7.2. *Given a large number $x \geq 1$, let $p \geq 2$ be a large prime such that $x < p$, and let $N = 2p$. Further, assume that $[x^{1/2}] = 2k$ is an even integer. If $n \leq x$ is an odd integer, then,*

1. $$\sum_{\substack{s \leq x^{1/2} \\ s-n \neq 0}} c_N(s-n) = \begin{cases} 0 & \text{if } [x^{1/2}] = 2k, \\ -1 & \text{if } [x^{1/2}] = 2k \pm 1. \end{cases}$$

2. $$\sum_{\substack{s \leq x^{1/2} \\ s^2-n \neq 0}} c_N(s^2-n) = \begin{cases} 0 & \text{if } [x^{1/2}] = 2k, \\ -1 & \text{if } [x^{1/2}] = 2k \pm 1. \end{cases}$$

Proof. (1) For a fixed odd integer $n \neq s^2 \leq x$, and $s \leq x^{1/2}$ such that $s-n \neq 0$, the finite sum $c_N(s-n) = (-1)^s$, see Lemma 7.1. Thus,

$$\sum_{\substack{s \leq x^{1/2} \\ s-n \neq 0}} c_N(s-n) = \sum_{s \leq x^{1/2}} (-1)^s = \begin{cases} 0 & \text{if } [x^{1/2}] = 2k, \\ -1 & \text{if } [x^{1/2}] = 2k \pm 1. \end{cases} \tag{27}$$

(2) The same proof applies to this case. ■

8 Gauss Sums

Theorem 8.1. (Gauss) *If $N \geq 1$ is an integer, then,*

$$\sum_{0 \leq s \leq N-1} e^{i2\pi s^2/N} = \begin{cases} \sqrt{N} & \text{if } N \equiv 1 \bmod 4, \\ 0 & \text{if } N \equiv 2 \bmod 4, \\ i\sqrt{N} & \text{if } N \equiv 3 \bmod 4, \\ (1+i)\sqrt{N} & \text{if } N \equiv 0 \bmod 4. \end{cases} \tag{28}$$

Proof. A proof based on finite Fourier transform appears in [2, Theorem I.1.1], and a proof based on the Poisson summation formula appears in [21, Corollary 9.16]. ■

Theorem 8.2. *If $N \geq 1$ is an integer, and $(n \mid N)$ is the quadratic symbol modulo N, then,*

$$\sum_{\substack{1 \leq s \leq N-1 \\ \gcd(s,N)=1}} \left(\frac{s}{N}\right) e^{i2\pi s/N} = \begin{cases} \sqrt{N} & \text{if } N \equiv 1 \bmod 4, \\ 0 & \text{if } N \equiv 2 \bmod 4, \\ i\sqrt{N} & \text{if } N \equiv 3 \bmod 4, \\ (1+i)\sqrt{N} & \text{if } N \equiv 0 \bmod 4. \end{cases} \tag{29}$$

Proof. Use the quadratic symbol (9) to remove the nonlinear term from the exponential sum:

$$\begin{aligned}\sum_{0\le s<N} e^{-i2\pi s^2/p} &= \sum_{0\le s<N}\left(1+\left(\frac{s}{N}\right)\right)e^{i2\pi s/N} \\ &= \sum_{0\le s<N} e^{i2\pi s/N}+\sum_{0\le s<N}\left(\frac{s}{N}\right)e^{i2\pi s/N}.\end{aligned} \tag{30}$$

The first term vanishes, and the second term is the same as Theorem 8.1. Further, the quadratic symbol

$$\left(\frac{s}{N}\right)=\begin{cases}\pm 1 & \text{if } \gcd(s,N)=1,\\ 0 & \text{if } \gcd(s,N)\neq 1,\end{cases} \tag{31}$$

and $(0\,|\,N)=0$. These observations complete the proof. ■

9 Euler Polynomial and Primes

The properties of the integers represented by the polynomial $f(t)=t^2+1\in\mathbb{Z}[t]$, such as squarefree values, almost prime values, and prime values, etc., are heavily studied in number theory. As early as 1760, Euler was developing the theory of prime values of polynomials. In fact, Euler computed an impressive large table of the prime values $p=n^2+1$, see [12, p. 123]. Probably, the prime values of polynomials was studied by other researchers before Euler. Later, circa 1910, Landau posed an updated question of the same problem about the primes values of this polynomial. A heuristic argument, based on circle methods, was demonstrated about two decades later. Surveys of the subsequent developments appear in [23, p. 342], [24, Section 19], and similar references.

Corollary 9.1. *Let $x\ge 1$ be a large number. Then*

$$\sum_{n\le x^{1/2}}\Lambda(n^2+1)\gg\frac{x^{1/2}}{2}+O\left(x^{1/2}e^{-c\sqrt{(\log x)^b}}\right), \tag{32}$$

where $c>0$ is an absolute constant.

Proof. Consider the polynomial $f(t)=4t^2+1\in\mathbb{Z}[t]$, where $q=4$ and $a=1$. Then,

$$\begin{aligned}\sum_{n\le x^{1/2}}\Lambda(n^2+1) &= \sum_{n\le x^{1/2}/2}\Lambda(4n^2+1)+O(\log x) \\ &\ge \sum_{\substack{n\le x^{1/2}/2\\ n\text{ odd}}}\Lambda(4n^2+1) \\ &\gg \frac{4}{2\varphi(4)}\frac{x^{1/2}}{2}+O\left(x^{1/2}e^{-c\sqrt{(\log x)^b}}\right),\end{aligned} \tag{33}$$

where $c>0$ is an absolute constant. The third line in (33) follows from Theorem 1.1. ■

The standard heuristic for the prime values of the polynomial $f(t)=t^2+1\in\mathbb{Z}[t]$ predicts the followings data.

Conjecture 9.1. ([16]) *Let $x\ge 1$ be a large number. Let Λ be the vonMangoldt function, and let $\chi(n)=(n\mid p)$ be the quadratic symbol modulo p. Then*

$$\sum_{n\le x^{1/2}}\Lambda\left(n^2+1\right)=c_f x^{1/2}+O\left(\frac{x^{1/2}}{\log x}\right), \tag{34}$$

where the density constant

$$c_f = \prod_{p \geq 3} \left(1 - \frac{\chi(-1)}{p-1}\right) = 1.37281346\ldots. \tag{35}$$

A list of the prime values of the polynomial $f(t) = t^2 + 1$ is archived in OEIS A002496.

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