# MACMAHON'S STATISTICS ON HIGHER-DIMENSIONAL PARTITIONS 

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#### Abstract

We study some combinatorial properties of higher-dimensional partitions which generalize plane partitions. We present a natural bijection between $d$-dimensional partitions and $d$-dimensional arrays of nonnegative integers. This bijection has a number of important applications. We introduce a statistic on $d$-dimensional partitions, called the corner-hook volume, whose generating function has the formula of MacMahon's conjecture. We obtain multivariable formulas whose specializations give analogues of various formulas known for plane partitions. We also introduce higher-dimensional analogues of dual Grothendieck polynomials which are quasisymmetric functions and whose specializations enumerate higher-dimensional partitions of a given shape. Finally, we show probabilistic connections with a directed last passage percolation model in $\mathbb{Z}^{d}$.


## 1. Introduction

Higher-dimensional partitions are classical combinatorial objects introduced by MacMahon over a century ago. While the concept itself is a straightforward generalization of the usual integer partitions, the problems related to it are very challenging. For (2-dimensional) plane partitions, MacMahon obtained his celebrated enumerative formulas [Mac16] (cf. [Sta99, Ch. 7]). For general $d$-dimensional partitions, he only conjectured a formula of the volume generating function, which was later computed to be incorrect [ABMM67].

Despite long interest and many connections to various fields including algebra, combinatorics, geometry, probability and statistical physics, the subject remains rather mysterious-very little is known about $d$-dimensional partitions for $d \geq 3$. See [ABMM67, Knu70, Gov13] on some computational and enumerative aspects; [MR03, BGP12, DG15] on asymptotic data and connections to physics; [BBS13, Nek17, CK18] on further aspects particularly related to the theory of Donaldson-Thomas invariants. (See also the remarks and references in final Sec. 8.)

At the same time, the theory of plane partitions has greatly developed, see [And98, Sta99, Krat16] and many references therein. Its success mainly comes from the theory of symmetric functions, especially by using the Robinson-Schensted-Knuth (RSK) correspondence and Schur polynomials. The lack of tools for higher-dimensional generalizations makes it difficult to approach them, and here one can try to develop analogous methods. This paper is in this direction.

Let us summarize our results.
1.1. Higher-dimensional partitions and matrices. Firstly, we present a natural bijection between $d$-dimensional arrays of nonnegative integers and $d$-dimensional partitions, see Sec. 3. Roughly speaking, any $d$-dimensional partition can be viewed as a matrix of largest paths for some source weight matrix. The bijection has nice properties which relate natural statistics for both objects. We then give a number of applications.
1.2. Corner-hook volume and interpretation of MacMahon's numbers. One of the main consequences of our bijection is the multivariable generating series presented in Theorem 4.2 whose specializations allow to explicitly compute generating functions for certain statistics on $d$-dimensional partitions. In particular, we introduce two statistics on $d$-dimensional partitions: corners cor $(\cdot)$ and corner-hook volume $|\cdot|_{\text {ch }}$ (see Sec. 4 and 5 for definitions) with generating functions shown below.

Theorem 1.1 (Corner-hook generating function, cf. Corollary 5.4). We have the following generating function

$$
\sum_{\pi} t^{\operatorname{cor}(\pi)} q^{|\pi|_{c h}}=\prod_{n=1}^{\infty}\left(1-t q^{n}\right)^{-\binom{n+d-2}{d-1}},
$$

where the sum runs over d-dimensional partitions $\pi$.
For $d=2$, this formula is equidistributed with Stanley's trace generating function [Sta99, Thm. 7.20.1] but the statistics are not identical. MacMahon conjectured [Mac16] that the generating function

$$
\sum_{n=0}^{\infty} m_{d}(n) q^{n}=\prod_{n=1}^{\infty}\left(1-q^{n}\right)^{-\binom{n+d-2}{d-1}}
$$

gives the volume generating function $\sum_{\pi} q^{|\pi|}$ for $d$-dimensional partitions. This was shown to be incorrect for $d \geq 3$ [ABMM67]. However, from Theorem 1.1 we obtain the following interpretation of MacMahon's numbers $m_{d}(n)$, thus 'correcting' his guess via the corner-hook volume statistic so that

$$
m_{d}(n)=\mid\left\{d \text {-dimensional partitions } \pi:|\pi|_{c h}=n\right\} \mid .
$$

More generally, we also prove results for generating functions over partitions with fixed shape.
Theorem 1.2 (Corner-hook generating function with fixed shape, cf. Theorem 5.2). Let $\rho$ be a shape of a fixed d-dimensional partition. We have the following generating function

$$
\sum_{\operatorname{sh}(\pi) \subseteq \rho} t^{\operatorname{cor}(\pi)} q^{|\pi|_{c h}}=\prod_{\left(i_{1}, \ldots, i_{d}\right) \in \rho}\left(1-t q^{i_{1}+\ldots+i_{d}-d+1}\right)^{-1}
$$

where the sum runs over d-dimensional partitions $\pi$ of shape $\rho$.
1.3. $d$-dimensional Grothendieck polynomials. To develop tools for studying $d$-dimensional partitions, one might be looking for analogues of Schur polynomials whose specializations allow to enumerate them. We work in a slightly different direction. In Sec. 6 we define higherdimensional analogues of dual Grothendieck polynomials. These new functions are indexed by shapes of $d$-dimensional partitions and in specializations they compute the number of such partitions. For $d=2$, they turn into the dual symmetric Grothendieck polynomials (indexed by partitions) known as $K$-theoretic analogues of Schur polynomials introduced in [LP07] (see also [Yel17, Yel19] for more on these functions).

Let us illustrate our results in the special case for (3-dimensional) solid partitions. We define the polynomials (see eq. (8)) $g_{\pi}(\mathbf{x} ; \mathbf{y} ; \mathbf{z})$ in three sets of variables indexed by plane partitions $\{\pi\}$. These polynomials enumerate solid partitions within a given shape, e.g. we have

$$
g_{[b] \times[c] \times[d]}\left(1^{a+1} ; 1^{b} ; 1^{c}\right)=\text { number of solid partitions inside the box }[a] \times[b] \times[c] \times[d] .
$$

We show that the following generating series identity holds.
Theorem 1.3 (Cauchy-type identity for 3d Grothendieck polynomials, cf. Corollary 6.5). We have

$$
\sum_{\pi} g_{\pi}(\mathbf{x} ; \mathbf{y} ; \mathbf{z})=\prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{1}{1-x_{i} y_{j} z_{k}}
$$

where the sum runs over plane partitions $\pi$ with shape inside the rectangle $b \times c$.
It is known that dual Grothendieck polynomials (for $d=2$ ) are symmetric (in $\mathbf{x}$ ). As we show, this is no longer the case for $d \geq 3$. However, we prove that these new functions are quasisymmetric, the next known class containing symmetric functions (see e.g. [Sta99, Ch. 7.19]).

Theorem 1.4 (cf. Theorem 6.9). We have: $g_{\pi}(\mathbf{x} ; \mathbf{y} ; \mathbf{z})$ is quasisymmetric in $\mathbf{x}$.
1.4. Last passage percolation in $\mathbb{Z}^{d}$. It turns out that these problems are closely related to the directed last passage percolation model with geometric weights in $\mathbb{Z}^{d}$ (see [Mar06] for a survey on this probabilistic model). We prove that $d$-dimensional Grothendieck polynomials naturally compute distribution formulas for this model (see Theorem 7.1). See Sec. 7 for details.

## 2. Preliminary definitions

We use the following basic notation: $\mathbb{N}$ is the set of nonnegative integers; $\mathbb{Z}_{+}$is the set of positive integers; $\left\{\mathbf{e}_{1}, \ldots, \mathbf{e}_{d}\right\}$ is the standard basis of $\mathbb{Z}^{d}$; and $[n]:=\{1, \ldots, n\}$.

A d-dimensional $\mathbb{N}$-matrix is an array $\left(a_{i_{1}, \ldots, i_{d}}\right)_{i_{1}, \ldots ., i_{d} \geq 1}$ of nonnegative integers with only finitely many nonzero elements. A $d$-dimensional partition is a $d$-dimensional $\mathbb{N}$-matrix $\left(\pi_{i_{1}, \ldots, i_{d}}\right)$ such that

$$
\pi_{i_{1}, \ldots, i_{d}} \geq \pi_{j_{1}, \ldots, j_{d}} \text { for } i_{1} \leq j_{1}, \ldots, i_{d} \leq j_{d}
$$

Let $\mathcal{M}^{(d)}$ be the set of $d$-dimensional $\mathbb{N}$-matrices and $\mathcal{P}^{(d)}$ be the set of $d$-dimensional partitions. For $\pi=\left(\pi_{i_{1}, \ldots, i_{d}}\right) \in \mathcal{P}^{(d)}$, the volume (or size) of $\pi$ denoted by $|\pi|$ is defined as

$$
|\pi|=\sum_{i_{1}, \ldots, i_{d}} \pi_{i_{1}, \ldots, i_{d}} .
$$

Any partition $\pi$ is uniquely determined by its diagram $D(\pi)$ which is the set

$$
D(\pi):=\left\{\left(i_{1}, \ldots, i_{d}, i\right) \in \mathbb{Z}_{+}^{d+1}: 1 \leq i \leq \pi_{i_{1}, \ldots, i_{d}}\right\} .
$$

The shape of $\pi$ denoted by $\operatorname{sh}(\pi)$ is the set

$$
\operatorname{sh}(\pi):=\left\{\left(i_{1}, \ldots, i_{d}\right) \in \mathbb{Z}_{+}^{d}: \pi_{i_{1}, \ldots, i_{d}}>0\right\} .
$$

Note that $\operatorname{sh}(\pi)$ is a diagram of some $(d-1)$-dimensional partition. Let

$$
\mathcal{M}\left(n_{1}, \ldots, n_{d}\right)=\left\{\left(a_{\mathbf{i}}\right): a_{\mathbf{i}} \in \mathbb{N}, \mathbf{i} \in\left[n_{1}\right] \times \cdots \times\left[n_{d}\right]\right\}
$$

be the set of $\left[n_{1}\right] \times \cdots \times\left[n_{d}\right] \mathbb{N}$-matrices and

$$
\mathcal{P}\left(n_{1}, \ldots, n_{d+1}\right):=\left\{\pi \in \mathcal{P}^{(d)}: D(\pi) \subseteq\left[n_{1}\right] \times \cdots \times\left[n_{d+1}\right]\right\}
$$

be the set of boxed $d$-dimensional partitions.
For $d=2,3$ partitions are called plane partitions and solid partitions. ${ }^{1}$

[^0]
## 3. A BiJection between $d$-Dimensional $\mathbb{N}$-matrices and partitions

3.1. Last passage matrix. A lattice path in $\mathbb{Z}^{d}$ is called directed if it uses only steps of the form $\mathbf{i} \rightarrow \mathbf{i}+\mathbf{e}_{\ell}$ for $\mathbf{i} \in \mathbb{Z}^{d}$ and $\ell \in[d]$. Given a $d$-dimensional $\mathbb{N}$-matrix $A=\left(a_{i_{1}, \ldots, i_{d}}\right)$, define the last passage times ${ }^{2}$

$$
G_{i_{1}, \ldots, i_{d}}:=\max _{\Pi:\left(i_{1}, \ldots, i_{d}\right) \rightarrow \infty^{d}} \sum_{\left(j_{1}, \ldots, j_{d}\right) \in \Pi} a_{j_{1}, \ldots, j_{d}}
$$

where the maximum is over directed lattice paths $\Pi$ which start at $\left(i_{1}, \ldots, i_{d}\right) \in \mathbb{Z}_{+}^{d}$. It is easy to see that the following recurrence relation holds

$$
\begin{equation*}
G_{\mathbf{i}}=a_{\mathbf{i}}+\max _{\ell \in[d]} G_{\mathbf{i}+\mathbf{e}_{\ell}}, \quad \mathbf{i} \in \mathbb{Z}_{+}^{d} \tag{1}
\end{equation*}
$$

Notice that the matrix $G=\left(G_{\mathbf{i}}\right)_{\mathbf{i} \in \mathbb{Z}_{+}^{d}} \in \mathcal{P}^{(d)}$ is a $d$-dimensional partition.
3.2. The bijection. Define the map $\Phi: \mathcal{M}^{(d)} \rightarrow \mathcal{P}^{(d)}$ as follows

$$
\begin{equation*}
\Phi: A \longmapsto G \tag{2}
\end{equation*}
$$

Let $\rho \subset \mathbb{Z}_{+}^{d}$ be a shape of some $d$-dimensional partition (or a diagram of a $(d-1)$-dimensional partition). Let

$$
\mathcal{P}(\rho, n):=\left\{\pi \in \mathcal{P}^{(d)}: \operatorname{sh}(\pi) \subseteq \rho, \pi_{1, \ldots, 1} \leq n\right\}
$$

be the set of $d$-dimensional partitions whose shape is a subset of $\rho$ and the largest entry is at most $n$. Let

$$
\mathcal{M}(\rho, n):=\left\{A=\left(a_{\mathbf{i}}\right) \in \mathcal{M}^{(d)}: a_{\mathbf{i}}>0 \Longrightarrow \mathbf{i} \in \rho, G_{1, \ldots, 1} \leq n\right\}
$$

be the set of $d$-dimensional $\mathbb{N}$-matrices whose support (i.e. the set of indices corresponding to positive entries) lies inside $\rho$ and the largest last passage time is at most $n$.

Theorem 3.1. The map $\Phi$ defines a bijection between the sets $\mathcal{M}(\rho, n)$ and $\mathcal{P}(\rho, n)$.
Proof. Let $A=\left(a_{\mathbf{i}}\right) \in \mathcal{M}(\rho, n)$. By construction of the map, it is not difficult to see that $\pi=\Phi(A) \in \mathcal{P}(\rho, n)$. Indeed, we have the largest last passage time $\pi_{1, \ldots, 1} \leq n$, and $\operatorname{sh}(\pi) \subseteq \rho$ since if $a_{\mathbf{i}}>0$ then $\mathbf{i} \in \rho$.

Conversely, given $\pi \in \mathcal{P}(\rho, n)$, to reconstruct the inverse map $\Phi^{-1}$, using the recurrence (1) we define the matrix $A=\left(a_{\mathbf{i}}\right)$ given by

$$
\begin{equation*}
a_{\mathbf{i}}=\pi_{\mathbf{i}}-\max _{\ell \in[d]} \pi_{\mathbf{i}+\mathbf{e}_{\ell}} \geq 0, \quad \mathbf{i} \in \mathbb{Z}_{+}^{d} \tag{3}
\end{equation*}
$$

Let $G=\left(G_{\mathbf{i}}\right)=\Phi(A)$. Let us check that $G=\pi$ and $A \in \mathcal{M}(\rho, n)$. Since $\operatorname{sh}(\pi) \subseteq \rho$ we have $a_{\mathbf{i}}=0$ for all $\mathbf{i} \notin \rho$ (in particular, $A \in \mathcal{M}(\rho, \infty)$ ). Hence $G_{\mathbf{i}}=\pi_{\mathbf{i}}=0$ for all $\mathbf{i} \notin \rho$. Consider the directed graph $\Gamma$ on the vertex set $\mathbf{i} \in \rho$ and edges $\mathbf{i} \rightarrow \mathbf{i}+\mathbf{e}_{\ell}$ (when $\mathbf{i}+\mathbf{e}_{\ell} \in \rho$ ) for $\ell \in[d]$. Then $\Gamma$ is acyclic (i.e. has no directed cycles). Notice that $a_{\mathbf{i}}=\pi_{\mathbf{i}}=G_{\mathbf{i}}$ if a vertex $\mathbf{i} \in \Gamma$ has no outgoing edges. Since $\Gamma$ is acyclic, we can sort its vertices in linear order $\left(\mathbf{i}^{(1)}, \ldots, \mathbf{i}^{(m)}\right)$ so that

[^1]

Figure 1. A plane partition $\pi \in \mathcal{P}^{(2)}$ whose $\operatorname{sh}(\pi)$ corresponds to the partition $(3,2)$; its boxed diagram presentation as a pile of cubes in $\mathbb{R}^{3}$; and boxes of this diagram which correspond to corners.
the edges go only in one direction $\mathbf{i}^{(\ell)} \rightarrow \mathbf{i}^{(k)}$ for $\ell<k$. We already noticed that $\pi_{\mathbf{i}^{(m)}}=G_{\mathbf{i}^{(m)}}$. Then inductively on $\ell=m-1, \ldots, 1$ we have

$$
\pi_{\mathbf{i}^{(\ell)}}=a_{\mathbf{i}^{(\ell)}}+\max _{\mathbf{i}^{(\ell)} \rightarrow \mathbf{i}^{(k)}} \pi_{\mathbf{i}^{(k)}}=a_{\mathbf{i}^{(\ell)}}+\max _{\mathbf{i}^{(\ell)} \rightarrow \mathbf{i}^{(k)}} G_{\mathbf{i}^{(k)}}=G_{\mathbf{i}^{(\ell)}} .
$$

Therefore, $\pi=G$. In particular, $G_{1, \ldots, 1} \leq n$ and hence $A \in \mathcal{M}(\rho, n)$.
Corollary 3.2. The map $\Phi$ defines a bijection between each of the following pairs of sets:
(i) $\mathcal{M}\left(\left[n_{1}\right] \times \cdots \times\left[n_{d}\right], n_{d+1}\right)$ and $\mathcal{P}\left(n_{1}, \ldots, n_{d+1}\right)$
(ii) $\mathcal{M}\left(n_{1}, \ldots, n_{d}\right)$ and $\mathcal{P}\left(n_{1}, \ldots, n_{d}, \infty\right)$
(iii) $\mathcal{M}(\rho, \infty)$ and $\mathcal{P}(\rho, \infty)$
(iv) $\mathcal{M}^{(d)}$ and $\mathcal{P}^{(d)}$.

Remark 1. The item (i) above states that the set of boxed $d$-dimensional partitions with diagrams inside the box $\left[n_{1}\right] \times \cdots \times\left[n_{d+1}\right]$ is equal to the number of $\left[n_{1}\right] \times \cdots \times\left[n_{d}\right] \mathbb{N}$-matrices whose largest last passage time is at most $n_{d+1}$.

Remark 2. For $d=2$, the map $\Phi$ gives a bijection between $\mathbb{N}$-matrices and plane partitions. This bijection is essentially equivalent (up to diagram rotations) to the one studied in [Yel19a, Yel19b]. Note that one can construct $d$-dimensional partitions $G$ dynamically using an insertion type procedure as in RSK. Note also that similar largest path (last passage time) properties hold for RSK as well, see [Pak01, Sag01].

## 4. Multivariate identities

4.1. Corners. Given a partition $\pi \in \mathcal{P}^{(d)}$, define the set of corners as follows

$$
\operatorname{Cor}(\pi):=\left\{\mathbf{i} \in \mathbb{Z}_{+}^{d+1}: \mathbf{i} \in D(\pi), \mathbf{i}+\mathbf{e}_{\ell} \notin D(\pi) \text { for all } \ell \in[d]\right\} .
$$

(Here $\left\{e_{\ell}\right\}$ is the standard basis in $\mathbb{Z}^{d+1}$.) Let $\operatorname{cor}(\pi):=|\operatorname{Cor}(\pi)|$ be the number of corners of $\pi$. Define also the set of top corners as follows

$$
\operatorname{Cr}(\pi):=\left\{\mathbf{i} \in \mathbb{Z}_{+}^{d+1}: \mathbf{i} \in D(\pi), \mathbf{i}+\mathbf{e}_{\ell} \notin D(\pi) \text { for all } \ell \in[d+1]\right\} \subseteq \operatorname{Cor}(\pi)
$$

Let $\operatorname{cr}(\pi):=|\operatorname{Cr}(\pi)|$ be the number of top corners of $\pi$. Note that the set of corners $\operatorname{Cr}(\pi)$ uniquely determines the partition $\pi$.

Example 4.1. Let $d=2$ and $\pi$ be the plane partition given in Fig. 1. We then have

$$
\begin{aligned}
\operatorname{Cor}(\pi) & =\{(i, j, k) \in D(\pi):(i+1, j, k),(i, j+1, k) \notin D(\pi)\} \\
& =\{(1,1,4),(1,3,1),(1,3,2),(2,2,1),(2,2,2),(2,2,3)\} \\
\operatorname{Cr}(\pi) & =\{(i, j, k) \in D(\pi):(i+1, j, k),(i, j+1, k),(i, j, k+1) \notin D(\pi)\} \\
& =\{(1,1,4),(1,3,2),(2,2,3)\}
\end{aligned}
$$

where corners in Fig. 1 correspond to local configurations $\boxtimes$ and top corners correspond to the configurations $\downarrow$.
4.2. Main formulas. For each $i \in[d]$, let $\mathbf{x}^{(i)}=\left(x_{1}^{(i)}, x_{2}^{(i)}, \ldots\right)$ be a countable set of indeterminate variables.

Theorem 4.2. Let $\rho \subset \mathbb{Z}_{+}^{d}$ be a fixed shape of a d-dimensional partition. We have the following multivariate generating function identities
(4) $\sum_{\substack{\pi \in \mathcal{P}(d),, \operatorname{sh}(\pi) \subseteq \rho}} \prod_{\left(i_{1}, \ldots, i_{d+1}\right) \in \operatorname{Cor}(\pi)} x_{i_{1}}^{(1)} \cdots x_{i_{d}}^{(d)}=\prod_{\substack{\left(i_{1}, \ldots, i_{d}\right) \in \rho}}\left(1-x_{i_{1}}^{(1)} \cdots x_{i_{d}}^{(d)}\right)^{-1}$
(5) $\sum_{\substack{\pi \in \mathcal{P}(d),, \operatorname{sh}(\pi)=\rho}} \prod_{\left.i_{1}, \ldots, i_{d+1}\right) \in \operatorname{Cor}(\pi)} x_{i_{1}}^{(1)} \cdots x_{i_{d}}^{(d)}=\prod_{\left(i_{1}, \ldots, i_{d}\right) \in \operatorname{Cr}(\rho)} x_{i_{1}}^{(1)} \cdots x_{i_{d}}^{(d)} \prod_{\left(i_{1}, \ldots, i_{d}\right) \in \rho}\left(1-x_{i_{1}}^{(1)} \cdots x_{i_{d}}^{(d)}\right)^{-1}$

It is convenient to define weights of matrices and partitions as follows. Given a matrix $A=\left(a_{i_{1}, \ldots, i_{d}}\right) \in \mathcal{M}^{(d)}$, we associate to it a multivariable monomial weight

$$
w_{A}:=\prod_{\left(i_{1}, \ldots, i_{d}\right) \in \mathbb{Z}_{+}^{d}}\left(x_{i_{1}}^{(1)} \cdots x_{i_{d}}^{(d)}\right)^{a_{i_{1}}, \ldots, i_{d}} .
$$

Given a partition $\pi \in \mathcal{P}^{(d)}$, we associate to it a multivariable monomial weight

$$
w(\pi):=\prod_{\left(i_{1}, \ldots, i_{d+1}\right) \in \operatorname{Cor}(\pi)} x_{i_{1}}^{(1)} \cdots x_{i_{d}}^{(d)} .
$$

Lemma 4.3. Let $A=\left(a_{\mathbf{i}}\right) \in \mathcal{M}^{(d)}$ and $\pi=\left(\pi_{\mathbf{i}}\right)=\Phi(A) \in \mathcal{P}^{(d)}$. Then $w_{A}=w(\pi)$.
Proof. Let us first show that

$$
\pi_{\mathbf{i}}-\max _{\ell \in[d]} \pi_{\mathbf{i}+\mathbf{e}_{\ell}}=\left|\left\{i_{d+1}:\left(\mathbf{i}, i_{d+1}\right) \in \operatorname{Cor}(\pi)\right\}\right|, \quad \mathbf{i} \in \mathbb{Z}_{+}^{d}
$$

Indeed, $\left(\mathbf{i}, i_{d+1}\right) \in \operatorname{Cor}(\pi)$ iff $i_{d+1}>\pi_{\mathbf{i}+\mathbf{e}_{\ell}}$ for all $\ell \in[d]$. From the description of $\Phi$ we then have the following equalities

$$
a_{\mathbf{i}}=\pi_{\mathbf{i}}-\max _{\ell \in[d]} \pi_{\mathbf{i}+\mathbf{e}_{\ell}}=\left|\left\{i_{d+1}:\left(\mathbf{i}, i_{d+1}\right) \in \operatorname{Cor}(\pi)\right\}\right|, \quad \mathbf{i}=\left(i_{1}, \ldots, i_{d}\right) \in \mathbb{Z}_{+}^{d}
$$

Now we have

$$
\begin{aligned}
w_{A} & =\prod_{\left(i_{1}, \ldots, i_{d}\right) \in \mathbb{Z}_{+}^{d}}\left(x_{i_{1}}^{(1)} \cdots x_{i_{d}}^{(d)}\right)^{a_{i_{1}}, \ldots, i_{d}} \\
& =\prod_{\left(i_{1}, \ldots, i_{d}\right) \in \mathbb{Z}_{+}^{d}}\left(x_{i_{1}}^{(1)} \cdots x_{i_{d}}^{(d)}\right)^{\left|\left\{i_{d+1}:\left(i_{1}, \ldots, i_{d}, i_{d+1}\right) \in \operatorname{Cor}(\pi)\right\}\right|} \\
& =\prod_{\left(i_{1}, \ldots, i_{d+1}\right) \in \operatorname{Cor}(\pi)} x_{i_{1}}^{(1)} \cdots x_{i_{d}}^{(d)} \\
& =w(\pi)
\end{aligned}
$$

which gives the needed.
Lemma 4.4. Let $A=\left(a_{\mathbf{i}}\right) \in \mathcal{M}(\rho, \infty)$ and $\pi=\left(\pi_{\mathbf{i}}\right)=\Phi(A) \in \mathcal{P}(\rho, \infty)$. The following are equivalent:
(a) $a_{\mathbf{i}}>0$ for all $\mathbf{i} \in \operatorname{Cr}(\rho)$
(b) $\operatorname{sh}(\pi)=\rho$.

Proof. Let $\mathbf{i} \in \operatorname{Cr}(\rho)$. Assume (a) holds. Since $A \in \mathcal{M}(\rho, \infty)$ we have $a_{\mathbf{i}+\mathbf{e}_{\ell}}=0$ for all $\ell \in[d]$. Therefore, $\pi_{\mathbf{i}}=a_{\mathbf{i}}>0$ and $\pi_{\mathbf{i}+\mathbf{e}_{\ell}}=0$. Hence $\operatorname{sh}(\pi)=\rho$.

Assume (b) holds. Then we have $\pi_{\mathbf{i}+\mathbf{e}_{\ell}}=0$ for all $\ell \in[d]$. Therefore, $a_{\mathbf{i}}=\pi_{\mathbf{i}}>0$.
Proof of Theorem 4.2. Firstly note that

$$
\begin{aligned}
\sum_{A=\left(a_{\mathbf{i}}\right) \in \mathcal{M}(\rho, \infty)} w_{A} & =\sum_{A=\left(a_{\mathbf{i}}\right) \in \mathcal{M}(\rho, \infty)} \prod_{\left(i_{1}, \ldots, i_{d}\right) \in \rho}\left(x_{i_{1}}^{(1)} \cdots x_{i_{d}}^{(d)}\right)^{a_{i_{1}, \ldots, i_{d}}} \\
& =\prod_{\left(i_{1}, \ldots, i_{d}\right) \in \rho}\left(1-x_{i_{1}}^{(1)} \cdots x_{i_{d}}^{(d)}\right)^{-1}
\end{aligned}
$$

On the other hand, using Theorem 3.1 and Lemma 4.3 we have

$$
\sum_{A=\left(a_{\mathbf{i}}\right) \in \mathcal{M}(\rho, \infty)} w_{A}=\sum_{\pi \in \mathcal{P}(\rho, \infty)} w(\pi)=\sum_{\pi \in \mathcal{P}^{(d)}, \operatorname{sh}(\pi) \subseteq \rho} \prod_{\left(i_{1}, \ldots, i_{d+1}\right) \in \operatorname{Cor}(\pi)} x_{i_{1}}^{(1)} \cdots x_{i_{d}}^{(d)}
$$

and hence the identity (4) follows.
Let $\overline{\mathcal{M}}(\rho, \infty)=\left\{A \in \mathcal{M}(\rho, \infty): \mathbf{i} \in \operatorname{Cr}(\rho) \Longrightarrow a_{\mathbf{i}}>0\right\}$. Similarly, note that

$$
\begin{aligned}
\sum_{A=\left(a_{\mathbf{i}}\right) \in \overline{\mathcal{M}}(\rho, \infty)} w_{A} & =\sum_{A=\left(a_{\mathbf{i}}\right) \in \mathcal{M}(\rho, \infty)} \prod_{\left(i_{1}, \ldots, i_{d}\right) \in \operatorname{Cr}(\rho)} x_{i_{1}}^{(1)} \cdots x_{i_{d}}^{(d)} \prod_{\left(i_{1}, \ldots, i_{d}\right) \in \rho}\left(x_{i_{1}}^{(1)} \cdots x_{i_{d}}^{(d)}\right)^{a_{i_{1}, \ldots, i_{d}}} \\
& =\prod_{\left(i_{1}, \ldots, i_{d}\right) \in \operatorname{Cr}(\rho)} x_{i_{1}}^{(1)} \cdots x_{i_{d}}^{(d)} \prod_{\left(i_{1}, \ldots, i_{d}\right) \in \rho}\left(1-x_{i_{1}}^{(1)} \cdots x_{i_{d}}^{(d)}\right)^{-1}
\end{aligned}
$$

On the other hand, using Lemma 4.4 we have

$$
\sum_{A=\left(a_{\mathbf{i}}\right) \in \overline{\mathcal{M}}(\rho, \infty)} w_{A}=\sum_{\pi \in \mathcal{P}(\rho, \infty), \operatorname{sh}(\pi)=\rho} w(\pi)=\sum_{\pi \in \mathcal{P}^{(d)}, \operatorname{sh}(\pi)=\rho} \prod_{\left(i_{1}, \ldots, i_{d+1}\right) \in \operatorname{Cor}(\pi)} x_{i_{1}}^{(1)} \cdots x_{i_{d}}^{(d)}
$$

and hence the identity (5) follows.
4.3. Some special cases. Let us list few immediate special cases of the above formulas.

Corollary 4.5 (Boxed case). We have

$$
\sum_{\pi \in \mathcal{P}\left(n_{1}, \ldots, n_{d}, \infty\right)} \prod_{\left(i_{1}, \ldots, i_{d+1}\right) \in \operatorname{Cor}(\pi)} x_{i_{1}}^{(1)} \cdots x_{i_{d}}^{(d)}=\prod_{i_{1}=1}^{n_{1}} \cdots \prod_{i_{d}=1}^{n_{d}}\left(1-x_{i_{1}}^{(1)} \cdots x_{i_{d}}^{(d)}\right)^{-1}
$$

Corollary 4.6 (Solid partitions, $d=3$ ). Let $\rho$ be a plane partition. We have

$$
\begin{aligned}
\sum_{\pi \in \mathcal{P}^{(3)}, \operatorname{sh}(\pi) \subseteq D(\rho)} \prod_{(i, j, k, \ell) \in \operatorname{Cor}(\pi)} x_{i} y_{j} z_{k} & =\prod_{(i, j, k) \in D(\rho)}\left(1-x_{i} y_{j} z_{k}\right)^{-1} \\
\sum_{\pi \in \mathcal{P}^{(3)}, \operatorname{sh}(\pi)=D(\rho)} \prod_{(i, j, k, \ell) \in \operatorname{Cor}(\pi)} x_{i} y_{j} z_{k} & =\prod_{(i, j, k) \in D(\rho)}\left(1-x_{i} y_{j} z_{k}\right)^{-1} \prod_{(i, j, k) \in \operatorname{Cr}(\rho)} x_{i} y_{j} z_{k} .
\end{aligned}
$$

Corollary 4.7 (Plane partitions, $d=2$ ). Let $\lambda$ be a partition. We have

$$
\begin{aligned}
& \sum_{\pi \in \mathcal{P}^{(2)}, \operatorname{sh}(\pi) \subseteq \lambda} \prod_{(i, j, k) \in \operatorname{Cor}(\pi)} x_{i} y_{j}=\prod_{(i, j) \in D(\lambda)}\left(1-x_{i} y_{j}\right)^{-1} \\
& \sum_{\pi \in \mathcal{P}^{(2)}, \operatorname{sh}(\pi)=\lambda} \prod_{(i, j, k) \in \operatorname{Cor}(\pi)} x_{i} y_{j}=\prod_{(i, j) \in D(\lambda)}\left(1-x_{i} y_{j}\right)^{-1} \prod_{(i, j) \in \operatorname{Cr}(\lambda)} x_{i} y_{j} .
\end{aligned}
$$

Remark 3. For $d=2$, the formula in the special rectangular case (up to rotation of diagrams of plane partitions) was proved in [Yel19b].

## 5. MacMahon's numbers and statistics

5.1. Corner-hook volume. Let $\pi \in \mathcal{P}^{(d)}$ be a $d$-dimensional partition. For each point $\left(i_{1}, \ldots, i_{d}\right)$, define the cohook length

$$
\operatorname{ch}\left(i_{1}, \ldots, i_{d}\right):=i_{1}+\ldots+i_{d}-d+1
$$

Define now the corner-hook volume statistics $|\cdot|_{c h}: \mathcal{P}^{(d)} \rightarrow \mathbb{N}$ computed as follows

$$
|\pi|_{c h}:=\sum_{\left(\mathbf{i}, i_{d+1}\right) \in \operatorname{Cor}(\pi)} \operatorname{ch}(\mathbf{i}) .
$$

Example 5.1. Let $d=2$ and $\pi$ be the plane partition given in Fig. 1. Recall that

$$
\begin{aligned}
\operatorname{Cor}(\pi) & =\{(i, j, k) \in D(\pi):(i+1, j, k),(i, j+1, k) \notin D(\pi)\} \\
& =\{(1,1,4),(1,3,1),(1,3,2),(2,2,1),(2,2,2),(2,2,3)\}
\end{aligned}
$$

and hence we have

$$
|\pi|_{c h}=(1+1-1)+(1+3-1)+(1+3-1)+(2+2-1)+(2+2-1)+(2+2-1)=16
$$

Theorem 5.2. Let $\rho \subset \mathbb{Z}_{+}^{d}$ be a fixed shape of a d-dimensional partition. We have the following generating functions

$$
\begin{aligned}
\sum_{\pi \in \mathcal{P}^{(d)}, \operatorname{sh}(\pi) \subseteq \rho} t^{\operatorname{cor}(\pi)} q^{|\pi|_{c h}} & =\prod_{\left(i_{1}, \ldots, i_{d}\right) \in \rho}\left(1-t q^{i_{1}+\cdots+i_{d}-d+1}\right)^{-1} \\
\sum_{\pi \in \mathcal{P}^{(d)}, \operatorname{sh}(\pi)=\rho} t^{\operatorname{cor}(\pi)} q^{|\pi|_{c h}} & =t^{\operatorname{cr}(\rho)} q^{|\rho|_{c r}} \prod_{\left(i_{1}, \ldots, i_{d}\right) \in \rho}\left(1-t q^{i_{1}+\cdots+i_{d}-d+1}\right)^{-1},
\end{aligned}
$$

where

$$
|\rho|_{c r}:=\sum_{\left(i_{1}, \ldots, i_{d}\right) \in \operatorname{Cr}(\rho)} \operatorname{ch}\left(i_{1}, \ldots, i_{d}\right)
$$

Proof. In Theorem 4.2 set $x_{i}^{(1)}=t q^{i}$ and $x_{i}^{(k)}=q^{i-1}$ for all $i \geq 1$ and $k \geq 2$.
Corollary 5.3 (Boxed version). We have

$$
\sum_{\pi \in \mathcal{P}\left(n_{1}, \ldots, n_{d}, \infty\right)} t^{\operatorname{cor}(\pi)} q^{|\pi|_{c h}}=\prod_{i_{1}=1}^{n_{1}} \cdots \prod_{i_{d}=1}^{n_{d}}\left(1-t q^{i_{1}+\cdots+i_{d}-d+1}\right)^{-1} .
$$

Corollary 5.4 (Full generating function). We have

$$
\sum_{\pi \in \mathcal{P}^{(d)}} t^{\operatorname{cor}(\pi)} q^{|\pi|_{c h}}=\prod_{n \geq 1}\left(1-t q^{n}\right)^{-\binom{n+d-2}{d-1}} .
$$

Corollary 5.5 (Interpretation of MacMahon's numbers). We have

$$
\sum_{\pi \in \mathcal{P}^{(d)}} q^{|\pi|_{c h}}=\prod_{n \geq 1}\left(1-q^{n}\right)^{-\binom{n+d-2}{d-1}}=\sum_{n=0}^{\infty} m_{d}(n) q^{n}
$$

and hence

$$
m_{d}(n)=\left|\left\{\pi \in \mathcal{P}^{(d)}:|\pi|_{c h}=n\right\}\right|,
$$

i.e. $m_{d}(n)$ is the number d-dimensional partitions whose corner-hook volume is $n$.

Corollary 5.6 (Pyramid partitions). Let $\Delta_{d}(m)$ be a d-dimensional partition whose diagram is $D\left(\Delta_{d}(m)\right)=\left\{\left(i_{1}, \ldots, i_{d+1}\right): \mathbb{Z}_{+}^{d+1}: i_{1}+\cdots+i_{d+1}-d \leq m\right\}$. We have

$$
\sum_{\pi \in \mathcal{P}\left(\Delta_{d-1}(m), \infty\right)} t^{\operatorname{cor}(\pi)} q^{|\pi|_{c h}}=\prod_{n=1}^{m}\left(1-t q^{n}\right)^{-\binom{n+d-2}{d-1}} .
$$

Corollary 5.7 ( $q=1$ specialization). We have

$$
\begin{aligned}
& \sum_{\pi \in \mathcal{P}^{(d)}, \operatorname{sh}(\pi) \subseteq \rho} t^{\operatorname{cor}(\pi)}=(1-t)^{-|\rho|}, \\
& \sum_{\pi \in \mathcal{P}^{(d)}, \operatorname{sh}(\pi)=\rho} t^{\operatorname{cor}(\pi)}=t^{\operatorname{cr}(\rho)}(1-t)^{-|\rho|} .
\end{aligned}
$$

Then the number of $\pi \in \mathcal{P}^{(d)}$ of shape $\rho$ with $k$ corners is equal to $\binom{k-\operatorname{cr}(\rho)+|\rho|-1}{|\rho|-1}$.
5.2. Solid partitions, $d=3$. Let us restate some of these results for solid partitions. Let $\pi \in \mathcal{P}^{(3)}$ be a solid partition. We then have

$$
|\pi|_{c h}=\sum_{(i, j, k, \ell) \in \operatorname{Cor}(\pi)}(i+j+k-2) .
$$

Corollary 5.8. Let $\rho$ be a fixed plane partition. We have

$$
\begin{aligned}
& \sum_{\pi \in \mathcal{P}^{(3)}, \operatorname{sh}(\pi) \subseteq \rho} t^{\operatorname{cor}(\pi)} q^{|\pi|_{c h}}=\prod_{(i, j, k) \in D(\rho)}\left(1-t q^{i+j+k-2}\right)^{-1} \\
& \sum_{\pi \in \mathcal{P}^{(3)}, \operatorname{sh}(\pi)=\rho} t^{\operatorname{cor}(\pi)} q^{|\pi|_{c h}}=t^{\operatorname{cr}(\rho)} q^{|\rho|_{c r}} \prod_{(i, j, k) \in D(\rho)}\left(1-t q^{i+j+k-2}\right)^{-1}
\end{aligned}
$$

and in particular the boxed version

$$
\sum_{\pi \in \mathcal{P}\left(n_{1}, n_{2}, n_{3}, \infty\right)} t^{\operatorname{cor}(\pi)} q^{|\pi|_{c h}}=\prod_{i=1}^{n_{1}} \prod_{j=1}^{n_{2}} \prod_{k=1}^{n_{3}}\left(1-t q^{i+j+k-2}\right)^{-1}
$$

5.3. Plane partitions, $d=2$. Similarly, let us restate some of these results for plane partitions. Let $\pi \in \mathcal{P}^{(2)}$ be a plane partition. We then have

$$
|\pi|_{c h}=\sum_{(i, j, k) \in \operatorname{Cor}(\pi)}(i+j-1) .
$$

Corollary 5.9. Let $\lambda$ be a fixed partition. We have

$$
\begin{aligned}
& \sum_{\pi \in \mathcal{P}^{(2)}, \operatorname{sh}(\pi) \subseteq \lambda} t^{\operatorname{cor}(\pi)} q^{|\pi|_{c h}}=\prod_{(i, j) \in D(\lambda)}\left(1-t q^{i+j-1}\right)^{-1} \\
& \sum_{\pi \in \mathcal{P}^{(2)}, \operatorname{sh}(\pi)=\lambda} t^{\operatorname{cor}(\pi)} q^{|\pi|_{c h}}=t^{\operatorname{cr}(\lambda)} q^{|\lambda|_{c r}} \prod_{(i, j) \in D(\lambda)}\left(1-t q^{i+j-1}\right)^{-1}
\end{aligned}
$$

and in particular the boxed version

$$
\sum_{\pi \in \mathcal{P}\left(n_{1}, n_{2} \infty\right)} t^{\operatorname{cor}(\pi)} q^{|\pi|_{c h}}=\prod_{i=1}^{n_{1}} \prod_{j=1}^{n_{2}}\left(1-t q^{i+j-1}\right)^{-1}
$$

Let us look on the last boxed formula. On the other hand, the following trace generating function is known for plane partitions (see e.g. [Sta99, Thm 7.20.1])

$$
\prod_{i=1}^{n_{1}} \prod_{j=1}^{n_{2}}\left(1-t q^{i+j-1}\right)^{-1}=\sum_{\pi \in \mathcal{P}\left(n_{1}, n_{2} \infty\right)} t^{\operatorname{tr}(\pi)} q^{|\pi|}
$$

where $\operatorname{tr}(\pi):=\sum_{i} \pi_{i, i}$ is the trace of a plane partition. Therefore, in this case we actually have the following equidistribution result.

Theorem 5.10 (Equidistribution of (tr, vol) and (cor, ch-vol) for plane partitions). We have

$$
\sum_{\pi \in \mathcal{P}\left(n_{1}, n_{2} \infty\right)} t^{\operatorname{cor}(\pi)} q^{|\pi|_{c h}}=\sum_{\pi \in \mathcal{P}\left(n_{1}, n_{2} \infty\right)} t^{\operatorname{tr}(\pi)} q^{|\pi|}
$$

Remark 4. Up to a variation of the $|\cdot|_{c h}$ statistic, this result was proved by the second author in [Yel19b]. We also have a direct bijective argument for (a stronger version of) this identity which is somewhat long and will be addressed elsewhere.

Remark 5. The formulas in Theorem 5.2 can be viewed as higher-dimensional analogues of the well-known formula

$$
\sum_{\operatorname{sh}(\pi)=\lambda} q^{|\pi|}=\prod_{(i, j) \in D(\lambda)}\left(1-q^{h_{\lambda}(i, j)}\right)^{-1}
$$

where $\lambda$ is a (usual) partition, $h_{\lambda}(i, j)=\lambda_{i}-i+\lambda_{j}^{\prime}-j+1$ are hook lengths, and the sum runs over reverse plane partitions $\pi$, see [Sta99, Ch. 7.22]. Its combinatorial proof is known as the Hillman-Grassl correspondence [HG76].

Remark 6. There are various enumeration and generating function formulas known for classes of symmetric plane partitions, see [Sta86]. Similarly, one can define classes of symmetries of diagrams for $d$-dimensional partitions. Are there any explicit corner-hook generating functions over symmetric $d$-dimensional partitions as in Theorem 5.2?
5.4. Other statistics. Theorem 4.2 is a source for many statistics over $d$-dimensional partitions, whose generating functions can be computed explicitly by taking appropriate specializations. For instance, another interesting statistic $|\cdot|_{c}: \mathcal{P}^{(d)} \rightarrow \mathbb{N}$ is given by

$$
|\pi|_{c}:=\sum_{\left(i_{1}, \ldots, i_{d+1}\right) \in \operatorname{Cor}(\pi)} i_{1}, \quad \pi \in \mathcal{P}^{(d)} .
$$

Then via the substitution $x_{i}^{(1)} \rightarrow q^{i}$ and $x_{i}^{(k)}=1$ for all $k \geq 2$ and $i \geq 1$ we obtain the following generating function

$$
\sum_{\pi \in \mathcal{P}\left(n_{1}, \ldots, n_{d}, \infty\right)} q^{|\pi|_{c}}=\prod_{i=1}^{n_{1}}\left(1-q^{i}\right)^{-n_{2} \cdots n_{d}}
$$

Another curious statistic is given by

$$
|\pi|_{p}:=\sum_{\left(i_{1}, \ldots, i_{d+1}\right) \in \operatorname{Cor}(\pi)}\left(i_{1}+2 i_{2}+\ldots+d i_{d}\right), \quad \pi \in \mathcal{P}^{(d)}
$$

for which via the substitution $x_{i}^{(k)}=q^{k i}$ for all $k, i \geq 1$, we obtain the following generating function

$$
\sum_{\pi \in \mathcal{P}^{(d)}} q^{|\pi|_{p}}=\prod_{n=1}^{\infty}\left(1-q^{n}\right)^{-p(n, d)},
$$

where $p(n, d)$ is the number of integer partitions of $n$ into $d$ distinct parts.

## 6. $d$-Dimensional Grothendieck polynomials

6.1. Definitions. Let $\pi$ be a $d$-dimensional partition. Define the set

$$
\operatorname{sh}_{1}(\pi):=\left\{\left(i_{2}, \ldots, i_{d+1}\right):\left(i_{1}, \ldots, i_{d+1}\right) \in D(\pi)\right\}
$$

which can be viewed as a shape of $\pi$ with respect to the first coordinate. Note that if $\pi \in$ $\mathcal{P}\left(n_{1}, \ldots, n_{d+1}\right)$, then $\operatorname{sh}_{1}(\pi)$ is a diagram of $(d-1)$-dimensional partition from $\mathcal{P}\left(n_{2}, \ldots, n_{d+1}\right)$. Alternatively, $\operatorname{sh}_{1}(\pi)$ is the diagram of the partition $\left(\pi_{1, i_{2}, \ldots, i_{d}}\right)$. For example, if $\pi$ is the plane partition in Fig. 1, then $\operatorname{sh}_{1}(\pi)$ corresponds to the partition $(4,3,2)$ which is the first row of $\pi$.

Throughout this section, let us assume that we have the sets of variables

$$
\mathbf{x}^{(i)}=\left(x_{1}^{(i)}, \ldots, x_{n_{i}}^{(i)}\right), \quad i \in[d] .
$$

Definition 6.1. Let $\rho$ be a ( $d-1$ )-dimensional partition from the set $\mathcal{P}\left(n_{2}, \ldots, n_{d+1}\right)$. Define the $d$-dimensional Grothendieck polynomials in $d$ sets of variables as follows

$$
\begin{equation*}
g_{\rho}\left(\mathbf{x}^{(1)} ; \ldots ; \mathbf{x}^{(d)}\right):=\sum_{\pi: \operatorname{sh}_{1}(\pi)=\rho} \prod_{\left(i_{1}, \ldots, i_{d+1}\right) \in \operatorname{Cor}(\pi)} x_{i_{1}}^{(1)} \cdots x_{i_{d}}^{(d)}, \tag{6}
\end{equation*}
$$

where the sum runs over $d$-dimensional partitions $\pi \in \mathcal{P}\left(n_{1}, \ldots, n_{d+1}\right)$ with $\operatorname{sh}_{1}(\pi)=\rho$ (here $\rho$ is identified with its diagram).

In the specialization $x_{i}^{(k)}=1$ for all $k \geq 2$, we simply denote these polynomials by $g_{\rho}(\mathbf{x})=$ $g_{\rho}\left(x_{1}, x_{2}, \ldots\right)$ in one set of variables $\mathbf{x}^{(1)}=\mathbf{x}=\left(x_{1}, \ldots, x_{n_{1}}\right)$ so that

$$
\begin{equation*}
g_{\rho}(\mathbf{x})=\sum_{\pi: \operatorname{sh}_{1}(\pi)=\rho} \prod_{i=1}^{n_{1}} x_{i}^{c_{i}(\pi)}, \text { where } c_{i}(\pi):=|\{\mathbf{i}:(i, \mathbf{i}) \in \operatorname{Cor}(\pi)\}| \tag{7}
\end{equation*}
$$

and the sum runs over $\pi \in \mathcal{P}\left(n_{1}, \ldots, n_{d+1}\right)$.

### 6.2. Examples.

Example 6.2. Consider the case $d=2$. Let $\lambda \in \mathcal{P}\left(n_{2}, n_{3}\right)$ be a partition and $\mathbf{x}^{(1)}=\mathbf{x}, \mathbf{x}^{(2)}=\mathbf{y}$. Then (7) becomes

$$
g_{\lambda}(\mathbf{x})=\sum_{\pi: \operatorname{sh}_{1}(\pi)=\lambda} \prod_{i=1}^{n_{1}} x_{i}^{c_{i}(\pi)}, \text { where } c_{i}(\pi)=|\{(j, k):(i, j, k) \in \operatorname{Cor}(\pi)\}|
$$

and the sum runs over plane partitions $\pi \in \mathcal{P}\left(n_{1}, n_{2}, n_{3}\right)$. One can see that this gives the dual symmetric Grothendieck polynomials defined in [LP07] (but phrased in a slightly different yet equivalent form). ${ }^{3}$ More generally, (6) becomes

$$
g_{\lambda}(\mathbf{x} ; \mathbf{y})=\sum_{\pi: \operatorname{sh}_{1}(\pi)=\lambda} \prod_{(i, j, k) \in \operatorname{Cor}(\pi)} x_{i} y_{j}
$$

which gives a generalized version as in [Yel19b] or by changing $\tilde{g}_{\lambda}(\mathbf{x} ; \mathbf{y})=\mathbf{y}^{\lambda} g_{\lambda}\left(\mathbf{x} ; \mathbf{y}^{-1}\right)$ the refined version introduced in [GGL16]. These polynomials are symmetric in the variables $\mathbf{x}$.
Example 6.3. Let $d=3,\left(n_{1}, n_{2}, n_{3}, n_{4}\right)=(3,2,2,2)$, and $\mathbf{x}^{(1)}=\mathbf{x}=\left(x_{1}, x_{2}, x_{3}\right), \mathbf{x}^{(2)}=\mathbf{y}=$ $\left(y_{1}, y_{2}\right), \mathbf{x}^{(3)}=\mathbf{z}=\left(z_{1}, z_{2}\right)$. Note that in this case, 3 -dimensional Grothendieck polynomials are indexed by plane partitions and defined as sums over solid partitions. Consider few examples.

(a) Let $\rho=$| 2 | 1 |
| :--- | :--- | . Then we have

$$
\begin{aligned}
g_{\rho}(\mathbf{x} ; \mathbf{y} ; \mathbf{z})=\left(x_{1}^{2} x_{2}\right. & \left.+x_{1}^{2} x_{3}+x_{1} x_{2}^{2}+x_{1} x_{3}^{2}+x_{2}^{2} x_{3}+x_{2} x_{3}^{2}+2 x_{1} x_{2} x_{3}\right) \cdot y_{1}^{3} z_{1}^{2} z_{2} \\
& +\left(x_{1}^{2}+x_{2}^{2}+x_{3}^{2}+x_{1} x_{2}+x_{1} x_{3}+x_{2} x_{3}\right) \cdot y_{1}^{2} z_{1} z_{2}
\end{aligned}
$$

which coincides with the ordinary dual Grothendieck polynomial indexed by the partition $\lambda=(2,1)$, i.e. in this case we have $g_{\lambda}(\mathbf{x})=g_{\rho}(\mathbf{x}, \mathbf{1}, \mathbf{1})$.

(b) Let $\rho=$| 1 | 1 |
| :--- | :--- |
| 1 | . | . Then we have

$$
\begin{aligned}
g_{\rho}(\mathbf{x} ; \mathbf{y} ; \mathbf{z})=\left(x_{1}^{2} x_{2}\right. & \left.+x_{1}^{2} x_{3}+x_{2}^{2} x_{3}\right) \cdot y_{1}^{2} y_{2} z_{1}^{2} z_{2}+2 x_{1} x_{2} x_{3} \cdot y_{1}^{2} y_{2} z_{1}^{2} z_{2} \\
& +\left(x_{1}^{2}+x_{2}^{2}+x_{3}^{2}+2 x_{1} x_{2}+2 x_{1} x_{3}+2 x_{2} x_{3}\right) \cdot y_{1} y_{2} z_{1} z_{2}
\end{aligned}
$$

${ }^{3}$ The polynomials $\left\{g_{\lambda}\right\}$ are usually defined using reverse plane partitions, see [LP07, Yel17, Yel19].
and in particular,

$$
g_{\rho}(\mathbf{x})=x_{1}^{2} x_{2}+x_{1}^{2} x_{3}+x_{2}^{2} x_{3}+2 x_{1} x_{2} x_{3}+x_{1}^{2}+x_{2}^{2}+x_{3}^{2}+2\left(x_{1} x_{2}+x_{1} x_{3}+x_{2} x_{3}\right)
$$

(c) Let $\rho=$| 2 | 1 |
| :--- | :--- |
| 1 | 1 | . Then we have

$$
\begin{aligned}
g_{\rho}(\mathbf{x}, \mathbf{y}, \mathbf{z}) & =\left(3 x_{1}^{2} x_{2} x_{3}+3 x_{1} x_{2}^{2} x_{3}+2 x_{1} x_{2} x_{3}^{2}+x_{1}^{2} x_{2}^{2}+x_{1}^{2} x_{3}^{2}+x_{2}^{2} x_{3}^{2}+x_{1}^{3} x_{2}+x_{1}^{3} x_{3}+x_{2}^{3} x_{3}\right) \cdot y_{1}^{3} y_{2} z_{1}^{3} z_{2} \\
& +\left(4 x_{1} x_{2} x_{3}+2 x_{1}^{2} x_{2}+2 x_{1}^{2} x_{3}+2 x_{2}^{2} x_{3}+3 x_{1} x_{2}^{2}+3 x_{1} x_{3}^{2}+3 x_{2} x_{3}^{2}+x_{1}^{3}+x_{2}^{3}+x_{3}^{3}\right) \cdot y_{1}^{2} y_{2} z_{1}^{2} z_{2}
\end{aligned}
$$

Let us illustrate few examples of solid partitions contributing to the last expansion. Each


picture here represents a solid partition as a filling of a diagram of some plane partition with numbers written on top of each box (to make entries of inner boxes visible, some facets are removed). On the left, we have $\operatorname{sh}_{1}(\pi)=\rho$. The next two are solid partitions $\pi^{(1)}$ and $\pi^{(2)}$ represented as fillings of diagrams of plane partitions $\operatorname{sh}\left(\pi^{(1)}\right)=$\begin{tabular}{|l|l|}
\hline 2 \& 1 <br>
\hline 2 \& 1 <br>
\hline 1 \& 1 <br>
\hline

 and $\operatorname{sh}\left(\pi^{(2)}\right)=$

\hline 2 \& 1 <br>
\hline 2 \& 1 <br>
\hline 2 \& <br>
\hline
\end{tabular} ; each has the weight $w\left(\pi^{(i)}\right)=x_{2}^{2} x_{3} \cdot y_{1}^{2} y_{2} z_{1}^{2} z_{2}$; and both have the same $\operatorname{sh}_{1}\left(\pi^{(i)}\right)=\rho(i=1,2)$ displayed on the left.

6.3. Properties. We now prove some properties of $d$-dimensional Grothendieck polynomials.

Theorem 6.4 (Cauchy-type identity). Let $\eta \in \mathcal{P}\left(n_{2}, \ldots, n_{d}\right)$ be a (d-2)-dimensional partition. Let $n \times \eta$ be a $(d-1)$-dimensional partition with the diagram $D(n \times \eta)=\{(i, \mathbf{i}): i \in[n], \mathbf{i} \in D(\eta)\}$. Then we have the following generating series:

$$
\sum_{\rho \in \mathcal{P}(\eta, \infty)} g_{\rho}\left(\mathbf{x}^{(1)} ; \ldots ; \mathbf{x}^{(d)}\right)=\prod_{\left(i_{1}, \ldots, i_{d}\right) \in D\left(n_{1} \times \eta\right)}\left(1-x_{i_{1}}^{(1)} \cdots x_{i_{d}}^{(d)}\right)^{-1} .
$$

Proof. Notice that we have

$$
\sum_{\rho \in \mathcal{P}(\eta, \infty)} g_{\rho}\left(\mathbf{x}^{(1)} ; \ldots ; \mathbf{x}^{(d)}\right)=\sum_{\rho \in \mathcal{P}(\eta, \infty)} \sum_{\operatorname{sh}_{1}(\pi)=\rho} w(\pi)=\sum_{\pi \in \mathcal{P}\left(n_{1} \times \eta, \infty\right)} w(\pi)
$$

On the other hand, from Theorem 4.2 we have

$$
\sum_{\pi \in \mathcal{P}\left(n_{1} \times \eta, \infty\right)} w(\pi)=\prod_{\left(i_{1}, \ldots, i_{d}\right) \in D\left(n_{1} \times \eta\right)}\left(1-x_{i_{1}}^{(1)} \cdots x_{i_{d}}^{(d)}\right)^{-1}
$$

which gives the result.

Corollary 6.5. We have

$$
\sum_{\rho \in \mathcal{P}\left(n_{2}, \ldots, n_{d}, \infty\right)} g_{\rho}\left(\mathbf{x}^{(1)} ; \ldots ; \mathbf{x}^{(d)}\right)=\prod_{i_{1}=1}^{n_{1}} \cdots \prod_{i_{d}=1}^{n_{d}}\left(1-x_{i_{1}}^{(1)} \cdots x_{i_{d}}^{(d)}\right)^{-1}
$$

Lemma 6.6 (Simple branching rule). We have

$$
g_{\pi}\left(1, x_{1}, \ldots, x_{n}\right)=\sum_{\rho \subseteq \pi} g_{\rho}\left(x_{1}, \ldots, x_{n}\right) .
$$

Proof. Given a plane partition $\tau$ with $\operatorname{sh}_{1}(\tau)=\pi$, it contributes to the l.h.s. the weight $\prod_{i=1}^{n} x_{i}^{c_{i+1}(\tau)}$ (see eq. (7)). Let us form the new partition $\rho \subseteq \pi$ with the diagram

$$
\{(i, \mathbf{i}):(i+1, \mathbf{i}) \in D(\tau)\}
$$

so that $\prod_{i=1}^{n} x_{i}^{c_{i+1}(\tau)}=\prod_{i=1}^{n} x_{i}^{c_{i}(\rho)}$ which contributes to the r.h.s. In other words, remove from $D(\tau)$ the points with the first coordinate 1 , then decrease by 1 the first coordinates for the remaining points. It is not difficult to see that this defines a proper weight-preserving bijection between both sides of the equation.

Denote $1^{k}=(1, \ldots, 1)$ with $k$ ones.
Proposition 6.7 (Boxed specialization). We have

$$
g_{\left[n_{2}\right] \times \cdots \times\left[n_{d+1}\right]}\left(1^{n_{1}+1}\right)=g_{\left[n_{2}\right] \times \cdots \times\left[n_{d+1}\right]}\left(1^{n_{1}+1} ; 1^{n_{2}} ; \ldots ; 1^{n_{d}}\right)=\left|\mathcal{P}\left(n_{1}, \ldots, n_{d+1}\right)\right|
$$

Proof. Denote $B=\left[n_{2}\right] \times \cdots \times\left[n_{d+1}\right]$. Let $\rho$ be a partition diagram inside $B$. From the definition of $g$ we immediately obtain that

$$
g_{\rho}\left(1^{n_{1}} ; \ldots ; 1^{n_{d}}\right)=\left|\left\{\pi \in \mathcal{P}\left(n_{1}, \ldots, n_{d+1}\right): \operatorname{sh}_{1}(\pi)=\rho\right\}\right| .
$$

Therefore, using the branching formula above we get

$$
\begin{aligned}
g_{B}\left(1^{n_{1}+1} ; 1^{n_{2}} ; \ldots ; 1^{n_{d}}\right) & =\sum_{\rho \subseteq B} g_{\rho}\left(1^{n_{1}} ; 1^{n_{2}} ; \ldots ; 1^{n_{d}}\right) \\
& =\sum_{\rho \subseteq B}\left|\left\{\pi \in \mathcal{P}\left(n_{1}, \ldots, n_{d+1}\right): \operatorname{sh}_{1}(\pi)=\rho\right\}\right| \\
& =\left|\mathcal{P}\left(n_{1}, \ldots, n_{d+1}\right)\right|
\end{aligned}
$$

which gives the needed.
6.4. Quasisymmetry. It is known that the dual Grothendieck polynomials $g_{\lambda}(\mathbf{x})$ are symmetric in $\mathbf{x}$ (in the case $d=2$ ). As Example 6.3 shows, the generalized polynomials $g_{\rho}$ are not necessarily symmetric for $d \geq 3$. However, as we show in this subsection, these polynomials are always quasisymmetric.

Definition 6.8. A polynomial $f \in \mathbb{Z}\left[x_{1}, \ldots, x_{n}\right]$ is called quasisymmetric if for all $1 \leq \ell_{1}<$ $\cdots<\ell_{k} \leq n, 1 \leq j_{1}<\cdots<j_{k} \leq n$, and $a_{1}, \ldots, a_{k} \in \mathbb{Z}_{+}$we have

$$
\left[x_{\ell_{1}}^{a_{1}} \cdots x_{\ell_{k}}^{a_{k}}\right] f=\left[x_{j_{1}}^{a_{1}} \cdots x_{j_{k}}^{a_{k}}\right] f
$$

where $\left[\mathbf{x}^{\alpha}\right] f$ denotes the coefficient of the monomial $\mathbf{x}^{\alpha}$ in $f$.
Theorem 6.9. We have: $g_{\rho}\left(\mathbf{x}^{(1)} ; \ldots ; \mathbf{x}^{(d)}\right)$ is quasisymmetric in the variables $\mathbf{x}^{(1)}$.

Proof. To simplify notation let us denote $\mathbf{x}^{(1)}=\mathbf{x}=\left(x_{1}, x_{2}, \ldots\right)$. We need to show that for all $a_{1}, \ldots, a_{k} \in \mathbb{Z}_{+}, 1 \leq \ell_{1}<\cdots<\ell_{k} \leq n_{1}, 1 \leq j_{1}<\cdots<j_{k} \leq n_{1}$ we have

$$
\left[x_{\ell_{1}}^{a_{1}} \cdots x_{\ell_{k}}^{a_{k}}\right] g_{\rho}=\left[x_{j_{1}}^{a_{1}} \cdots x_{j_{k}}^{a_{k}}\right] g_{\rho} .
$$

Let $L$ and $R$ be the sets of $d$-dimensional partitions which contribute to the l.h.s. and r.h.s. respectively. We are going to construct a weight-preserving bijection $\phi: L \rightarrow R$.

Let $\pi \in L$ for which we have $\pi \in \mathcal{P}\left(n_{1}, \ldots, n_{d+1}\right)$ with $\operatorname{sh}_{1}(\pi)=D(\rho)$ and $w(\pi)=x_{\ell_{1}}^{a_{1}} \cdots x_{\ell_{k}}^{a_{k}} \times$ $w^{\prime}$, where $w^{\prime}$ is the the remaining product which does not contain the variables $\mathbf{x}$.

For a matrix $X=\left(x_{\mathbf{i}}\right)_{\mathbf{i} \in \mathbb{Z}_{+}^{d}}$, define the submatrices $X^{(\ell)}=\left(x_{\ell, \mathbf{i}}\right)_{\mathbf{i} \in \mathbf{Z}_{+}^{d-1}}$. Let $|X|$ denotes the sum of the entries of $X$.

Let $A=\left(a_{\mathbf{i}}\right)=\Phi^{-1}(\pi) \in \mathcal{M}\left(n_{1}, \ldots, n_{d}\right)$. Note that $A^{(\ell)} \in \mathcal{M}\left(n_{2}, \ldots, n_{d}\right)$ for $\ell \in\left[n_{1}\right]$. Since $\Phi$ preserves weights, i.e. $w_{A}=w(\pi)$ (see Lemma 4.3) we must have $A^{(\ell)} \neq \mathbf{0}$ iff $\ell \in\left\{\ell_{1}, \ldots, \ell_{k}\right\}$. We then have

$$
w(\pi)=w_{A}=\prod_{i=1}^{n_{1}} x_{i}^{\left|A^{(i)}\right|} \prod_{\mathbf{i}=\left(i_{2}, \ldots, i_{d}\right)}\left(x_{i_{2}}^{(2)} \cdots x_{i_{d}}^{(d)}\right)^{a_{i, \mathbf{i}}}=\prod_{i=1}^{k} x_{\ell_{i}}^{a_{i}} \times w^{\prime} .
$$

Let us now construct another matrix $B=\left(b_{\mathbf{i}}\right) \in \mathcal{M}\left(n_{1}, \ldots, n_{d}\right)$ so that $B^{(j)} \neq \mathbf{0}$ iff $j \in$ $\left\{j_{1}, \ldots, j_{k}\right\}$ and $B^{\left(j_{i}\right)}=A^{\left(\ell_{i}\right)}$ for all $i \in[k]$. Let $\pi^{\prime}=\Phi(B)$. We then clearly have

$$
w\left(\pi^{\prime}\right)=w_{B}=\prod_{i=1}^{k} x_{j_{i}}^{a_{i}} \times w^{\prime}
$$

Let us show that $\operatorname{sh}_{1}\left(\pi^{\prime}\right)=D(\rho)=\operatorname{sh}_{1}(\pi)$. Recall that $\operatorname{sh}_{1}\left(\pi^{\prime}\right)$ is the diagram of the partition $\left(\pi_{1, i_{2}, \ldots, i_{d}}^{\prime}\right)$. By definition of $\Phi$, each entry $\pi_{1, i_{2}, \ldots, i_{d}}^{\prime}$ is the largest weight directed path from $\left(1, i_{2}, \ldots, i_{d}\right)$ to $\left(n_{1}, \ldots, n_{d}\right)$ through the matrix $B$. Similarly, each entry $\pi_{1, i_{2}, \ldots, i_{d}}$ is the largest weight directed path from $\left(1, i_{2}, \ldots, i_{d}\right)$ to $\left(n_{1}, \ldots, n_{d}\right)$ through the matrix $A$. We then have

$$
\begin{aligned}
\pi_{1, i_{2}, \ldots, i_{d}} & =\max _{\Pi:\left(1, i_{2}, \ldots, i_{d}\right) \rightarrow\left(n_{1}, \ldots, n_{d}\right)} \sum_{(\ell, \mathbf{i}) \in \Pi, \ell \in\left\{\ell_{1}, \ldots, \ell_{k}\right\}} a_{\mathbf{j}} \\
& =\sum_{\Pi:\left(1, i_{2}, \ldots, i_{d}\right) \rightarrow\left(n_{1}, \ldots, n_{d}\right)} \sum_{(j, \mathbf{i}) \in \Pi, j \in\left\{j_{1}, \ldots, j_{k}\right\}} b_{\mathbf{j}} \\
& =\pi_{1, i_{2}, \ldots, i_{d}}^{\prime} .
\end{aligned}
$$

Hence $\pi^{\prime} \in R$, we can set $\phi: \pi \mapsto \pi^{\prime}$ and it is a well-defined bijection between $L$ and $R$.
Let us define the boxed polynomials

$$
F_{\left(n_{1}, \ldots, n_{d+1}\right)}\left(\mathbf{x}^{(1)} ; \ldots ; \mathbf{x}^{(d)}\right):=\sum_{\pi \in \mathcal{P}\left(n_{1}, \ldots, n_{d+1}\right)} \prod_{\left(i_{1}, \ldots, i_{d+1}\right) \in \operatorname{Cor}(\pi)} x_{i_{1}}^{(1)} \cdots x_{i_{d}}^{(d)}
$$

which is a bounded version of the Cauchy product as by Theorem 4.2 we have

$$
\lim _{n_{d+1} \rightarrow \infty} F_{\left(n_{1}, \ldots, n_{d+1}\right)}\left(\mathbf{x}^{(1)} ; \ldots ; \mathbf{x}^{(d)}\right)=\prod_{i_{1}=1}^{n_{1}} \cdots \prod_{i_{d}=1}^{n_{d}}\left(1-x_{i_{1}}^{(1)} \cdots x_{i_{d}}^{(d)}\right)^{-1}
$$

These polynomials can also be expanded as follows:

$$
F_{\left(n_{1}, \ldots, n_{d+1}\right)}\left(\mathbf{x}^{(1)} ; \ldots ; \mathbf{x}^{(d)}\right)=\sum_{\rho \in \mathcal{P}\left(n_{2}, \ldots, n_{d+1}\right)} \sum_{\operatorname{sh}_{1}(\pi)=\rho} w(\pi)=\sum_{\rho \in \mathcal{P}\left(n_{2}, \ldots, n_{d+1}\right)} g_{\rho}\left(\mathbf{x}^{(1)} ; \ldots ; \mathbf{x}^{(d)}\right) .
$$

Corollary 6.10 (Full quasisymmetry of boxed polynomials). We have: $F_{\left(n_{1}, \ldots, n_{d+1}\right)}$ is quasisymmetric in each set of the variables $\mathbf{x}^{(1)} ; \ldots ; \mathbf{x}^{(d)}$ independently.

Proof. The quasisymmetry in $\mathbf{x}^{(1)}$ is immediate from the previous theorem. The same holds for any other set of variables by noting that the definitions of $\operatorname{Cor}(\pi)$ and weights $\pi$ are symmetric in the first $d$ coordinates and hence we may repeat the proof by 'rotation', i.e. moving any coordinate as the first one.

Definition 6.11. Let $A=\left(a_{i_{1}, \ldots, i_{d}}\right) \in \mathcal{M}\left(n_{1}, \ldots, n_{d}\right)$. For each $\ell \in[d]$, consider the matrices $B_{i}^{(\ell)}=\left(a_{i_{1}, \ldots, i_{d}}\right)_{i_{\ell}=i}$, i.e. submatrices of $A$ with fixed $\ell$-th coordinate. Define the vectors

$$
s_{\ell}(A):=\left(\left|B_{1}^{(\ell)}\right|,\left|B_{2}^{(\ell)}\right|, \ldots\right)
$$

where $|B|$ denotes the sum of entries of $B$. For example, if $d=2$, then $s_{1}(A)$ is the vector of row sums of $A$, and $s_{2}(A)$ is the column sums of $A$. Let us also say that $A$ is a packed matrix if for each $\ell \in[d]$, the sequence $s_{\ell}(A)$ does not contain zeros between its positive entries. Denote by pack $(A)$ the packed matrix formed from $A$ by removing its zero submatrices $B_{i}^{(\ell)}=\mathbf{0}$.

For a composition $\alpha=\left(\alpha_{1}, \ldots, \alpha_{k}\right) \in \mathbb{Z}_{+}^{k}$, recall the monomial quasisymmetric functions

$$
M_{\alpha}(\mathbf{x}):=\sum_{i_{1}<\ldots<i_{k}} x_{i_{1}}^{\alpha_{1}} \cdots x_{i_{k}}^{\alpha_{k}} .
$$

Note that they form a basis of the algebra of quasisymmetric functions.
It is easy to see that

$$
F_{\left(n_{1}, \ldots, n_{d}, \infty\right)}=\sum_{A \in \mathcal{M}\left(n_{1}, \ldots, n_{d}\right)} w_{A}=\sum_{\alpha^{(1)}, \ldots, \alpha^{(d)}} m_{\alpha^{(1)}, \ldots, \alpha^{(d)}}\left(\mathbf{x}^{(1)}\right)^{\alpha^{(1)}} \cdots\left(\mathbf{x}^{(d)}\right)^{\alpha^{(d)}}
$$

where $m_{\alpha^{(1)}, \ldots, \alpha^{(d)}}$ is the number of $A \in \mathcal{M}\left(n_{1}, \ldots, n_{d}\right)$ with $s_{\ell}(A)=\alpha^{(\ell)} \in \mathbb{N}^{n_{\ell}}$. The following result is a finite boxed version of this expansion.

Theorem 6.12 (Monomial basis expansion of boxed polynomials). We have

$$
F_{\left(n_{1}, \ldots, n_{d+1}\right)}=\sum_{\alpha^{(1)}, \ldots, \alpha^{(d)}} m_{\alpha^{(1)}, \ldots, \alpha^{(d)}}^{\left(n_{d+1}\right)} M_{\alpha^{(1)}}\left(\mathbf{x}^{(1)}\right) \cdots M_{\alpha^{(d)}}\left(\mathbf{x}^{(d)}\right)
$$

where the sum runs over compositions $\alpha^{(1)}, \ldots, \alpha^{(d)}$ such that $\left|\alpha^{(i)}\right|=\left|\alpha^{(j)}\right|$ for all $i, j$, and the coefficient $m_{\alpha^{(1)}, \ldots, \alpha^{(d)}}^{\left(n_{d+1}\right)}$ is equal to the number of packed matrices $A \in \mathcal{M}\left(\left[n_{1}\right] \times \cdots \times\left[n_{d}\right], n_{d+1}\right)$ such that $s_{\ell}(A)=\alpha^{(\ell)}$ for all $\ell \in[d]$.
Proof. Let $P \in \mathcal{M}\left(\left[n_{1}\right] \times \cdots \times\left[n_{d}\right], n_{d+1}\right)$ be a packed matrix and let $M(P)$ be the set of matrices $A \in \mathcal{M}\left(\left[n_{1}\right] \times \cdots \times\left[n_{d}\right], n_{d+1}\right)$ such that $\operatorname{pack}(A)=P$. Let $s_{\ell}(P)=\alpha^{(\ell)}$. Then (by an argument as in Theorem 6.9) it is not difficult to obtain that we have

$$
\sum_{A \in M(P)} w_{A}=m_{\alpha^{(1)}, \ldots, \alpha^{(d)}}^{\left(n_{d+1}\right)} M_{\alpha^{(1)}}\left(\mathbf{x}^{(1)}\right) \cdots M_{\alpha^{(d)}}\left(\mathbf{x}^{(d)}\right)
$$

Therefore, we obtain

$$
\begin{aligned}
F_{\left(n_{1}, \ldots, n_{d+1}\right)} & =\sum_{\pi \in \mathcal{P}\left(n_{1}, \ldots, n_{d+1}\right)} w(\pi) \\
& =\sum_{A \in \mathcal{M}\left(\left[n_{1}\right] \times \cdots \times\left[n_{d}\right], n_{d+1}\right)} w_{A} \\
& =\sum_{P \text { packed }} \sum_{A \in M(P)} w_{A} \\
& =\sum_{\alpha^{(1)}, \ldots, \alpha^{(d)}} m_{\alpha^{(1)}, \ldots, \alpha^{(d)}}^{\left(n_{d+1}\right)} M_{\alpha^{(1)}}\left(\mathbf{x}^{(1)}\right) \cdots M_{\alpha^{(d)}}\left(\mathbf{x}^{(d)}\right)
\end{aligned}
$$

as needed.
Remark 7. For $d=2$, packed matrices appear in the algebra of matrix quasisymmetric functions, see [DHT02].
6.5. Dual Grothendieck polynomials, $d=2$. Recall that in this case (see Example 6.2), we get the following definition of polynomials $g_{\lambda}(\mathbf{x} ; \mathbf{y})$ indexed by partitions $\lambda$. We define

$$
g_{\lambda}(\mathbf{x} ; \mathbf{y}):=\sum_{\pi: \operatorname{sh}_{1}(\pi)=\lambda} \prod_{(i, j, k) \in \operatorname{Cor}(\pi)} x_{i} y_{j}
$$

where the sum runs over plane partitions $\pi$. The polynomials $g_{\lambda}(\mathbf{x} ; \mathbf{y})$ are generalizations of dual Grothendieck polynomials which correspond to the specialization $g_{\lambda}(\mathbf{x})=g_{\lambda}(\mathbf{x} ; \mathbf{1})$. In fact, $g_{\lambda}(\mathbf{x} ; \mathbf{y})$ is symmetric in $\mathbf{x}$. The Cauchy-type identity in Corollary 6.5 becomes

$$
\sum_{\lambda \in \mathcal{P}\left(n_{2}, \infty\right)} g_{\lambda}(\mathbf{x} ; \mathbf{y})=\prod_{i=1}^{n_{1}} \prod_{j=1}^{n_{2}} \frac{1}{1-x_{i} y_{j}}
$$

which was proved in [Yel19a, Yel19b]. The boxed specialization formula in Proposition 6.7 becomes the following

$$
g_{\left[n_{2}\right] \times\left[n_{3}\right]}\left(1^{n_{1}+1}\right)=\left|\mathcal{P}\left(n_{1}, n_{2}, n_{3}\right)\right|,
$$

the number of plane partitions inside the box $\left[n_{1}\right] \times\left[n_{2}\right] \times\left[n_{3}\right]$, for which there is also the famous MacMahon boxed product formula

$$
\left|\mathcal{P}\left(n_{1}, n_{2}, n_{3}\right)\right|=\prod_{i=1}^{n_{1}} \prod_{j=1}^{n_{2}} \prod_{k=1}^{n_{3}} \frac{i+j+k-1}{i+j+k-2} .
$$

Using determinantal formulas for dual Grothendieck polynomials [Yel17] we also have the following 'coincidence' formula (see [Yel19a, Yel19b]) connecting them with the Schur polynomials $\left\{s_{\lambda}\right\}$ as follows

$$
g_{\left[n_{2}\right] \times\left[n_{3}\right]}(\mathbf{x})=s_{\left[n_{2}\right] \times\left[n_{3}\right]}\left(\mathbf{x}, 1^{n_{2}-1}\right) .
$$

6.6. 3d Grothendieck polynomials, $d=3$. In this case, we get the following definition of polynomials $g_{\rho}(\mathbf{x} ; \mathbf{y} ; \mathbf{z})$ indexed by plane partitions $\rho$. We define

$$
\begin{equation*}
g_{\rho}(\mathbf{x} ; \mathbf{y} ; \mathbf{z}):=\sum_{\pi: \operatorname{sh}_{1}(\pi)=\rho} \prod_{(i, j, k, \ell) \in \operatorname{Cor}(\pi)} x_{i} y_{j} z_{k} \tag{8}
\end{equation*}
$$

where the sum runs over solid partitions $\pi \in \mathcal{P}\left(n_{1}, n_{2}, n_{3}, n_{4}\right)$. Note also that if $\rho$ satisfies $D(\rho)=\{(1, i, j):(i, j) \in D(\lambda)\}$ where $\lambda$ is a partition, we then have $g_{\rho}(\mathbf{x} ; \mathbf{y} ; \mathbf{1})=g_{\lambda}(\mathbf{x} ; \mathbf{y})$ reduces to the 2 d case discussed above. The polynomials $g_{\rho}(\mathbf{x} ; \mathbf{y} ; \mathbf{z})$ are quasisymmetric in $\mathbf{x}$. Then Cauchy-type identity in Corollary 6.5 becomes

$$
\sum_{\rho \in \mathcal{P}\left(n_{2}, n_{3}, \infty\right)} g_{\rho}(\mathbf{x} ; \mathbf{y} ; \mathbf{z})=\prod_{i=1}^{n_{1}} \prod_{j=1}^{n_{2}} \prod_{k=1}^{n_{3}}\left(1-x_{i} y_{j} z_{k}\right)^{-1}
$$

The boxed specialization formula becomes the following

$$
g_{\left[n_{2}\right] \times\left[n_{3}\right] \times\left[n_{4}\right]}\left(1^{n_{1}+1}\right)=\left|\mathcal{P}\left(n_{1}, n_{2}, n_{3}, n_{4}\right)\right|,
$$

the number of solid partitions inside the box $\left[n_{1}\right] \times\left[n_{2}\right] \times\left[n_{3}\right] \times\left[n_{4}\right]$.
Remark 8 (On higher-dimensional Schur polynomials and SSYT). Note that the $d$-dimensional Grothendieck polynomials $g_{\rho}(\mathbf{x})$ are inhomogeneous. It is well known that for $d=2$ we have $g_{\lambda}=s_{\lambda}+$ lower degree terms. By analogy, the top degree homogeneous component of $g_{\rho}(\mathbf{x})$ denoted by $s_{\rho}(\mathbf{x})$ can be viewed as a higher-dimensional analogue of Schur polynomials. It sums over a subset of $d$-dimensional partitions which are analogous to semistandard Young tableaux (SSYT) for the case $d=2$. By Theorem 6.9, $\left\{s_{\rho}\right\}$ are also quasisymmetric polynomials. Are there any interesting properties of these functions and tableaux?

## 7. Last passage percolation in $\mathbb{Z}^{d}$

In this section we consider a directed last passage percolation model with geometric weights and show its connections with $d$-dimensional Grothendieck polynomials studied in the previous section.

Let $W=\left(w_{\mathbf{i}}\right)_{\mathbf{i} \in \mathbb{Z}_{+}^{d}}$ be a random matrix with i.i.d. entries $w_{\mathbf{i}}$ which have geometric distribution with parameter $q \in(0,1)$, i.e.

$$
\operatorname{Prob}\left(w_{\mathbf{i}}=k\right)=(1-q) q^{k}, \quad k \in \mathbb{N} .
$$

Define the last passage times as follows

$$
G(\mathbf{i})=G(\mathbf{1} \rightarrow \mathbf{i})=\max _{\Pi: 1 \rightarrow \mathbf{i}} \sum_{\mathbf{j} \in \Pi} w_{\mathbf{j}}, \quad \mathbf{i} \in \mathbb{Z}_{+}^{d},
$$

where the maximum is over directed lattice paths $\Pi$ from $(1, \ldots, 1)$ to i. Using Kingman's subadditivity theorem, one can show that there is a deterministic limit shape $\varphi: \mathbb{R}_{\geq 0}^{d} \rightarrow \mathbb{R}_{\geq 0}$ (see [Mar06]) such that as $n \rightarrow \infty$ we have a.s. convergence

$$
\frac{1}{n} G(\lfloor n \mathbf{x}\rfloor) \rightarrow \varphi(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}_{\geq 0}^{d} .
$$

The case $d=2$ is exactly solvable and $\varphi(x, y)=(x+y+2 \sqrt{q x y}) /(1-q)$; moreover, the fluctuations around the shape are of order $n^{1 / 3}$ and tend to the Tracy-Widom distribution [Joh00]. However, much less is known for $d \geq 3$.

Now we are going to show that $d$-dimensional Grothendieck polynomials naturally appear in distribution formulas for this model.

Theorem 7.1. Let $n_{1}, \ldots, n_{d} \in \mathbb{Z}_{+}$and $\rho \in \mathcal{P}\left(n_{2}, \ldots, n_{d}, \infty\right)$ be a (d-1)-dimensional partition. Denote $\mathbf{n}=\left(n_{2}+1, \ldots, n_{d}+1\right)$ and $N=n_{1} \cdots n_{d}$. We have the following joint distribution formula

$$
\operatorname{Prob}\left(G\left(n_{1}, \mathbf{n}-\mathbf{i}\right)=\rho_{\mathbf{i}}: \mathbf{i} \in\left[n_{2}\right] \times \cdots \times\left[n_{d}\right]\right)=(1-q)^{N} g_{\rho}(\underbrace{q, \ldots, q}_{n_{1} \text { times }}) .
$$

Proof. Let us flip and truncate the matrix $W$ to get $W^{\prime}=\left(w_{\mathbf{i}}^{\prime}\right)=\left(w_{\left(n_{1}+1, \mathbf{n}\right)-\mathbf{i}}\right)_{\mathbf{i} \in\left[n_{1}\right] \times \cdots \times\left[n_{d}\right]}$.
Let $\pi=\left(\pi_{\mathbf{i}}\right) \in \mathcal{P}\left(n_{1}, \ldots, n_{d}, \infty\right)$ and $\left(a_{\mathbf{i}}\right)=\Phi^{-1}(\pi)$. We obtain

$$
\operatorname{Prob}\left(W^{\prime}=\Phi^{-1}(\pi)\right)=\prod_{\mathbf{i} \in\left[n_{1}\right] \times \cdots \times\left[n_{d}\right]} \operatorname{Prob}\left(w_{\mathbf{i}}^{\prime}=a_{\mathbf{i}}\right)=(1-q)^{N} q^{S(\pi)},
$$

where $S(\pi)=\sum_{\mathbf{i}} a_{\mathbf{i}}$. Note that from (7) we have

$$
\begin{aligned}
(1-q)^{N} g_{\rho}(\underbrace{q, \ldots, q}_{n_{1} \text { times }}) & =(1-q)^{N} \sum_{\pi: \operatorname{sh}_{1}(\pi)=\rho} q^{c_{1}(\pi)+\ldots+c_{n_{1}}(\pi)} \\
& =\sum_{\pi: \operatorname{sh}_{1}(\pi)=\rho}(1-q)^{N} q^{S(\pi)} \\
& =\sum_{\pi: \operatorname{sh}_{1}(\pi)=\rho} \operatorname{Prob}\left(W^{\prime}=\Phi^{-1}(\pi)\right)
\end{aligned}
$$

where the sum runs over $\pi \in \mathcal{P}\left(n_{1}, \ldots, n_{d}, \infty\right)$. Observe that we have $\Phi\left(W^{\prime}\right)=(G(\mathbf{i}))_{\mathbf{i} \in \mathbb{Z}_{+}^{d}}$. Therefore, now we get

$$
\begin{aligned}
\operatorname{Prob}\left(G\left(n_{1}, \mathbf{n}-\mathbf{i}\right)=\rho_{\mathbf{i}}: \mathbf{i} \in\left[n_{2}\right] \times \cdots \times\left[n_{d}\right]\right) & =\sum_{\pi: \operatorname{sh}_{1}(\pi)=\rho} \operatorname{Prob}\left(\Phi\left(W^{\prime}\right)=\pi\right) \\
& =(1-q)^{N} g_{\rho}(\underbrace{q, \ldots, q}_{n_{1} \text { times }})
\end{aligned}
$$

as needed.
Corollary 7.2 (Single point distribution formula). We have

$$
\operatorname{Prob}\left(G\left(n_{1}, \ldots, n_{d}\right) \leq n\right)=(1-q)^{N} g_{\left[n_{2}\right] \times \cdots \times\left[n_{d}\right] \times[n]}(1, \underbrace{q, \ldots, q}_{n_{1} \text { times }})
$$

Proof. Follows by combining the theorem with Lemma 6.6.
Corollary 7.3 (The case $d=2$ ). Let $\lambda \in \mathcal{P}\left(n_{2}, \infty\right)$ be a partition. We have

$$
\operatorname{Prob}\left(G\left(n_{1}, n_{2}+1-i\right)=\lambda_{i}: i \in\left[n_{2}\right]\right)=(1-q)^{n_{1} n_{2}} g_{\lambda}(\underbrace{q, \ldots, q}_{n_{1} \text { times }}) .
$$

Remark 9. This formula (which shows that dual symmetric Grothendieck polynomials arise naturally in the last passage percolation model) was proved in [Yel19a] and in more general case with different parameters in [Yel20]. Note that in this case we can obtain many determinantal formulas.

Remark 10. Theorem 7.1 suggests a probability distribution on the set $\mathcal{P}\left(n_{2}, \ldots, n_{d}, \infty\right)$ of ( $d-1$ )-dimensional partitions defined as follows:

$$
\operatorname{Prob}_{g}(\rho):=(1-q)^{n_{1} \cdots n_{d}} g_{\rho}(\underbrace{q, \ldots, q}_{n_{1} \text { times }}), \quad \rho \in \mathcal{P}\left(n_{2}, \ldots, n_{d}, \infty\right) .
$$

## 8. Concluding remarks and open questions

8.1. After defining plane partitions in EC2 [Sta99, Ch. 7.20], Richard Stanley writes:
"... It now seems obvious to define $r$-dimensional partitions for any $r \geq 1$.
However, almost nothing significant is known for $r \geq 3$."
Few more remarks and references on the subject can be found in an early survey [Sta71] (on the theory of plane partitions). For more recent works, see [MR03, BGP12, Gov13, DG15].
8.2. Asymptotics. MacMahon's numbers $m_{d}(n)$ have the following asymptotics [BGP12]

$$
\lim _{n \rightarrow \infty} n^{-d /(1+d)} \log m_{d}(n)=\frac{1+d}{d}(d \zeta(1+d))^{1 /(1+d)}
$$

where $\zeta$ is the Riemann zeta function (which is computed based on the explicit formula for the generating function). It was conjectured in [BGP12] and (for solid partitions) in [MR03] supported by numerical experiments, that $p_{d}(n)$, the number of $d$-dimensional partitions of volume (size) $n$, has exactly the same asymptotics. However, later computations reported in [DG15] suggest that this is not the case (for $d=3$ ) and that $p_{3}(n)$ is asymptotically larger than $m_{3}(n)$ (despite the fact that $m_{3}(n)=p_{3}(n)$ for $n \leq 5$ and $m_{3}(n)>p_{3}(n)$ for the next many values of $n$ [ABMM67, DG15]; cf. the sequences A000293, A000294 in [OEIS]). See also [Ekh12] and a useful resource [Gov] for more related data. Given our interpretation for $m_{d}(n)$ (Corollary 5.5), is it possible to compare them with $p_{d}(n)$ ?
8.3. $d$-dimensional Grothendieck polynomials. Are there any (algebraic, determinantal) formulas for $d$-dimensional Grothendieck polynomials? They will be important for at least two applications: enumeration of boxed higher-dimensional partitions, and computing distribution formulas (or performing asymptotic analysis) for the last passage percolation problem discussed above. Note that for $d=2$, there are several determinantal formulas (Jacobi-Trudi, bialternant types) known, see [Yel17, AY20].

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[^0]:    ${ }^{1}$ In some literature, there is $a+1$ shift in dimensions, when partitions are associated with their diagrams.

[^1]:    ${ }^{2}$ We use terminology related to probabilistic model of last passage percolation, see Sec. 7 .

