

ON SCHUBERT VARIETIES OF COMPLEXITY ONE

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ABSTRACT. Let B be a Borel subgroup of $\mathrm{GL}_n(\mathbb{C})$ and \mathbb{T} a maximal torus contained in B . Then \mathbb{T} acts on $\mathrm{GL}_n(\mathbb{C})/B$ and every Schubert variety is \mathbb{T} -invariant. We say that a Schubert variety is of complexity k if a maximal \mathbb{T} -orbit in X_w has codimension k . In this paper, we discuss topology, geometry, and combinatorics related to Schubert varieties of complexity one.

1. INTRODUCTION

The flag manifold $\mathcal{F}\ell(\mathbb{C}^n)$ is the homogeneous space $\mathrm{GL}_n(\mathbb{C})/B$, where B is the set of all upper triangular matrices in $\mathrm{GL}_n(\mathbb{C})$. The left action of B on $\mathcal{F}\ell(\mathbb{C}^n)$ has finitely many orbits BwB/B , where w is a permutation in \mathfrak{S}_n , and the Schubert variety X_w is the (Zariski) closure of the B -orbit BwB/B . Most Schubert varieties are not smooth and they are desingularized to Bott–Samelson varieties, see [2, 8].

Let \mathbb{T} be the set of all diagonal matrices in $\mathrm{GL}_n(\mathbb{C})$. Then \mathbb{T} is isomorphic to the torus $(\mathbb{C}^*)^n$ and acts on $\mathcal{F}\ell(\mathbb{C}^n)$ by the left multiplication, and every Schubert variety X_w is a \mathbb{T} -invariant irreducible subvariety of $\mathcal{F}\ell(\mathbb{C}^n)$. We say that a Schubert variety X_w is of complexity k with respect to the action of \mathbb{T} (or simply, X_w is of complexity k) if a maximal \mathbb{T} -orbit in X_w has codimension k . In this paper, we are interested in the Schubert varieties of complexity one.

There were several studies on Schubert varieties of complexity zero (i.e., toric Schubert varieties) and related combinatorics. It is known that X_w is of complexity zero if and only if a reduced decomposition of w consists of distinct letters, and in this case X_w is smooth and isomorphic to a Bott–Samelson variety ([10], [17]). On the other hand, permutation patterns are related to the form of reduced decompositions. A reduced decomposition of w consists of distinct letters if and only if w avoids the patterns 321 and 3412, see [24]. It was also shown in [23] that a reduced decomposition of w consists of distinct letters if and only if the Bruhat interval $[e, w]$ is isomorphic to the Boolean algebra $\mathfrak{B}_{\ell(w)}$ of rank $\ell(w)$, where $\ell(w)$ is the length of w . The *Bruhat interval polytope* $Q_{v,w}$, introduced in [25], is the convex hull of the points $(u(1), \dots, u(n))$ in \mathbb{R}^n for all $v \leq u \leq w$. The Schubert variety X_w is a smooth toric variety if and only if $Q_{e,w}$ is combinatorially equivalent to the $\ell(w)$ -dimensional cube $I^{\ell(w)}$, see [21].

Theorem 1.1. [10, 17, 23, 24, 21] *The following are equivalent:*

- (0) X_w is a toric variety (i.e., of complexity zero).
- (1) X_w is a smooth toric variety.

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- (2) w avoids the patterns 321 and 3412.
- (3) A reduced decomposition of w consists of distinct letters.
- (4) X_w is isomorphic to a Bott–Samelson variety.
- (5) The Bruhat interval $[e, w]$ is isomorphic to $\mathfrak{B}_{\ell(w)}$, the Boolean algebra of rank $\ell(w)$.
- (6) The Bruhat interval polytope $Q_{e,w}$ is combinatorially equivalent to the $\ell(w)$ -dimensional cube.

Recall that for a decomposition $\underline{w} = s_{i_1} \dots s_{i_\ell}$ of a permutation w , the Bott–Samelson variety $Z_{\underline{w}}$ is defined by the orbit space of $\overline{Bs_{i_1}B} \times \dots \times \overline{Bs_{i_\ell}B}$ by the right action of B^ℓ in (2.2). Then $Z_{\underline{w}}$ has an iterated $\mathbb{C}P^1$ -bundle structure because $\overline{Bs_{i_k}B}/B \cong \mathbb{C}P^1$. Note that $\overline{Bs_{i_k}B}/B = X_{s_{i_k}}$ and $\overline{Bs_{i_k}B}/B \cong \mathcal{F}\ell(\mathbb{C}^2) = \mathbb{C}P^1$. A Bott–Samelson variety $Z_{\underline{w}}$ is not a toric variety in general, but it is diffeomorphic to a toric variety, called a Bott manifold. We refer the reader to [13] for more details.

In this article, we study an analog of the equivalent statements (1)~(6) in Theorem 1.1 for Schubert varieties of complexity one. Whereas every toric Schubert variety is smooth, not every Schubert variety of complexity one is smooth. For example, the Schubert varieties X_{3214} and X_{3412} are of complexity one, but X_{3214} is smooth and X_{3412} is singular.

Jantzen [15] generalized the notion of Bott–Samelson varieties. For an ordered tuple of permutations (w_1, \dots, w_r) , the variety $Z_{(w_1, \dots, w_r)}$ is the orbit space of $\overline{Bw_1B} \times \dots \times \overline{Bw_rB}$ by the right action of B^r in (4.1). Unfortunately, there was no name for the variety $Z_{(w_1, \dots, w_r)}$ in [15], but now, it is called a *generalized Bott–Samelson variety* (cf. [4] and [22]). When X_{w_i} is a flag manifold for every $i = 1, \dots, r$, the variety $Z_{(w_1, \dots, w_r)}$ is called a *flag Bott–Samelson variety*, see [11].

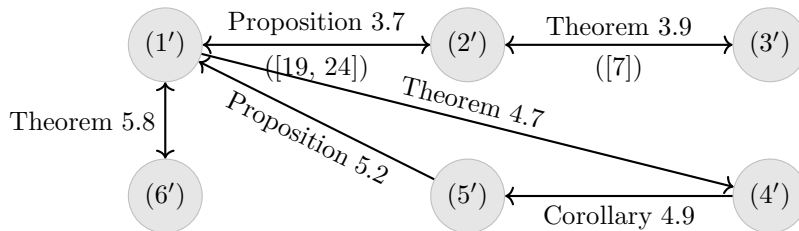
Theorem 1.2. *For a permutation w in \mathfrak{S}_n , the following are equivalent:*

- (1') X_w is smooth and of complexity one.
- (2') w contains the pattern 321 exactly once and avoids the pattern 3412.
- (3') There exists a reduced decomposition of w containing $s_i s_{i+1} s_i$ as a factor and no other repetitions.
- (4') X_w is isomorphic to a flag Bott–Samelson variety $Z_{(w_1, \dots, w_r)}$ such that $r = \ell(w) - 2$, $w_k = s_j s_{j+1} s_j$ for some $1 \leq k \leq r$, $w_i = s_{j_i}$ for $i \neq k$, and $j_1, \dots, j_{k-1}, j_{k+1}, \dots, j_r, j, j+1$ are pairwise distinct.
- (5') The Bruhat interval $[e, w]$ is isomorphic to $\mathfrak{S}_3 \times \mathfrak{B}_{\ell(w)-3}$.
- (6') The Bruhat interval polytope $Q_{e,w}$ is combinatorially equivalent to the product of the hexagon and the cube $I^{\ell(w)-3}$.

Theorem 1.3. *For a permutation w in \mathfrak{S}_n , the following are equivalent:*

- (1'') X_w is singular and of complexity one.
- (2'') w contains the pattern 3412 exactly once and avoids the pattern 321.
- (3'') There exists a reduced decomposition of w containing $s_{i+1} s_i s_{i+2} s_{i+1}$ as a factor and no other repetitions.
- (4'') X_w is isomorphic to a generalized Bott–Samelson variety $Z_{(w_1, \dots, w_r)}$ such that $r = \ell(w) - 3$, $w_k = s_{j+1} s_j s_{j+2} s_{j+1}$ for some $1 \leq k \leq r$, $w_i = s_{j_i}$ for $i \neq k$, and $j_1, \dots, j_{k-1}, j_{k+1}, \dots, j_r, j, j+1, j+2$ are pairwise distinct.
- (5'') The Bruhat interval $[e, w]$ is isomorphic to $[e, 3412] \times \mathfrak{B}_{\ell(w)-4}$.
- (6'') The Bruhat interval polytope $Q_{e,w}$ is combinatorially equivalent to the product of $Q_{e,3412}$ and the cube $I^{\ell(w)-4}$.

The equivalence between the first two statements (respectively, the second and the third statements) in Theorems 1.2 and 1.3 is an immediate consequence of [19] and [24] (respectively, [7]). The following diagram shows how we prove the main theorems in the paper. We prove Theorems 1.2 and 1.3 in parallel.



Like as a Bott–Samelson variety is diffeomorphic to a Bott manifold having a higher rank torus action, a flag Bott–Samelson variety is diffeomorphic to a *flag Bott manifold* which admits a higher rank torus action. Whereas a Bott manifold is a toric variety, a flag Bott manifold is not a toric variety in general, but it becomes a GKM manifold. In addition, we will see that every smooth Schubert variety of complexity one is diffeomorphic to a flag Bott manifold.

This paper is organized as follows. Section 2 contains basic notions and facts about symmetric groups, Schubert varieties, Bott–Samelson varieties and Bott towers. In Section 3, we introduce the various relation between the pattern avoidance of a permutation and the complexity of a Schubert variety, and see the equivalence among the first three statements in Theorems 1.2 and 1.3. In Section 4, we introduce the notions of flag Bott–Samelson varieties and generalized Bott–Samelson varieties, and prove the implications $(1') \Rightarrow (4') \Rightarrow (5')$ and $(1'') \Rightarrow (4'') \Rightarrow (5'')$ in Theorems 1.2 and 1.3, respectively. In Section 5, we study the properties of Bruhat intervals and Bruhat interval polytopes related to Schubert varieties of complexity one, and then complete proofs of Theorems 1.2 and 1.3. In Section 6, we introduce the notion of flag Bott manifolds, and then show that every smooth Schubert variety of complexity one is diffeomorphic to a flag Bott manifold.

2. PRELIMINARIES

In this section, we first prepare basic facts about symmetric groups and Schubert varieties from [3], and then see the relation among Schubert varieties, Bott–Samelson varieties, and Bott towers.

Let $G = \mathrm{GL}_n(\mathbb{C})$ and B the set of upper triangular matrices in G . We denote by \mathbb{T} the set of diagonal matrices in G . Then, $\mathbb{T} \cong (\mathbb{C}^*)^n$. The homogeneous space G/B is a smooth projective variety can be identified with the set

$$\mathcal{Fl}(\mathbb{C}^n) = \{(\{0\} \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_n = \mathbb{C}^n) \mid \dim_{\mathbb{C}} V_i = i \text{ for } i = 1, \dots, n\}$$

of chains of subspaces of \mathbb{C}^n . The Weyl group of G is identified with the symmetric group \mathfrak{S}_n on the set $[n] := \{1, 2, \dots, n\}$. For an element $w \in \mathfrak{S}_n$, we use the one-line notation

$$w = w(1)w(2)\cdots w(n).$$

In this one-line notation, the identity element e is presented by $e = 1\ 2\ \cdots\ n$.

We denote the set of transpositions by

$$(2.1) \quad T = \{(i, j) \mid 1 \leq i < j \leq n\},$$

which are permutations on $[n]$ swapping i and j . The *simple transpositions* s_i are the transpositions of the form

$$s_i := (i, i + 1), \quad \text{for } i = 1, \dots, n - 1.$$

Since \mathfrak{S}_n is generated by simple transpositions, every $w \in \mathfrak{S}_n$ can be expressed as a product of simple transpositions. If $w = s_{i_1} \cdots s_{i_\ell}$ and ℓ is minimal among all such expressions, then ℓ is called the *length* of w (written $\ell(w) = \ell$) and the expression $s_{i_1} \cdots s_{i_\ell}$ is called a *reduced decomposition* (or *reduced expression* or *reduced word*) for w . We denote by $R(w)$ the set consisting of all reduced decompositions of w . A consecutive substring of a reduced decomposition is called a *factor*. For instance, since

$$R(321) = \{s_1 s_2 s_1, s_2 s_1 s_2\} \text{ and } R(3412) = \{s_2 s_3 s_1 s_2, s_2 s_1 s_3 s_2\},$$

no reduced decomposition for 3412 contains $s_i s_{i+1} s_i$ as a factor unlike 321.

The Bruhat order on \mathfrak{S}_n is defined by $v \leq w$ if a reduced decomposition of v is a substring of some reduced decomposition of w . Then \mathfrak{S}_n with the *Bruhat order* is a graded poset, with rank function given by length. Figure 1 gives the Hasse diagram for the Bruhat order on \mathfrak{S}_4 .

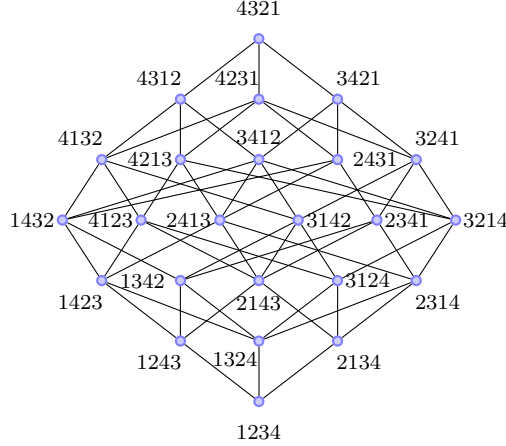


FIGURE 1. The Bruhat order on \mathfrak{S}_4 .

The complex torus \mathbb{T} acts on G/B by the left multiplication, and the set of \mathbb{T} -fixed points is identified with \mathfrak{S}_n . More precisely, each element $w \in \mathfrak{S}_n$ corresponds to a coordinate flag given by

$$\{\{0\} \subsetneq \langle \mathbf{e}_{w(1)} \rangle \subsetneq \langle \mathbf{e}_{w(1)}, \mathbf{e}_{w(2)} \rangle \subsetneq \cdots \subsetneq V_n = \mathbb{C}^n\}$$

where $\mathbf{e}_1, \dots, \mathbf{e}_n$ are the standard basis vectors in \mathbb{C}^n . We denote by wB this standard coordinate flag. It is well-known that $\mathcal{F}\ell(\mathbb{C}^n)$ has a Bruhat decomposition

$$\mathcal{F}\ell(\mathbb{C}^n) = \bigsqcup_{w \in \mathfrak{S}_n} BwB/B.$$

Moreover, the B -orbit BwB/B is isomorphic to $\mathbb{C}^{\ell(w)}$ and called a *Schubert cell*. The (Zariski) closure of BwB/B is the *Schubert variety* X_w , and each Schubert

variety decomposes into Schubert cells:

$$X_w = \bigsqcup_{v \leq w} BvB/B.$$

Note that most Schubert varieties are singular, and they are desingularized using Bott–Samelson varieties. Let w be a permutation and consider a decomposition $\underline{w} = (i_1, \dots, i_\ell)$ of w (not necessarily reduced). The *Bott–Samelson variety* associated with \underline{w} , denoted $Z_{\underline{w}}$, is the quotient of $\overline{Bs_{i_1}B} \times \dots \times \overline{Bs_{i_\ell}B}$ by the action of $B^\ell := \underbrace{B \times \dots \times B}_\ell$ given by:

$$(2.2) \quad (b_1, \dots, b_\ell) \cdot (p_1, \dots, p_\ell) := (p_1 b_1, b_1^{-1} p_2 b_2, \dots, b_{\ell-1}^{-1} p_\ell b_\ell)$$

for $(b_1, \dots, b_\ell) \in B^\ell$ and $(p_1, \dots, p_\ell) \in \prod_{k=1}^\ell \overline{Bs_{i_k}B}$. Then $Z_{\underline{w}}$ is smooth and it has an iterated $\mathbb{C}P^1$ -bundle structure because $\overline{Bs_{i_k}B}/B \cong \mathbb{C}P^1$. Moreover, the left multiplication of B on $\overline{Bs_{i_1}B}$ induces an action of B on $Z_{\underline{w}}$ and we have a B -equivariant map

$$p_{\underline{w}}: Z_{\underline{w}} \rightarrow G/B$$

defined by $p_{\underline{w}}(p_1, \dots, p_\ell) = p_1 \dots p_\ell B$. If \underline{w} is a reduced decomposition of w , then $p_{\underline{w}}$ gives a resolution of singularities for X_w . See [2, 8] for details.

A Bott–Samelson variety $Z_{\underline{w}}$ is not a toric variety in general, but it is diffeomorphic to a toric variety, called a Bott manifold.

Definition 2.1. [13] A *Bott tower* is an iterated $\mathbb{C}P^1$ -bundle:

$$\begin{array}{ccccccc} \mathcal{B}_\ell & \xrightarrow{\pi_\ell} & \mathcal{B}_{\ell-1} & \xrightarrow{\pi_{\ell-1}} & \dots & \xrightarrow{\pi_2} & \mathcal{B}_1 & \xrightarrow{\pi_1} & \mathcal{B}_0, \\ \parallel & & & & & & \parallel & & \parallel \\ P(\underline{\mathbb{C}} \oplus \xi_\ell) & & & & & & \mathbb{C}P^1 & & \{\text{a point}\} \end{array}$$

where each \mathcal{B}_k is the complex projectivization of the Whitney sum of a holomorphic line bundle ξ_k over \mathcal{B}_{k-1} and the trivial bundle $\underline{\mathbb{C}}$. Each \mathcal{B}_k is called a *Bott manifold* (of height k).

Let γ_j be the tautological line bundle over \mathcal{B}_j and $\gamma_{j,i}$ the pullback of γ_j by the projection $\pi_i \circ \dots \circ \pi_{j+1}: \mathcal{B}_i \rightarrow \mathcal{B}_j$ for $i > j$. For convenience, we define $\gamma_{j,j} = \gamma_j$. Then for each $k = 2, \dots, \ell$, there exist $a_{j,k} \in \mathbb{Z}$ for $1 \leq j < k$ such that

$$\xi_k = \bigotimes_{1 \leq j < k} \gamma_{j,k-1}^{\otimes a_{j,k}}$$

in Definition 2.1 since the Picard number of \mathcal{B}_k is k (cf. [14, Exercise II.7.9]). Each Bott tower is determined by the list of the integers $a_{j,k}$ ($1 \leq j < k \leq \ell$), and we visualize them as an upper triangular matrix

$$\begin{pmatrix} 0 & a_{1,2} & a_{1,3} & \dots & a_{1,\ell} \\ & 0 & a_{2,3} & \dots & a_{2,\ell} \\ & & \ddots & \ddots & \vdots \\ & & & 0 & a_{\ell-1,\ell} \\ & & & & 0 \end{pmatrix}.$$

Note that each Bott manifold \mathcal{B}_ℓ is a smooth projective toric variety whose fan is determined by the above matrix. The moment polytope of \mathcal{B}_ℓ is combinatorially equivalent to the ℓ -dimensional cube I^ℓ .

Theorem 2.2 ([13, Proposition 3.10]). *For $w \in \mathfrak{S}_n$, let $\underline{w} = (i_1, i_2, \dots, i_\ell)$ be a reduced decomposition. Then the Bott–Samelson variety $Z_{\underline{w}}$ is diffeomorphic to a Bott manifold \mathcal{B}_ℓ determined by the integers*

$$(2.3) \quad a_{j,k} = \langle \mathbf{e}_{i_j} - \mathbf{e}_{i_{j+1}}, \mathbf{e}_{i_k} - \mathbf{e}_{i_{k+1}} \rangle$$

for $1 \leq j < k \leq \ell$. Here, $\mathbf{e}_1, \dots, \mathbf{e}_{n+1}$ are the standard basis vectors in \mathbb{R}^{n+1} and $\langle \cdot, \cdot \rangle$ is the standard inner product in \mathbb{R}^{n+1} .

Note that most Schubert varieties are neither smooth nor toric. However toric Schubert varieties are smooth and they are Bott manifolds. For a toric Schubert variety X_w , every reduced decomposition of w consists of distinct letters, and hence for the associated Bott manifold the integers $a_{j,k}$ in (2.3) are either 0 or -1 .

3. PATTERN AVOIDANCE AND THE COMPLEXITY

In this section, we define the complexity of a Schubert variety using the notion of complexity of a torus action, and see the relation between the complexity of a Schubert variety and patterns of a permutation. We also show the equivalence among the first three statements in Theorems 1.2 and 1.3.

Let X be a smooth complex projective algebraic variety having an action of algebraic torus $\mathbb{T} = (\mathbb{C}^*)^n$. When the maximal \mathbb{T} -orbit in X has codimension k , we call the number k the complexity of the action.

Every Schubert variety X_w is a \mathbb{T} -invariant irreducible subvariety of $\mathcal{F}\ell(\mathbb{C}^n)$. Note that the \mathbb{T} -fixed point set of X_w is the set of coordinate flags uB for $u \leq w$. That is, there is a bijection between $(X_w)^\mathbb{T}$ and the Bruhat interval

$$[e, w] := \{v \in \mathfrak{S}_n \mid v \leq w\}.$$
¹

Using the Plücker embedding, we get a moment map $\mu: \mathcal{F}\ell(\mathbb{C}^n) \rightarrow \mathbb{R}^n$ which sends

$$uB \mapsto (u^{-1}(1), \dots, u^{-1}(n)),$$

and the image $\mu(\mathcal{F}\ell(\mathbb{C}^n))$ is a simple convex polytope

$$\text{Perm}_{n-1} := \text{Conv}\{(w(1), \dots, w(n)) \in \mathbb{R}^n \mid w \in \mathfrak{S}_n\},$$

called the *permutohedron*.

The notion of Bruhat interval polytopes was introduced in [25] as a generalization of the notion of permutohedra. For two elements v and w in \mathfrak{S}_n with $v \leq w$ in Bruhat order, the Bruhat interval polytope $Q_{v,w}$ is the convex polytope given by the convex hull of the points $(u(1), \dots, u(n)) \in \mathbb{R}^n$ for $v \leq u \leq w$. Then for a Schubert variety X_w , the moment map image $\mu(X_w)$ becomes the Bruhat interval polytope

$$Q_{e,w^{-1}} := \text{Conv}\{(u(1), \dots, u(n)) \in \mathbb{R}^n \mid u \leq w^{-1}\},$$

and moreover, the images $\mu(uB)$ for all $u \leq w$ are the vertices of the Bruhat interval polytope $Q_{e,w^{-1}}$.

Proposition 3.1 (cf. [25, Proposition 6.20] and references therein). *The vertices of $Q_{e,w^{-1}}$ are the points $(u(1), \dots, u(n)) \in \mathbb{R}^n$ for all $u \leq w^{-1}$.*

We refer the readers to [21] for more details on moment maps and the correspondence between Schubert varieties and Bruhat interval polytopes.

¹For v, w in \mathfrak{S}_n with $v < w$ in Bruhat order, the Bruhat interval $[v, w]$ is the subposet of $(\mathfrak{S}_n, <)$ consisting of all the permutations u with $v \leq u \leq w$, and the Bruhat interval $[e, w]$ is also known as the principal order ideal of w , see [23].

Remark 3.2 (cf. [21, Remark 4.4]). For every $w \in \mathfrak{S}_n$ with $n \leq 4$, $Q_{v,w}$ and $Q_{v^{-1},w^{-1}}$ are combinatorially equivalent. However, for $w = 35412$ in \mathfrak{S}_5 , the Bruhat interval polytopes $Q_{e,w}$ and $Q_{e,w^{-1}}$ are not combinatorially equivalent. In fact, their f -vectors are different:

$$f(Q_{e,35412}) = (60, 123, 82, 19, 1) \text{ and } f(Q_{e,45132}) = (60, 122, 81, 19, 1),$$

so $Q_{e,w}$ has one more edge than $Q_{e,w^{-1}}$. These vectors are computed by a computer program SageMath.

Note that the complex dimension of a maximal \mathbb{T} -orbit in X_w is the same as the real dimension of the moment map image $\mu(X_w)$. Hence we can define the *complexity* of X_w as follows:

$$c(w) = \dim_{\mathbb{C}} X_w - \dim_{\mathbb{R}} Q_{e,w^{-1}} = \ell(w) - \dim_{\mathbb{R}} Q_{e,w^{-1}}.$$

See [21, Section 6] for more details. For example, $c(3142) = 3 - 3 = 0$ and $c(4132) = 4 - 3 = 1$, see Figure 2.

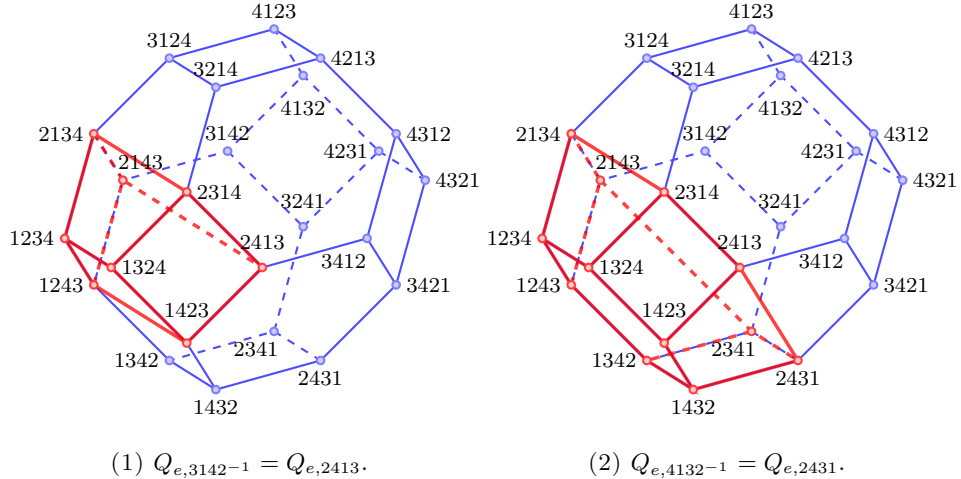


FIGURE 2. Moment map images of X_{3142} and X_{4132} .

For a permutation $w \in \mathfrak{S}_n$, the dimension of the polytope $Q_{e,w}$ is related to a reduced decomposition of w . The *support* of w is the set of distinct letters appearing in a reduced decomposition of w , and we denote it by $\text{supp}(w)$. In fact, $\text{supp}(w)$ is the same as the set of atoms in the Bruhat interval $[e, w]$.

It follows from [20, Corollary 7.13] that the dimension of a Bruhat interval polytope $Q_{e,w}$ is determined by the number of edges emanating from the vertex $(1, 2, \dots, n)$, and moreover, by [20, Remark 7.5(5)], that number is the same as the cardinality of $\text{supp}(w)$. Indeed, the vertices connecting to the vertex $(1, 2, \dots, n)$ by an edge are presented by

$$(1, 2, \dots, i - 1, i + 1, i, i + 2, \dots, n)$$

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for all $s_i \in \text{supp}(w)$. Therefore, the complexity of X_w is determined by the support of w :

$$c(w) = \ell(w) - |\text{supp}(w)|.$$

It means that $c(w)$ equals the number of repeated letters in a reduced decomposition of w . In [24], Tenner denoted this quantity by $\text{rep}(w)$ and found a relation with patterns in w .

Definition 3.3. For $w \in \mathfrak{S}_n$ and $p \in \mathfrak{S}_k$ with $k \leq n$, we say that the permutation w contains the pattern p if there exists a sequence $1 \leq i_1 < \cdots < i_k \leq n$ such that $w(i_1) \cdots w(i_k)$ is in the same relative order as $p(1) \cdots p(k)$. If w does not contain p , then we say that w avoids p , or is p -avoiding.

For example, the permutation 4231 in \mathfrak{S}_4 has the pattern 321 twice. Let $[321; 3412](w)$ be the number of distinct 321- and 3412-patterns in a permutation w . Then we can interpret Theorem 2.17 in [24] in terms of the complexity of X_w .

Theorem 3.4. [24, Theorem 2.17] *For a permutation w in \mathfrak{S}_n , we have*

- (1) $c(w) = 0$ if and only if $[321; 3412](w) = 0$, and
- (2) $c(w) = 1$ if and only if $[321; 3412](w) = 1$.

Example 3.5. For $w \in \mathfrak{S}_4$, $[321; 3412](w) = 1$ if and only if w is one of the following permutations

$$1432, 3214, 4132, 4213, 2431, 3241, 3412.$$

The first six permutations contains the 321-pattern once, and the last one avoids it.

Recall that not every Schubert variety is smooth, and Lakshmibai and Sandhya characterized the smoothness of Schubert varieties in terms of pattern avoidance.

Theorem 3.6. [19] *For a permutation $w \in \mathfrak{S}_n$, the Schubert variety X_w is smooth if and only if w avoids the patterns 3412 and 4231.*

Combining Theorems 3.4 and 3.6, we obtain the following proposition. It shows the equivalence between the first two statements in Theorems 1.2 and 1.3.

Proposition 3.7. *For $w \in \mathfrak{S}_n$, the following hold:*

- (1) X_w is a toric variety if and only if w avoids the patterns both 321 and 3412;
- (2) X_w is smooth and of complexity one if and only if w has the pattern 321 exactly once and avoids the pattern 3412; and
- (3) X_w is singular and of complexity one if and only if w has the pattern 3412 exactly once and avoids the pattern 321.

In general, $c(w) \leq [321; 3412](w)$ and the equality holds only when w avoids every pattern in the set

$$\{4321, 34512, 45123, 35412, 43512, 45132, 45213, 53412, 45312, 45231\},$$

see [24, Theorem 3.2]. For example, the permutation 4321 has four distinct 321-patterns but $c(4321) = 3$. In the set above, every pattern except 4321 has the pattern 3412. Since w avoids the pattern 3412 when X_w is smooth, we get the following proposition.

Proposition 3.8. *For a smooth Schubert variety X_w , the complexity of X_w is less than or equal to the number of distinct 321-patterns in w , and the equality holds if and only if w avoids 4321.*

Since $c(w)$ equals the number of repeated letters in a reduced decomposition of w , Theorem 3.4(1) implies that a permutation $w \in \mathfrak{S}_n$ avoids both 321- and 3412-patterns if and only if every reduced decomposition of w consists of distinct letters. Daly characterized the reduced decomposition of the permutations satisfying $[321; 3412](w) = 1$ as follows.

Theorem 3.9. [7] *For a permutation $w \in \mathfrak{S}_n$, the following hold:*

- (1) *w contains exactly one 321 pattern and avoids 3412 if and only if there exists a reduced decomposition of w containing $s_i s_{i+1} s_i$ as a factor and no other repetitions.*
- (2) *w contains exactly one 3412 pattern and avoids 321 if and only if there exists a reduced decomposition of w containing $s_{i+1} s_{i+2} s_{i+1}$ as a factor and no other repetitions.*

The above theorem shows the equivalence between the second and the third statements in Theorems 1.2 and 1.3.

It is also shown in [6] that there is a one-to-one correspondence between

$$\{w \in \mathfrak{S}_n \mid w \text{ contains exactly one 321 pattern and avoids 3412}\}$$

and

$$\{w \in \mathfrak{S}_{n+1} \mid w \text{ contains exactly one 3412 pattern and avoids 321}\},$$

and the cardinality of the set is given in A001871 in OEIS [1]. Thus we obtain that

$$\begin{aligned} & \#\{X_w \subseteq \mathcal{F}\ell(\mathbb{C}^n) \mid X_w \text{ is smooth and of complexity one}\} \\ &= \#\{X_{w'} \subseteq \mathcal{F}\ell(\mathbb{C}^{n+1}) \mid X_{w'} \text{ is singular and of complexity one}\}. \end{aligned}$$

4. FLAG BOTT–SAMELSON VARIETIES AND GENERALIZED BOTT–SAMELSON VARIETIES

In this section, we recall flag Bott–Samelson varieties from [11] and generalized Bott–Samelson varieties from [15].

Let $G = \mathrm{GL}_n(\mathbb{C})$. For a subset $I \subseteq [n] := \{1, \dots, n\}$, we define the subgroup W_I of W by

$$W_I := \langle s_i \mid i \in I \rangle.$$

Then the *parabolic subgroup* P_I of G corresponding to I is defined by

$$P_I = \overline{B w_I B} \subseteq G,$$

where w_I is the longest element in W_I .

Definition 4.1 ([11, Definition 2.1]). Let $\mathcal{I} = (I_1, \dots, I_r)$ be a sequence of subsets of $[n]$. The *flag Bott–Samelson variety* $Z_{\mathcal{I}}$ is defined by the orbit space

$$Z_{\mathcal{I}} := (P_{I_1} \times \cdots \times P_{I_r}) / \Theta,$$

where the right action Θ of $B^r := \underbrace{B \times \cdots \times B}_r$ on $\prod_{k=1}^r P_{I_k}$ is defined by

$$(4.1) \quad \Theta((p_1, \dots, p_r), (b_1, \dots, b_r)) = (p_1 b_1, b_1^{-1} p_2 b_2, \dots, b_{r-1}^{-1} p_r b_r)$$

for $(p_1, \dots, p_r) \in \prod_{k=1}^r P_{I_k}$ and $(b_1, \dots, b_r) \in B^r$.

Note that $P_{I_k}/B \cong \mathcal{F}\ell(\mathbb{C}^{|I_k|})$ for $k = 1, \dots, r$. Hence if $|I_1| = \dots = |I_r| = 1$, then $Z_{\mathcal{I}}$ becomes a Bott–Samelson variety. Each Bott–Samelson variety can be described as an iterated $\mathbb{C}P^1$ -bundle, whereas each flag Bott–Samelson variety can be described as an iterated bundle whose fiber at each stage is a flag manifold.

We recall properties of flag Bott–Samelson varieties from [11, Proposition 2.3]. The flag Bott–Samelson variety $Z_{\mathcal{I}}$ is a smooth projective variety and admits a nice decomposition of affine cells. For $(w_1, \dots, w_r) \in \prod_{k=1}^r W_{I_k}$, we define $Z'_{(w_1, \dots, w_r)}$ and $Z_{(w_1, \dots, w_r)}$ in $Z_{\mathcal{I}}$ by

$$\begin{aligned} Z'_{(w_1, \dots, w_r)} &:= (Bw_1B \times \dots \times Bw_rB)/\Theta, \\ Z_{(w_1, \dots, w_r)} &:= (\overline{Bw_1B} \times \dots \times \overline{Bw_rB})/\Theta. \end{aligned}$$

We call $Z_{(w_1, \dots, w_r)}$ a *generalized Bott–Samelson variety*, see [15] and also [4, 22]. Then $Z'_{(w_1, \dots, w_r)}$ is an open dense subset of the generalized Bott–Samelson variety $Z_{(w_1, \dots, w_r)}$ and we have that

$$Z'_{(w_1, \dots, w_r)} \simeq \mathbb{C}^{\sum_{k=1}^r \ell(w_k)}.$$

For $w_k \in W_{I_k}$, since $\overline{Bw_kB} = \bigsqcup_{\substack{v \in W_{I_k} \\ v \leq w_k}} BvB$, we have that

$$(4.2) \quad Z_{(w_1, \dots, w_r)} = \bigsqcup_{\substack{(v_1, \dots, v_r) \in \prod_{k=1}^r W_{I_k} \\ v_k \leq w_k \text{ for } k=1, \dots, r}} Z'_{(v_1, \dots, v_r)}.$$

Note that $Z_{\mathcal{I}} = Z_{(w_{I_1}, \dots, w_{I_r})}$, where $\mathcal{I} = (I_1, \dots, I_r)$ and w_{I_k} is the longest element in W_{I_k} for $k = 1, \dots, r$.

Example 4.2. Let $\mathcal{I} = (\{1, 2\}, \{3\})$. Then we have that

$$\begin{aligned} Z_{\mathcal{I}} &= Z'_{(e, e)} \sqcup Z'_{(s_1, e)} \sqcup Z'_{(s_2, e)} \sqcup Z'_{(s_1 s_2, e)} \sqcup Z'_{(s_2 s_1, e)} \sqcup Z'_{(s_1 s_2 s_1, e)} \\ &\quad \sqcup Z'_{(e, s_3)} \sqcup Z'_{(s_1, s_3)} \sqcup Z'_{(s_2, s_3)} \sqcup Z'_{(s_1 s_2, s_3)} \sqcup Z'_{(s_2 s_1, s_3)} \sqcup Z'_{(s_1 s_2 s_1, s_3)}. \end{aligned}$$

Each $Z'_{(w_1, w_2)}$ is isomorphic to an affine cell as follows:

$$\begin{aligned} Z'_{(e, e)} &\simeq \mathbb{C}^0, & Z'_{(s_1, e)} &\simeq Z'_{(s_2, e)} \simeq Z'_{(e, s_3)} \simeq \mathbb{C}^1, \\ Z'_{(s_1 s_2, e)} &\simeq Z'_{(s_2 s_1, e)} \simeq Z'_{(s_1, s_3)} \simeq Z'_{(s_2, s_3)} \simeq \mathbb{C}^2, \\ Z'_{(s_1 s_2 s_1, e)} &\simeq Z'_{(s_1 s_2, s_3)} \simeq Z'_{(s_2 s_1, s_3)} \simeq \mathbb{C}^3, & Z'_{(s_1 s_2 s_1, s_3)} &\simeq \mathbb{C}^4. \end{aligned}$$

The multiplication map

$$p_{\mathcal{I}}: Z_{\mathcal{I}} \rightarrow G/B, \quad [p_1, \dots, p_r] \mapsto p_1 \cdots p_r B$$

is well-defined because of the definition of the right action Θ . The maximal torus \mathbb{T} acts on a generalized Bott–Samelson variety $Z_{(w_1, \dots, w_r)}$ by

$$t \cdot [g_1, \dots, g_r] = [tg_1, g_2, \dots, g_r].$$

We note that the multiplication map $p_{\mathcal{I}}$ is \mathbb{T} -equivariant.

Proposition 4.3 ([11, Proposition 2.7]). *Let $(w_1, \dots, w_r) \in \prod_{k=1}^r W_{I_k}$. Suppose that $w_1 \cdots w_r = v$ and $\ell(w_1) + \dots + \ell(w_r) = \ell(v)$. Then the multiplication map $p_{\mathcal{I}}$ induces a birational morphism:*

$$p_{\mathcal{I}}|_{Z_{(w_1, \dots, w_r)}}: Z_{(w_1, \dots, w_r)} \rightarrow X_v.$$

Indeed, we have an isomorphism between dense open subsets:

$$Z'_{(w_1, \dots, w_r)} \xrightarrow{\sim} BvB/B.$$

Example 4.4. Let $G = \mathrm{GL}_3(\mathbb{C})$, and let $\mathcal{I} = (\{1\}, \{2\}, \{1\})$. Then the flag Bott–Samelson variety $Z_{\mathcal{I}}$ has the decomposition:

$$\begin{aligned} Z_{\mathcal{I}} = & Z'_{(e,e,e)} \sqcup Z'_{(s_1,e,e)} \sqcup Z'_{(e,s_2,e)} \sqcup Z'_{(s_1,s_2,e)} \\ & \sqcup Z'_{(e,e,s_1)} \sqcup Z'_{(s_1,e,s_1)} \sqcup Z'_{(e,s_2,s_1)} \sqcup Z'_{(s_1,s_2,s_1)}. \end{aligned}$$

By Proposition 4.3, the multiplication map $p_{\mathcal{I}}$ induces the isomorphism $Z'_{(w_1,w_2,w_3)} \cong Bw_1w_2w_3B/B$ except for $(w_1, w_2, w_3) = (s_1, e, s_1)$. Moreover, one can see that the multiplication map $p_{\mathcal{I}}$ is injective on $Z_{\mathcal{I}} \setminus (Z'_{(s_1,e,s_1)} \sqcup Z'_{(s_1,e,e)} \sqcup Z'_{(e,e,s_1)})$.

Corollary 4.5. *Suppose that I_1, \dots, I_r are pairwise disjoint subsets of $[n]$. Then the multiplication map $p_{\mathcal{I}}$ with $\mathcal{I} = (I_1, \dots, I_r)$ induces an isomorphism between $Z_{(w_1, \dots, w_r)}$ and its image $X_{w_1 \dots w_r}$ as \mathbb{T} -varieties, where w_k is an element in W_{I_k} .*

Proof. We first note that $X_{w_1 \dots w_r} = \bigsqcup_{v \leq w_1 \dots w_r} BvB/B$. Since I_1, \dots, I_r are pairwise disjoint and $w_k \in W_{I_k}$, we have that

$$v \leq w_1 \dots w_r \iff v = v_1 \dots v_r, \quad \text{and } v_k \leq w_k \text{ in } W_{I_k}.$$

Therefore, we get that

$$(4.3) \quad X_{w_1 \dots w_r} = \bigsqcup_{\substack{(v_1, \dots, v_r) \in \prod_{k=1}^r W_{I_k} \\ v_k \leq w_k \text{ for } k=1, \dots, r}} Bv_1 \dots v_r B/B.$$

This implies that the Schubert variety $X_{w_1 \dots w_r}$ can be decomposed into affine cells indexed by elements in $\prod_{k=1}^r [e, w_k]$, where $[e, w_k]$ is a Bruhat interval in W_{I_k} . Moreover, since I_1, \dots, I_r are pairwise disjoint, we know that $\ell(v_1 \dots v_r) = \ell(v_1) + \dots + \ell(v_r)$. Hence by Proposition 4.3, the multiplication map $p_{\mathcal{I}}$ induces a \mathbb{T} -equivariant isomorphism

$$Z'_{(v_1, \dots, v_r)} \cong Bv_1 \dots v_r B/B$$

for each $(v_1, \dots, v_r) \in \prod_{k=1}^r W_{I_k}$. Hence by combining the above isomorphism with equations (4.2) and (4.3), the result follows. \square

Example 4.6. Continuing Example 4.2, the multiplication map $p_{\mathcal{I}}$ induces the isomorphisms:

$$\begin{aligned} Z'_{(e,e)} &\cong BeB/B, & Z'_{(s_1,e)} &\cong Bs_1B/B, \\ Z'_{(s_2,e)} &\cong Bs_2B/B, & Z'_{(s_1s_2,e)} &\cong Bs_1s_2B/B, \\ Z'_{(s_2s_1,e)} &\cong Bs_2s_1B/B, & Z'_{(s_1s_2s_1,e)} &\cong Bs_1s_2s_1B/B, \\ Z'_{(e,s_3)} &\cong Bs_3B/B, & Z'_{(s_1,s_3)} &\cong Bs_1s_3B/B, \\ Z'_{(s_2,s_3)} &\cong Bs_2s_3B/B, & Z'_{(s_1s_2,s_3)} &\cong Bs_1s_2s_3B/B, \\ Z'_{(s_2s_1,s_3)} &\cong Bs_2s_1s_3B/B, & Z'_{(s_1s_2s_1,s_3)} &\cong Bs_1s_2s_1s_3B/B \end{aligned}$$

Since we have the decomposition $X_w = \bigsqcup_{v \leq w} BvB/B$, we get an isomorphism $Z_{\mathcal{I}} = Z_{(s_1s_2s_1,s_3)} \cong X_{s_1s_2s_1s_3}$. Indeed, $\{1, 2\} \cap \{3\} = \emptyset$.

Theorem 4.7. *Every Schubert variety X_w of complexity one is \mathbb{T} -equivariantly isomorphic to a generalized Bott–Samelson variety. If X_w is of complexity one and smooth, then it is \mathbb{T} -equivariantly isomorphic to a flag Bott–Samelson variety.*

Proof. By Proposition 3.7 and Theorem 3.9, if X_w is smooth, then there exists a reduced decomposition $\underline{w} = s_{i_1} \dots s_{i_\ell}$ of w containing $s_i s_{i+1} s_i$ as a factor and no other repetitions. On the other hand, if X_w is singular, then there exists a reduced

decomposition $\underline{w} = s_{i_1} \cdots s_{i_\ell}$ of w containing $s_{i+1}s_i s_{i+2}s_{i+1}$ as a factor and no other repetitions. Hence there is a reduced decomposition \underline{w} such that

- (1) if X_w is smooth, then $(i_q, i_{q+1}, i_{q+2}) = (i, i+1, i)$ for some $1 \leq q \leq \ell - 2$; and
- (2) if X_w is singular, then $(i_q, i_{q+1}, i_{q+2}, i_{q+3}) = (i+1, i, i+2, i+1)$ for some $1 \leq q \leq \ell - 3$.

Now we define a sequence $\mathcal{I} = (I_1, \dots, I_r)$ as follows:

- (1) If X_w is smooth, then $r = \ell - 2$ and we set

$$I_k = \begin{cases} \{i_k\} & \text{if } 1 \leq k < q, \\ \{i, i+1\} & \text{if } k = q, \\ \{i_{k+2}\} & \text{if } k > q. \end{cases}$$

- (2) If X_w is singular, then $r = \ell - 3$ and we set

$$I_k = \begin{cases} \{i_k\} & \text{if } 1 \leq k < q, \\ \{i, i+1, i+2\} & \text{if } k = q, \\ \{i_{k+3}\} & \text{if } k > q. \end{cases}$$

Then, in any case, the subsets I_1, \dots, I_r are pairwise disjoint. Moreover, if X_w is smooth, then the concatenation of longest elements in W_{I_k} is the same as w . Hence by Corollary 4.5, the Schubert variety X_w is \mathbb{T} -equivariantly isomorphic to the flag Bott–Samelson variety $Z_{\mathcal{I}}$.

If X_w is singular, then we set

$$w_k = \begin{cases} s_{i_k} & \text{if } 1 \leq k < q, \\ s_{i+1}s_i s_{i+2}s_{i+1} & \text{if } k = q, \\ s_{i_{k+3}} & \text{if } q < k \leq r. \end{cases}$$

Then $(w_1, \dots, w_r) \in \prod_{k=1}^r W_{I_k}$ and $w = w_1 \dots w_r$. Hence by Corollary 4.5, X_w is \mathbb{T} -equivariantly isomorphic to the generalized Bott–Samelson variety $Z_{(w_1, \dots, w_r)}$. \square

The above theorem proves the implications (1') \Rightarrow (4') and (1'') \Rightarrow (4'') in Theorems 1.2 and 1.3, respectively.

The following proposition can be checked easily and we omit the proof.

Proposition 4.8. *There is a natural bijection between the set of \mathbb{T} -fixed points in $Z_{(w_1, \dots, w_r)}$ and the product of Bruhat intervals*

$$[e, w_1] \times \cdots \times [e, w_r].$$

If a Schubert variety X_w and a generalized Bott–Samelson variety $Z_{(w_1, \dots, w_r)}$ are isomorphic as \mathbb{T} -varieties, then they have the same set of \mathbb{T} -fixed points. Hence we get the following.

Corollary 4.9. *If a Schubert variety X_w is isomorphic to a generalized Bott–Samelson variety $Z_{(w_1, \dots, w_r)}$ as \mathbb{T} -varieties, then the Bruhat interval $[e, w]$ is isomorphic to the product of Bruhat intervals $\prod_{k=1}^r [e, w_k]$.*

The above corollary shows the implications (4') \Rightarrow (5') and (4'') \Rightarrow (5'') in Theorems 1.2 and 1.3, respectively.

5. BRUHAT INTERVALS AND BRUHAT INTERVAL POLYTOPES

In this section, we study the properties of the Bruhat interval $[e, w]$ and the Bruhat interval polytope $Q_{e,w}$ for a permutation w with $c(w) \leq 1$. We will complete proofs of Theorems 1.2 and 1.3.

Recall that the following seven permutations in \mathfrak{S}_4

$$1432, 3214, 4132, 4213, 2431, 3241, 3412$$

satisfy $[321; 3412](w) = 1$ (in Example 3.5). One can easily check that

- If w is either 1432 or 3214, then the Bruhat interval $[e, w]$ is isomorphic to \mathfrak{S}_3 and the Bruhat interval polytope $Q_{e,w}$ is combinatorially equivalent to the hexagon Perm_2 .
- If w is one of the permutations 4132, 4213, 2431 and 3241, then the Bruhat interval $[e, w]$ is isomorphic to $\mathfrak{S}_3 \times \mathfrak{B}_1$ and the Bruhat interval polytope $Q_{e,w}$ is combinatorially equivalent to the hexagonal prism, $\text{Perm}_2 \times I$.
- If $w = 3412$, then the Bruhat interval $[e, w]$ and $Q_{e,w}$ are given in Figure 3. Thus $[e, w]$ is isomorphic to neither $\mathfrak{S}_3 \times \mathfrak{B}_1$ nor \mathfrak{B}_4 , and $Q_{e,w}$ is combinatorially equivalent to neither the hexagonal prism nor a 4-cube.

Here, \mathfrak{B}_ℓ denotes the Boolean algebra of length ℓ . The above situation happens in general and we have the following lemma that is obvious but plays a role in our argument.

Lemma 5.1. *For any positive integer i , the following hold:*

- (1) *if $w = s_i s_{i+1} s_i$, then $[e, w]$ is isomorphic to \mathfrak{S}_3 as a poset, and $Q_{e,w}$ is combinatorially equivalent to the hexagon Perm_2 ; and*
- (2) *if $w = s_{i+1} s_i s_{i+2} s_{i+1}$, then $[e, w]$ is isomorphic to $[e, 3412]$ as a poset, and $Q_{e,w}$ is combinatorially equivalent to $Q_{e,3412}$.*

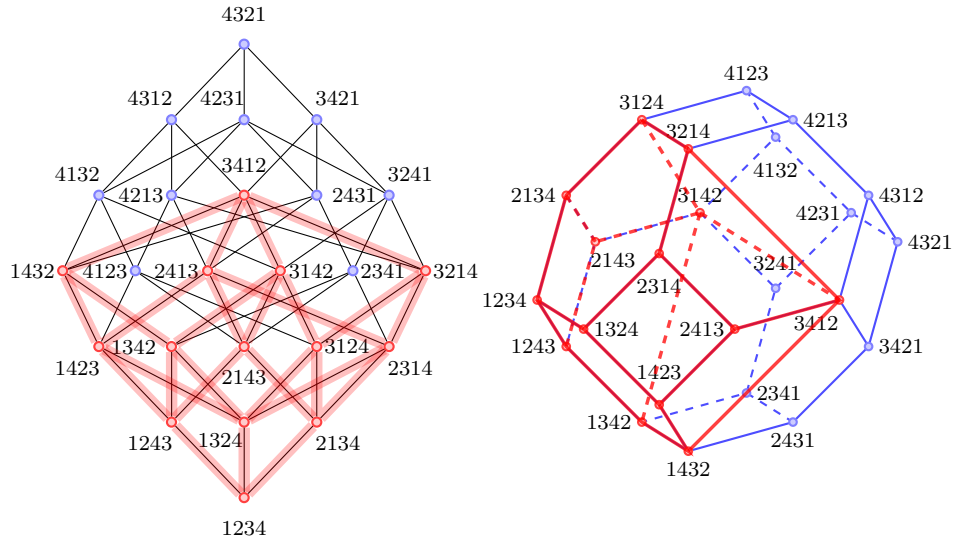


FIGURE 3. The Bruhat interval $[e, 3412]$ and the Bruhat interval polytope $Q_{e,3412}$.

Therefore, if $w = s_{i+1}s_i s_{i+2}s_{i+1}$ for some positive integer i , then $[e, w]$ is isomorphic to neither $\mathfrak{S}_3 \times \mathfrak{B}_1$ nor \mathfrak{B}_4 , and $Q_{e,w}$ is combinatorially equivalent to neither the hexagonal prism nor a 4-cube.

Proposition 5.2. *For a permutation w in \mathfrak{S}_n , the following hold:*

- (1) *If the Bruhat interval $[e, w]$ is isomorphic to $\mathfrak{S}_3 \times \mathfrak{B}_{\ell(w)-3}$ (as a poset), then X_w is smooth and of complexity one.*
- (2) *If the Bruhat interval $[e, w]$ is isomorphic to $[e, 3412] \times \mathfrak{B}_{\ell(w)-4}$ (as a poset), then X_w is singular and of complexity one.*

Note that the converse of each statement in the above proposition is also true by Theorem 4.7 and Corollary 4.9.

Proof of Proposition 5.2. Let us prove the first statement. Assume that the Bruhat interval $[e, w]$ is isomorphic to a poset $\mathfrak{S}_3 \times \mathfrak{B}_{\ell-3}$. Note that

- (1) $\mathfrak{S}_3 \times \mathfrak{B}_{\ell-3}$ has $\ell - 1$ atoms;
- (2) $\mathfrak{S}_3 \times \mathfrak{B}_{\ell-3}$ has rank ℓ ; and
- (3) every subinterval of $\mathfrak{S}_3 \times \mathfrak{B}_{\ell-3}$ is isomorphic to $\mathfrak{S}_3 \times \mathfrak{B}_k$ ($k < \ell - 3$) or $\mathfrak{B}_{k'}$ ($k' \leq \ell - 1$).

From (1) and (2) above, we get $c(w) = 1$. If X_w is not smooth, then w contains 3412 exactly once and avoids 321. Then by Theorem 3.9, there is a reduced decomposition \underline{w} of w such that

$$\underline{w} = s_{i_1} \cdots s_{i_{k-1}} s_{i_k+1} s_{i_k} s_{i_k+2} s_{i_k+1} s_{i_k+3} \cdots s_{i_{\ell-1}} \text{ and } i_{k+1} = i_k + 1, i_{k+2} = i_k + 2.$$

However, the interval $[e, s_{i_k+1}s_{i_k}s_{i_k+2}s_{i_k+1}]$ is a subinterval of neither $\mathfrak{S}_3 \times \mathfrak{B}_{\ell-3}$ nor \mathfrak{B}_ℓ by (3). Therefore, X_w is smooth.

Now we prove the second statement. Note that $[e, 3412] \times \mathfrak{B}^{\ell-4}$ and $\mathfrak{S}_3 \times \mathfrak{B}^{\ell-3}$. Since we already showed that X_w is smooth and $c(w) = 1$ if and only if $[e, w]$ is isomorphic to $\mathfrak{S}_3 \times \mathfrak{B}_{\ell-3}$ in the above, it is enough to show that $c(w) = 1$. Since the poset $[e, 3412] \times \mathfrak{B}_{\ell-4}$ has $\ell - 1$ atoms and is of rank ℓ , if $[e, w]$ is isomorphic to $[e, 3412] \times \mathfrak{B}_{\ell-4}$, then we get $c(w) = 1$. This proves the proposition. \square

Therefore, the first five statements in Theorems 1.2 and 1.3 are equivalent.

Now we determine the combinatorial type of the Bruhat interval polytope $Q_{e,w}$ when $c(w) = 1$. The combinatorial aspects of Bruhat interval polytopes are well-studied in [25]. Every face of a Bruhat interval polytope is itself a Bruhat interval polytope. However, for a subinterval $[x, y]$ of an interval $[v, w]$, $Q_{x,y}$ may not be a face of $Q_{v,w}$. For a subinterval $[x, y]$ of $[v, w]$, we introduce a directed graph $G_{x,y}^{v,w}$ which will be used to determine whether $Q_{x,y}$ is a face of $Q_{v,w}$.

Let $v \leq w$ in \mathfrak{S}_n . For $u \in [v, w]$, we define

$$\begin{aligned} \overline{T}(u, [v, w]) &:= \{(i, j) \in T \mid u < u(i, j) \leq w, \ell(u(i, j)) - \ell(u) = 1\}, \\ \underline{T}(u, [v, w]) &:= \{(i, j) \in T \mid v \leq u(i, j) < u, \ell(u) - \ell(u(i, j)) = 1\}, \end{aligned}$$

where T is the set of transpositions in \mathfrak{S}_n , see (2.1). We first construct a labelled graph $G_{x,y}$ on $[n] = \{1, \dots, n\}$ having an edge between the vertices a and b if and only if $(a, b) \in \overline{T}(x, [x, y])$.

Theorem 5.3. [25] *Let $[x, y] \subseteq [v, w]$. We define the graph $G_{x,y}^{v,w}$ as follows:*

- (1) *The vertices of $G_{x,y}^{v,w}$ are $\{1, 2, \dots, n\}$, with vertices i and j identified if they are in the same connected component of the graph $G_{x,y}$.*

- (2) There is a directed edge $i \rightarrow j$ if $(i, j) \in \overline{T}(y, [v, w])$.
(3) There is a directed edge $j \rightarrow i$ if $(i, j) \in \underline{T}(x, [v, w])$.

Then the Bruhat interval polytope $Q_{x,y}$ is a face of the Bruhat interval polytope $Q_{v,w}$ if and only if the graph $G_{x,y}^{v,w}$ is a directed acyclic graph.

Remark 5.4. The above theorem used Lemma 4.18 in [25]. Unfortunately, the proof of the lemma had a gap. Recently, the gap has been corrected by Caselli, D'Adderio and Marietti, see [5, Remarks 5.5].

Example 5.5. Let $w = 1432$, $x = 1324$, and $y = 1342$. Then we have $\overline{T}(x, [x, y]) = \{(3, 4)\}$, $\overline{T}(y, [e, w]) = \{(2, 3)\}$, and $\underline{T}(x, [e, w]) = \{(2, 3)\}$. Hence $G_{x,y}^{e,w}$ is not acyclic, and thus $Q_{x,y}$ is not a face of $Q_{e,w}$. See Figure 4.



FIGURE 4. Examples of $G_{x,y}$ and $G_{x,y}^{v,w}$.

We will use the following fact. Let $a, b \in \mathfrak{S}_n$ which do not have s_r in their supports. Then we have that

$$(5.1) \quad a \leq b \iff s_r a \leq s_r b \iff a s_r \leq b s_r.$$

In particular,

$$(5.2) \quad \ell(b) - \ell(a) = \ell(s_r b) - \ell(s_r a) = \ell(b s_r) - \ell(a s_r).$$

Hence

$$(5.3) \quad \overline{T}(a, [a, b]) = \overline{T}(s_r a, [s_r a, s_r b]).$$

For $\widehat{w} = s_r w$ or ws_r , in order to see which subinterval of $[v, \widehat{w}]$ gives a face of $Q_{v, \widehat{w}}$, we prepare the following lemma.

Lemma 5.6. Let $u, v, w \in \mathfrak{S}_n$ with $v \leq u \leq w$ and $s_r \notin \text{supp}(w)$. Then

- (1) $\underline{T}(u, [v, s_r w]) = \underline{T}(u, [v, w])$ and $\underline{T}(u, [v, ws_r]) = \underline{T}(u, [v, w])$;
- (2) $\overline{T}(u, [v, s_r w]) = \overline{T}(u, [v, w]) \cup \{(u^{-1}(r), u^{-1}(r+1))\}$;
- (3) $\overline{T}(u, [v, ws_r]) = \overline{T}(u, [v, w]) \cup \{(r, r+1)\}$;
- (4) $\overline{T}(s_r u, [v, s_r w]) = \overline{T}(u, [v, w])$;
- (5) $\underline{T}(s_r u, [v, s_r w]) = \underline{T}(u, [v, w]) \cup \{(u^{-1}(r), u^{-1}(r+1))\}$;
- (6) $\overline{T}(us_r, [v, ws_r]) = \{(s_r(i), s_r(j)) \mid (i, j) \in \overline{T}(u, [v, w])\}$; and
- (7) $\underline{T}(us_r, [v, ws_r]) = \{(s_r(i), s_r(j)) \mid (i, j) \in \underline{T}(u, [v, w])\} \cup \{(r, r+1)\}$.

Proof. The first statement is clear from the definition of $\underline{T}(u, [v, w])$ and the assumption $s_r \notin \text{supp}(w)$. In (2)~(5), the inclusion (\supseteq) is clear and hence it is enough to check the inclusion (\subseteq).

In (2) and (3), for $(i, j) \in \overline{T}(u, [v, s_r w])$ (respectively, $(i, j) \in \overline{T}(u, [v, ws_r])$), if $s_r \in \text{supp}(u(i, j))$, then $u(i, j) = s_r u = u(u^{-1}(r), u^{-1}(r+1))$ (respectively, $u(i, j) = us_r = u(r, r+1)$); otherwise, $(i, j) \in \overline{T}(u, [v, w])$. This proves (2) and (3).

In (4) and (5), we use (5.1) and (5.2). For $(i, j) \in \overline{T}(s_r u, [v, s_r w])$ in (4), since $s_r u < s_r u(i, j) \leq s_r w$ and $\ell(s_r u(i, j)) - \ell(s_r u) = 1$, we have $u < u(i, j) \leq w$ and $\ell(u(i, j)) - \ell(u) = 1$. This proves that $(i, j) \in \overline{T}(u, [v, w])$. In (5), if $(i, j) \in \underline{T}(s_r u, [v, s_r w])$, then $s_r u(i, j) = u$ or $s_r v \leq s_r u(i, j) < s_r u$. Note that $s_r u(u^{-1}(r), u^{-1}(r+1)) = u$. Thus if $s_r u(i, j) = u$, then $(i, j) = (u^{-1}(r), u^{-1}(r+1))$; otherwise, $(i, j) \in \underline{T}(u, [v, w])$.

Let us prove (6). It follows from (5.1) and (5.2) that there exists $z \in [us_r, ws_r]$ such that $\ell(z) - \ell(us_r) = 1$ if and only if there exists $z' \in [u, w]$ such that $z = z' s_r$ and $\ell(z') - \ell(u) = 1$. Here, z' is of the form $z' = u(i, j)$ for some $(i, j) \in \overline{T}(u, [v, w])$. Since $z = u(i, j) s_r = us_r(s_r(i), s_r(j))$, we get

$$\overline{T}(us_r, [v, ws_r]) \supseteq \{(s_r(i), s_r(j)) \mid (i, j) \in \overline{T}(u, [v, w])\}.$$

On the other hand, if $z' s_r = us_r(i', j')$ for $(i', j') \in \overline{T}(us_r, [v, ws_r])$, then $z' = us_r(i', j') s_r = u(s_r(i'), s_r(j'))$ and $z' \leq w$. Hence there exists $(i, j) \in \overline{T}(u, [v, w])$ such that $(i', j') = (s_r(i), s_r(j))$. This proves (6).

Finally, we prove (7). If $(i, j) \in \underline{T}(us_r, [v, ws_r])$, then $us_r(i, j) = u$ or $vs_r \leq us_r(i, j) < us_r$. If $us_r(i, j) = u$, then $(i, j) = (r, r+1)$. If $vs_r \leq us_r(i, j) < us_r$, then there exists $u' \in [v, w]$ such that $us_r(i, j) = u' s_r$ and $u' < u$. Then $u' = us_r(i, j) s_r = u(s_r(i), s_r(j))$. Note that

$$\ell(u) - \ell(u') = \ell(us_r) - \ell(u' s_r) = \ell(us_r) - \ell(us_r(i, j)) = 1.$$

Hence $(s_r(i), s_r(j)) \in \underline{T}(u, [v, w])$. Thus, we get

$$\underline{T}(us_r, [v, ws_r]) \subseteq \{(s_r(i), s_r(j)) \mid (i, j) \in \underline{T}(u, [v, w])\} \cup \{(r, r+1)\}.$$

On the other hand, $(r, r+1) \in \underline{T}(us_r, [v, ws_r])$ clearly. For $(i, j) \in \underline{T}(u, [v, w])$, since $v \leq u(i, j) < u$ and $\ell(u) - \ell(u(i, j)) = 1$, we get $vs_r \leq u(i, j) s_r < us_r$ and $\ell(us_r) - \ell(u(i, j) s_r) = 1$ by (5.1) and (5.2). Note that $u(i, j) s_r = us_r(s_r(i), s_r(j))$. Therefore, $(s_r(i), s_r(j)) \in \underline{T}(us_r, [v, ws_r])$. This proves (7). \square

Proposition 5.7. *For v, w in \mathfrak{S}_n with $v \leq w$, if $s_r \notin \text{supp}(w)$, then both $Q_{v, s_r w}$ and Q_{v, ws_r} are combinatorially equivalent to the polytope $Q_{v, w} \times I$.*

Proof. Note that for $u \in \mathfrak{S}_n$ with $u \leq s_r w$ (respectively, $u \leq ws_r$), if $s_r \in \text{supp}(u)$, then there is a unique u' such that $s_r u' = u$ (respectively, $u' s_r = u$) because $s_r \notin \text{supp}(w)$. We set

$$\tilde{u} = \begin{cases} u & \text{if } s_r \notin \text{supp}(u), \\ u' & \text{if } s_r \in \text{supp}(u) \end{cases} \quad \text{and} \quad \widehat{w} = s_r w \text{ or } ws_r.$$

Then what we have to prove is that for any $[x, y] \subseteq [v, \widehat{w}]$,

$$Q_{x, y} \text{ is a face of } Q_{v, \widehat{w}} \text{ if and only if } Q_{\tilde{x}, \tilde{y}} \text{ is a face of } Q_{v, w},$$

which is equivalent to

$$(*) \quad G_{x, y}^{v, \widehat{w}} \text{ is acyclic if and only if } G_{\tilde{x}, \tilde{y}}^{v, w} \text{ is acyclic}$$

by Theorem 5.3. We will prove $(*)$ in the following.

Note that for each $[x, y] \subseteq [v, \widehat{w}]$, there are three possibilities:

- (i) $(\tilde{x}, \tilde{y}) = (x, y)$, i.e., $s_r \notin \text{supp}(y)$,
- (ii) $(\tilde{x}, \tilde{y}) = (x', y')$, i.e., $s_r \in \text{supp}(x)$, and
- (iii) $(\tilde{x}, \tilde{y}) = (x, y')$, i.e., $s_r \notin \text{supp}(x)$ but $s_r \in \text{supp}(y)$.

Since $s_r \notin \text{supp}(w)$, the graph $G_{\tilde{x}, \tilde{y}}$ has no directed edge between $[r]$ and $[n] \setminus [r]$. We prove (*) by showing that $G_{x,y}^{v, \hat{w}}$ is the directed graph $G_{\tilde{x}, \tilde{y}}^{v, w}$ with one directed edge added between $[r]$ and $[n] \setminus [r]$ in (i) and (ii) and with a pair of vertices in $[r]$ and $[n] \setminus [r]$ identified in (iii).

Case 1: $\hat{w} = s_r w$. Note that in cases (i) and (ii), $\overline{T}(x, [x, y]) = \overline{T}(\tilde{x}, [\tilde{x}, \tilde{y}])$ by (5.3). Hence the vertex set of $G_{x,y}^{v, \hat{w}}$ is the same as that of $G_{\tilde{x}, \tilde{y}}^{v, w}$. In case (i), it follows from (1) and (2) of Lemma 5.6 that

$$\underline{T}(x, [v, s_r w]) = \underline{T}(\tilde{x}, [v, w])$$

and

$$\overline{T}(y, [v, s_r w]) = \overline{T}(\tilde{y}, [v, w]) \cup \{(y^{-1}(r), y^{-1}(r+1))\}.$$

Hence the graph $G_{x,y}^{v, \hat{w}}$ is obtained from $G_{\tilde{x}, \tilde{y}}^{v, w}$ by adding a directed edge from $y^{-1}(r) \in [r]$ to $y^{-1}(r+1) \in [n] \setminus [r]$. In case (ii), it follows from (4) and (5) of Lemma 5.6 that

$$\overline{T}(y, [v, s_r w]) = \overline{T}(\tilde{y}, [v, w])$$

and

$$\underline{T}(x, [v, s_r w]) = \underline{T}(\tilde{x}, [v, w]) \cup \{((\tilde{x})^{-1}(r), (\tilde{x})^{-1}(r+1))\}.$$

Hence the graph $G_{x,y}^{v, \hat{w}}$ is obtained from $G_{\tilde{x}, \tilde{y}}^{v, w}$ by adding a directed edge from $(\tilde{x})^{-1}(r+1) \in [n] \setminus [r]$ to $(\tilde{x})^{-1}(r) \in [r]$. In case (iii), by (1) and (4) of Lemma 5.6, we have

$$\underline{T}(x, [v, s_r w]) = \underline{T}(\tilde{x}, [v, w]) \text{ and } \overline{T}(y, [v, s_r w]) = \overline{T}(\tilde{y}, [v, w]).$$

Since

$$\overline{T}(x, [x, y]) = \overline{T}(\tilde{x}, [\tilde{x}, \tilde{y}]) \cup \{(x^{-1}(r), x^{-1}(r+1))\}$$

by Lemma 5.6(2), the graph $G_{x,y}^{v, \hat{w}}$ is obtained from $G_{\tilde{x}, \tilde{y}}^{v, w}$ by identifying the vertices $x^{-1}(r) \in [r]$ and $x^{-1}(r+1) \in [n] \setminus [r]$. This proves Case 1.

Case 2: $\hat{w} = w s_r$. In case (i), since $\overline{T}(x, [x, y]) = \overline{T}(\tilde{x}, [\tilde{x}, \tilde{y}])$, the vertex set of $G_{x,y}^{v, \hat{w}}$ is the same as that of $G_{\tilde{x}, \tilde{y}}^{v, w}$. It follows from (1) and (3) of Lemma 5.6 that

$$\underline{T}(x, [v, w s_r]) = \underline{T}(\tilde{x}, [v, w])$$

and

$$\overline{T}(y, [v, w s_r]) = \overline{T}(\tilde{y}, [v, w]) \cup \{(r, r+1)\}.$$

Hence the graph $G_{x,y}^{v, \hat{w}}$ is obtained from $G_{\tilde{x}, \tilde{y}}^{v, w}$ by adding a directed edge from r to $r+1$. In case (ii), since $\tilde{x}(i, j) s_r = \tilde{x} s_r(s_r(i), s_r(j)) = x(s_r(i), s_r(j))$, we get

$$\overline{T}(x, [x, y]) = \{(s_r(i), s_r(j)) \mid (i, j) \in \overline{T}(\tilde{x}, [\tilde{x}, \tilde{y}])\}$$

similarly to the proof of (6) in Lemma 5.6. Hence $G_{x,y}$ is obtained from $G_{\tilde{x}, \tilde{y}}$ by interchanging the labelling of the vertices r and $r+1$. By (6) and (7) of Lemma 5.6, we have

$$\overline{T}(y, [v, w s_r]) = \{(s_r(i), s_r(j)) \mid (i, j) \in \overline{T}(\tilde{y}, [v, w])\}$$

and

$$\underline{T}(x, [v, w s_r]) = \{(s_r(i), s_r(j)) \mid (i, j) \in \underline{T}(\tilde{x}, [v, w])\} \cup \{(r, r+1)\}.$$

Thus the graph $G_{x,y}^{v, \hat{w}}$ is obtained from $G_{\tilde{x}, \tilde{y}}^{v, w}$ by interchanging the labelling of the vertices r and $r+1$ and then adding a directed edge from $r+1$ to r . In case (iii), it follows from (1) and (6) of Lemma 5.6 that

$$\underline{T}(x, [v, w s_r]) = \underline{T}(\tilde{x}, [v, w])$$

and

$$\overline{T}(y, [v, ws_r]) = \{(s_r(i), s_r(j)) \mid (i, j) \in \overline{T}(\tilde{y}, [v, w])\}.$$

Note that

$$\overline{T}(x, [x, y]) = \overline{T}(\tilde{x}, [\tilde{x}, \tilde{y}]) \cup \{(r, r+1)\}$$

by Lemma 5.6(3). Hence the graph $G_{x,y}$ is obtained from $G_{\tilde{x},\tilde{y}}$ by adding an edge between the vertices r and $r+1$. Therefore, the graph $G_{x,y}^{v,w}$ is obtained from $G_{\tilde{x},\tilde{y}}^{v,w}$ by identifying the vertices r and $r+1$. \square

The above proposition implies that $Q_{e,w}$ is combinatorially equivalent to the cube $I^{\ell(w)}$ if X_w is of complexity zero, i.e., a reduced decomposition of w consists of distinct letters.

Theorem 5.8. *For a permutation w in \mathfrak{S}_n , the following hold:*

- (1) *the Schubert variety X_w is smooth and of complexity one if and only if $Q_{e,w}$ is combinatorially equivalent to the polytope $\text{Perm}_2 \times I^{\ell(w)-3}$, and*
- (2) *the Schubert variety X_w is singular and of complexity one if and only if $Q_{e,w}$ is combinatorially equivalent to the polytope $Q_{e,3412} \times I^{\ell(w)-4}$.*

Proof. Since both the polytopes $\text{Perm}_2 \times I^{\ell(w)-3}$ and $Q_{e,3412} \times I^{\ell(w)-4}$ are $\ell(w)-1$ dimensional, we get $c(w) = 1$. Hence it suffices to prove the ‘only if’ part in each statement.

If a Schubert variety X_w is smooth and of complexity one, then w has a reduced decomposition \underline{w} of the form:

$$\underline{w} = s_{i_1} \cdots s_{i_{k-1}} s_{i_k} s_{i_{k+1}} s_{i_k} s_{i_{k+2}} \cdots s_{i_{\ell-1}} \quad \text{and} \quad i_{k+1} = i_k + 1.$$

Hence the polytope $Q_{e,w}$ is combinatorially equivalent to $Q_{e,s_{i_k} s_{i_{k+1}} s_{i_k}} \times I^{\ell(w)-3} \cong \text{Perm}_2 \times I^{\ell(w)-3}$ by Lemma 5.1 and Proposition 5.7. If X_w is singular and of complexity one, then w has a reduced decomposition \underline{w} of the form:

$$\underline{w} = s_{i_1} \cdots s_{i_{k-1}} s_{i_{k+1}} s_{i_k} s_{i_{k+2}} s_{i_{k+1}} s_{i_{k+3}} \cdots s_{i_{\ell-1}} \quad \text{and} \quad i_{k+1} = i_k + 1, i_{k+2} = i_k + 2.$$

Hence $Q_{e,w}$ is combinatorially equivalent to $Q_{e,s_{i_k+1} s_{i_k} s_{i_{k+2}} s_{i_{k+1}}} \times I^{\ell(w)-4} \cong Q_{e,3412} \times I^{\ell(w)-4}$ by Lemma 5.1 and Proposition 5.7. \square

The first statement in the above theorem shows the equivalence between (1') and (6') in Theorem 1.2, and hence it proves Theorem 1.2. The second statement in the above theorem shows the equivalence between (1'') and (6'') in Theorem 1.3, and hence it proves Theorem 1.3.

Note that for a permutation w in \mathfrak{S}_n , the following hold:

- (1) w avoids the patterns 3412 and 4231 if and only if w^{-1} avoids those patterns, and
- (2) $[321; 3412](w) = 1$ if and only if $[321; 3412](w^{-1}) = 1$.

Hence X_w is smooth and of complexity one if and only if $X_{w^{-1}}$ is smooth and of complexity one.

Corollary 5.9. *For $w \in \mathfrak{S}_n$, if $[321; 3412](w) = 1$, then the Bruhat interval polytopes $Q_{e,w}$ and $Q_{e,w^{-1}}$ are combinatorially equivalent.*

Therefore, for a Schubert variety X_w of complexity one, X_w is smooth if and only if the moment polytope $Q_{e,w^{-1}}$ is simple.

6. FLAG BOTT MANIFOLDS

Like as a Bott–Samelson variety is diffeomorphic to a Bott manifold with a higher rank torus action, a flag Bott–Samelson variety is diffeomorphic to a *flag Bott manifold* with a higher rank torus action. Whereas a Bott manifold is a toric variety, a flag Bott manifold is not a toric variety in general, but it becomes a GKM manifold, see [18]. We have previously observed in Theorem 4.7 that there is an isomorphism between a smooth Schubert variety and a flag Bott–Samelson variety. Using the diffeomorphism between a flag Bott–Samelson variety and a certain flag Bott manifold, we will provide a formula for the cohomology ring of a smooth Schubert variety of complexity one.

Recall that for a holomorphic vector bundle E over a complex manifold M , there is an associated flag-bundle $\mathcal{F}\ell(E)$ obtained from E by replacing each fiber E_p by the full flag manifold $\mathcal{F}\ell(E_p)$.

Definition 6.1 ([18, Definition 2.1]). A *flag Bott tower* is an iterated flag-bundle

$$\mathcal{F}_r \xrightarrow{\pi_r} \mathcal{F}_{r-1} \xrightarrow{\pi_{r-1}} \cdots \xrightarrow{\pi_2} \mathcal{F}_1 \xrightarrow{\pi_1} \mathcal{F}_0 = \{\text{a point}\}$$

of manifolds $\mathcal{F}_k = \mathcal{F}\ell\left(\underline{\mathbb{C}} \oplus \bigoplus_{m=1}^{n_k} \xi_k^{(m)}\right)$, where $\xi_k^{(m)}$ is a holomorphic line bundle over \mathcal{F}_{k-1} for each $1 \leq m \leq n_k$ and $1 \leq k \leq r$. Each \mathcal{F}_k is called a *flag Bott manifold* (of height k).

Because we are considering an iterated full flag-bundle, there is an isomorphism

$$\psi: \mathbb{Z}^{n_1} \times \cdots \times \mathbb{Z}^{n_r} \rightarrow \text{Pic}(\mathcal{F}_r).$$

Therefore, there is a sequence

$$\left(\mathbf{a}_{1,k}^{(m)}, \mathbf{a}_{2,k}^{(m)}, \dots, \mathbf{a}_{k-1,k}^{(m)} \right)$$

of integer vectors with $\mathbf{a}_{j,k}^{(m)} \in \mathbb{Z}^{n_j}$ which maps to $\xi_k^{(m)}$ via ψ . Hence a set $\{\mathbf{a}_{j,k}^{(m)}\}_{1 \leq m \leq n_k, 1 \leq j < k \leq r}$ of integer vectors determines a flag Bott manifold.

Theorem 6.2. [11, Theorem 4.10] *Let $\mathcal{I} = (I_1, \dots, I_r)$ be a sequence of subsets of $[n]$, where $I_k = \{u_{k,1}, \dots, u_{k,n_k}\}$ for $1 \leq k \leq r$. Assume that each I_k is an interval. Then the flag Bott–Samelson variety $Z_{\mathcal{I}}$ is diffeomorphic to a flag Bott manifold \mathcal{F}_r determined by the vectors*

$$\mathbf{a}_{j,k}^{(m)} = (\mathbf{a}_{j,k}^{(m)}(1), \dots, \mathbf{a}_{j,k}^{(m)}(n_j)) \in \mathbb{Z}^{n_j} \quad (1 \leq m \leq n_k \text{ and } 1 \leq j < k \leq r)$$

where

$$\mathbf{a}_{j,k}^{(m)}(p) = \langle \mathbf{e}_{u_{j,p}} - \mathbf{e}_{u_{j,n_j+1}}, \mathbf{e}_{u_{k,m}} - \mathbf{e}_{u_{k,n_k+1}} \rangle$$

for $1 \leq p \leq n_j$. Here, $\mathbf{e}_{u_{k,1}}, \dots, \mathbf{e}_{u_{k,n_k+1}}$ are the standard basis vectors of \mathbb{R}^{n_k+1} for $1 \leq k \leq r$.

Theorem 6.3. *Let X_w be a smooth Schubert variety of complexity one. Then X_w is diffeomorphic to a flag Bott manifold of height $\ell(w) - 2$*

$$\mathcal{F}_{\ell(w)-2} \xrightarrow{\pi_{\ell(w)-2}} \mathcal{F}_{\ell(w)-3} \xrightarrow{\pi_{\ell(w)-3}} \cdots \xrightarrow{\pi_2} \mathcal{F}_1 \xrightarrow{\pi_1} \mathcal{F}_0 = \{\text{a point}\},$$

where $\mathcal{F}_q \rightarrow \mathcal{F}_{q-1}$ is a $\mathcal{F}\ell(\mathbb{C}^3)$ -bundle for some $1 \leq q \leq \ell(w) - 2$ and $\mathcal{F}_k \rightarrow \mathcal{F}_{k-1}$ is a $\mathbb{C}P^1$ -bundle for every $k \neq q$. Furthermore, the iterated bundle structure is completely determined by a reduced decomposition of w .

Proof. Let $\ell(w) = \ell$. From the proof of Theorem 4.7, we may assume that X_w is isomorphic to a flag Bott–Samelson variety $Z_{\mathcal{I}}$, where $\mathcal{I} = (I_1, \dots, I_{\ell-2})$ of length $\ell - 2$ consists of the sets

$$I_k = \begin{cases} \{i_k\} & \text{if } 1 \leq k < q, \\ \{i, i+1\} & \text{if } k = q, \\ \{i_{k+2}\} & \text{if } k > q. \end{cases}$$

Note that $\underline{w} = (i_1, \dots, i_{q-1}, i, i+1, i, i_{q+3}, \dots, i_{\ell})$. Thus X_w is diffeomorphic to a flag Bott manifold of height $\ell - 2$ by Theorem 6.2. Furthermore, for each $1 \leq k \leq \ell - 2$, the manifold \mathcal{F}_k is determined as follows.

- (1) If $1 \leq k < q$, then $n_k = 1$ and $\mathcal{F}_k = \mathcal{F}\ell(\underline{\mathbb{C}} \oplus \xi_k)$, where ξ_k is a holomorphic line bundle determined by a sequence $(\mathbf{a}_{1,k}^{(1)}, \dots, \mathbf{a}_{k-1,k}^{(1)})$ of integers, where

$$\mathbf{a}_{j,k}^{(1)} = \langle \mathbf{e}_{i_j} - \mathbf{e}_{i_{j+1}}, \mathbf{e}_{i_k} - \mathbf{e}_{i_{k+1}} \rangle \in \mathbb{Z}.$$

- (2) If $k = q$, then $n_j = 1$ for $j < k = q$ and $n_k = 2$ and $\mathcal{F}_k = \mathcal{F}\ell(\underline{\mathbb{C}} \oplus \xi_k^{(1)} \oplus \xi_k^{(2)})$, where $\xi_k^{(1)}$ and $\xi_k^{(2)}$ determined by sequences $(\mathbf{a}_{1,q}^{(1)}, \dots, \mathbf{a}_{q-1,q}^{(1)})$ and $(\mathbf{a}_{1,q}^{(2)}, \dots, \mathbf{a}_{q-1,q}^{(2)})$ of integers, respectively. Here, we have that

$$\mathbf{a}_{j,q}^{(1)} = \langle \mathbf{e}_{i_j} - \mathbf{e}_{i_{j+1}}, \mathbf{e}_i - \mathbf{e}_{i+2} \rangle \in \mathbb{Z},$$

$$\mathbf{a}_{j,q}^{(2)} = \langle \mathbf{e}_{i_j} - \mathbf{e}_{i_{j+1}}, \mathbf{e}_{i+1} - \mathbf{e}_{i+2} \rangle \in \mathbb{Z}.$$

- (3) If $k > q$, then $n_k = 1$ and $\mathcal{F}_k = \mathcal{F}\ell(\underline{\mathbb{C}} \oplus \xi_k)$, where ξ_k is a holomorphic line bundle determined by a sequence $(\mathbf{a}_{1,k}^{(1)}, \dots, \mathbf{a}_{k-1,k}^{(1)})$ of integer vectors, where

$$\mathbf{a}_{j,k}^{(1)} = \begin{cases} \langle \mathbf{e}_{i_j} - \mathbf{e}_{i_{j+1}}, \mathbf{e}_{i_{k+2}} - \mathbf{e}_{i_{k+2+1}} \rangle & \text{if } j < q, \\ \langle \langle \mathbf{e}_i - \mathbf{e}_{i+2}, \mathbf{e}_{i_{k+2}} - \mathbf{e}_{i_{k+2+1}} \rangle, \langle \mathbf{e}_{i+1} - \mathbf{e}_{i+2}, \mathbf{e}_{i_{k+2}} - \mathbf{e}_{i_{k+2+1}} \rangle \rangle & \text{if } j = q, \\ \langle \mathbf{e}_{i_{j+2}} - \mathbf{e}_{i_{j+2+1}}, \mathbf{e}_{i_{k+2}} - \mathbf{e}_{i_{k+2+1}} \rangle & \text{if } j > q. \end{cases}$$

This proves the theorem. \square

Combining Theorem 2.2 with Theorems 4.7 and 6.3, we conclude that every smooth Schubert variety of complexity ≤ 1 is isomorphic to a flag Bott–Samelson variety, and hence it is diffeomorphic to a flag Bott manifold whose height is determined by the length $\ell(w)$ and the complexity $c(w)$. Therefore, using [16, Corollary 4.4], we provide the cohomology ring $H^*(X_w; \mathbb{Z})$ of a smooth Schubert variety X_w with $c(w) \leq 1$.

Corollary 6.4. *Let X_w be a smooth Schubert variety of complexity one. Suppose that $\underline{w} = s_{i_1} \dots s_{i_{q-1}} s_i s_{i+1} s_i s_{i_{q+3}} \dots s_{i_{\ell}}$ is a reduced decomposition of w . Then the cohomology ring $H^*(X_w; \mathbb{Z})$ is given as follows:*

$$H^*(X_w; \mathbb{Z}) \cong \mathbb{Z}[y_{j,1}, \dots, y_{j,n_j+1} \mid 1 \leq j \leq \ell - 2] / \langle I_1, \dots, I_{\ell-2} \rangle.$$

Here, $y_{j,k}$'s are degree two elements and I_k is the ideal given as follows.

$$\begin{aligned}
\text{For } k < q: \quad I_k &= (1 - y_{k,1})(1 - y_{k,2}) - \left(1 - \sum_{j=1}^{k-1} \mathbf{a}_{j,k}^{(1)} y_{j,1}\right), \\
\text{for } k = q: \quad I_q &= (1 - y_{q,1})(1 - y_{q,2})(1 - y_{q,3}) \\
&\quad - \left(1 - \sum_{j=1}^{q-1} \mathbf{a}_{j,q}^{(1)} y_{j,1}\right) \left(1 - \sum_{j=1}^{q-1} \mathbf{a}_{j,q}^{(2)} y_{j,1}\right), \\
\text{for } k > q: \quad I_k &= (1 - y_{k,1})(1 - y_{k,2}) \\
&\quad - \left(1 - \sum_{\substack{1 \leq j \leq k-1, \\ j \neq q}} \mathbf{a}_{j,k}^{(1)} y_{j,1} - (\mathbf{a}_{q,k}^{(1)}(1)y_{q,1} + \mathbf{a}_{q,k}^{(1)}(2)y_{q,2})\right).
\end{aligned}$$

Example 6.5. Suppose that $w = s_1 s_2 s_3 s_4 s_3$. Then the Schubert variety X_w is smooth of complexity one. By Theorem 6.2, the Schubert variety X_w is diffeomorphic to a flag Bott manifold of height 3 with $n_1 = n_2 = 1$ and $n_3 = 2$ which is determined by the following integer vectors:

$$\begin{aligned}
\mathbf{a}_{1,2}^{(1)} &= \langle \mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_2 - \mathbf{e}_3 \rangle = -1, \\
\mathbf{a}_{1,3}^{(1)} &= \langle \mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_3 - \mathbf{e}_5 \rangle = 0, \quad \mathbf{a}_{1,3}^{(2)} = \langle \mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_4 - \mathbf{e}_5 \rangle = 0, \\
\mathbf{a}_{2,3}^{(1)} &= \langle \mathbf{e}_2 - \mathbf{e}_3, \mathbf{e}_3 - \mathbf{e}_5 \rangle = -1, \quad \mathbf{a}_{2,3}^{(2)} = \langle \mathbf{e}_2 - \mathbf{e}_3, \mathbf{e}_4 - \mathbf{e}_5 \rangle = 0.
\end{aligned}$$

Therefore, by Corollary 6.4, the cohomology ring of X_w is

$$H^*(X_w; \mathbb{Z}) \cong \mathbb{Z}[y_{j,1}, \dots, y_{j,n_j+1} \mid 1 \leq j \leq 3] / \langle I_1, \dots, I_3 \rangle,$$

where

$$\begin{aligned}
I_1 &= (1 - y_{1,1})(1 - y_{1,2}) - 1, \\
I_2 &= (1 - y_{2,1})(1 - y_{2,2}) - (1 + y_{1,1}), \\
I_3 &= (1 - y_{3,1})(1 - y_{3,2})(1 - y_{3,3}) - (1 + y_{2,1}).
\end{aligned}$$

This gives relations among generators:

$$\begin{aligned}
y_{1,1} + y_{1,2} &= y_{1,1}y_{1,2} = 0, \\
y_{2,1} + y_{2,2} + y_{1,1} &= y_{2,1}y_{2,2} = 0, \\
y_{3,1} + y_{3,2} + y_{3,3} + y_{2,1} &= y_{3,1}y_{3,2} + y_{3,1}y_{3,3} + y_{3,2}y_{3,3} = y_{3,1}y_{3,2}y_{3,3} = 0.
\end{aligned}$$

Therefore, by setting $y_1 = y_{1,1}$, $y_2 = y_{2,1}$, $y_3 = y_{3,1}$, $y_4 = y_{3,2}$, we have that

$$H^*(X_w; \mathbb{Z}) \cong \mathbb{Z}[y_1, \dots, y_4] / I,$$

where I is an ideal generated by

$$y_1^2, y_2(y_1 + y_2), (y_2 + y_3)y_3 + (y_2 + y_3 + y_4)y_4, y_3y_4(y_2 + y_3 + y_4).$$

We enclose this section by mentioning other studies on the cohomology rings of smooth Schubert varieties. Indeed, the cohomology rings of smooth Schubert varieties are studied in [12] (also, see [9]). We will demonstrate their result for a specific permutation 23541, which is the one considered in Example 6.5. First we recall from [9] that this permutation is related to a partition as follows. For

a given partition $\lambda = (0 \leq \lambda_1 \leq \dots \leq \lambda_n)$, one may associate a permutation $w = w(1) \dots w(n)$ by the recursive rule

$$w(i) = \max(\{1, \dots, \lambda_i\} \setminus \{w(1), \dots, w(i-1)\}).$$

For example, if $\lambda = (2, 3, 5, 5, 5)$, then the corresponding permutation is 23541. We denote by X_λ the Schubert variety given by the permutation coming from λ . The presentation for the cohomology ring of X_λ is given as follows:

$$(6.1) \quad H^*(X_\lambda; \mathbb{Z}) \cong \mathbb{Z}[x_1, \dots, x_n] / \langle h_{\lambda_i - i + 1}(i) \mid 1 \leq i \leq n \rangle$$

where $h_m(N)$ is the complete homogeneous symmetric function:

$$h_m(N) := h_m(x_1, \dots, x_N) = \sum_{1 \leq i_1 \leq \dots \leq i_m \leq N} x_{i_1} \cdots x_{i_m}.$$

For the partition $\lambda = (2, 3, 5, 5, 5)$, the ideal $J := \langle h_{\lambda_i - i + 1}(i) \mid 1 \leq i \leq n \rangle$ on the right hand side of (6.1) is generated by the complete homogeneous symmetric polynomials

$$\begin{aligned} h_2(1) &= x_1^2, \\ h_2(2) &= x_1^2 + (x_1 + x_2)x_2, \\ h_3(3) &= (x_1 + x_2 + x_3)x_1^2 + (x_1x_2 + x_2^2)(x_2 + x_3) + (x_1 + x_2 + x_3)x_3^2, \\ h_2(4) &= x_1^2 + (x_1 + x_2)x_2 + (x_1 + x_2 + x_3)x_3 + (x_1 + x_2 + x_3 + x_4)x_4, \\ h_1(5) &= x_1 + x_2 + x_3 + x_4 + x_5. \end{aligned}$$

Moreover, we get that $\mathbb{Z}[x_1, \dots, x_5]/J \cong \mathbb{Z}[x_1, \dots, x_4]/J'$, where

$$J' = \langle x_1^2, (x_1 + x_2)x_2, (x_1 + x_2 + x_3)x_3^2, (x_1 + x_2 + x_3)x_3 + (x_1 + x_2 + x_3 + x_4)x_4 \rangle.$$

By sending $y_1 \mapsto -x_1$, $y_2 \mapsto x_1 + x_2$, $y_3 \mapsto x_3$, $y_4 \mapsto x_4$, we obtain an isomorphism $\mathbb{Z}[x_1, \dots, x_5]/I \cong \mathbb{Z}[y_1, \dots, y_5]/J'$ between the two cohomology ring representations. For instance, we have that

$$\begin{aligned} (y_2 + y_3)y_3 + (y_2 + y_3 + y_4)y_4 &\mapsto (x_1 + x_2 + x_3)x_3 + (x_1 + x_2 + x_3 + x_4)x_4, \\ y_3y_4(y_2 + y_3 + y_4) & \\ \mapsto x_3x_4(x_1 + x_2 + x_3 + x_4) & \\ = x_3((x_1 + x_2 + x_3)x_3 + (x_1 + x_2 + x_3 + x_4)x_4) - (x_1 + x_2 + x_3)x_3^2. & \end{aligned}$$

Remark 6.6. The notion of flag Bott–Samelson variety can be defined in a general Lie type, where G is a simply-connected semisimple algebraic group over \mathbb{C} . Furthermore, Proposition 4.3, Corollary 4.5, and Theorem 6.2 are still true in a general Lie type.

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