ABSOLUTE IRREDUCIBILITY OF THE BINOMIAL POLYNOMIALS

ROSWITHA RISSNER AND DANIEL WINDISCH

Abstract. In this paper we investigate the factorization behaviour of the binomial polynomials $\binom{x}{n} = \frac{x(x-1)\cdots(x-n+1)}{n!}$ and their powers in the ring of integer-valued polynomials $\mathrm{Int}(\mathbb{Z})$. While it is well-known that the binomial polynomials are irreducible elements in $\mathrm{Int}(\mathbb{Z})$, the factorization behaviour of their powers has not yet been fully understood. We fill this gap and show that the binomial polynomials are absolutely irreducible in $\mathrm{Int}(\mathbb{Z})$, that is, $\binom{x}{n}^m$ factors uniquely into irreducible elements in $\mathrm{Int}(\mathbb{Z})$ for all $m \in \mathbb{N}$. By reformulating the problem in terms of linear algebra and number theory, we show that the question can be reduced to determining the rank of, what we call, the valuation matrix of n. A main ingredient in computing this rank is the following number-theoretical result for which we also provide a proof: If $n > 10$ and n, $n-1, \ldots, n-(k-1)$ are composite integers, then there exists a prime number $p > 2k$ that divides one of these integers.

1. Introduction

In this work, our main objects of interest are the so-called *binomial polynomials*

$$\binom{x}{n} = \frac{x(x-1)\cdots(x-n+1)}{n!}$$

for integers $n \geq 1$ as elements of $\mathrm{Int}(\mathbb{Z}) = \{f \in \mathbb{Q}[x] \mid f(\mathbb{Z}) \subseteq \mathbb{Z}\}$, where $\mathbb{Z}$ denotes the ring of integers and $\mathbb{Q}$ is the field of rational numbers.

It is well-known that $n!$ divides the product of any n consecutive integers, and therefore $\binom{x}{n}$ indeed is an integer-valued polynomial for all $n \geq 1$. Moreover, the binomial polynomials are known to be irreducible in $\mathrm{Int}(\mathbb{Z})$, cf. the survey of Cahen and Chabert [3]. A non-zero non-unit a in a (commutative) integral domain D is said to be *irreducible* if $a = bc$ implies that either b or c is a unit for all b, $c \in D$.

However, the irreducibility of a does not tell anything about the factorization behaviour of the powers a^m. An irreducible element a is said to be *absolutely irreducible* (or a *strong atom*) if a^m factors uniquely into irreducible elements for all integers $m \geq 1$. There are plenty of non-absolutely irreducible elements in $\mathrm{Int}(\mathbb{Z})$, as it was shown lately by Nakato [12]. Moreover, the recent work of Frisch and Nakato [6] gives a criterion for the absolute irreducibility of integer-valued polynomials with square-free denominators over $\mathbb{Z}$. In particular, it follows from their Theorem 2 that, for all prime

R. Rissner is supported by the Austrian Science Fund (FWF): P 28466.
D. Windisch is supported by the Austrian Science Fund (FWF): P 30934.

numbers p, the binomial polynomial $\binom{x}{p}$ is absolutely irreducible (see [6, Example 2.6]).

In the present paper we show that the binomial polynomial $\binom{x}{n} \in \operatorname{Int}(\mathbb{Z})$ is absolutely-irreducible for all $n \geq 1$. Our approach also covers the case that $n = p$ is a prime number, and hence serves in this special case also as an alternative proof to the one in [6].

The binomial polynomials play a central role in the study of $\operatorname{Int}(\mathbb{Z})$ as they form a so-called regular $\mathbb{Z}$-module basis, that is, a basis which contains exactly one polynomial of each degree. Implicitly, this fact was already applied by Newton who used integer-valued polynomials to interpolate integer-valued functions on $\mathbb{Z}$.

Our viewpoint on the binomial polynomials in this paper relates to the whole area of investigating non-unique factorizations. When mathematicians first explored that factorizations into irreducible elements do not have to be unique in general, they considered this behaviour as pathological and passed over to unique factorizations of ideals into prime ideals in Dedekind domains. It is a movement of the last few decades that non-unique factorizations are viewed in their own right. Since then, the machinery for their investigation has been developed mostly in the direction of Krull domains and monoids, including for instance the concept of the divisor class group of a Krull domain or monoid. For an introduction to this topic, we refer to the textbook of Geroldinger and Halter-Koch [8].

However, almost nothing is known in the area of non-unique factorizations in non-Noetherian Prüfer domains. In this context, general rings of integer-valued polynomials

$$\operatorname{Int}(D) = \{f \in K[x] \mid f(D) \subseteq D\}$$

where D is an integral domain with quotient field K are of interest. The integral domain $\operatorname{Int}(D)$ has been studied very intensively during the last decades and is a standard source for examples and counterexamples. For instance, it is well-known that $\operatorname{Int}(D)$ is a non-Noetherian Prüfer domain provided that D is a Dedekind domain with finite residue fields. A profound introduction to the theory of integer-valued polynomial is given in the textbook of Cahen and Chabert [2], and a more recent survey is their work [3].

Coming back to factorization theory, Frisch [4] proved that in $\operatorname{Int}(\mathbb{Z})$ arbitrary sets of lengths can be realized. These results were generalized by Frisch, Nakato and Rissner [7] to rings of integer-valued polynomials on Dedekind domains with infinitely many maximal ideal which are all of finite index. Both papers use a very specific type of irreducible elements of $\operatorname{Int}(D)$ to realize the aimed sets of lengths.

On the one hand, recognizing irreducible elements in $\operatorname{Int}(D)$ is in general far from being trivial. A partial answer for $\operatorname{Int}(\mathbb{Z})$ using an algorithmic approach can be found in [1]. On the other hand, the behaviour of products of general irreducible elements in rings of integer-valued polynomials is not at all understood. Our result on the absolute irreducibility of the binomial polynomials leads to a better understanding of one important class of irreducible elements in $\operatorname{Int}(\mathbb{Z})$. It also broadens the knowledge of a different aspect of rings of integer-valued polynomials. It has been already mentioned that $\mathrm{Int}(D)$ is a non-Noetherian Prüfer domain and is therefore non-Krull, when D is a Dedekind domain with finite residue fields. Nevertheless, Reinhart [13] showed that if D is any factorial domain, then $\mathrm{Int}(D)$ is *monadically Krull*, i.e., the so-called *monadic submonoid*

$$[\![f]\!] = \{g \in \mathrm{Int}(D) \mid g \text{ divides } f^n \text{ for some } n \in \mathbb{N}\}$$

is a Krull monoid for each $f \in \mathrm{Int}(D)$. Frisch [5] extended this result to integer-valued polynomial rings on Krull domains. Moreover, Reinhart [14] showed that for a factorial domain D the divisor class group of a monadic submonoid $[\![f]\!]$ of $\mathrm{Int}(D)$ is free Abelian for every $f \in D[x] \setminus \{0\}$. For general polynomials $f \in \mathrm{Int}(D)$, the structure of the divisor class group of the monadic submonoid generated by f is not known, but it can be easily seen that the following two assertions are equivalent for an irreducible $f \in \mathrm{Int}(D)$:

- The polynomial $f \in \mathrm{Int}(D)$ is absolutely irreducible.
- The monoid $[\![f]\!]$ is factorial, i.e., its divisor class group is trivial.

Therefore our main result gives a description of a whole new family of divisor class groups of monadic submonoids of $\mathrm{Int}(\mathbb{Z})$.

This paper is structured as follows. In Section 2 we present our two main results. Theorem 1 states that the binomial polynomials are absolutely irreducible in $\mathrm{Int}(\mathbb{Z})$. A second result, Theorem 2, is a number-theoretic result which we need in the proof of Theorem 1 but is interesting in its own right. It states that given a sequence of k consecutive composite integers such that the largest integer is greater than 10, then one of the numbers of this sequence has a prime divisor $p > 2k$.

The remaining paper is dedicated to the proofs of the main results. In Section 3 we explain the strategy for the proof of Theorem 1 and introduce the necessary notation. The main idea is to rephrase the question of absolute irreducibility of the binomial polynomial $\binom{x}{n}$ in terms of linear algebra and number theory. For this purpose, we introduce, what we call, the *valuation matrix* $\mathbf{A}_n$ (Definition 3.10) and show that $\binom{x}{n}$ is absolutely irreducible in $\mathrm{Int}(\mathbb{Z})$ if $\mathrm{rank}(\mathbf{A}_n) = n - 1$ (Proposition 3.15). This point of view also motivates the content of Section 4, where we present our number-theoretic toolbox, including a proof of Theorem 2. Finally, in Section 5 we prove that $\mathrm{rank}(\mathbf{A}_n) = n-1$ holds for all $n \in \mathbb{N}$ which is the final piece for proving the absolute irreducibility of the binomial polynomial $\binom{x}{n}$.

2. Results

In this section, we present the main results of this paper. The subsequent sections are dedicated to their proofs.

Theorem 1. *The binomial polynomial $\binom{x}{n}$ is absolutely irreducible in* $\mathrm{Int}(\mathbb{Z})$ *for all $n \in \mathbb{N}$.*

The proof of Theorem 1 is a consequence of the results of Sections 3 and 5. To be more specific, the theorem is the summarized statements of Corollaries 5.12 and 5.15 and Remark 3.3.

Corollary 2.1. *The monadic submonoid $[\![\binom{x}{n}]\!]$ of* $\mathrm{Int}(\mathbb{Z})$ *is factorial for all $n \geq 1$. In particular, it has trivial divisor class group.*

The theory developed in Section 5 heavily depends on some number-theoretic results which are built up in Section 4. Next to well-known facts which are collected there, we also prove the following theorem in this section.

Theorem 2. *Let $n > 10$ be an integer and P the maximal prime number with $P \leq n$.*

If $2 \leq k \leq n - P$ then there exists a prime number $p > 2k$ which divides one of the numbers n, $n-1$, $\dots$, $n-(k-1)$.

3. Absolute irreducibility of binomial polynomials and the valuation matrix

In this section we introduce the *valuation matrix* $\mathbf{A}_n$ which is associated to the binomial polynomial $\binom{x}{n}$ and explain how the question of its absolute irreducibility can be answered by determining the rank of $\mathbf{A}_n$. First, we discuss the notion of absolute irreducibility.

Definition 3.1. Let D be an integral domain and $b \in D$ be an irreducible element. We say b is *absolutely irreducible* if b^m factors uniquely into irreducible elements for every $m \in \mathbb{N}$.

Remark 3.2. Let D be an integral domain and $b \in D$ be an irreducible element. A straight forward verification shows that the following assertions are equivalent:

(1) b is absolutely irreducible.
(2) For every non-negative integer m and for all f, $g \in D$ with $b^m = f \cdot g$, there exist non-negative integers k, ℓ and units u, $v \in D$ such that $f = ub^k$ and $g = vb^\ell$.

Remark 3.3. Since x^m factors uniquely in $\mathrm{Int}(\mathbb{Z})$ for all $m \geq 1$, it follows immediately that $\binom{x}{1} = x$ is absolutely irreducible. This covers the case $n = 1$.

Remark 3.4. For the rest of this work, fix a positive integer $n \geq 2$. Our goal is to show that for every positive integer m and f, $g \in \mathrm{Int}(\mathbb{Z})$ the following property holds:

$$\binom{x}{n}^m = f \cdot g \implies f = \pm\binom{x}{n}^k \text{ and } g = \pm\binom{x}{n}^\ell \text{ with } k, \ell \in \mathbb{N}_0.$$

Once this is shown, it follows by Remark 3.2 that $\binom{x}{n} \in \mathrm{Int}(\mathbb{Z})$ is absolutely irreducible.

Our first step is to give a precise description of f and g exploiting the fact that $\mathbb{Q}[x]$ is a UFD.

Proposition 3.5. *Let n, $m \geq 2$ and f, $g \in \mathrm{Int}(\mathbb{Z})$ with $\binom{x}{n}^m = f \cdot g$.*

Then there exist k_i and $\ell_i \in \mathbb{N}_0$ with $k_i + \ell_i = m$ for $0 \leq i \leq n-1$ such that

$$f = \pm\prod_{i=0}^{n-1}\left(\frac{x-i}{n-i}\right)^{k_i} \text{ and } g = \pm\prod_{i=0}^{n-1}\left(\frac{x-i}{n-i}\right)^{\ell_i}$$

holds.

Proof. Since both sides of the equality $\binom{x}{n}^m = f \cdot g$ factor uniquely in $\mathbb{Q}[x]$ into irreducible elements, it follows that

$$f = q_1 \prod_{i=0}^{n-1} (x-i)^{k_i} \text{ and } g = q_2 \prod_{i=0}^{n-1} (x-i)^{\ell_i} \tag{1}$$

for $q_1, q_2 \in \mathbb{Q}$ and non-negative integers $k_0, \ldots, k_{n-1}$ and $\ell_0, \ldots, \ell_{n-1}$ with $k_i + \ell_i = m$ for all $i \in \{0, \ldots, n-1\}$.

Evaluating $\binom{x}{n}^m = f \cdot g$ at $x = n$ implies $f(n) \cdot g(n) = 1$. Moreover, as $f, g \in \mathrm{Int}(\mathbb{Z})$, it follows that $f(n) = \pm 1$ and $g(n) = \pm 1$. This observation together with Equations (1) imply that

$$\frac{1}{q_1} = \pm \prod_{i=0}^{n-1} (n-i)^{k_i} \text{ and } \frac{1}{q_2} = \pm \prod_{i=0}^{n-1} (n-i)^{\ell_i}.$$

The assertion follows. □

Remark 3.6. Let $k_0, \ldots, k_{n-1}$ and $\ell_0, \ldots, \ell_{n-1} \in \mathbb{N}_0$ be the exponents of the factors f and g of $\binom{x}{n}^m$, cf. Proposition 3.5. If $k_0 = k_1 = \cdots = k_{n-1}$ (and hence $\ell_0 = \ell_1 = \cdots = \ell_{n-1}$), then $f = \binom{x}{n}^{k_0}$ and $g = \binom{x}{m}^{\ell_0}$.

Having Remark 3.4 in mind, we aim to show that, for all possible factors f and g, the corresponding exponents satisfy $k_0 = k_1 = \cdots = k_{n-1}$ and $\ell_0 = \ell_1 = \cdots = \ell_{n-1}$ in order to prove that the binomial polynomial $\binom{x}{n}$ is absolutely irreducible.

We reformulate the task at hand into a homogeneous system of linear equations for which the vectors $(k_0, \ldots, k_{n-1})^t$ and $(\ell_0, \ldots, \ell_{n-1})^t$ are solutions. Given the form of possible factors f and g of $\binom{x}{n}^m$ in $\mathrm{Int}(\mathbb{Z})$ (see Proposition 3.5), it is natural to rephrase the integer-valued condition in terms of p-adic valuations.

Notation 3.7. For $w \in \mathbb{Q}$ and a prime number p, we denote by $\mathbf{v}_p(w)$ the p-adic valuation of w.

Remark 3.8. Let $k_0, \ldots, k_{n-1}$ and $\ell_0, \ldots, \ell_{n-1}$ be the exponents of the factors f and g of $\binom{x}{n}^m$, cf. Proposition 3.5. Since f and g are integer-valued polynomials, it follows that

$$\mathbf{v}_p(f(s)) = \sum_{j=0}^{n-1} (\mathbf{v}_p(s-j) - \mathbf{v}_p(n-j)) k_j \geq 0 \text{ and}$$
$$\mathbf{v}_p(g(s)) = \sum_{j=0}^{n-1} (\mathbf{v}_p(s-j) - \mathbf{v}_p(n-j)) \ell_j \geq 0$$

for all $s \in \mathbb{Z}$ and all $p \in \mathbb{P}$.

For our purposes, it turns out to be sufficient to consider p-adic valuations for prime numbers $p \leq n$ and integers of the form $s = n + r$ for $r \in \{1, \ldots, p - r_{n,p} - 1\}$ where $r_{n,p}$ is the uniquely determined integer with $0 \leq r_{n,p} \leq p - 1$ and $n \equiv r_{n,p} \mod p$. However, there are two cases for which this range of r is not sufficient, that is, when $n = 2^s$ with $s > 1$ and $p = 2$ we also need $r = 2$ and in case $n = 9$ and $p = 3$ we also need $r = 3$ and $r = 4$. This motivates the following notation which we use throughout the remainder of this paper.

Notation 3.9. For $n \in \mathbb{N}$, we use the following notation.

(1) $\mathcal{P}_n = \{p \mid 0 < p \leq n \text{ prime number}\}$
(2) For $p \in \mathcal{P}_n$, let $0 \leq r_{n,p} < p$ be the uniquely determined integer with $n \equiv r_{n,p} \mod p$.
(3) For $p \in \mathcal{P}_n$, we set

$$\mathcal{R}_{n,p} = \begin{cases} \{1,2\} & \text{if } n = 2^s \text{ with } s > 1 \text{ and } p = 2 \\ \{1,2,3,4\} & \text{if } n = 9 \text{ and } p = 3 \\ \{r \mid 1 \leq r \leq p - r_{n,p} - 1\} & \text{else} \end{cases}$$

As mentioned above, our goal is to reformulate the question whether the binomial polynomial $\binom{x}{n}$ is absolutely irreducible as a homogeneous equation system.

Definition 3.10. For $n \geq 2$, we define the *valuation matrix* $\mathbf{A}_n$ of n by

$$\mathbf{A}_n = (\mathbf{v}_p(n + r - j) - \mathbf{v}_p(n - j))_{\substack{p \in \mathcal{P}_n, r \in \mathcal{R}_{n,p} \\ 0 \leq j \leq n-1}}$$

cf. Notation 3.9.

Remark 3.11. Let $\mathbf{k} = (k_0, \ldots, k_{n-1})^t$ and $\mathbf{l} = (\ell_0, \ldots, \ell_{n-1})^t \in \mathbb{N}_0^n$ be (the vectors of) the exponents of the factors f and g of $\binom{x}{n}^m$, cf. Proposition 3.5. The condition that f and g are integer-valued immediately implies that

$$\mathbf{A}_n\mathbf{k} \geq 0 \text{ and } \mathbf{A}_n\mathbf{l} \geq 0,$$

cf. Remark 3.8.

Our next step is to show that the exponent vectors $\mathbf{k} = (k_0, \ldots, k_{n-1})^t$ and $\mathbf{l} = (\ell_0, \ldots, \ell_{n-1})^t \in \mathbb{N}_0^n$ of possible integer-valued factors f and g of $\binom{x}{n}^m$ are actually solutions to the homogeneous equation system $\mathbf{A}_n x = 0$, cf. Remark 3.11. Before we prove this in Proposition 3.15, we need the following lemma which states that the row sums of $\mathbf{A}_n$ equal zero.

Lemma 3.12. *Let $n \in \mathbb{N}$, $p \in \mathcal{P}_n$ and $r \in \mathcal{R}_{n,p}$, cf. Notation 3.9. Then*

$$\sum_{j=0}^{n-1} (\mathbf{v}_p(n + r - j) - \mathbf{v}_p(n - j)) = 0$$

Proof. Let $p \in \mathcal{P}_n$ and q and $r_{n,p}$ be the uniquely determined integers such that $n = qp + r_{n,p}$ and $0 \leq r_{n,p} \leq p - 1$. First, we assume that $0 \leq r \leq p - r_{n,p} - 1$ (this covers all cases except the one where n is a power of 2 and $r = p = 2$ and the one where $r \in \{3, 4\}$, $p = 3$ and $n = 9$). Then for all $0 \leq j \leq n - 1$ at most one of the numbers $n + r - j$ and $n - j$ is divisible by p. Since

$$2 = n + 1 - (n - 1) \leq n + r - j \leq n + (p - r_{n,p} - 1) = qp + (p - 1),$$

it follows that $n + r - j = kp$ if and only if $k \in \{1, \ldots, q\}$ and hence

$$\sum_{j=0}^{n-1} \mathbf{v}_p(n + r - j) = \sum_{k=1}^{q} \mathbf{v}_p(kp).$$

Similarly, since $1 \leq n - j \leq n = qp + r_{n,p}$, it follows that

$$\sum_{j=0}^{n-1} \mathbf{v}_p(n-j) = \sum_{k=1}^{q} \mathbf{v}_p(kp)$$

which proves the assertion in this case.

Next, we discuss the case where $n = 2^s$ is a power of 2 with $s > 1$ and $r = p = 2$. We want to show that $\sum_{j=0}^{n-1}(\mathbf{v}_2(n+2-j) - \mathbf{v}_2(n-j)) = 0$.

Let $v_j = \mathbf{v}_2(n+2-j) - \mathbf{v}_2(n-j)$. As n is even, $v_j = 0$ whenever j is odd. Moreover, for $j \in 4\mathbb{Z}$, $\mathbf{v}_2(n+2-j) = \mathbf{v}_2(n-2-j) = 1$ and $\mathbf{v}_2(n-j) > 1$. Therefore, for $j \in 4\mathbb{Z}$,

$$v_j + v_{j+2} = \mathbf{v}_2(n+2-j) - \mathbf{v}_2(n-j) + \mathbf{v}_2(n-j) - \mathbf{v}_2(n-2-j) = 0$$

which implies that

$$\sum_{j=0}^{n-1} v_j = \sum_{\substack{j=0 \\ j \in 2\mathbb{Z}}}^{n-1} v_j = \sum_{\substack{j=0 \\ j \in 4\mathbb{Z}}}^{n-1} v_j + v_{j+2} = 0.$$

Finally, it remains to discuss the case where $n = 9$, $p = 3$ and $r = 3$, 4. The two rows are

$$\begin{pmatrix} -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & -1 & 2 & 0 & -1 & 1 & 0 \end{pmatrix}.$$

An easy computation verifies the claim and the assertion follows. □

Lemma 3.12 is key to show that the exponent vectors $\mathbf{k} = (k_0, \ldots, k_{n-1})^t$ and $\mathbf{l} = (\ell_0, \ldots, \ell_{n-1})^t$ of potential factors f and g of $\binom{x}{n}^m$ can be viewed as the solutions of the homogeneous equation system $\mathbf{A}_n x = 0$, or equivalently, as elements of $\ker(\mathbf{A}_n)$.

Proposition 3.13. *Let n, $m \geq 2$ be integers and $\mathbf{k} = (k_0, \ldots, k_{n-1})^t$ and $\mathbf{l} = (\ell_0, \ldots, \ell_{n-1})^t \in \mathbb{N}_0^n$ such that*

$$\prod_{i=0}^{n-1} \left(\frac{x-i}{n-i}\right)^{k_i}, \prod_{i=0}^{n-1} \left(\frac{x-i}{n-i}\right)^{\ell_i} \in \mathrm{Int}(\mathbb{Z})$$

and $k_j + \ell_j = m$ for all $0 \leq j \leq n-1$.

Then $\mathbf{k}$, $\mathbf{l} \in \ker(\mathbf{A}_n)$.

Proof. The integer-valued condition in the hypothesis implies that $\mathbf{A}_n\mathbf{k} \geq 0$ and $\mathbf{A}_n\mathbf{l} \geq 0$, cf. Remark 3.11.

Moreover, by Lemma 3.12,

$$\begin{aligned} 0 &= m \cdot \sum_{j=0}^{n-1} (\mathbf{v}_p(n+r-j) - \mathbf{v}_p(n-j)) \\ &= \sum_{j=0}^{n-1} (\mathbf{v}_p(n+r-j) - \mathbf{v}_p(n-j)) \cdot (k_j + l_j) \\ &= \mathbf{A}_n(\mathbf{k} + \mathbf{l}) = \mathbf{A}_n\mathbf{k} + \mathbf{A}_n\mathbf{l} \end{aligned}$$

holds and therefore $\mathbf{A}_n\mathbf{k} = \mathbf{A}_n\mathbf{l} = 0$. □

Remark 3.14. It follows from Lemma 3.12 that $(k)_{0\le j\le n-1}$ are elements of $\ker(\mathbf{A}_n)$ for all $k \in \mathbb{Q}$. In particular, this implies that $(1)_{0\le j\le n-1} \in \ker(\mathbf{A}_n)$, or equivalently, $\operatorname{rank}(\mathbf{A}_n) < n$.

Moreover, note that the exponent vector of $\binom{x}{n}^k$ (written in the form of Proposition 3.5) is a scalar multiple of the element $(1)_{0\le j\le n-1}$.

This brings us to the main result of this section which states that, in order to show that $\binom{x}{n}$ is absolutely irreducible, it suffices to prove that the rank of the valuation matrix $\mathbf{A}_n$ of n is exactly $n-1$.

Proposition 3.15. *Let $n \ge 2$ be an integer and $\mathbf{A}_n$ the valuation matrix of n.*

If $\operatorname{rank}(\mathbf{A}_n) = n-1$*, then $\binom{x}{n}$ is absolutely irreducible.*

Proof. Let $m \ge 2$ be an integer and assume that $f, g \in \operatorname{Int}(\mathbb{Z})$ such that $\binom{x}{n}^m = f \cdot g$. According to Propositions 3.5 and 3.13 it follows that

$$f = \pm \prod_{i=0}^{n-1} \left(\frac{x-i}{n-i}\right)^{k_i} \text{ and } g = \pm \prod_{i=0}^{n-1} \left(\frac{x-i}{n-i}\right)^{\ell_i}$$

where $(k_0, \dots, k_{n-1})^t, (\ell_0, \dots, \ell_{n-1})^t \in \ker(\mathbf{A}_n) \cap \mathbb{N}_0^n$.

Since $\dim \ker(\mathbf{A}_n) = n - \operatorname{rank}(\mathbf{A}_n) = 1$ and $(1)_{0\le j\le n-1} \in \ker(\mathbf{A}_n)$ by Lemma 3.12, it follows that $\ker(\mathbf{A}_n) = \operatorname{span}_{\mathbb{Q}}\{(1)_{0\le j\le n}\}$. Therefore, $k_0 = k_1 = \dots = k_{n-1}$ and $\ell_0 = \ell_1 = \dots = \ell_{n-1}$ which implies that $f = \binom{x}{n}^{k_0}$ and $g = \binom{x}{n}^{\ell_0}$.

It follows that $\binom{x}{n}$ is absolutely irreducible by Remark 3.4. □

To be able to prove that $\operatorname{rank}(\mathbf{A}_n) = n-1$ for all n in Section 5, we first need to show Theorem 2 which is part of our number-theoretic toolbox.

4. Number-theoretic toolbox

The goal of this section is to prove Theorem 2. Given an integer $n > 10$, let P be the maximal prime number with $P \le n$. We show that for every $2 \le k \le n-P$ there exists a prime number $p > 2k$ which divides one of the numbers $n, n-1, \dots, n-k+1$. Note that the condition $k \le n-P$ is equivalent to $n, n-1, \dots, n-k+1$ being composite numbers.

The literature provides us a collection of number-theoretic facts which, putting the pieces together, give a proof of Theorem 2. We split the proof into cases and present partial results on their own. We start with the case for large n.

Proposition 4.1. *Let $2 \le k < n$ be positive integers with $n \ge 4{,}021{,}520$.*

If $n, n-1, \dots, n-k+1$ are composite numbers, then one of them has a prime divisor $p > 2k$.

For the proof we use the following facts from the literature.

Fact 4.2 (Bertrand's postulate)**.** *For all integers $n \ge 4$, there exists a prime number p with $\frac{n}{2} < p < n$.*

Fact 4.3 ([15, Theorem 12])**.** *Let m be an integer with $m \ge 2{,}010{,}760$. Then there exists a prime number p such that $m < p < (1 + \frac{1}{16597})m$.*

Fact 4.4 ([9, Theorem 1])**.** *Let m and $k \geq 2$ be positive integers such that $m > \max\{k+13, \frac{279}{262}k\}$.*

Then the product $m(m+1)\cdots(m+k-1)$ has a prime factor greater than $2k$.

Proof. Let P be the largest prime number with $P \leq n$. First, we prove the following

Claim. $P+1 > \max\{(n-P)+13, \frac{279}{262}(n-P)\}$

Assume for a moment that the claim holds. By hypothesis, the numbers $n, n-1, \ldots, n-k+1$ are composite numbers which implies that $n-k+1 > P$ or, equivalently, $n-P \geq k$. Therefore

$$n-k+1 \geq P+1 > \max\{(n-P)+13, \frac{279}{262}(n-P)\} \geq \max\{k+13, \frac{279}{262}k\}$$

and the assertion follows from Fact 4.4 with $m = n-k+1$.

It remains to prove the claim. Let Q be the smallest prime number with $Q > n$. According to Fact 4.3 it follows that $P > \frac{n}{2} \geq 2,010,760$ and $Q < (1+\frac{1}{16597})P$.

Therefore, $2P-n > 2P-Q > (1-\frac{1}{16597})P > 12$ which implies that $P+1 > (n-P)+13$. Moreover, since $2(1+\frac{1}{16597}) < 1+\frac{279}{262}$ it follows that $\frac{279}{262}n - 1 < 2n < 2Q < (1+\frac{279}{262})P$ and hence $P+1 > \frac{279}{262}(n-P)$. This completes the proof of the claim. □

It remains to discuss the case where $n < 4,021,520$. Here, we can use a result of Laishram and Shorey [10]. They proved Grimm's conjecture for sequences of consecutive positive integers whose minimum does not exceed a certain bound.

Fact 4.5. [10, Theorem 1] *Let $k < n$ such that $n-k+1 \leq 1.9 \cdot 10^{10}$.*

If $n, n-1, \ldots, n-k+1$ are composite numbers then there exist pairwise distinct prime numbers $p_0, p_1, \ldots, p_{k-1}$ with $p_i \mid n-i$ for all $0 \leq i \leq k-1$.

Remark 4.6. Let p_k denote the k-th prime number. It is easily seen that $p_k > 2k$ for $k \geq 5$. Therefore, under the hypothesis of Fact 4.5, there exists a prime number $p > 2k$ which divides one of the numbers $n, n-1, \ldots, n-k+1$.

We treat the cases $k = 2, 3, 4$ in the proof below where we use the following facts.

Fact 4.7 (Catalan's conjecture, [11])**.** *Mihăilescu proved Catalan's conjecture which states that the only consecutive positive integers which are non-prime prime powers are 8 and 9.*

Fact 4.8 (Pillai's conjecture for difference 2, [16])**.** *The only non-prime prime powers p^x and q^y less than 10^{18} with $p^x - q^y = 2$ are 25 and 27.*

Finally, we restate the desired theorem and give a complete proof.

Theorem 2. *Let $n > 10$ be an integer and P the maximal prime number with $P \leq n$.*

If $2 \leq k \leq n-P$ then there exists a prime number $p > 2k$ which divides one of the numbers $n, n-1, \ldots, n-(k-1)$.

Remark 4.9. It is immediately clear that n has a prime divisor $p > 2$ $(k = 1)$ if and only if n is not a power of 2.

It turns out that we need to work around the lack of other prime divisors in case $n = 2^x$, cf. Definition 3.10.

Proof. For $n \geq 4,021,520$, the assertion follows from Proposition 4.1.

For $n < 4,021,520$ and $k \geq 5$, the assertion follows from Fact 4.5, cf. Remark 4.6.

We treat the cases $k = 2$, 3, 4 separately.

Case $k = 2$**:** Since n and $n - 1$ are composite numbers and not both of them are prime powers by Fact 4.7, one of them has two distinct prime divisors. Since 2 divides exactly one of the numbers n and $n - 1$ it follows that $n(n - 1)$ has three distinct prime divisors, one of which is necessarily at least $5 \geq 2 \cdot 2$.

Case $k = 3$**:** We use a similar argument as above. If $n = 27$ then $p = 13 > 2 \cdot 3$ divides $n - 1 = 26$. By Fact 4.8, we can therefore assume that at most one of the numbers n, $n-1$ and $n-2$ is a prime power. Clearly, one of these numbers is divisible by 3. Without restriction we assume that $3 \mid n$. Moreover, either both n and $n - 2$ are even or $n - 1$ is even. We visualize the two cases in Table 1. At most one of the numbers n, $n - 1$ and $n - 2$

n	$n-1$	$n-2$
2,3		2
3	2	

TABLE 1. Distribution of 2 and 3 as prime factors of n, $n-1$ and $n - 2$.

is a prime power. Therefore, at least two of them have two distinct prime divisors. In terms of Table 1 this means it is possible to fill the columns with additional prime numbers representing the respective divisors. The primes 2 and 3 are already covered, so we can only use prime numbers greater or equal 5 for this purpose and each of them can be used at most one. Hence we need at least two more distinct prime numbers which implies that one of them is at least $7 > 2 \cdot 3$.

Case $k = 4$**:** If $27 \in \{n, n-1\}$, then $p = 13 > 2 \cdot 4$ divides $n-1$ or $n-2$. We can therefore assume that either at most one of the numbers n, $n - 1$, $n - 2$, $n - 3$ is a prime power or n and $n - 3$ are the only prime powers among them.

Similar to the previous case, we visualize the possible cases for the prime divisors 2 and 3 in Table 2. Without restriction, we assume that n is even. At most two of the numbers are divisible by 3. Moreover, we assume here that at most one of the numbers n, $n-1$, $n-2$ and $n-3$ is a prime power. The other case where n and $n - 3$ are the only prime powers among them follows in the same way. Analogously to above, three of the four columns contain two distinct prime divisors. Therefore, in each case there are at least three more pairwise distinct prime divisor of of n, $n - 1$, $n - 2$ and $n - 3$, all at least 5. It follows that one of them is at least $11 > 2 \cdot 4$. □

n	$n-1$	$n-2$	$n-3$
2,3		2	3
2	3	2	
2		2,3	

TABLE 2. Distribution of 2 and 3 as prime factors of n, $n-1$, $n-2$ and $n-3$.

5. rank($\mathbf{A}_n$) AND THE p-BLOCKS

In this section we show that $\operatorname{rank}(\mathbf{A}_n) = n-1$ for all $n \geq 2$. Once this is shown, it follows from Proposition 3.15 that $\binom{x}{n}$ is absolutely irreducible.

Our strategy is to show that the columns $j \in \{0, 1, \ldots, n-1\} \setminus \{n-P\}$ are linearly independent where $P = \max \mathcal{P}_n$ is the maximal prime number which is less than or equal n. To do so, we split these $n-1$ columns of $\mathbf{A}_n$ into two groups, namely the "outer" $2(n-P)$ columns indexed with $\{0, \ldots, n-P-1\} \cup \{P, P+1, \ldots, n-1\}$ and the "inner" $2P-n-1$ columns indexed with $\{n-P+1, \ldots, P-1\}$. According to the next lemma, $n-P < P-1$ which implies the that we can always partition the $n-1$ columns in this way.

Lemma 5.1. *Let $n \geq 2$ be an integer and $P = \max \mathcal{P}_n$ be the largest prime number which is less than or equal n (cf. Notation 3.9).*

Then $P-1 > n-P$.

Proof. According to Bertrand's postulate (see Fact 4.2), there is a prime number q with $P < q < 2P$. Since $P = \max \mathcal{P}_n$ is the maximal prime number at most n, it follows that $n < q < 2P$. Therefore, $n < 2P-1$ which is equivalent to $n-P < P-1$. □

Next, we introduce the $\mathbb{Q}$-spans of groups of columns of a matrix that are of interest for our investigation. This is done slightly more general as the two column groups of $\mathbf{A}_n$ we mentioned above. On the one hand, we want to switch between the whole matrix $\mathbf{A}_n$ and certain submatrices which we introduce below (the p-blocks). On the other hand, to show that the outermost $2(n-P)$ columns or $\mathbf{A}_n$ are linearly independent, we use an inductive argument for which the definition below is convenient.

Definition 5.2. Let $n \geq 2$ and ℓ be positive integers and $C \in \mathbb{Q}^{\ell \times n}$ with columns $c_0, \ldots, c_{n-1}$. Further, let $P = \max \mathcal{P}_n$ be the largest prime number less than or equal to n, cf. Notation 3.9.

(1) We denote by $\mathcal{I}(C) = \operatorname{span}\{c_{n-P+1}, c_{n-P+1}, \ldots, c_{P-1}\}$ be the span of the "inner" $2P-n-1$ columns of C.

(2) For $0 \leq s < t \leq n-P-1$, we denote by

$$\mathcal{O}_{[s,t]}(C) = \operatorname{span}\{c_j \mid s \leq j \leq t \text{ and } n-(t+1) \leq j \leq n-(s+1)\}$$

be the $\mathbb{Q}$-span of the $2(t-s+1)$ columns in the range between s and t in the left half of C and the columns $n-(t+1)$ and $n-(s+1)$ in the left half of C (the same range of columns on the "left" and the "right" side of C).

For a visualization of the generating columns of these spans, see Figure 1.

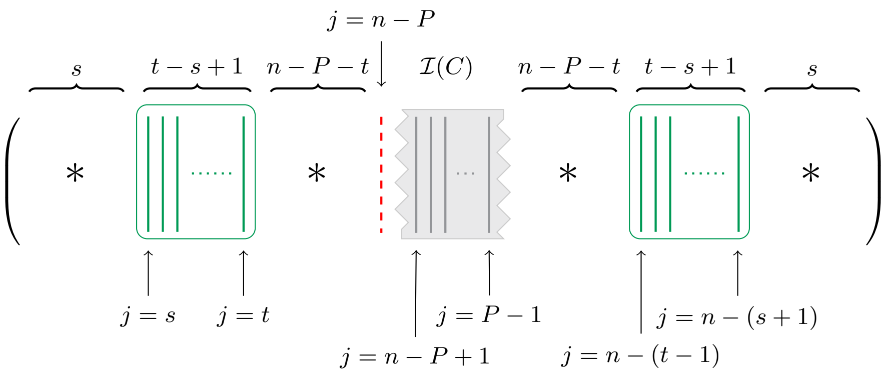

FIGURE 1. $\mathcal{I}(C)$ is spanned by the columns in the grey zigzag-area; $\mathcal{O}_{[s,t]}(C)$ is spanned by the columns in the green rectangles

Remark 5.3. Let $n \in \mathbb{N}$ and $P = \max \mathcal{P}_n$, cf. Notation 3.9.

Our goal is to show that

(i) $\dim \mathcal{O}_{[0,n-P-1]}(\mathbf{A}_n) = 2(n-P)$,
(ii) $\dim \mathcal{I}(\mathbf{A}_n) = 2P - n - 1$ and
(iii) $\mathcal{I}(\mathbf{A}_n) \cap \mathcal{O}_{[0,n-P-1]}(\mathbf{A}_n) = \mathbf{0}$.

Together this then implies that the sum $\mathcal{I}(\mathbf{A}_n) + \mathcal{O}_{[0,n-P-1]}(\mathbf{A}_n)$ is direct and

$$\dim \left(\mathcal{I}(\mathbf{A}_n) \oplus \mathcal{O}_{[0,n-P-1]}(\mathbf{A}_n)\right) = 2(n-P) + (2P-n-1) = n-1.$$

Since $n > \operatorname{rank}(\mathbf{A}_n)$ by Remark 3.14, it further follows that

$$n > \operatorname{rank}(\mathbf{A}_n) \geq \dim \left(\mathcal{I}(\mathbf{A}_n) \oplus \mathcal{O}_{[0,n-P-1]}(\mathbf{A}_n)\right) = n-1$$

and hence

$$\operatorname{rank}(\mathbf{A}_n) = n-1.$$

Then, by Proposition 3.15, $\binom{x}{n}$ is absolutely irreducible.

Note that Assertion (i) is shown in Proposition 5.14 and Assertions (ii) and (iii) are proven in Corollary 5.10 below.

For our further investigation, it turns out to be useful to split the valuation matrix $\mathbf{A}_n$ into blocks of rows, each block corresponding to a prime number $p \in \mathcal{P}_n$.

5.1. The structure of a p-block.

Definition 5.4. For $n \in \mathbb{N}$ and $p \in \mathcal{P}_n$, we define the *p-block* $\mathbf{B}_{n,p}$ as the $|\mathcal{R}_{n,p}| \times n$ integer matrix defined by

$$\mathbf{B}_{n,p} = (\mathbf{v}_p(n+r-j) - \mathbf{v}_p(n-j))_{\substack{r \in \mathcal{R}_{n,p} \\ 0 \leq j \leq n-1}}$$

For our purposes, we focus on the distribution of zero and non-zero entries of leftmost p and rightmost $p-1$ columns of the p-blocks. In this sense $\mathbf{B}_{n,p}$ has a "structure" which is described in Propositions 5.6 and 5.7.

Remark 5.5. Note that in the exceptional cases where $\mathcal{R}_{n.p} \neq \{r \mid 1 \leq r \leq p - r_{n,p} - 1\}$, Propositions 5.6 and 5.7 only give information about the first $p - r_{n,p} - 1$ rows of the corresponding p-blocks. However, in all situations where apply the two propositions below, we can rule out these exceptional cases.

Proposition 5.6. *Let $n \in \mathbb{N}$, $p \in \mathcal{P}_n$ (cf. Notation 3.9) and $\mathbf{B}_{n,p} = (b_{r,j})$ the corresponding p-block.*

For $0 \leq j \leq p-1$ and $1 \leq r \leq p - r_{n,p} - 1$ the following holds:

$$b_{r,j} = \begin{cases} -\mathbf{v}_p(n - r_{n,p}) & j = r_{n,p} \\ \mathbf{v}_p(n - r_{n,p}) & j = r + r_{n,p} \\ 0 & \textit{else.} \end{cases}$$

where, as usual, $0 \leq r_{n,p} < p$ with $n \equiv r_{n,p} \mod p$.

A visualization of the statement in Proposition 5.6 can be found in Figure 2.

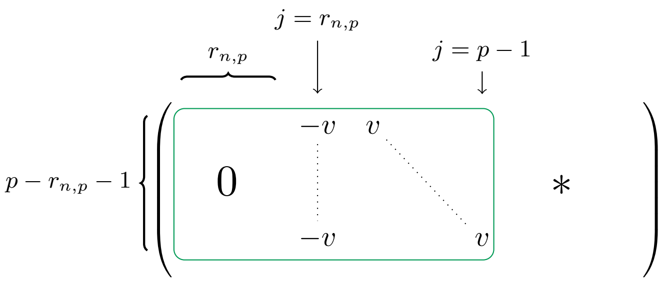

FIGURE 2. Leftmost p columns of (the first $p - r_{n,p} - 1$ rows of) $\mathbf{B}_{n,p}$ where $v = \mathbf{v}_p(n - r_{n,p})$, cf. Proposition 5.6

Proof. Let $0 \leq j \leq p-1$. In this range of j it follows from the definition of $r_{n,p}$ that

$$\mathbf{v}_p(n-j) \neq 0 \iff j = r_{n,p}$$

holds. Moreover, $\mathbf{v}_p(n+r-j) \neq 0$ if and only if $p \mid r_{n,p}+r-j$, or equivalently, $j \equiv r_{n,p} + r \mod p$. However, since $1 \leq r \leq p - r_{n,p} - 1$, it follows that $1 + r_{n,p} \leq r + r_{n,p} \leq p - 1$ and therefore $\mathbf{v}_p(n + r - j) \neq 0$ if and only if $j = r_{n,p} + r$.

Therefore, all entries of first $r_{n,p}$ columns of $\mathbf{B}_{n,p}$ are zero and the entries of the $r_{n,p}+1$-st column equal $-\mathbf{v}_p(n - r_{n,p}) \neq 0$ and the remaining columns $r_{n,p} + 1 \leq j \leq p-1$ are zero except for the entry in row $r = j - r_{n,p}$ which amounts to $\mathbf{v}_p(n - r_{n,p})$. The assertion follows. □

Proposition 5.6 describes the zero and non-zero entries of the leftmost p columns of the p-block $\mathbf{B}_{n,p}$. Next, we have a closer look at the rightmost $p-1$ columns.

Proposition 5.7. *Let $n \in \mathbb{N}$, $p \in \mathcal{P}_n$ (cf. Notation 3.9) and $\mathbf{B}_{n,p} = (b_{r,j})$ the corresponding p-block.*

For $n-(p-1) \leq j \leq n-1$ and $r \in \mathcal{R}_{n,p}$ the following holds:

$$b_{r,j} = \begin{cases} 1 & j = n-p+r \\ 0 & \textit{else.} \end{cases}$$

A visualization of the statement in Proposition 5.7 can be found in Figure 3.

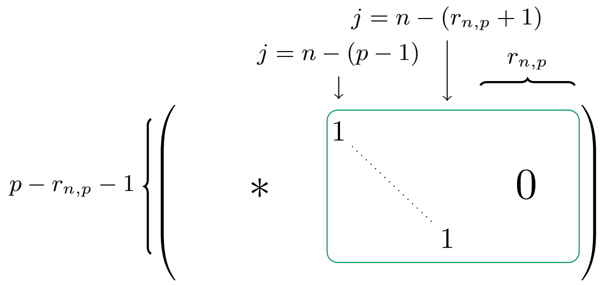

FIGURE 3. Rightmost $p-1$ columns of (the first $p-r_{n,p}-1$ rows of) $\mathbf{B}_{n,p}$, cf. Proposition 5.7

Proof. For $n-(p-1) \leq j \leq n-1$, it follows that $1 \leq n-j \leq p-1$ and therefore $\mathbf{v}_p(n-j) = 0$. Moreover, $\mathbf{v}_p(n+r-j) \neq 0$ is equivalent to $p \mid n-j+r$. However, given the ranges of $n-j$ and r, this is the case if and only if $n-j+r = p$ in which case $\mathbf{v}_p(n+r-j) = 1$. The assertion follows. □

5.2. **The inner columns of $\mathbf{A}_n$.** In this section we prove the results concerning the "inner" columns of $\mathbf{A}_n$, namely we show that the goals (ii) and (iii) formulated in Remark 5.3 hold, see Corollary 5.10. Moreover, this leads to the special case of Theorem 1 where $n = P$ is a prime number, see Corollary 5.12 below.

In this section we exploit the structure of the P-block $\mathbf{B}_{n,P}$ where $P = \max \mathcal{P}_n$ is the maximal prime number which is less than or equal to n.

Remark 5.8. Let $n \geq 2$ and $P = \max \mathcal{P}_n$, cf. Notation 3.9. If $n = 2^s$ with $s \geq 2$, then $P > 2$ and if $n = 9$, then $P = 7 > 3$. Therefore, $\mathcal{R}_{n,P} = \{r \mid 1 \leq r \leq P - r_{n-P-1}\}$.

Hence, the P-block $\mathbf{B}_{n,P}$ always consists of $P - r_{n,P} - 1$ rows and Propositions 5.6 and 5.7 refer to the whole P-block, cf. Remark 5.5.

Proposition 5.9. *Let $n \geq 2$ be an integer and $P = \max \mathcal{P}_n$ be the largest prime number which is less than or equal to n.*

Then

(1) $\dim \mathcal{I}(\mathbf{B}_{n,P}) = 2P - n - 1$
(2) $\mathcal{O}_{[0,n-P-1]}(\mathbf{B}_{n,P}) = \mathbf{0}$

Proof. By Lemma 5.1, $n - P < P - 1$ holds. In particular, it follows that $r_{n,P} = n - P$. Moreover, $\mathcal{R}_{n,P} = \{1, 2, \ldots, 2P - n - 1\}$, cf. Remark 5.8. By Propositions 5.6 and 5.7 the P-block $\mathbf{B}_{n,P}$ is of the following form, see also Figure 4:

(a) All columns indexed with $j \in \{0, \dots, n-P-1\} \cup \{P, \dots, P+1\}$ are zero columns.
(b) Each entry of the column $j = r_{n,P} = n - P$ is equal -1.
(c) Each column indexed with $n - P + 1 \le j \le P - 1$ contains exactly one non-zero entry, namely, the entry in row $r = j + P - n$ is equal 1.

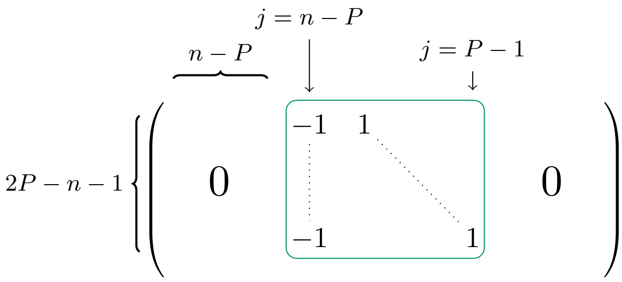

FIGURE 4. $\mathbf{B}_{n,P}$

It immediately follows from (a), that $\mathcal{O}_{[0,n-P-1]}(\mathbf{B}_{n,P}) = \mathbf{0}$. Moreover, as $\mathcal{I}(\mathbf{B}_{n,P})$ is spanned by $2P - n - 1$ vectors, the dimension is bounded by $2P-n-1$. By (c), the submatrix of $\mathbf{B}_{n,P}$ consisting of columns $n-P+1 \le j \le P-1$ is the identity matrix of dimension $P - 1 - (n - P - 1) + 1 = 2P - n - 1$. As all these columns are elements of $\mathcal{I}(\mathbf{B}_{n,P})$ it follows that $\dim \mathcal{I}(\mathbf{B}_{n,P}) = 2P - n - 1$. ☐

Corollary 5.10. *Let $n \ge 2$ be an integer and $P = \max \mathcal{P}_n$ be the largest prime number less than or equal to n, cf. Notation 3.9.*
Then
(i) $\dim \mathcal{I}(\mathbf{A}_n) = 2P - n - 1$ *and*
(ii) $\mathcal{O}_{[0,n-P-1]}(\mathbf{A}_n) \cap \mathcal{I}(\mathbf{A}_n) = \mathbf{0}$.

Note that the assertions in Corollary 5.10 are exactly the Goals (ii) and (iii) of Remark 5.3.

Proof. Since $\mathcal{I}(\mathbf{A}_n)$ is spanned by $2P-n-1$ vectors, $\dim \mathcal{I}(\mathbf{A}_n) \le 2P-n-1$ holds. Moreover, as the projection $\pi : \mathcal{I}(\mathbf{A}_n) \to \mathcal{I}(\mathbf{B}_{n,P})$ is an epimorphism, it follows that $\dim \mathcal{I}(\mathbf{A}_n) = 2P - n - 1$ by Proposition 5.9.

For the second assertion, let $v \in \mathcal{O}_{[0,n-P-1]}(\mathbf{A}_n) \cap \mathcal{I}(\mathbf{A}_n)$. Let a_{n-P+1}, $\dots$, a_{P-1} denote the columns of $\mathbf{A}_n$ which span $\mathcal{I}(\mathbf{A}_n)$. Then there exist $\lambda_{n-P+1}, \dots, \lambda_{P-1} \in \mathbb{Q}$ such that

$$v = \sum_{j=n-P+1}^{P-1} \lambda_j a_j \in \mathcal{O}_{[0,n-P-1]}(\mathbf{A}_n) \cap \mathcal{I}(\mathbf{A}_n). \tag{2}$$

Then

$$\pi(v) = \sum_{j=n-P+1}^{P-1} \lambda_j b_j^{(P)} \in \mathcal{O}_{[0,n-P-1]}(\mathbf{B}_{n,P}) \cap \mathcal{I}(\mathbf{B}_{n,P}) \tag{3}$$

where $b_j^{(P)}$ denotes the j-th column of $\mathbf{B}_{n,P}$ for $n - P + 1 \le j \le P - 1$. By Proposition 5.9, $\mathcal{O}_{[0,n-P-1]}(\mathbf{B}_{n,P}) = \mathbf{0}$ and the columns $b_j^{(P)}$ are linearly independent. Hence $\pi(v) = \mathbf{0}$ and Equation (7) implies that $\lambda_j = 0$ for $n - P + 1 \leq j \leq P - 1$. Therefore, by plugging into Equation (2), it follows that $v = 0$ which completes the proof. □

It follows from Corollary 5.10 that the "inner" $2P - n - 1$ columns of $\mathbf{A}_n$, that is, the columns which span $\mathcal{I}(\mathbf{A}_n)$ are linearly independent. This immediately implies the next corollary.

Corollary 5.11. *Let $n \geq 2$ be an integer and $\mathbf{A}_n$ its valuation matrix. Then* $\operatorname{rank}(\mathbf{A}_n) \geq 2P - n - 1$.

As shown below, the results so far imply the absolute irreducibility of $\binom{x}{n}$ in the special case where $n = P$ is a prime number. Note that this has already been shown by Frisch and Nakato [6, Example 2.6] and our proof only serves as an alternative.

Corollary 5.12. *Let P be a prime number, then $\binom{x}{P}$ is absolutely irreducible.*

Proof. If $n = P$, then $\operatorname{rank}(\mathbf{A}_P) \geq 2P - P - 1 = P - 1$ by Corollary 5.11. Since $\operatorname{rank}(\mathbf{A}_P) < P$ by Remark 3.14, it follows $\operatorname{rank}(\mathbf{A}_P) = P - 1$. According to Proposition 3.15, $\binom{x}{P}$ is absolutely irreducible. □

5.3. **The outer columns of $\mathbf{A}_n$.** For a composite number $n \geq 2$, the span of the "outer" columns of $\mathbf{A}_n$ is not trivial. The goal of this section is to show that $\dim \mathcal{O}_{[0,n-P-1]}(\mathbf{A}_n) = 2(n - P)$, see Proposition 5.14. This is the final ingredient to prove that $\operatorname{rank}(\mathbf{A}_n) = n - 1$ which then further implies that $\binom{x}{n}$ is absolutely irreducible, cf. Remark 5.3.

As above, we have to exploit the structure of certain p-blocks to reach this goal. In contrast to the arguments in the previous subsection, we need to find more than one suitable p-block. Which choices to make is explained in detail in the proof of Proposition 5.14. The next proposition gives information about the dimension of $\mathcal{O}_{[r_{n,p},k-1]}(\mathbf{B}_{n,p})$ for certain choices of k.

Proposition 5.13. *Let $n \geq 2$ be a composite number, $p \in \mathcal{P}_n$ (cf. Notation 3.9) be a prime number and $\mathbf{B}_{n,p}$ the corresponding p-block.*
If $r_{n,p} + 1 \leq k \leq \min\{n - P, \frac{p + r_{n,p} - 1}{2}\}$, then
(1) $\dim \mathcal{O}_{[r_{n,p},k-1]}(\mathbf{B}_{n,p}) = 2(k - r_{n,p})$ *and*
(2) $\mathcal{O}_{[0,r_{n,p}-1]}(\mathbf{B}_{n,p}) = \mathbf{0}$.

Proof. We first treat the special case $n = 9$ and $p = 3$. Figure 5 displays $\mathbf{B}_{9,3}$. Since $r_{9,3} = 0$, the second assertion of the proposition follows trivially. Moreover, since $k \in \{1, 2\}$, the first assertion can be verified by direct computation.

For the remainder of the proof, we assume that $n \neq 9$ or $p \neq 3$. It follows from $r_{n,p} + 1 \leq k \leq \min\{n - P, \frac{p + r_{n,p} - 1}{2}\}$ that $r_{n,p} \leq p - 3$ and $2k \leq p + r_{n,p} - 1$. In particular, $p > 2$ holds and it follows that the p-block $\mathbf{B}_{n,p}$ has $p - r_{n,p} - 1 \geq 2(k - r_{n,p})$ rows, cf. Remark 5.5.

Moreover, since $k \leq \frac{p + r_{n,p} - 1}{2} \leq p - 1$, we can apply Propositions 5.6 and 5.7 to describe the outermost left k and right k columns of the (whole) p-block $\mathbf{B}_{n,p}$, depicted in Figure 6. They are of the following form:

$$\mathbf{B}_{9,3} = \begin{pmatrix} -2 & 2 & \mathcal{I}(\mathbf{B}_{9,3}) & 1 & 0 \\ -2 & 0 & & 0 & 1 \\ -1 & 0 & & 0 & 0 \\ -2 & 1 & & 1 & 0 \end{pmatrix}.$$

FIGURE 5. The 3-block $\mathbf{B}_{9,3}$ of 9.

(a) The columns indexed with $j \in \{0, \ldots, r_{n,p}-1\} \cup \{n-r_{n,p}, \ldots, n-1\}$ are zero columns (these are the $r_{n,p}$ left outermost and $r_{n,p}$ right outermost columns).
(b) The entries of column $j = r_{n,p}$ are all equal $-\mathbf{v}_p(n - r_{n,p}) \neq 0$.
(c) The columns $r_{n,p} + 1 \leq j \leq k - 1$ have exactly one non-zero entry, $\mathbf{v}_p(n - r_{n,p})$, in row $r = j - r_{n,p}$, that is, the first $k - (r_{n,p} + 1)$ rows of these columns form a diagonal matrix with $\mathbf{v}(n - r_{n,p})$ on the diagonal and all entries below are zero.
(d) The columns $n - k \leq j \leq n - (r_{n,p} + 1)$ contain exactly one non-zero entry, namely 1, in row $r = j - (n - p)$, that is, the lower $k - r_{n,p}$ rows of these columns form the identity matrix and all entries above are zero.
(e) Since $k < \frac{p+r_{n,p}}{2}$ by assumption, the first $k-(r_{n,p}+1)$ rows and the last $k - r_{n,p}$ rows are disjoint and there is at least one row in the part in the middle, namely there is at least one row indexed with $k - r_{n,p} \leq r \leq p - k - 1$.

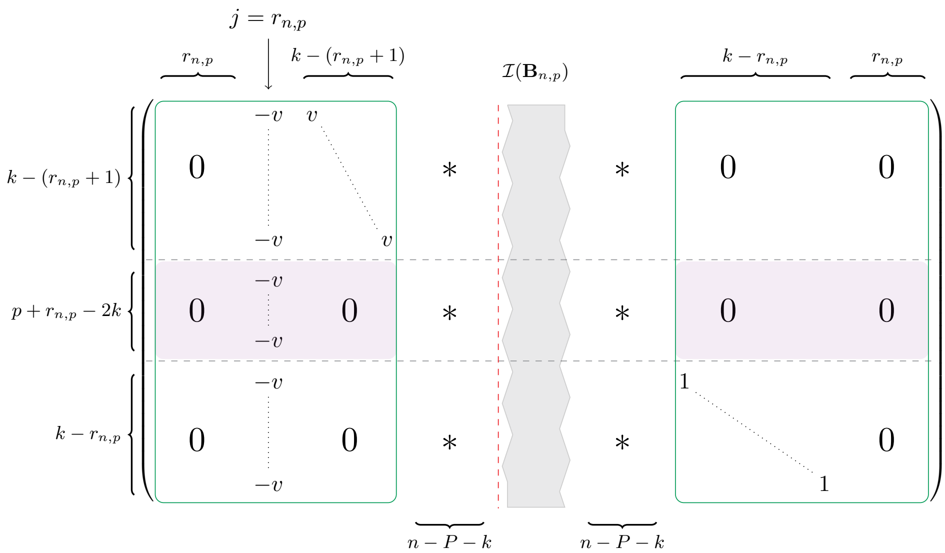

FIGURE 6. p-block $\mathbf{B}_{n,p}$ with $p + r_{n,p} > 2k$

It immediately follows, that $\mathcal{O}_{[0,r_{n,p}-1]}(\mathbf{B}_{n,p}) = \mathbf{0}$ since it is spanned only by zero columns. It remains to show that $\dim \mathcal{O}_{[r_{n,p},k-1]}(\mathbf{B}_{n,p}) = 2(k-r_{n,p})$.

Let b_j denote the j-th column of $\mathbf{B}_{n,p}$ and assume that there are $\lambda_j \in \mathbb{Q}$ with $j \in \{r_{n,p}, \dots, k-1\} \cup \{n-k, \dots, n-(r_{n,p}+1)\}$ such that

$$\sum_{j=r_{n,p}}^{k-1} \lambda_j b_j + \sum_{j=n-k}^{n-(r_{n,p}+1)} \lambda_j b_j = 0 \tag{4}$$

Since $b_{r_{n,p}}$ is the only column which has a non-zero entry in the rows indexed with $k - r_{n,p} \leq r \leq p-k-1$, Equation (4) implies that $\lambda_{r_{n,p}} = 0$. Now for each of the remaining rows, there is exactly one column with a non-zero entry in this (and no other) row. With the same reasoning we can conclude that $\lambda_j = 0$ for $r_{n,p}+1 \leq j \leq k-1$ and $n-k \leq j \leq n-(r_{n,p}+1)$. Therefore, these $2(k - r_{n,p})$ columns of $\mathbf{B}_{n,p}$ are linearly independent. □

Proposition 5.14. *Let $n \geq 2$ be a composite integer and $\mathbf{A}_n$ its valuation matrix.*

Then $\dim \mathcal{O}_{[0,n-P-1]}(\mathbf{A}_n) = 2(n-P)$.

Proof. We treat the cases $n = 9$ and $n = 10$ first as we want to exclude them below. For $n = 9$, observe that $P = 7$ and hence $n - P = 2$. It follows from Proposition 5.13, that $\dim \mathcal{O}_{[0,1]}(\mathbf{B}_{9,3}) = 4$, see also Figure 5 for direct verification. Since the projection $\mathcal{O}_{[0,1]}(\mathbf{A}_9) \to \mathcal{O}_{[0,1]}(\mathbf{B}_{9,3})$ is an epimorphism, it follows that $\dim \mathcal{O}_{[0,1]}(\mathbf{A}_9) = 4$.

The valuation matrix $\mathbf{A}_{10}$ is displayed in Figure 7. A direct computation verifies that $\operatorname{rank}(\mathcal{O}_{[0,2]}(\mathbf{A}_{10})) = 6$.

$$\mathcal{I}(\mathbf{A}_{10})$$

$$\mathbf{A}_{10} = \left(\begin{array}{ccc|c|ccc} -1 & 1 & -3 & \cdots & 2 & -1 & 1 \\ \hline 0 & -2 & 2 & \cdots & -1 & 1 & 0 \\ \hline -1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 0 & 1 & \cdots & 1 & 0 & 0 \\ -1 & 0 & 0 & \cdots & 0 & 1 & 0 \\ -1 & 0 & 0 & \cdots & 0 & 0 & 1 \\ \hline & 0 & & \cdots & & 0 & \end{array}\right) \begin{array}{l} \}\ \mathbf{B}_{10,2} \\ \}\ \mathbf{B}_{10,3} \\ \\ \}\ \mathbf{B}_{10,5} \\ \\ \\ \}\ \mathbf{B}_{10,7} \end{array} .$$

FIGURE 7. $\mathcal{O}_{[0,2]}(\mathbf{A}_{10})$ has dimension 6.

From now on, assume that $n \geq 2$ is a composite integer with $n \neq 9$ and $n \neq 10$. We prove by induction on $1 \leq k \leq n - P$ that

$$\dim \mathcal{O}_{[0,k-1]}(\mathbf{A}_n) = 2k$$

holds.

Base case $k = 1$. Since the projection $\mathcal{O}_{[0,0]}(\mathbf{A}_n) \to \mathcal{O}_{[0,0]}(\mathbf{B}_{n,p})$ is an epimorphims, it follows that $2 \geq \dim \mathcal{O}_{[0,0]}(\mathbf{A}_n) \geq \dim \mathcal{O}_{[0,0]}(\mathbf{B}_{n,p})$ for all $p \in \mathcal{P}_n$. Therefore, it suffices to show that there always exists a prime number p with $\dim \mathcal{O}_{[0,0]}(\mathbf{B}_{n,p}) = 2$.

If $n = 2^s$ is a proper power of 2, then $\mathcal{R}_{2^s,2} = \{1, 2\}$ by definition which implies that $\mathbf{B}_{2^s,2}$ has two rows. Moreover, the outermost columns of $\mathbf{B}_{2^s,2}$ are displayed in Figure 8. As $\mathbf{v}_2(n) = s \geq 2$, it follows that $\dim \mathcal{O}_{[0,0]}(\mathbf{B}_{2^2,2}) = 2$.

$$\mathbf{B}_{2^s,2} = \begin{pmatrix} \mathbf{v}_2(n) & \overset{\mathcal{I}(\mathbf{B}_{2^s,2})}{\cdots} & 1 \\ 1-\mathbf{v}_2(n) & \cdots & 0 \end{pmatrix}.$$

FIGURE 8. $\mathbf{B}_{2^s,2}$

If n is not a power of 2, then n has a prime divisor $p \geq 3$. It follows that $r_{n,p} = 0$ and hence $1 = r_{n,p} + 1 \leq k = 1 \leq \min\{n - P, p - 1\}$. Therefore, we can apply Proposition 5.13 and conclude that $\dim \mathcal{O}_{[0,0]}(\mathbf{B}_{n,p}) = 2$. This completes the proof of the base case.

Observe that the induction base case also covers the cases $n = 4$, $n = 6$ and $n = 8$ because in each of these cases $n - P = 1$. Therefore, we already handled all cases where $n \leq 10$.

We proceed with the **induction step** and prove the following claim.

Claim. There exists $p \in \mathcal{P}_n$ with $r_{n,p} \leq k - 1$ such that

(1) $\dim \mathcal{O}_{[r_{n,p},k-1]}(\mathbf{A}_n) = 2(k - r_{n,p})$ and
(2) $\mathcal{O}_{[0,r_{n,p}-1]}(\mathbf{A}_n) \cap \mathcal{O}_{[r_{n,p},k-1]}(\mathbf{A}_n) = \mathbf{0}$

Assume for a moment that this claim holds. Since $r_{n,p} < k$ we can apply the induction hypothesis, that is, $\dim \mathcal{O}_{[0,r_{n,p}-1]}(\mathbf{A}_n) = 2r_{n,p}$ and hence

$$\begin{aligned}\dim \mathcal{O}_{[0,k-1]}(\mathbf{A}_n) &= \dim \left(\mathcal{O}_{[0,r_{n,p}-1]}(\mathbf{A}_n) \oplus \mathcal{O}_{[r_{n,p},k-1]}(\mathbf{A}_n)\right) \\ &= \dim \mathcal{O}_{[0,r_{n,p}-1]}(\mathbf{A}_n) + \dim \mathcal{O}_{[r_{n,p},k-1]}(\mathbf{A}_n) \\ &= 2r_{n,p} + 2(k - r_{n,p}) \\ &= 2k\end{aligned}$$

and the assertion follows.

Next, we prove the claim. By Theorem 2 (for which we needed to assume that $n > 10$), there exists $p \in \mathcal{P}_n$ such that $p > 2k$ and $r_{n,p} \leq k - 1$.

Therefore $r_{n,p} + 1 \leq k \leq \min\{n - P, \frac{p-1}{2}\} \leq \min\{n - P, \frac{p+r_{n,p}-1}{2}\}$ and hence, by Proposition 5.13, $\dim \mathcal{O}_{[r_{n,p},k-1]}(\mathbf{B}_{n,p}) = 2(k - r_{n,p})$ holds.

Let $\pi : \mathcal{O}_{[r_{n,p},k-1]}(\mathbf{A}_n) \to \mathcal{O}_{[r_{n,p},k-1]}(\mathbf{B}_{n,p})$ be the projection. As π is an epimorphism, it follows that $\dim \mathcal{O}_{[r_{n,p},k-1]}(\mathbf{A}_n) \geq \dim \mathcal{O}_{[r_{n,p},k-1]}(\mathbf{B}_{n,p})$. Hence (1) of the claim above follows.

To prove (2) of the claim, assume that $v \in \mathcal{O}_{[0,r_{n,p}-1]}(\mathbf{A}_n) \cap \mathcal{O}_{[r_{n,p},k-1]}(\mathbf{A}_n)$ and let $\mathcal{J}_0 = \{j \mid 0 \leq j \leq r_{n,p} - 1 \text{ or } n - r_{n,p} \leq j \leq n - 1\}$ and $\mathcal{J}_1 = \{j \mid r_{n,p} \leq j \leq k-1 \text{ or } n-k \leq j \leq n-(r_{n,p}+1)\}$ be the indices of the columns of $\mathbf{A}_n$ which span $\mathcal{O}_{[0,r_{n,p}-1]}(\mathbf{A}_n)$ and $\mathcal{O}_{[r_{n,p},k-1]}(\mathbf{A}_n)$, respectively.

For $0 \leq j \leq n-1$, let a_j with denote the columns of $\mathbf{A}_n$ and $\lambda_j \in \mathbb{Q}$ such that

$$v = \sum_{j \in \mathcal{J}_0} \lambda_j a_j = \sum_{j \in \mathcal{J}_1} \lambda_j a_j \in \mathcal{O}_{[0,r_{n,p}-1]}(\mathbf{A}_n) \cap \mathcal{O}_{[r_{n,p},k-1]}(\mathbf{A}_n). \tag{5}$$

Then

$$\pi(v) = \sum_{j \in \mathcal{J}_0} \lambda_j b_j^{(p)} = \sum_{j \in \mathcal{J}_1} \lambda_j b_j^{(p)} \in \mathcal{O}_{[0,r_{n,p}-1]}(\mathbf{B}_{n,p}) \cap \mathcal{O}_{[r_{n,p},k-1]}(\mathbf{B}_{n,p}) \tag{6}$$

where $b_j^{(p)}$ denotes the j-th column of $\mathbf{B}_{n,p}$. However, by Proposition 5.13, $\mathcal{O}_{[0,r_{n,p}-1]}(\mathbf{B}_{n,P}) = \mathbf{0}$ and hence Equation (6) reduces to

$$0 = \pi(v) = \sum_{j \in \mathcal{J}_1} \lambda_j b_j^{(p)}. \tag{7}$$

As the columns $b_j^{(p)}$ with $j \in \mathcal{J}_1$ of $\mathbf{B}_{n,p}$ are linearly independent by Proposition 5.13, it follows that $\lambda_j = 0$ for all $j \in \mathcal{J}_1$. Therefore, plugging into Equation (5), it follows that $v = 0$ which completes the proof of the claim. □

Proposition 5.14 together with Corollary 5.10 imply that $\mathrm{rank}(\mathbf{A}_n) = n-1$ for all composite numbers $n \geq 2$, cf. Remark 5.3. Using Proposition 3.15 yields the corollary below.

Corollary 5.15. *Let $n \geq 2$ be a composite number. Then $\binom{x}{n}$ is absolutely irreducible.*

References

[1] Austin Antoniou, Sarah Nakato, and Roswitha Rissner, *Irreducible polynomials in* Int(ℤ), ITM Web Conf. Volume 20, 2018 International Conference on Mathematics (ICM 2018) Recent Advances in Algebra, Numerical Analysis, Applied Analysis and Statistics, 2018.

[2] Paul-Jean Cahen and Jean-Luc Chabert, *Integer-valued polynomials*, Mathematical Surveys and Monographs, vol. 48, American Mathematical Society, Providence, RI, 1997.

[3] ———, *What you should know about integer-valued polynomials*, Amer. Math. Monthly **123** (2016), no. 4, 311–337.

[4] Sophie Frisch, *A construction of integer-valued polynomials with prescribed sets of lengths of factorizations*, Monatsh. Math. **171** (2013), no. 3-4, 341–350.

[5] ———, *Relative polynomial closure and monadically Krull monoids of integer-valued polynomials*, Multiplicative ideal theory and factorization theory, Springer Proc. Math. Stat., vol. 170, Springer, [Cham], 2016, pp. 145–157.

[6] Sophie Frisch and Sarah Nakato, *A graph-theoretic criterion for absolute irreducibility of integer-valued polynomials with square-free denominator*, Comm. Algebra **48** (2020), no. 9, 3716–3723.

[7] Sophie Frisch, Sarah Nakato, and Roswitha Rissner, *Sets of lengths of factorizations of integer-valued polynomials on Dedekind domains with finite residue fields*, J. Algebra **528** (2019), 231–249.

[8] Alfred Geroldinger and Franz Halter-Koch, *Non-unique factorizations*, Pure and Applied Mathematics (Boca Raton), vol. 278, Chapman & Hall/CRC, Boca Raton, FL, 2006, Algebraic, combinatorial and analytic theory.

[9] Shanta Laishram and T. N. Shorey, *The greatest prime divisor of a product of consecutive integers*, Acta Arith. **120** (2005), no. 3, 299–306.

[10] ———, *Grimm's conjecture on consecutive integers*, Int. J. Number Theory **2** (2006), no. 2, 207–211.

[11] Preda Mihăilescu, *Primary cyclotomic units and a proof of Catalan's conjecture*, J. Reine Angew. Math. **572** (2004), 167–195.

[12] Sarah Nakato, *Non-absolutely irreducible elements in the ring of integer-valued polynomials*, Comm. Algebra **48** (2020), no. 4, 1789–1802.

[13] Andreas Reinhart, *On monoids and domains whose monadic submonoids are Krull*, Commutative algebra, Springer, New York, 2014, pp. 307–330.

[14] ———, *On the divisor-class group of monadic submonoids of rings of integer-valued polynomials*, Commun. Korean Math. Soc. **32** (2017), no. 2, 233–260.

[15] Lowell Schoenfeld, *Sharper bounds for the Chebyshev functions $\theta(x)$ and $\psi(x)$. II*, Math. Comp. **30** (1976), no. 134, 337–360.

[16] N. J. A. Sloane, *The on-line encyclopedia of integer sequences*, Sequence A076427.

(Roswitha Rissner) Institut für Mathematik, Alpen-Adria-Universität Klagenfurt, Universitätsstrasse 65-67, 9020 Klagenfurt am Wörthersee, Austria
E-mail address: roswitha.rissner@aau.at

(Daniel Windisch) Institut für Analysis und Zahlentheorie, Technische Universität Graz, Kopernikusgasse 24/II, 8010 Graz, Austria
E-mail address: dwindisch@math.tugraz.at