Symbolic dynamical scales: modes, orbitals, and transversals

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Abstract

We study classes of musical scales obtained from shift spaces from symbolic dynamics through the "distinguished symbol rule", which yields scales in any n-TET tuning system. The modes are thought as elements of orbit equivalence classes of cyclic shift actions on languages, and we study their orbitals and transversals. We present explicit formulations of the generating functions that allow us to deduce the orbital and transversal dimensions of classes of musical scales generated by vertex shifts, for all n, in particular for the 12-TET tuning system. For this, we use first return loop systems obtained from quotients of zeta functions, and integer compositions as the combinatorial class representing all musical scales. We develop the following case studies: three zero entropy symbolic systems arising from substitutions, namely the Thue-Morse, the Fibonacci, and the Fagenbaum scales, the golden mean scales, and a shift of finite type that is not a vertex shift.

1 Introduction

1.1 Symbolic sequential scales

A *symbolic sequential scale* is a musical scale obtained from a (mathematical) symbolic sequence, according to certain (mathematical) rule. For example, the *standard Thue-Morse scales* introduced in [21] and defined as certain sets of scales obtained from the renowned Thue-Morse binary sequence, with coding rule the binary representation of scales. This rule can be generalized to sequences over larger alphabets, for example as the *distinguished symbol rule*, that likewise generates scales on every n-TET tuning system; it consists on coupling a block that occurs in the sequence with a "symbolic chromatic scale" of the same length, that is, a block formed with increasingly ordered elements of a set of "symbolic notes", and then letting the scale defined by the rule be formed with the notes carrying the distinguished initial symbol of the corresponding block. The rule can be applied to sets of symbolic sequences.

Our goal here is to present a general formalism to study symbolic sequential scales of this type, with the number of notes on each scale as a parameter, together with their modes, incorporating techniques from both symbolic dynamics [31] and analytic combinatorics [16].

1.2 Shift actions

We consider *shift spaces* as sets of symbolic sequences, that is, closed subsets $X \subseteq \mathcal{A}^{\mathbb{Z}}$ of (bi)infinite sequences over an alphabet $\mathcal{A}$, together with actions $\sigma\colon \mathbb{Z} \curvearrowright X$ by translations, induced by the left shift automorphism $\sigma\colon \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$. Shift spaces have several characterizations, for instance, as sets of sequences that avoid the elements of a subset $\mathcal{F} \subseteq \mathcal{A}^*$ of (finite) *forbidden* configurations. A shift space X is also characterized by its *language* $\mathcal{L}(X) \subseteq \mathcal{A}^*$, which is defined as the union of the *admissible* configurations that occur in its sequences. Thus, they can be constructed, for example, by specifying either its language (e.g., as the admisible blocks in a sequence, like the Thue-Morse sequence), or a set of forbidden configurations (e.g., the distinguished symbol rule will certainly yield admissible scales with no half tones between consecutive notes if the forbidden set is like $\mathcal{F} \supseteq \{a^2 : a \in \mathcal{A}\}$). *Shifts of finite type* (SFTs) are shift spaces defined by *finite* forbidden sets $\mathcal{F} \Subset \mathcal{A}^*$, and they can always be conjugated to *vertex shifts*, which are shift spaces defined by forbidden sets like $\mathcal{F} \subseteq \mathcal{A}^2$. Vertex shifts are well understood symbolic dynamical systems, they posses matrix representations that provide algebraic and analytic tools to study dynamic properties, like entropy, periodic points and their zeta functions, etc., and some of these tools can be extrapolated through conjugacies to study dynamical properties of SFTs and further, e.g. zeta functions of sofic shifts [11, 33]. SFTs are used as models of more general dynamical systems (e.g. through Markov partitions [10] and universality properties [24]), and they are also used in other areas, like in knot theory [38].

1.3 Cyclic shift actions

The modes of symbolic sequential scales are thought as instances of α-orbit equivalence classes of actions $\alpha\colon \mathbb{Z} \curvearrowright \mathcal{S}^*$ induced by the *cyclic* left shift combinatorial automorphism $\alpha\colon \mathcal{S}^* \to \mathcal{S}^*$ on *finite* sequences over some countable alphabet $\mathcal{S}$ (e.g., a binary alphabet, or a set of symbolic notes, or the positive integers, etc.). *Orbitals* are unions of α-orbit equivalence classes, and they are the subsets upon which α acts. In general, arbitrary subsets of $\mathcal{S}^*$ are not orbitals, like the language of a shift space (in fact, the language of a shift space is an orbital if and only if the space is a full shift). Likewise, if the distinguished symbol rule is formalized as a block function $\varphi\colon \mathcal{L}(X) \to \mathcal{S}^*$ valued on symbolic sequences over some alphabet $\mathcal{S}$ that represent musical scales in a way that the α-orbits correspond to the modes of the scales (for eample, see (3.1)), then $\varphi\big(\mathcal{L}(X)\big)$ is not, in general, an orbital. Thus we consider orbitals generated by subsets $B \subseteq \mathcal{S}^*$ as unions of the α-orbit equivalence classes of their elements, and a *transversal* is a set of representatives of the generated orbital.

Remark 1.1. If B is a set of musical scales, like $\varphi(\mathcal{L}(X))$, then the cardinality of a transversal is the number of "essentially different" scales an instrumentalist would have to learn to play any scale in B, together with all its modes, for a total number of scales that corresponds to the cardinality of its generated orbital.

We will refer to these cardinalities as *transversal* and *orbital dimensions*. Since any set B decomposes into a sequence $(B_n)_{n\geq 0}$ with $B_n \triangleq B \cap \mathcal{S}^n$, there are *transversal* and *orbital* generating functions $\dim_{\mathsf{T}}^{B}(z)$ and $\dim_{\mathsf{O}}^{B}(z)$, respectively.

1.4 Main result

Thus we aim to find transversal and orbital generating functions of classes of musical scales generated by shift spaces. We use integer compositions as the combinatorial model of the class of all musical scales (that is, above, $\mathcal{S}$ will be the set $\mathbb{N}_{>0}$ of positive integers), not only because its α-orbits represent the modes of the scales (as *wheels*), but also because it is an ubiquitous combinatorial class in analytic combinatorics: its elements are represented as sequences of positive integers, and their generating functions, including bivariate versions marking several parameters like the number of summands, are well understood [16]. With them (see Theorem 2), and the interplay of the σ-action on sequences and the α-action on languages, it is possible to make the main formulation that is required to deduce transversal and orbital generating functions of musical scales generated by vertex shifts (with reference to Remark 1.1):

Theorem 1. *Let $X \subseteq \mathcal{A}^{\mathbb{Z}}$ be an irreducible vertex shift and choose a distinguished symbol $\mathfrak{s} \in \mathcal{A}$. Then there is a set $\mathcal{K}(\mathfrak{s}) \subseteq \mathbb{N}_{>0}$ of positive integers (see* (3.8)*) that yields decompositions of the transversal and orbital generating functions of all musical scales in $\varphi(\mathcal{L}(X,\mathfrak{s}))$, where $\mathcal{L}(X,\mathfrak{s})$ is the language of all admissible words in X that start with $\mathfrak{s}$. These decompositions are*

$$\dim_{\mathsf{T}}^{\varphi(X,\mathfrak{s})}(z) = W^{\mathcal{K}(\mathfrak{s})}(z) + a^{\mathcal{K}(\mathfrak{s})}(z), \tag{1.2}$$

$$\dim_{\mathsf{O}}^{\varphi(X,\mathfrak{s})}(z) = C^{\mathcal{K}(\mathfrak{s})}(z) + b^{\mathcal{K}(\mathfrak{s})}(z), \tag{1.3}$$

where the four generating functions on the right hand sides above are as follows:

1. *$C^{\mathcal{K}(\mathfrak{s})}(z)$ and $W^{\mathcal{K}(\mathfrak{s})}(z)$ are the generating functions of integer compositions and wheels, respectively, both with summands in $\mathcal{K}(\mathfrak{s})$ (see* (2.3) *and* (2.5) *evaluated at $u = 1$).*

2. *$a^{\mathcal{K}(\mathfrak{s})}(z)$ is the generating function of aperiodic compositions, also with summands in $\mathcal{K}(\mathfrak{s})$, except for the last one that belongs to the complement $\mathcal{K}(\mathfrak{s})^{\mathsf{c}}$ and is bounded above by an element of $\mathcal{K}(\mathfrak{s})$ (see* (3.9)*).*

3. *The orbital generating function $b^{\mathcal{K}(\mathfrak{s})}(z)$ associated to the class represented by $a^{\mathcal{K}(\mathfrak{s})}(z)$ is, in fact, the corresponding cumulative generating function with respect to the number of notes (see* (3.12)*).*

The proof follows from the definitions and the formulations in the rest of the paper, which is organized as follows. In section 2 we declare the class of all musical scales as combinatorially isomorphic to the class of integer compositions, and define their modes, orbitals, transversals, and their dimensions. In section 3 we address shift spaces and the classes of musical scales they define through the distinguished symbol rule. We recall periodic points and zeta functions, and then focus on shifts of finite type, vertex shifts, and loop systems. Finally we settle the decompositions above and discuss their extrapolation to SFTs. Our formalism is general for all n-TET tuning systems, and adapts for finite values of n to numerical procedures for exact computations, for example when $n = 12$, which is of most interest from the musical point of view. In section 4 we develop five examples as case studies. The first three are classes of *substitutive scales*, that is, scales defined by substitutive subshifts, namely the Thue-Morse, Fibonacci, and Fagenbaum scales. Then we elaborate in detail one example of vertex shift scales to illustrate the methods in sections 2 and 3 that yield Theorem 1, namely the golden mean scales. Finally we sketch how to methods can be adapted to proper SFTs (i.e. an SFT that is not a vertex shift). In the last section 5 we make final remarks and conclusions, with respect to other works, possible generalizations, and further applications.

2 Scales, modes, orbitals, and transversals

2.1 Musical scales and integer compositions

A musical scale in 12-TET tuning system can be coded by a sequence of integers in which each term is the number of half tones within consecutive notes of the scale. For example, the chromatic and major scales, that are coded with a binary alphabet $\mathcal{B} = \{\circ, \bullet\}$ as $\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ\circ$ and $\circ\bullet\circ\bullet\circ\circ\bullet\circ\bullet\circ\bullet\circ$, correspond to $(1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$ and $(2, 2, 1, 2, 2, 2, 1)$ in the integer sequence representation, respectively. Observe that in both cases the sums of the entries yields 12. This is a general phenomena that, though elementary by definition, we state formally:

Theorem 2. *In n-TET tuning system, the musical scales are in bijective correspondence with the set of ordered sequences of positive integers that add up to n. In other words, the set of all musical scales (in any tuning system) is combinatorially isomorphic to the combinatorial class of integer compositions.*

Let $\mathbb{N}_{>0} \triangleq \{1, 2, 3, \ldots\}$ denote the set of positive integers. Let

$$\mathcal{C} \triangleq \textsc{Seq}(\mathbb{N}_{>0}) = \bigcup_{k=0}^{\infty} \mathbb{N}_{>0}^{k} \tag{2.1}$$

denote the class of integer compositions, and henceforth think of its elements as musical scales. For every integer $n \geq 0$, let $\mathcal{C}_n \subset \mathcal{C}$ be the compositions of n, i.e. $\mathcal{C}_n$ denotes the set of scales in n-TET tuning system. Then $C_n \triangleq \#\mathcal{C}_n = 2^{n-1}$, which is consistent with the binary representation of musical scales, and thus the ordinary generating function of all musical scales is the rational function[1]

$$C(z) \triangleq \sum_{n=0}^{\infty} C_n z^n = \frac{1-z}{1-2z}. \tag{2.2}$$

More generally, for any subset $\mathcal{K} \subseteq \mathbb{N}_{>0}$ of positive integers, the class $\mathcal{C}^{\mathcal{K}} \subseteq \mathcal{C}$ of all integer compositions with summands in $\mathcal{K}$ has generating function

$$C^{\mathcal{K}}(z) \triangleq \sum_{n=0}^{\infty} C_n^{\mathcal{K}} z^n = \frac{1}{1 - \sum\limits_{k\in\mathcal{K}} z^k}, \tag{2.3}$$

where $C_n^{\mathcal{K}} \triangleq \#(\mathcal{C}^{\mathcal{K}} \cap \mathcal{C}_n)$. Henceforth, for any set of finite sequences $\mathcal{Y}$, like $\mathcal{C}$, we will be considering the *length* of the sequences as a parameter, $\ell\colon \mathcal{Y} \to \mathbb{N}$ (for example, when coded by integer compositions, the major and chromatic scales have lengths 7 and 12, respectively, but on the other hand, both have length 12 in binary code). Thus, the bivariate generating function $C^{\mathcal{K}}(z,u)$ where u marks the length of the compositions is

$$C^{\mathcal{K}}(z,u) \triangleq \sum_{n,m\geq 0} C_{n,m}^{\mathcal{K}} z^n u^m = \frac{1}{1 - u\sum\limits_{k\in\mathcal{K}} z^k} \tag{2.4}$$

with $C_{n,m}^{\mathcal{K}} \triangleq \#\{w \in \mathcal{C}_n^{\mathcal{K}} : \ell(w) = m\}$.

2.2 Modes, orbitals, transversals, and their dimensions

Let $\mathcal{A}$ be a countable alphabet and then let $\mathcal{A}^* \triangleq \bigcup_{k\geq 0} \mathcal{A}^k$, where

$$\mathcal{A}^k \triangleq \underbrace{\mathcal{A} \times \cdots \times \mathcal{A}}_{k \text{ times}}.$$

Let $\alpha\colon \mathbb{Z} \curvearrowright \mathcal{A}^*$ be the cyclic left shift action induced by the combinatorial isomorphism $\alpha\colon \mathcal{A}^* \to \mathcal{A}^*$ defined for every $w = (w_1, \ldots, w_k) \in \mathcal{A}^k$ by $\alpha(w) \triangleq (w_2, \ldots, w_k, w_1) \in \mathcal{A}^k$, for all $k \geq 1$. The α-*orbit* of $w \in \mathcal{A}^*$ is

$$\mathcal{O}_\alpha(w) \triangleq \{\alpha^j(w) \ : \ \forall\, j \in \mathbb{Z}\}.$$

The set of α-orbits forms a partition of $\mathcal{A}^*$ induced by the α-orbit equivalence relation $\overset{\alpha}{\sim}$. The representation of musical scales by integer compositions is such that the α-orbit

[1] Combinatorially, integer compositions are sequences of positive integers (see (2.1)), and on the right hand side of (2.2) we already see the form of Pólya's quasi-inverse operator that corresponds to sequence constructions.

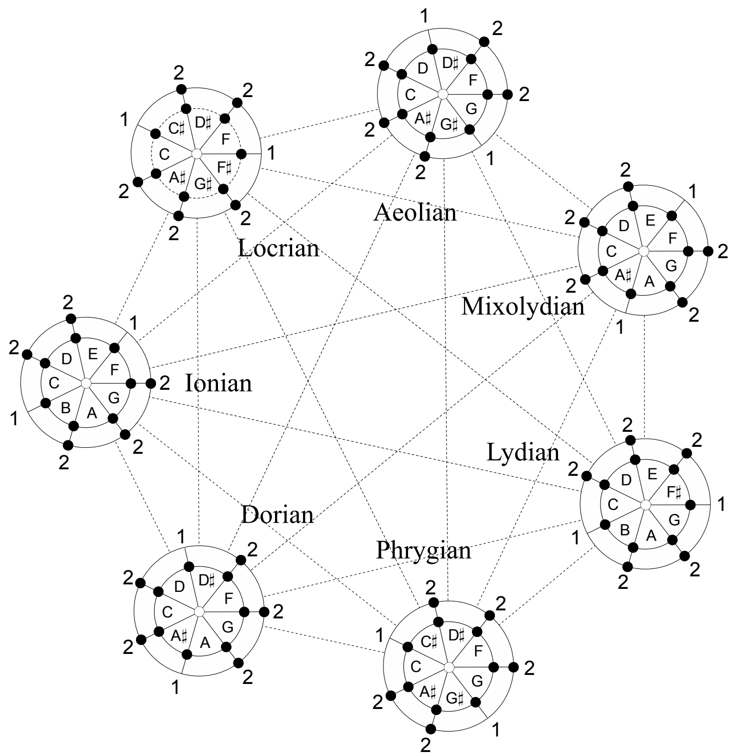


Figure 1: The diatonic wheel $(2, 2, 1, 2, 2, 2, 1)$ in C: its size is 12, its length is 7, it is aperiodic, thus it consists of 7 modes.

equivalence class of an integer composition $\mathbf{w} \in \mathcal{C}$, i.e. the elements of its α-orbit $\mathcal{O}_\alpha(\mathbf{w})$, are the *modes* of the corresponding scale[2], and thus, in this case, we write $\textsc{modes}(\mathbf{w}) \triangleq \mathcal{O}_\alpha(\mathbf{w})$. For any subset $B \subseteq \mathcal{A}^*$, let

$$\mathcal{O}_\alpha(B) \triangleq \bigcup_{w \in B} \mathcal{O}_\alpha(w),$$

and similarly, if $B \subseteq \mathcal{C}$, then we write $\textsc{modes}(B) \triangleq \mathcal{O}_\alpha(B)$. Now, since $\mathcal{A}^* / \overset{\alpha}{\sim}$ is the combinatorial class of *cycles* of elements of $\mathcal{A}$, the class of all musical scales, modulo their modes, is the class $\mathcal{W}$ of *cyclic* compositions of positive integers, the so called *wheels* (see Figure 1). Two musical scales are *essentially different* if they are different as wheels.

[2]This is not generally the case. For example, in the binary representation of musical scales, the α-orbits do not always correspond to the modes of the scales.

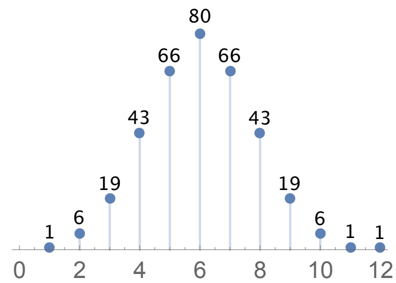


Figure 2: Number of notes versus number of essentially different scales for the 12-TET tuning system. This is in general represented for every n by the coefficient $[z^n]W(z,u)$, where $W(z,u)$ is the bivariate version (2.5) of $W(z)$ (when $\mathcal{K} = \mathbb{N}_{>0}$). Thus, for example, $[z^{12}]W(z,u) = u + 6u^2 + 19u^3 + 43u^4 + 66u^5 + 80u^6 + 66u^7 + 43u^8 + 19u^9 + 6u^{10} + u^{11} + u^{12}$. The limiting distribution of the number of notes is gaussian as $n \to \infty$.

Therefore, the generating function of all musical scales, modulo their modes, is

$$\begin{aligned} W(z) &\triangleq \sum_{k=1}^{\infty} W_n z^n \\ &= \sum_{k=1}^{\infty} \frac{\phi(k)}{k} \log \left(1 - \frac{z^k}{1-z^k}\right)^{-1} \\ &= z + 2z^2 + 3z^3 + 5z^4 + 7z^5 + 13z^6 + \dots, \end{aligned}$$

with $\phi \colon \mathbb{N}_{>0} \to \mathbb{N}_{>0}$ the Euler totient function, that is, $\phi(n) \triangleq \#\{k \leq n : \gcd(n,k) = 1\}$. More generally, the bivariate generating function of the class $\mathcal{W}^{\mathcal{K}}$ of wheels with summands in $\mathcal{K}$, with u marking the length of the wheels (i.e. the number of notes in the scales), is

$$W^{\mathcal{K}}(z,u) \triangleq \sum_{n,m=1}^{\infty} W^{\mathcal{K}}_{n,m} z^n u^m = \sum_{k=1}^{\infty} \frac{\phi(k)}{k} \log \frac{1}{1 - \sum\limits_{j \in \mathcal{K}} u^k z^{jk}}. \tag{2.5}$$

For example, in 12-TET tuning system, there are 351 essentially different musical scales, and their distribution according to the number of notes is illustrated in Figure 2.

A set $A \subseteq \mathcal{A}^*$ is *independent* if any pair of distinct elements of A belong to distinct α-orbit equivalence classes, i.e. $\mathcal{O}_\alpha(v) \cap \mathcal{O}_\alpha(w) = \varnothing$ for all $v, w \in A$ with $v \neq w$. Two subsets $A, B \subseteq \mathcal{A}^*$ are *mutually independent* if $\mathcal{O}_\alpha(A) \cap \mathcal{O}_\alpha(B) = \varnothing$ (each set A and B may or may not be independent). A *transversal* of A is a maximal independent subset $T_A \subseteq A$. Any nonempty set $A \neq \varnothing$ possesses at least one transversal $T_A \subseteq A$, and any two transversals of A have the same cardinality, the *transversal dimension*

$$\dim_{\mathsf{T}}(A) \triangleq \#T_A.$$

Clearly, $A \subseteq \mathcal{O}_\alpha(A) = \mathcal{O}_\alpha(T)$ and $\mathcal{O}_\alpha(T') \subsetneq \mathcal{O}_\alpha(T)$ for any transversal $T \subseteq A$ and any proper (independent) subset $T' \subsetneq T$. Hence, if $A \subseteq \mathcal{C}$ is a set of integer compositions, then the transversal dimension is the number of essentially different scales an instrumentalist would have to learn to play any scale in $\textsc{modes}(A)$. The *orbital dimension* of A is

$$\dim_{\mathsf{O}}(A) \triangleq \#\mathcal{O}_\alpha(A).$$

Again for subsets $A \subseteq \mathcal{C}$ of integer compositions, the orbital dimension $\dim_{\mathsf{O}}(A) = \#\textsc{modes}(A)$ is the total number of scales that an instrumentalist can play with the elements of (a transversal $T \subseteq A$ of) A and their modes. Thus, the orbital dimension $\dim_{\mathsf{O}}(w) \triangleq \#\mathcal{O}_\alpha(w)$ of $w \in \mathcal{A}^*$ is bounded above by $\ell(w)$. Moreover, the former divides the later, i.e. $\dim_{\mathsf{O}}(w)|\ell(w)$, thus there is an integer $k = k(w) \geq 1$ such that $\dim_{\mathsf{O}}(w) \cdot k = \ell(w)$, and then let the *period* of w be defined as $\mathrm{per}(w) \triangleq k$. If $\mathrm{per}(w) = 1$, then w is *aperiodic*. The orbital dimension of A is therefore computed with any α-transversal $T \subseteq A$ through the equality

$$\dim_{\mathsf{O}}(A) = \sum_{w \in T} \frac{\dim_{\mathsf{O}}(w)}{\mathrm{per}(w)}.$$

3 Symbolic dynamical scales

3.1 Shift spaces and their musical scales

A *shift space* $X \subseteq \mathcal{A}^{\mathbb{Z}}$ is determined by a set of *forbidden blocks* $\mathcal{F} \subseteq \mathcal{A}^*$, that is, $X = \mathsf{X}_{\mathcal{F}}$ where

$$\mathsf{X}_{\mathcal{F}} \triangleq \{x = (x_n)_{n\in\mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}} : \forall w \in \mathcal{F},\ \forall k \in \mathbb{Z},\ x_{[k,k+\ell(w)-1]} \neq w\}$$

(above, and henceforth, for any sequence x, let $x_{[i,j]} \triangleq x_i \dots x_j \triangleq (x_i, \dots, x_j)$), and is accompanied by the *left shift $\mathbb{Z}$-action* $\sigma \colon \mathbb{Z} \curvearrowright X$ induced by the automorphism

$$\sigma(x)_n \triangleq x_{n+1} \ \ \forall\, x = (x_n)_{n\in\mathbb{Z}} \in X,\ \forall n \in \mathbb{Z}.$$

The *language* of a shift space X is $\mathcal{L}(X) \triangleq \bigcup\limits_{n\geq 0} \mathcal{L}_n(X) \subseteq \mathcal{A}^*$, where

$$\mathcal{L}_n(X) \triangleq \{x_{[1,n]} \in \mathcal{A}^n : x \in X\},$$

and also, for every symbol $\mathfrak{s} \in \mathcal{A}$, let $\mathcal{L}(X, \mathfrak{s}) \triangleq \bigcup_{n\geq 1} \mathcal{L}_n(X, \mathfrak{s})$, where

$$\mathcal{L}_n(X, \mathfrak{s}) \triangleq \{x_{[1,n]} \in \mathcal{L}_n(X) : x_1 = \mathfrak{s}\}.$$

X is *irreducible* if for every $u, w \in \mathcal{L}(X)$, there exists $v \in \mathcal{L}(X)$ such that $uvw \in \mathcal{L}(X)$.

The *distinguished symbol rule* $\varphi \colon \mathcal{L}(X) \to \mathcal{C}$ is defined for each $w = w_1 \dots w_n \in \mathcal{L}_n(X)$ as certain composition $\varphi(w) \in \mathcal{C}_n$ of $n = \ell(w)$, as follows. Let $\mathfrak{s} \triangleq w_1$ and then let

$1 = n_1 < n_2 < \ldots < n_{r(w)} \leq n$ be the coordinates where $\mathfrak{s}$ occurs in w, that is, $w_j = \mathfrak{s}$ if and only if $j = n_i$ for some $i = 1, \ldots, r(w)$. Then the composition of n that w induces has length $\ell\big(\varphi(w)\big) = r(w)$ and is defined by

$$\varphi(w) \triangleq (\underbrace{n_2 - n_1}_{k_1}, \underbrace{n_3 - n_2}_{k_2}, \ldots, \underbrace{n_{r(w)} - n_{r(w)-1}}_{k_{r(w)-1}}, \underbrace{n - n_{r(w)} + 1}_{k_{r(w)}}). \tag{3.1}$$

For every $\mathfrak{s} \in \mathcal{A}$ and $n \geq 1$, let

$$\mathcal{C}_n^{(X,\mathfrak{s})} \triangleq \varphi\big(\mathcal{L}_n(X,\mathfrak{s})\big) \qquad \text{and} \qquad \mathcal{C}_n^{(X)} \triangleq \varphi\big(\mathcal{L}_n(X)\big)$$

and also let

$$\mathcal{C}^{(X,\mathfrak{s})} \triangleq \varphi\big(\mathcal{L}(X,\mathfrak{s})\big) \qquad \text{and} \qquad \mathcal{C}^{(X)} \triangleq \varphi\big(\mathcal{L}(X)\big).$$

Define the generating functions

$$C^{(X,\mathfrak{s})}(z) \triangleq \sum_{n\geq 0} C_n^{(X,\mathfrak{s})} z^n \qquad \text{and} \qquad C^{(X)}(z) \triangleq \sum_{n\geq 0} C_n^{(X)} z^n$$

where $C_n^{(X,\mathfrak{s})} \triangleq \#\mathcal{C}_n^{(X,\mathfrak{s})}$ and $C_n^{(X)} \triangleq \#\mathcal{C}_n^{(X)}$. Our foremost concern here are the transversal and orbital (bivariate) generating functions

$$\dim_{\mathsf{T}}^{(X,\mathfrak{s})}(z,u) \triangleq \sum_{n,m\geq 0} \dim_{\mathsf{T}}\big(\mathcal{C}_{n,m}^{(X,\mathfrak{s})}\big) z^n u^m,$$
$$\dim_{\mathsf{T}}^{(X)}(z,u) \triangleq \sum_{n,m\geq 0} \dim_{\mathsf{T}}\big(\mathcal{C}_{n,m}^{(X)}\big) z^n u^m,$$

and

$$\dim_{\mathsf{O}}^{(X,\mathfrak{s})}(z,u) \triangleq \sum_{n,m\geq 0} \dim_{\mathsf{O}}\big(\mathcal{C}_{n,m}^{(X,\mathfrak{s})}\big) z^n u^m,$$
$$\dim_{\mathsf{O}}^{(X)}(z) \triangleq \sum_{n,m\geq 0} \dim_{\mathsf{O}}\big(\mathcal{C}_{n,m}^{(X)}\big) z^n u^m.$$

3.2 Periodic points and zeta functions

The *σ-orbit* of $x \in X$ is $\mathcal{O}_\sigma(x) \triangleq \{\sigma^n(x) : n \in \mathbb{Z}\} \subseteq X$. For every $n \geq 1$, a point $x \in X$ is *n-periodic* if $\sigma^n(x) = x$, and if x is n-periodic, then there exists the *minimal period* $n_x \geq 1$ of x, namely, the cardinality of its orbit $n_x \triangleq \#\mathcal{O}_\sigma(x)$, and moreover, $n_x | n$. Let $P_n(X) \triangleq \{x \in X : \sigma^n(x) = x\}$ and $Q_n(x) \triangleq \{x \in P_n(X) : n_x = n\}$ be the sets of n-periodic points and minimal n-periodic points, respectively, and also let $p_n(X) \triangleq \#P_n(X)$

and $q_n(X) \triangleq \#Q_n(X)$. Recall that the relationship between $p_n(X)$ and $q_n(X)$ is through Möbius inversion,

$$p_n(X) = \sum_{k|n} q_k(X) \qquad \text{and} \qquad q_n(X) = \sum_{k|n} \mu\left(\frac{n}{k}\right) p_k(X), \tag{3.2}$$

where $\mu\colon \mathbb{N}_{>0} \to \{-1, 0, 1\}$ is the Möbius function defined by

$$\mu(n) \triangleq \begin{cases} 0 & \text{if there exists } p \geq 2 \text{ such that } p^2|n, \text{ and} \\ (-1)^r & \text{if } n = p_1 \cdots p_r \text{ with } p_1, p_2, \ldots, p_r \geq 2 \text{ distinct prime numbers.} \end{cases}$$

The *dynamic zeta function* of X is

$$\zeta_X(z) \triangleq \exp\left(\sum_{n=1}^{\infty} \frac{p_n(X)}{n} z^n\right) = \prod_{n\geq 1} \frac{1}{(1-z^n)^{q_n(X)/n}}.$$

3.3 Shifts of finite type, vertex shifts, and loop systems

A *shift of finite type* $X \subseteq \mathcal{A}^{\mathbb{Z}}$ is a shift space $X = \mathsf{X}_{\mathcal{F}}$ that can be defined by a *finite* set of forbidden blocks $\mathcal{F} \Subset \mathcal{A}^*$, and in this case define $m \triangleq \max\{\ell(w) : w \in \mathcal{F}\} - 1$ and say that X is *m-step* (since it is always possible to find a set of $(m+1)$-blocks $\mathcal{F}' \subseteq \mathcal{A}^{m+1}$ such that $X = \mathsf{X}_{\mathcal{F}'}$). A *vertex shift space* is a 1-step shift of finite type. Let $X = \mathsf{X}_{\mathcal{F}} \subseteq \mathcal{A}^{\mathbb{Z}}$ be a vertex shift space defined by a set of forbidden 2-blocks $\mathcal{F} \subseteq \mathcal{A}^2$. Let A be the square $\{0,1\}$-matrix indexed by $\mathcal{A}$ and defined by the rule $A(i,j) = 1$ if and only if $ij \notin \mathcal{F}$. Then $X = \widehat{\mathsf{X}}_A$, where

$$\widehat{\mathsf{X}}_A \triangleq \{x = (x_n)_{n\in\mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}} : \forall n \in \mathbb{Z},\ A(x_n, x_{n+1}) \neq 0\}.$$

The matrix representation of vertex shifts yields expressions that can be useful to study transversal and orbital dimensions. For example, the dynamic zeta function is obtained through

$$\zeta_{\widehat{\mathsf{X}}_A}(z) = \frac{1}{\det(I - zA)} \tag{3.3}$$

(observe that $\det(I - zA) = z^{\#\mathcal{A}} \chi_A(z^{-1})$, where $\chi_A(z)$ is the characteristic polynomial of the matrix A). From here we can get

$$p_n(\widehat{\mathsf{X}}_A) = \frac{1}{(n-1)!} \frac{d^n}{dz^n} \log \zeta_{\widehat{\mathsf{X}}_A}(z)\bigg|_{z=0} = \operatorname{trace}(A^n)$$

and also $q_n(\widehat{\mathsf{X}}_A)$ by Möbius inversion (3.2). The following result follows.

Theorem 3 (Transversal and orbital dimensions of languages of vertex shifts)**.** *The nth transversal dimension of the language of a vertex shift is*

$$\dim_{\mathsf{T}}\big(\mathcal{L}_n(\widehat{\mathsf{X}}_A)\big) = \sum_{\substack{i,j\in\mathcal{A}\\ A_{j,i}=0}} A_{i,j}^{n-1} + \sum_{k|n} \frac{q_k(\widehat{\mathsf{X}}_A)}{k} \tag{3.4}$$

and the corresponding nth orbital dimension is

$$\dim_{\mathsf{O}}\big(\mathcal{L}_n(\widehat{\mathsf{X}}_A)\big) = n \sum_{\substack{i,j\in\mathcal{A}\\ A_{j,i}=0}} A_{i,j}^{n-1} + p_n(\widehat{\mathsf{X}}_A). \tag{3.5}$$

Now, for studying musical scales arising from languages of vertex shift spaces through the distinguished symbol rule, consider the *first return loop system* to a given symbol $\mathfrak{s}\in\mathcal{A}$, defined by the generating function

$$f^{(\mathfrak{s})}(z) \triangleq \sum_{k=1}^{\infty} f_k^{(\mathfrak{s})} z^k$$

where

$$f_k^{(\mathfrak{s})} \triangleq \#\{w = w_0,\dots w_k \in \mathcal{L}_{k+1}(X) : w_0 w_k = \mathfrak{s} \neq w_j \quad \forall\ j\neq 0,k\}. \tag{3.6}$$

The power series $f^{(\mathfrak{s})}(z)$ is obtained through the equation

$$1 - f^{(\mathfrak{s})}(z) = \frac{\zeta_{\widehat{\mathsf{X}}_B}(z)}{\zeta_{\widehat{\mathsf{X}}_A}(z)}, \tag{3.7}$$

where B is the square $\{0,1\}$-matrix indexed by $\mathcal{A}\setminus\{\mathfrak{s}\}$ and obtained from A by removing the row and column indexed by $\mathfrak{s}$.

3.4 Generating functions for distinguished symbol rule on vertex shifts

Here we proof Theorem 1. Let

$$\mathcal{K}(\mathfrak{s}) \triangleq \{k\geq 1 : f_k^{(\mathfrak{s})} \neq 0\} \tag{3.8}$$

and also denote its complement by $\mathcal{K}(\mathfrak{s})^{\mathsf{c}} \triangleq \mathbb{N}_{>0}\setminus\mathcal{K}(\mathfrak{s})$. According to (3.1) and (3.6), if $w\in\mathcal{L}(X,\mathfrak{s})$, then $\varphi(w) = (k_1,k_2,\dots,k_{\ell(\varphi(w))})$ is a composition of $\ell(w)$, with summands in $\mathcal{K}(\mathfrak{s})$, except perhaps for the last summand $k_{\ell(\varphi(w))}$. Suppose that this is the case, that is, $k_{\ell(\varphi(w))}\in\mathcal{K}(\mathfrak{s})^{\mathsf{c}}$. Since X is irreducible, there exists $v\in\mathcal{L}(X,\mathfrak{s})$ such that v also ends in $\mathfrak{s}$ and w is a prefix of v, that is, $v_{\ell(v)} = \mathfrak{s}$ and

$$v = w v_{[\ell(w)+1,\ell(v)]},$$

thus $k_{\ell(\varphi(w))}$ is bounded above by an element of $\mathcal{K}(\mathfrak{s})$. Let

$$a^{\mathcal{K}(\mathfrak{s})}(z) \triangleq \sum_{n\geq 1} a_n^{\mathcal{K}(\mathfrak{s})} z^n$$

be the generating function of this subclass which is described in item 2 of Theorem 1. Then

$$a_n^{\mathcal{K}(\mathfrak{s})} = \sum_{\substack{k\in\mathcal{K}(\mathfrak{s})^{\mathsf{c}} \\ \exists k'\in\mathcal{K}(\mathfrak{s}),\ k'>k}} C_{n-k}^{\mathcal{K}(\mathfrak{s})} \tag{3.9}$$

(for any subset $\mathcal{K}\neq\varnothing$ of positive integers, $C_0^{\mathcal{K}} \triangleq 1$). If we also let

$$b^{\mathcal{K}(\mathfrak{s})}(z) \triangleq \sum_{n\geq 1} b_n^{\mathcal{K}(\mathfrak{s})} z^n$$

be the generating function of the corresponding orbital, then, by independence, there is a decomposition of the form (1.2) and (1.3), as described in Theorem 1, we just need to justify that $b^{\mathcal{K}(\mathfrak{s})}(z)$ is the cumulative generating function of the subclass represented by $a^{\mathcal{K}(\mathfrak{s})}(z)$, with respect to the number of notes. This follows from the fact that the elements represented by $a^{\mathcal{K}(\mathfrak{s})}(z)$ are aperiodic. To be explicit, write the bivariate coefficients

$$\dim_{\mathsf{T}}\big(\mathcal{C}_{n,m}^{(X,\mathfrak{s})}\big) = W_{n,m}^{\mathcal{K}(\mathfrak{s})} + a_{n,m}^{\mathcal{K}(\mathfrak{s})} \qquad \text{and} \qquad \dim_{\mathsf{O}}\big(\mathcal{C}_{n,m}^{(X,\mathfrak{s})}\big) = C_{n,m}^{\mathcal{K}(\mathfrak{s})} + b_{n,m}^{\mathcal{K}(\mathfrak{s})}. \tag{3.10}$$

Then, for every $n, m \geq 1$, we have

$$a_{n,m}^{\mathcal{K}(\mathfrak{s})} = \sum_{\substack{k\in\mathcal{K}(\mathfrak{s})^{\mathsf{c}} \\ \exists k'\in\mathcal{K}(\mathfrak{s}),\ k'>k}} C_{n-k,m-1}^{\mathcal{K}(\mathfrak{s})}.$$

By aperiodicity, the corresponding orbital dimension is

$$b_{n,m}^{\mathcal{K}(\mathfrak{s})} = m \cdot a_{n,m}^{\mathcal{K}(\mathfrak{s})}.$$

Thus, if we define

$$a^{\mathcal{K}(\mathfrak{s})}(z,u) \triangleq \sum_{n,m\geq 1} a_{n,m}^{\mathcal{K}(\mathfrak{s})} z^n u^m \tag{3.11}$$

and

$$b^{\mathcal{K}(\mathfrak{s})}(z,u) \triangleq \sum_{n,m\geq 1} b_{n,m}^{\mathcal{K}(\mathfrak{s})} z^n u^m,$$

then we observe that

$$b^{\mathcal{K}(\mathfrak{s})}(z) = \frac{\partial}{\partial u} a^{\mathcal{K}(\mathfrak{s})}(z,u)|_{u=1}, \tag{3.12}$$

and in fact

$$b^{\mathcal{K}(\mathfrak{s})}(z,u) = u\frac{\partial}{\partial u}a^{\mathcal{K}(\mathfrak{s})}(z,u). \tag{3.13}$$

Hence $b^{\mathcal{K}(\mathfrak{s})}(z)$ is the cumulative generating function of the number of summands in the class of compositions represented by $a^{\mathcal{K}(\mathfrak{s})}(z)$. This settles Theorem 1, and also gives decompositions of the bivariate transversal and orbital generating functions,

$$\dim_{\mathsf{T}}^{\varphi(X,\mathfrak{s})}(z,u) = W^{\mathcal{K}(\mathfrak{s})}(z,u) + a^{\mathcal{K}(\mathfrak{s})}(z,u)$$

and

$$\dim_{\mathsf{O}}^{\varphi(X,\mathfrak{s})}(z,u) = C^{\mathcal{K}(\mathfrak{s})}(z,u) + b^{\mathcal{K}(\mathfrak{s})}(z,u).$$

To determine the transversal and orbital dimensions of the whole set of vertex shift scales $\mathcal{C}^{(X)} \triangleq \varphi\big(\mathcal{L}(X)\big) = \cup_{\mathfrak{s}\in\mathcal{A}}\mathcal{C}^{(X,\mathfrak{s})}$, it is required to take into account the intersections between each pair of symbols, otherwise multiple counting may occur. The analysis can be done one symbol at the time, adding only new contributions to the cumulative counting.

3.5 SFT scales

Suppose that $X \subseteq \mathcal{A}^{\mathbb{Z}}$ is an irreducible M-step shift of finite type for some $M > 1$. The *higher block presentation* $X^{[M]}$ may be defined as the vertex shift represented by a directed graph with vertex set $\mathcal{L}_M(X)$, and for every pair $u, v \in \mathcal{L}_M(X)$, (u,v) is an edge from u to v if and only if $u_{[2,M]} = v_{[1,M-1]}$. Therefore, studying musical scales generated by arbitrary SFTs through the distinguished symbol rule is equivalent to studying musical scales generated by vertex shifts through the *distinguished set of symbols* rule.

For the distinguished symbol rule on vertex shifts, once a symbol is fixed, there is a *formal power series* that determines the induced musical scales (see (3.8)). For the distinguished set of symbols rule, there is a *matrix* over formal power series, indexed by the set of distinguished symbols, that determines the induced musical scales. More precisely, let $X \subseteq \mathcal{A}^{\mathbb{Z}}$ be a vertex shift defined by a square $\{0,1\}$-matrix A indexed by $\mathcal{A}$, and let $\mathcal{D} \subset \mathcal{A}$ be a subset of distinguished symbols. For every pair $\mathfrak{s}, \mathfrak{t} \in \mathcal{D}$, let

$$f^{(\mathfrak{s},\mathfrak{t})}(z) \triangleq \sum_{n=1}^{\infty} f_n^{(\mathfrak{s},\mathfrak{t})} z^n$$

be the generating function that represents the set of paths from $\mathfrak{s}$ to $\mathfrak{t}$ that do not cross $\mathcal{D}$ (except for the end points). The coefficients are

$$f_k^{(\mathfrak{s},\mathfrak{t})} \triangleq \#\{w = w_0, \dots w_k \in \mathcal{L}_{k+1}(X) : w_0 = \mathfrak{s},\ w_k = \mathfrak{t},\ w_j \in \mathcal{A}\setminus\mathcal{D}\ \ \forall\ j \neq 0, k\}$$

and thus

$$f^{(\mathfrak{s},\mathfrak{t})}(z) = A_{\mathfrak{s},\mathfrak{t}}z + \sum_{n=2}^{\infty}\sum_{j,k\in\mathcal{A}\setminus\mathcal{D}} A_{\mathfrak{s},j}B_{j,k}^{n-2}A_{k,\mathfrak{t}}z^n$$

where B is the matrix obtained from A by removing the rows and columns indexed by elements of $\mathcal{D}$. The auxiliary matrix is

$$\left(f^{(\mathfrak{s},\mathfrak{t})}(z)\right)_{\mathfrak{s},\mathfrak{t}\in\mathcal{D}}. \tag{3.14}$$

There is no closed form, such as (3.7), for $f^{(\mathfrak{s},\mathfrak{t})}(z)$. Nevertheless, we will see with an example how we can make use of (3.14) in our problem.

4 Examples

First we analyze three classes of scales arising from substitutons and then we consider vertex shift scales.

4.1 Substitutive scales

Here we analyze the Thue-Morse, Fibonacci, and Feigenbaum scales. First some notation: The *language* of a sequence $\mathbf{x}$ in $\mathcal{A}^{\mathbb{N}}$, or $\mathcal{A}^{\mathbb{Z}}$, is defined as $\mathcal{L}(\mathbf{x}) \triangleq \bigcup_{n\geq 0} \mathcal{L}_n(\mathbf{x})$, where

$$\mathcal{L}_n(\mathbf{x}) \triangleq \{w = w_1 \dots w_n \in \mathcal{A}^n : \exists k,\ \mathbf{x}_{[k,k+n-1]} = w\}$$

are the *admissible* n-blocks of $\mathbf{x}$. Also, for each $\mathfrak{s} \in \mathcal{A}$, let

$$\mathcal{L}(\mathbf{x},\mathfrak{s}) = \{w \in \mathcal{L}(\mathbf{x}) \ : \ w_1 = \mathfrak{s}\} \qquad \text{and} \qquad \mathcal{L}_n(\mathbf{x},\mathfrak{s}) \triangleq \mathcal{L}(\mathbf{x},\mathfrak{s}) \cap \mathcal{A}^n.$$

Let $\mathcal{B} \triangleq \{\circ, \bullet\}$ be a binary alphabet.

Example 4 (Thue-Morse scales). The iterated morphism defined by the rules $\circ \mapsto \circ\bullet$ and $\bullet \mapsto \bullet\circ$, starting from $\circ$, yields the Thue-Morse sequence

$$\mathfrak{m} \triangleq \circ\bullet\bullet\circ\bullet\circ\circ\bullet\bullet\circ\circ\bullet\circ\bullet\bullet\circ\bullet\circ\circ\bullet\circ\bullet\bullet\circ\circ\bullet\bullet\circ\bullet\circ\circ\bullet\cdots \in \mathcal{B}^{\mathbb{N}}$$

(see [3, 30] for more on the Thue-Morse sequence and substitution systems in general). The set of *standard* (or *admisible*) Thue-Morse scales in 12-TET tuning system reported in [21], as integer compositions, is $M \triangleq \mathcal{C}_{12}^{(\mathfrak{m})} = \varphi\big(\mathcal{L}_{12}(\mathfrak{m})\big) \subset \mathcal{C}_{12}$ and consists of the following 18 elements:

$$\begin{array}{lll}
\mathbf{m}^{(1)} \triangleq \overset{(3,\ 2,\ 1,\ 3,\ 1,\ 2)}{\circ\bullet\bullet\circ\bullet\circ\circ\bullet\bullet\circ\circ\bullet} & \mathbf{m}^{(2)} \triangleq \overset{(3,\ 2,\ 1,\ 2,\ 3,\ 1)}{\circ\bullet\bullet\circ\bullet\circ\circ\bullet\circ\bullet\bullet\circ} & \mathbf{m}^{(3)} \triangleq \overset{(3,\ 1,\ 3,\ 2,\ 1,\ 2)}{\circ\bullet\bullet\circ\circ\bullet\bullet\circ\bullet\circ\circ\bullet} \\
\mathbf{m}^{(4)} \triangleq \overset{(2,\ 1,\ 3,\ 1,\ 2,\ 3)}{\circ\bullet\circ\circ\bullet\bullet\circ\circ\bullet\circ\bullet\bullet} & \mathbf{m}^{(5)} \triangleq \overset{(2,\ 1,\ 2,\ 3,\ 1,\ 3)}{\circ\bullet\circ\circ\bullet\circ\bullet\bullet\circ\circ\bullet\bullet} & \mathbf{m}^{(6)} \triangleq \overset{(1,\ 3,\ 2,\ 1,\ 2,\ 3)}{\circ\circ\bullet\bullet\circ\bullet\circ\circ\bullet\circ\bullet\bullet} \\
\mathbf{m}^{(7)} \triangleq \overset{(1,\ 3,\ 1,\ 2,\ 3,\ 2)}{\circ\circ\bullet\bullet\circ\circ\bullet\circ\bullet\bullet\circ\bullet} & \mathbf{m}^{(8)} \triangleq \overset{(1,\ 2,\ 3,\ 1,\ 3,\ 2)}{\circ\circ\bullet\circ\bullet\bullet\circ\circ\bullet\bullet\circ\bullet} & \mathbf{m}^{(9)} \triangleq \overset{(1,\ 3,\ 1,\ 2,\ 3,\ 1,\ 1)}{\circ\circ\bullet\bullet\circ\circ\bullet\circ\bullet\bullet\circ\circ} \\
\mathbf{m}^{(10)} \triangleq \overset{(3,\ 1,\ 2,\ 3,\ 2,\ 1)}{\circ\bullet\bullet\circ\circ\bullet\circ\bullet\bullet\circ\bullet\circ} & \mathbf{m}^{(11)} \triangleq \overset{(2,\ 3,\ 1,\ 3,\ 2,\ 1)}{\circ\bullet\circ\bullet\bullet\circ\circ\bullet\bullet\circ\bullet\circ} & \mathbf{m}^{(12)} \triangleq \overset{(3,\ 1,\ 2,\ 3,\ 1,\ 2)}{\circ\bullet\bullet\circ\circ\bullet\circ\bullet\bullet\circ\circ\bullet} \\
\mathbf{m}^{(13)} \triangleq \overset{(1,\ 2,\ 3,\ 2,\ 1,\ 2,\ 1)}{\circ\circ\bullet\circ\bullet\bullet\circ\bullet\circ\circ\bullet\circ} & \mathbf{m}^{(14)} \triangleq \overset{(1,\ 2,\ 3,\ 2,\ 1,\ 3)}{\circ\circ\bullet\circ\bullet\bullet\circ\bullet\circ\circ\bullet\bullet} & \mathbf{m}^{(15)} \triangleq \overset{(2,\ 3,\ 2,\ 1,\ 2,\ 2)}{\circ\bullet\circ\bullet\bullet\circ\bullet\circ\circ\bullet\circ\bullet} \\
\mathbf{m}^{(16)} \triangleq \overset{(2,\ 3,\ 2,\ 1,\ 3,\ 1)}{\circ\bullet\circ\bullet\bullet\circ\bullet\circ\circ\bullet\bullet\circ} & \mathbf{m}^{(17)} \triangleq \overset{(1,\ 3,\ 2,\ 1,\ 3,\ 1,\ 1)}{\circ\circ\bullet\bullet\circ\bullet\circ\circ\bullet\bullet\circ\circ} & \mathbf{m}^{(18)} \triangleq \overset{(2,\ 1,\ 2,\ 3,\ 2,\ 1,\ 1)}{\circ\bullet\circ\circ\bullet\circ\bullet\bullet\circ\bullet\circ\circ}
\end{array}$$

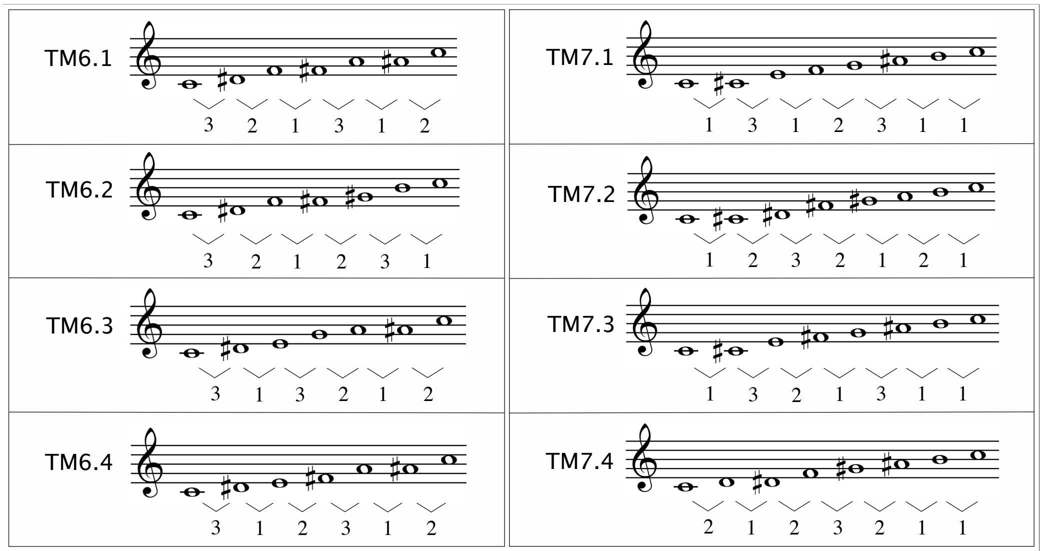


Figure 3: A transversal of admissible (standard) Thue-Morse scales.

Observe that

$$\mathcal{O}_\alpha(\mathbf{m}^{(1)}) = \{\mathbf{m}^{(1)}, \mathbf{m}^{(4)}, \mathbf{m}^{(7)}, \mathbf{m}^{(10)}, \mathbf{m}^{(14)}, \mathbf{m}^{(16)}\}$$
$$\mathcal{O}_\alpha(\mathbf{m}^{(2)}) = \{\mathbf{m}^{(2)}, \mathbf{m}^{(5)}, \mathbf{m}^{(8)}, \mathbf{m}^{(11)}, \mathbf{m}^{(3)}, \mathbf{m}^{(6)}\},$$

and also that the set

$$M \setminus \left(\mathcal{O}_\alpha(\mathbf{m}^{(1)}) \cup \mathcal{O}_\alpha(\mathbf{m}^{(2)})\right) = \{\mathbf{m}^{(9)}, \mathbf{m}^{(12)}, \mathbf{m}^{(13)}, \mathbf{m}^{(15)}, \mathbf{m}^{(17)}, \mathbf{m}^{(18)}\}$$

is independent. Therefore, the transversals of M (contained in M) are the sets T formed by $M \setminus \left(\mathcal{O}_\alpha(\mathbf{m}^{(1)}) \cup \mathcal{O}_\alpha(\mathbf{m}^{(2)})\right)$ and one element from each of the two α-orbits $\mathcal{O}_\alpha(\mathbf{m}^{(1)})$ and $\mathcal{O}_\alpha(\mathbf{m}^{(2)})$. In particular,

$$T_M \triangleq \{\mathbf{m}^{(1)}, \mathbf{m}^{(2)}\} \cup \Big(M \setminus \left(\mathcal{O}_\alpha(\mathbf{m}^{(1)}) \cup \mathcal{O}_\alpha(\mathbf{m}^{(2)})\right)\Big)$$

is a transversal of M, which is shown in music notation in Figure 3. Thus, there are only

$$\dim_{\mathsf{T}}(M) = 8$$

essentially different Thue-Morse scales. Now, the elements of T_M are aperiodic except for scale TM6.3 that has length 6 and period 2. Scales TM6.1, TM6.2, and TM6.4 have all length 6 too. All the scales with label TM7 have length 7. Hence the dimension of the space of modes of M is

$$\dim_{\mathsf{O}}(M) = 1 \times \frac{6}{2} + 3 \times 6 + 4 \times 7 = 49.$$

Note that the set M is an example of a subset $A \subset \mathcal{C}$ with $T \subsetneq A \subsetneq \textsc{modes}(A)$ for a (equivalently every) transversal T of A.

$\diamond$

Example 5 (Fibonacci scales)*.* The substitution rule $\circ \mapsto \circ\bullet$ and $\bullet \mapsto \circ$ iterated over $\circ$ yields the *infinite Fibonacci word*

$$\mathfrak{f} \triangleq \circ\bullet\circ\circ\bullet\circ\bullet\circ\circ\bullet\circ\circ\bullet\circ\bullet\circ\circ\bullet\circ\bullet\circ\circ\bullet\circ\circ\bullet\circ\bullet\circ\circ\bullet\circ\circ\bullet\ldots.$$

Definition 6*.* For every $k \geq 1$, let $\{F_n^{(k)}\}_{n\geq 0}$ be the *k-Fibonacci sequence* defined by

$$F_0^{(k)} \triangleq 1, \quad F_1^{(k)} \triangleq k, \quad \text{and} \quad F_{n+2}^{(k)} \triangleq F_n^{(k)} + F_{n+1}^{(k)}.$$

For example, if $w_0 = \circ$, $w_1 = \circ\bullet$, and $w_{n+2} = w_{n+1}w_n$ for every $n \geq 0$, then $\mathfrak{f} = \lim_{n\to\infty} w_n$ and the sequence $\{\ell(w_n)\}_{n\geq 0} = \{1, 2, 3, 5, 8, \ldots\}$ is 1-Fibonacci.

Again, let us focus on the 12-TET tuning system. It is well known that $\mathfrak{f}$ is *Sturmian* (i.e. it has minimal subword complexity, see e.g. [3, 30]), and thus there are 13 admisible 12-blocks. Unlike the Thue-Morse sequence, there will be two types of scales according to the distinguished symbol $\mathfrak{s} \in \mathcal{B}$, and together they will form the set $F \subset \mathcal{C}$ of admissible Fibonacci scales. The $\circ$-admissible Fibonacci scales $F_\circ \triangleq \varphi\big(\mathcal{L}_{12}(\mathfrak{f}, \circ)\big)$ are

$$\mathbf{f}^{(1)} \triangleq \overset{(1,\ 2,\ 1,\ 2,\ 2,\ 1,\ 2,\ 1)}{\circ\circ\bullet\circ\circ\bullet\circ\bullet\circ\circ\bullet\circ} \qquad \mathbf{f}^{(2)} \triangleq \overset{(1,\ 2,\ 2,\ 1,\ 2,\ 1,\ 2,\ 1)}{\circ\circ\bullet\circ\bullet\circ\circ\bullet\circ\circ\bullet\circ} \qquad \mathbf{f}^{(3)} \triangleq \overset{(1,\ 2,\ 2,\ 1,\ 2,\ 2,\ 1,\ 1)}{\circ\circ\bullet\circ\bullet\circ\circ\bullet\circ\bullet\circ\circ}$$

$$\mathbf{f}^{(4)} \triangleq \overset{(2,\ 1,\ 2,\ 1,\ 2,\ 2,\ 1,\ 1)}{\circ\bullet\circ\circ\bullet\circ\circ\bullet\circ\bullet\circ\circ} \qquad \mathbf{f}^{(5)} \triangleq \overset{(2,\ 1,\ 2,\ 2,\ 1,\ 2,\ 1,\ 1)}{\circ\bullet\circ\circ\bullet\circ\bullet\circ\circ\bullet\circ\circ} \qquad \mathbf{f}^{(6)} \triangleq \overset{(2,\ 1,\ 2,\ 2,\ 1,\ 2,\ 2)}{\circ\bullet\circ\circ\bullet\circ\bullet\circ\circ\bullet\circ\bullet}$$

$$\mathbf{f}^{(7)} \triangleq \overset{(2,\ 2,\ 1,\ 2,\ 1,\ 2,\ 2)}{\circ\bullet\circ\bullet\circ\circ\bullet\circ\circ\bullet\circ\bullet} \qquad \mathbf{f}^{(8)} \triangleq \overset{(2,\ 2,\ 1,\ 2,\ 2,\ 1,\ 2)}{\circ\bullet\circ\bullet\circ\circ\bullet\circ\bullet\circ\circ\bullet}$$

and the $\bullet$-admissible Fibonacci scales $F_\bullet \triangleq \varphi\big(\mathcal{L}_{12}(\mathfrak{m}, \bullet)\big)$ are

$$\mathbf{f}^{(9)} \triangleq \overset{(3,\ \ 3,\ \ 2,\ \ 3,\ \ 1)}{\bullet\circ\circ\bullet\circ\circ\bullet\circ\bullet\circ\circ\bullet} \qquad \mathbf{f}^{(10)} \triangleq \overset{(3,\ \ 2,\ \ 3,\ \ 3,\ \ 1)}{\bullet\circ\circ\bullet\circ\bullet\circ\circ\bullet\circ\circ\bullet} \qquad \mathbf{f}^{(11)} \triangleq \overset{(3,\ \ 2,\ \ 3,\ \ 2,\ \ 2)}{\bullet\circ\circ\bullet\circ\bullet\circ\circ\bullet\circ\bullet\circ}$$

$$\mathbf{f}^{(12)} \triangleq \overset{(2,\ \ 3,\ \ 3,\ \ 2,\ \ 2)}{\bullet\circ\bullet\circ\circ\bullet\circ\circ\bullet\circ\bullet\circ} \qquad \mathbf{f}^{(13)} \triangleq \overset{(2,\ \ 3,\ \ 2,\ \ 3,\ \ 2)}{\bullet\circ\bullet\circ\circ\bullet\circ\bullet\circ\circ\bullet\circ}\ .$$

We observe that when $\mathfrak{s} = \circ$, $\alpha(\mathbf{f}^{(1)}) = \mathbf{f}^{(5)}$, $\alpha^{-1}(\mathbf{f}^{(6)}) = \mathbf{f}^{(8)}$, and the set

$$T_\circ \triangleq \{\mathbf{f}^{(1)}, \mathbf{f}^{(2)}, \mathbf{f}^{(3)}, \mathbf{f}^{(4)}, \mathbf{f}^{(7)}, \mathbf{f}^{(8)}\}$$

is independent, and when $\mathfrak{s} = \bullet$, $\alpha^{-1}(\mathbf{f}^{(11)}) = \mathbf{f}^{(13)}$ and the set

$$T_\bullet \triangleq \{\mathbf{f}^{(9)}, \mathbf{f}^{(10)}, \mathbf{f}^{(11)}, \mathbf{f}^{(12)}\}$$

is independent. Thus, both $T_\circ$ and $T_\bullet$ are transversals of $F_\circ$ and $F_\bullet$, respectively, and since $T_\circ$ and $T_\bullet$ are mutually independent, we obtain $T_F \triangleq T_\circ \cup T_\bullet$ as an admissible transversal of Fibonacci scales, its elements are shown in Figure 4. Thus

$$\dim_{\mathsf{T}}(F) = \dim_{\mathsf{T}}(F_\circ) + \dim_{\mathsf{T}}(F_\bullet) = 6 + 4 = 10.$$

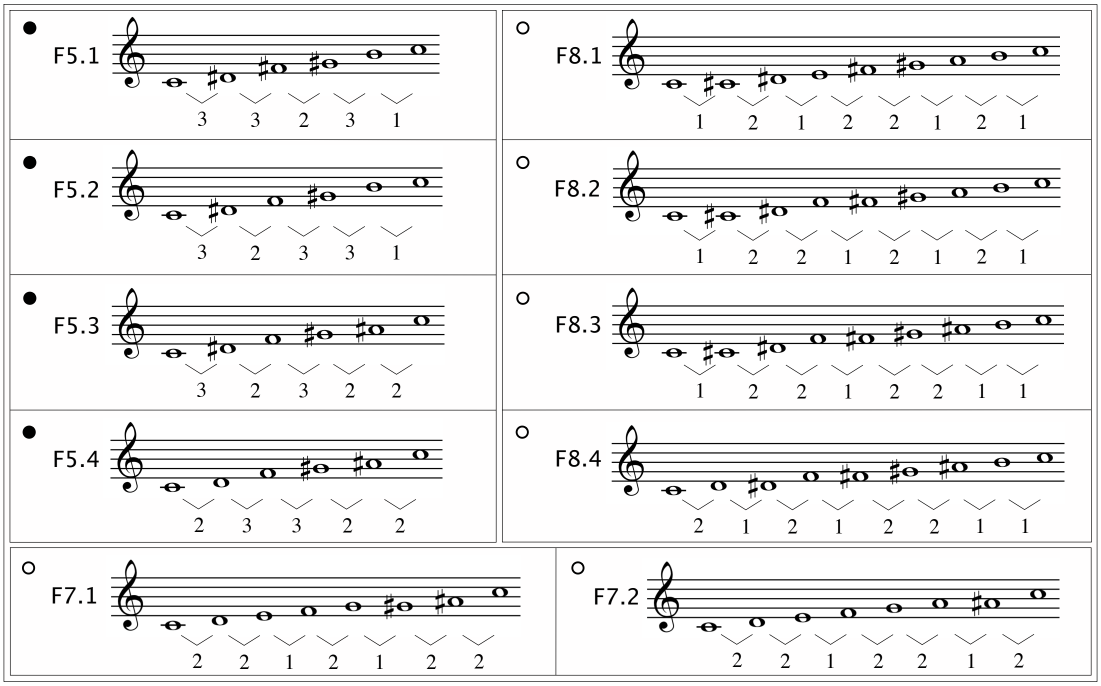


Figure 4: A transversal of admissible Fibonacci scales.

Since every element of T_F is aperiodic, the dimension of the space of modes of the Fibonacci scales is

$$\dim_{\bigcirc}(F) = (4 \times 8 + 2 \times 7) + (4 \times 5) + = 66.$$

Several well known scales are Fibonacci scales, for example, $\alpha^2(\mathbf{f}^{(6)})$ is the major scale (thus all the scales in the diatonic wheel are Fibonacci, see Figure 1), and $\alpha^3(\mathbf{f}^{(11)})$ is the pentatonic scale. Also note that M and F are mutually independent.

$\diamond$

Example 7 (Feigenbaum scales). The substitution rule $\circ \mapsto \bullet\bullet$ and $\bullet \mapsto \bullet\circ$ iterated over $\bullet$ yields the *Feigenbaum sequence*

$$\mathfrak{g} = \bullet\circ\bullet\bullet\bullet\circ\bullet\circ\bullet\circ\bullet\bullet\bullet\circ\bullet\bullet\bullet\circ\bullet\bullet\bullet\circ\bullet\circ\bullet\circ\bullet\bullet\bullet\circ\bullet\circ\ldots.$$

The elements of $G_\circ \triangleq \varphi\big(\mathcal{L}_{12}(\mathfrak{g},\circ)\big)$ are

$$\mathbf{g}^{(1)} \triangleq \circ\overset{(2,\ 2,\ 4,\ 2,\ 2)}{\bullet\circ\bullet\circ\bullet\bullet\bullet\circ\bullet\circ\bullet} \qquad \mathbf{g}^{(2)} \triangleq \circ\overset{(2,\ 2,\ 4,\ 4)}{\bullet\circ\bullet\circ\bullet\bullet\bullet\circ\bullet\bullet\bullet} \qquad \mathbf{g}^{(3)} \triangleq \circ\overset{(2,\ 4,\ 2,\ 2,\ 2)}{\bullet\circ\bullet\bullet\bullet\circ\bullet\circ\bullet\circ\bullet}$$

$$\mathbf{g}^{(4)} \triangleq \circ\overset{(2,\ 4,\ 4,\ 2)}{\bullet\circ\bullet\bullet\bullet\circ\bullet\bullet\bullet\circ\bullet} \qquad \mathbf{g}^{(5)} \triangleq \circ\overset{(4,\ 2,\ 2,\ 4)}{\bullet\bullet\bullet\circ\bullet\circ\bullet\circ\bullet\bullet\bullet} \qquad \mathbf{g}^{(6)} \triangleq \circ\overset{(4,\ 4,\ 2,\ 2)}{\bullet\bullet\bullet\circ\bullet\bullet\bullet\circ\bullet\circ\bullet}$$

$$\mathbf{g}^{(7)} \triangleq \circ\overset{(4,\ 4,\ 4)}{\bullet\bullet\bullet\circ\bullet\bullet\bullet\circ\bullet\bullet\bullet}$$

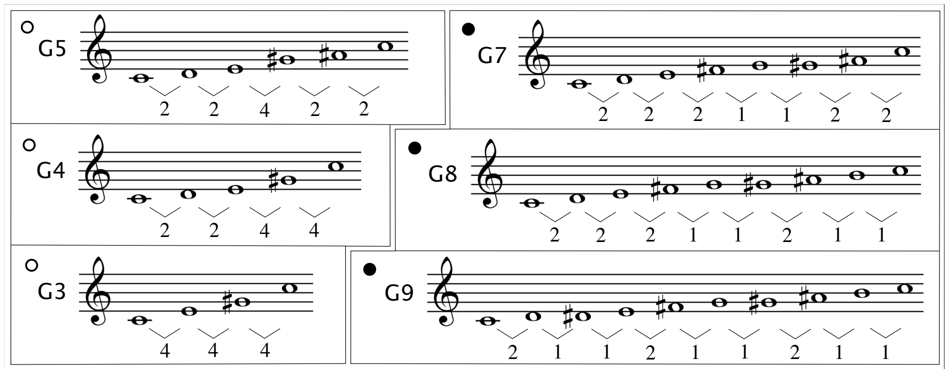


Figure 5: A transversal of admissible Feigenbaum scales.

and then the elements of $G_\bullet \triangleq \varphi\big(\mathcal{L}_{12}(\mathfrak{g}, \bullet)\big)$ are

$$\mathbf{g}^{(8)} \triangleq \overset{(2,\ 2,\ 2,\ 1,\ 1,\ 2,\ 2)}{\bullet\circ\bullet\circ\bullet\circ\bullet\bullet\bullet\circ\bullet\circ} \qquad \mathbf{g}^{(9)} \triangleq \overset{(2,\ 2,\ 2,\ 1,\ 1,\ 2,\ 1,\ 1)}{\bullet\circ\bullet\circ\bullet\circ\bullet\bullet\bullet\circ\bullet\bullet} \qquad \mathbf{g}^{(10)} \triangleq \overset{(2,\ 2,\ 1,\ 1,\ 2,\ 2,\ 2)}{\bullet\circ\bullet\circ\bullet\bullet\bullet\circ\bullet\circ\bullet\circ}$$

$$\mathbf{g}^{(11)} \triangleq \overset{(2,\ 2,\ 1,\ 1,\ 2,\ 1,\ 1,\ 2)}{\bullet\circ\bullet\circ\bullet\bullet\bullet\circ\bullet\bullet\bullet\circ} \qquad \mathbf{g}^{(12)} \triangleq \overset{(2,\ 1,\ 1,\ 2,\ 2,\ 2,\ 1,\ 1)}{\bullet\circ\bullet\bullet\bullet\circ\bullet\circ\bullet\circ\bullet\bullet} \qquad \mathbf{g}^{(13)} \triangleq \overset{(2,\ 1,\ 1,\ 2,\ 1,\ 1,\ 2,\ 2)}{\bullet\circ\bullet\bullet\bullet\circ\bullet\bullet\bullet\circ\bullet\circ}$$

$$\mathbf{g}^{(14)} \triangleq \overset{(2,\ 1,\ 1,\ 2,\ 1,\ 1,\ 2,\ 1,\ 1)}{\bullet\circ\bullet\bullet\bullet\circ\bullet\bullet\bullet\circ\bullet\bullet} \qquad \mathbf{g}^{(15)} \triangleq \overset{(1,\ 2,\ 2,\ 2,\ 1,\ 1,\ 2,\ 1)}{\bullet\bullet\circ\bullet\circ\bullet\circ\bullet\bullet\bullet\circ\bullet} \qquad \mathbf{g}^{(16)} \triangleq \overset{(1,\ 2,\ 1,\ 1,\ 2,\ 2,\ 2,\ 1)}{\bullet\bullet\circ\bullet\bullet\bullet\circ\bullet\circ\bullet\circ\bullet}$$

$$\mathbf{g}^{(17)} \triangleq \overset{(1,\ 2,\ 1,\ 1,\ 2,\ 1,\ 1,\ 2,\ 1)}{\bullet\bullet\circ\bullet\bullet\bullet\circ\bullet\bullet\bullet\circ\bullet} \qquad \mathbf{g}^{(18)} \triangleq \overset{(1,\ 1,\ 2,\ 2,\ 2,\ 1,\ 1,\ 2)}{\bullet\bullet\bullet\circ\bullet\circ\bullet\circ\bullet\bullet\bullet\circ} \qquad \mathbf{g}^{(19)} \triangleq \overset{(1,\ 1,\ 2,\ 1,\ 1,\ 2,\ 2,\ 2)}{\bullet\bullet\bullet\circ\bullet\bullet\bullet\circ\bullet\circ\bullet\circ}$$

$$\mathbf{g}^{(20)} \triangleq \overset{(1,\ 1,\ 2,\ 1,\ 1,\ 2,\ 1,\ 1,\ 2)}{\bullet\bullet\bullet\circ\bullet\bullet\bullet\circ\bullet\bullet\bullet\circ}$$

Observe that $\mathbf{g}^{(3)} \in \mathcal{O}_\alpha(\mathbf{g}^{(1)})$ with $\ell(\mathbf{g}^{(1)}) = 5$, $\mathbf{g}^{(4)}, \mathbf{g}^{(5)}, \mathbf{g}^{(6)} \in \mathcal{O}_\alpha(\mathbf{g}^{(2)})$ with $\ell(\mathbf{g}^{(2)}) = 4$, $\ell(\mathbf{g}^{(7)}) = 3$, $\mathbf{g}^{(10)} \in \mathcal{O}_\alpha(\mathbf{g}^{(8)})$ with $\ell(\mathbf{g}^{(8)}) = 7$, $\mathbf{g}^{(11)}, \mathbf{g}^{(12)}, \mathbf{g}^{(13)}, \mathbf{g}^{(15)}, \mathbf{g}^{(16)}, \mathbf{g}^{(18)}, \mathbf{g}^{(19)} \in \mathcal{O}_\alpha(\mathbf{g}^{(9)})$ with $\ell(\mathbf{g}^{(9)}) = 8$, and $\mathbf{g}^{(17)}, \mathbf{g}^{(20)} \in \mathcal{O}_\alpha(\mathbf{g}^{(14)})$ with $\ell(\mathbf{g}^{(14)}) = 9$. All are aperiodic except for $\mathbf{g}^{(7)}$ and $\mathbf{g}^{(14)}$ that are 3-periodic. Thus

$$T_\circ = \{\mathbf{g}^{(1)}, \mathbf{g}^{(2)}, \mathbf{g}^{(7)}\}$$

and

$$T_\bullet = \{\mathbf{g}^{(8)}, \mathbf{g}^{(9)}, \mathbf{g}^{(14)}\}$$

are transversals of $G_\circ$ and $G_\bullet$, respectively, and they are mutually independent (each scale has a different number of notes), thus $T_G \triangleq G_\circ \cup G_\bullet$ is a transversal of Feigenbaum sclaes, its elements are show in Figure 5. We conclude that

$$\dim_{\mathsf{T}}(G) = \dim_{\mathsf{T}}(G_\circ) + \dim_{\mathsf{T}}(G_\bullet) = 3 + 3 = 6$$

and

$$\dim_{\mathsf{O}}(G) = \dim_{\mathsf{O}}(G_\circ) + \dim_{\mathsf{O}}(G_\bullet) = \left(\frac{3}{3} + 4 + 5\right) + \left(7 + 8 + \frac{9}{3}\right) = 10 + 18 = 28.$$

M, F, and G are mutually independent.

$\diamond$

Problem 8. Find the transversal and orbital generating functions of the Thue-Morse, the Fibonacci, and the Feigenbaum scales.

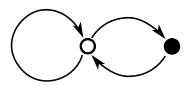
4.2 Symbolic dynamical scales of finite type

Here we use the material from sections 2 and 3.

Example 9 (Golden mean scales). Consider the *golden mean shift* $X \triangleq \mathsf{X}_{\mathcal{F}} \subseteq \mathcal{B}^{\mathbb{Z}}$ that is defined as the subshift that results from the forbidden set of blocks $\mathcal{F} = \{\bullet\bullet\}$. Thus $X = \widehat{\mathsf{X}}_A$ where

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

The matrix A is the adjacency matrix of the following directed graph:



Transversals and orbitals of the golden mean language. First let us illustrate the use of Theorem 3. The counting sequence of the language of the golden mean shift is 2-Fibonacci, that is, $\#\mathcal{L}_n(X) = F_n^{(2)}$ for all $n \geq 0$. On the other hand, the zeta function

$$\zeta_X(z) = \frac{1}{1 - z - z^2} = \sum_{n=0}^{\infty} F_n^{(1)} z^n$$

is the generating function of the 1-Fibonacci sequence, and the periodic counting sequence $\{p_n(X) = F_{n-1}^{(3)}\}_{n\geq 1}$ is 3-Fibonacci. For transversal dimensions, with reference to (3.4), we first see that

$$\sum_{\substack{i,j\in\mathcal{A} \\ A_{j,i}=0}} A_{i,j}^{n-1} = A_{\bullet,\bullet}^{n-1}$$

represents return loops of length n that begin and end at $\bullet$, but these are in fact sequences of *first* return loops to $\bullet$. Using (3.7) and (3.3) we get the generating function of the system of first return loops to $\bullet$,

$$f^{(\bullet)}(z) = \frac{z^2}{1 - z},$$

and deduce that

$$\begin{aligned}\sum_{n=1}^{\infty} A_{\bullet,\bullet}^{n-1} z^n &= z + z\frac{f^{(\bullet)}(z)}{1 - f^{(\bullet)}(z)} = \frac{z(1 - z)}{1 - z - z^2} \\ &= z + z^3 + z^4 + 2z^5 + 3z^6 + 5z^7 + 8z^8 + 13z^9 + \ldots,\end{aligned}$$

in particular, for $n \geq 3$, $A^{n}_{\bullet,\bullet} = F^{(2)}_{n-3}$ is 2-Fibonacci. Next, the sequence $\{q_n(X)\}_{n\geq 1}$, obtained from $\{p_n(X) = F^{(3)}_{n-1}\}_{n\geq 1}$ by Möbius inversion, defines a minimal periodic generating function

$$\begin{aligned} q^{(X)}(z) &\triangleq \sum_{n\geq 1} q_n(X) z^n \\ &= z + z^2 + z^3 + z^4 + 2z^5 + 2z^6 + 4z^7 + 5z^8 + 8z^9 + \dots \end{aligned}$$

that corresponds to A006206 in [39], described as the number of *aperiodic* binary necklaces with no subsequence ••, excluding the necklace •. The coefficients of the generating function

$$\begin{aligned} \overline{q}^{(X)}(z) &\triangleq \sum_{n\geq 1} \left(\sum_{k|n} \frac{q_k(X)}{k} \right) z^n \\ &= z + 2z^2 + 2z^3 + 3z^4 + 3z^5 + 5z^6 + 5z^7 + 8z^8 + 10z^9 + \dots \end{aligned} \tag{4.1}$$

correspond to 000358 in [39], which is described as the number of necklaces with no subsequence ••, excluding the necklace •. Thus the transversal generating function of the language of X has the form

$$\dim^{(X)}_{\mathsf{T}}(z) = \frac{z(1-z)}{1-z-z^2} + \overline{q}^{(X)}(z).$$

For example, according to (3.4),

$$\begin{aligned}
\dim_{\mathsf{T}}\big(\mathcal{L}_1(X)\big) &= 1+1 = 2, & &\{\circ\} \cup \{\bullet\} \\
\dim_{\mathsf{T}}\big(\mathcal{L}_2(X)\big) &= 0+2 = 2, & &\{\circ\,\circ, \circ\,\bullet\} \\
\dim_{\mathsf{T}}\big(\mathcal{L}_3(X)\big) &= 1+2 = 3, & &\{\circ\,\circ\,\circ, \circ\,\circ\,\bullet\} \cup \{\bullet\,\circ\,\bullet\} \\
\dim_{\mathsf{T}}\big(\mathcal{L}_4(X)\big) &= 1+3 = 4, & &\{\circ\,\circ\,\circ\,\circ, \circ\,\circ\,\circ\,\bullet, \circ\,\bullet\,\circ\,\bullet\} \cup \{\bullet\,\circ\,\circ\,\bullet\} \\
\dim_{\mathsf{T}}\big(\mathcal{L}_5(X)\big) &= 2+3 = 5, & &\{\circ\,\circ\,\circ\,\circ\,\circ, \circ\,\circ\,\circ\,\circ\,\bullet, \circ\,\circ\,\bullet\,\circ\,\bullet\} \cup \{\bullet\,\circ\,\circ\,\circ\,\bullet, \bullet\,\circ\,\bullet\,\circ\,\bullet\} \\
\dim_{\mathsf{T}}\big(\mathcal{L}_6(X)\big) &= 3+5 = 8, & &\left\{ \begin{matrix} \circ\,\circ\,\circ\,\circ\,\circ\,\circ, & \circ\,\circ\,\circ\,\circ\,\circ\,\bullet, \\ \circ\,\circ\,\circ\,\bullet\,\circ\,\bullet, & \circ\,\circ\,\bullet\,\circ\,\circ\,\bullet, \\ \circ\,\bullet\,\circ\,\bullet\,\circ\,\bullet & \end{matrix} \right\} \cup \left\{ \begin{matrix} \bullet\,\circ\,\circ\,\circ\,\circ\,\bullet, \\ \bullet\,\circ\,\circ\,\bullet\,\circ\,\bullet, \\ \bullet\,\circ\,\bullet\,\circ\,\circ\,\bullet \end{matrix} \right\} \\
&\vdots
\end{aligned}$$

For orbital dimensions, with reference to (3.5), first we see that

$$\begin{aligned} \sum_{n=1}^{\infty} n A^{n-1}_{\bullet,\bullet} z^{n-1} &= \frac{d}{dz}\left(\frac{z(1-z)}{1-z-z^2}\right) = \frac{1-2z+2z^2}{(1-z-z^2)^2} \\ &= 1 + 3z^2 + 4z^3 + 10z^4 + 18z^5 + 35z^6 + 64z^7 + 117z^8 + 210z^9 + \dots, \end{aligned}$$

which corresponds to A006490 in [39]. We already know that $p_n(X)$ is the 3-Fibonacci sequence $F^{(3)}_{n-1}$. Thus

$$\begin{aligned}\dim_{\mathsf{O}}^{\mathcal{L}(X)}(z) &= z\frac{1-2z+2z^2}{(1-z-z^2)^2} + z\frac{1+2z}{1-z-z^2} = \frac{z(2-z-z^2-2z^3)}{(1-z-z^2)^2}\\ &= 2z+3z^2+7z^3+11z^4+21z^5+36z^6+64z^7+111z^8+193z^9+\dots\end{aligned}$$

(the corresponding sequence of coefficients has no record in [39]).

Admissible golden mean scales. Now we look at the image

$$\varphi\big(\mathcal{L}(X)\big) \subset \mathcal{C}$$

that corresponds to the *admissible golden mean scales*. First, using again (3.7) and (3.3), we get the generating functions of the loop systems, namely

$$f^{(\circ)}(z) = z + z^2 \tag{4.2}$$

$$f^{(\bullet)}(z) = \frac{z^2}{1-z} = z^2 + z^3 + z^4 + \dots. \tag{4.3}$$

Then the $\circ$-admissible golden mean scales $\mathcal{C}^{(X,\circ)} \triangleq \varphi\big(\mathcal{L}(X,\circ)\big)$ are all the integer compositions with summands in $\mathcal{K}(\circ) = \{1,2\}$ (see (4.2)), that is, all the scales with no more than two tone measures of difference between consecutive notes, as expected (in this case we have $a^{\mathcal{K}(\circ)}(z,u) = 0$, and thus also $b^{\mathcal{K}(\circ)}(z,u) = 0$, because any element in $\mathcal{K}(\circ)^{\mathsf{c}}$ is not bounded above by any element of $\mathcal{K}(\circ)$). The corresponding generating function for this class of integer compositions, according to (2.3), is

$$C^{(X,\circ)}(z) = C^{\mathcal{K}(\circ)}(z) = \frac{1}{1-z-z^2}, \tag{4.4}$$

thus $C^{(X,\circ)}_n = F^{(1)}_n$ is 1-Fibonacci. For example, for 12-TET tuning system, the total number of $\circ$-admissible golden mean scales is

$$C^{(X,\circ)}_{12} = F^{(1)}_{12} = 233.$$

The set $\mathcal{C}^{(X,\circ)}_{12}$ is too large to list. With the purpose of having a smaller context that helps illustrating and verifying the claims, let us imagine, for instance, that we are doing scales over a small set of notes, say over 5 notes (e.g. over a pentatonic scale). Combinatorially, the model is that of a 5-TET tuning system. We thus have

$$C^{(X,\circ)}_5 = F^{(1)}_5 = 8, \qquad \mathcal{C}^{(X,\circ)}_5 = \left\{ \begin{array}{c} \overset{(1,1,1,1,1)}{\circ\circ\circ\circ\circ}, \overset{(1,1,1,2)}{\circ\circ\circ\circ\bullet}, \overset{(1,1,2,1)}{\circ\circ\circ\bullet\circ}, \overset{(1,2,1,1)}{\circ\circ\bullet\circ\circ}, \\ \\ \overset{(2,1,1,1)}{\circ\bullet\circ\circ\circ}, \overset{(1,2,2)}{\circ\circ\bullet\circ\bullet}, \overset{(2,1,2)}{\circ\bullet\circ\circ\bullet}, \overset{(2,2,1)}{\circ\bullet\circ\bullet\circ} \end{array} \right\}. \tag{4.5}$$

We also have the bivariate version of (4.4), with u marking the number of notes, namely

$$C^{(X,\circ)}(z,u) = C^{\mathcal{K}(\circ)}(z,u) = \frac{1}{1-uz-uz^2}.$$

For the $\bullet$-admissible golden mean scales $\mathcal{C}^{(X,\bullet)} \triangleq \varphi\big(\mathcal{L}(X,\bullet)\big)$, first observe that the class of integer compositions with summands in $\mathcal{K}(\bullet) = \{2,3,4,\ldots\}$ (see (4.3)) has ordinary generating function

$$C^{\mathcal{K}(\bullet)}(z) = \frac{1-z}{1-z-z^2}.$$

The bivariate version of $C^{\mathcal{K}(\bullet)}(z)$, with the variable u marking the number of notes, is

$$C^{\mathcal{K}(\bullet)}(z,u) = \frac{1-z}{1-z-uz^2}.$$

In addition, in this case, the last summand in the elements of $\mathcal{C}^{(X,\bullet)}$ *is* allowed to be $1 \notin \mathcal{K}(\bullet)$ (for example, in 12-TET tuning system, the binary admisible 12-block $\bullet\circ\bullet\circ\bullet\circ\bullet\circ\bullet\circ\circ\bullet$ yields the $\bullet$-admissible golden mean scale $(2,2,2,2,3,1)$). Therefore, the corresponding generating function is

$$C^{(X,\bullet)}(z) = C^{\mathcal{K}(\bullet)}(z) + zC^{\mathcal{K}(\bullet)}(z) = \frac{1-z^2}{1-z-z^2}, \tag{4.6}$$

and we also get its bivariate version,

$$C^{(X,\bullet)}(z,u) = C^{\mathcal{K}(\bullet)}(z,u) + uzC^{\mathcal{K}(\bullet)}(z,u) = \frac{(1+uz)(1-z)}{1-z-uz^2}.$$

Thus $C_0^{(X,\bullet)} = 1$ and for $n \geq 1$ the sequence of coefficients $C_n^{(X,\bullet)} = F_{n-1}^{(1)}$ is 1-Fibonacci. For example, for the 12-TET and 5-TET tuning systems we have

$$\begin{aligned} &C_{12}^{(X,\bullet)} = F_{11}^{(1)} = 144, \\ &C_5^{(X,\bullet)} = F_4^{(1)} = 5, \qquad \mathcal{C}_5^{(X,\bullet)} = \left\{ \bullet \overset{(5)}{\circ\circ}\circ\circ, \bullet \circ \overset{(4,1)}{\circ\circ} \bullet, \bullet \circ \overset{(3,2)}{\circ\bullet} \circ, \bullet \circ \overset{(2,3)}{\bullet\circ} \circ, \bullet \circ \overset{(2,2,1)}{\bullet\circ} \bullet \right\}. \end{aligned} \tag{4.7}$$

Remark 4.8. The only elements that are both $\circ$-admissible and $\bullet$-admissible golden mean scales are the compositions of even $n = 2d$ and odd $n = 2d+1$ integers of the form

$$(\underbrace{2,2,\ldots,2}_{d\text{-times}}) \qquad \text{and} \qquad (\underbrace{2,2,\ldots,2}_{d\text{-times}},1). \tag{4.9}$$

Thus, combining (4.4) and (4.6), we conclude that the generating function of *admissible* golden mean scales is

$$\begin{aligned}C^{(X)}(z) &= C^{(X,\circ)}(z) + C^{(X,\bullet)}(z) - \frac{1+z}{1-z^2} = \frac{1-z+z^3}{1-2z+z^3}\\ &= 1+z+2z^2+4z^3+7z^4+12z^5+20z^6+33z^7+54z^8+88z^9+143z^{10}+\ldots\end{aligned}$$

(the corresponding coefficients are, essentially, A000071 in [39]), and we also get its bivariate version,

$$C^{(X)}(z,u) = C^{(X,\circ)}(z,u) + C^{(X,\bullet)}(z,u) - \frac{1+uz}{1-uz^2}.$$

Then, for every $n \geq 1$ we have $C_n^{(X)} = F_n^{(1)} + F_{n-1}^{(1)} - 1$. For example, the number of admissible golden means scales in 12-TET and 5-TET tuning systems are (for the later see (4.5) and (4.7))

$$C_{12}^{(X)} = 233 + 144 - 1 = 376,$$

$$C_5^{(X)} = 8+5-1 = 12, \qquad \mathcal{C}_5^{(X)} = \left\{ \begin{array}{c} \overset{(1,1,1,1,1)}{\circ\circ\circ\circ\circ}, \overset{(1,1,1,2)}{\circ\circ\circ\circ\bullet}, \overset{(1,1,2,1)}{\circ\circ\circ\bullet\circ}, \overset{(1,2,1,1)}{\circ\circ\bullet\circ\circ}, \\ \\ \overset{(2,1,1,1)}{\circ\bullet\circ\circ\circ}, \overset{(1,2,2)}{\circ\circ\bullet\circ\bullet}, \overset{(2,1,2)}{\circ\bullet\circ\circ\bullet}, \overset{(2,2,1)}{\circ\bullet\circ\bullet\circ}, \\ \\ \overset{(5)}{\bullet\circ\circ\circ\circ}, \overset{(4,1)}{\bullet\circ\circ\circ\bullet}, \overset{(3,2)}{\bullet\circ\circ\bullet\circ}, \overset{(2,3)}{\bullet\circ\bullet\circ\circ} \end{array} \right\}.$$

Transversal and orbital dimensions of golden mean scales. First look at the case when $\mathfrak{s} = \circ$. With the ordinary form $W^{\mathcal{K}}(z) \triangleq W^{\mathcal{K}}(z,1)$ of (2.5), we obtain the first summand in the right hand side of (1.2),

$$W^{\mathcal{K}(\circ)}(z) = z+2z^2+2z^3+3z^4+3z^5+5z^6+5z^7+8z^8+10z^9+15z^{10}+19z^{11}+31z^{12}+\ldots$$

with coefficients forming again the sequence A000358 in [39] (that is, in this case, we have $W^{\mathcal{K}(\circ)}(z) = \overline{q}^{(X)}(z)$, see (4.1)). Since $a^{\mathcal{K}(\circ)}(z) = b^{\mathcal{K}(\circ)}(z) = 0$,

$$\dim_{\mathsf{T}}^{\varphi(X,\circ)}(z) = W^{\mathcal{K}(\circ)}(z) \tag{4.10}$$

and

$$\dim_{\mathsf{O}}^{\varphi(X,\circ)}(z) = C^{\mathcal{K}(\circ)}(z) \tag{4.11}$$

(see (4.4)). In particular, for the 12-TET and 5-TET tuning system, the transversal dimensions are

$$\begin{aligned}\dim_{\mathsf{T}}\big(\varphi\big(\mathcal{L}_{12}(X,\circ)\big)\big) &= W_{12}^{\mathcal{K}(\circ)}(z) = 31\\ \dim_{\mathsf{T}}\big(\varphi\big(\mathcal{L}_{5}(X,\circ)\big)\big) &= W_{5}^{\mathcal{K}(\circ)}(z) = 3, \qquad T_5^{(X,\circ)} \triangleq \left\{ \overset{(1,1,1,1,1)}{\circ\circ\circ\circ\circ}, \overset{(1,1,1,2)}{\circ\circ\circ\circ\bullet}, \overset{(1,2,2)}{\circ\circ\bullet\circ\bullet} \right\}\end{aligned} \tag{4.12}$$

and the corresponding orbital dimensions are

$$\begin{aligned}
\dim_{\mathsf{O}}\big(\varphi\big(\mathcal{L}_{12}(X,\circ)\big)\big) &= C_{12}^{\mathcal{K}(\circ)}(z) = 233 \\
\dim_{\mathsf{O}}\big(\varphi\big(\mathcal{L}_{5}(X,\circ)\big)\big) &= C_{5}^{\mathcal{K}(\circ)}(z) = 8 \qquad \text{(see again (4.5))}.
\end{aligned}$$

Now suppose that $\mathfrak{s} = \bullet$ and proceed similarly. For the first summand in the right hand side of (3.10), use (2.5) to obtain the generating function

$$W^{\mathcal{K}(\bullet)}(z) = z^2 + z^3 + 2z^4 + 2z^5 + 4z^6 + 4z^7 + 7z^8 + 9z^9 + 14z^{10} + 18z^{11} + 30z^{12} + \dots$$

with coefficients forming the sequence A032190 in [39] (modulo a shift), which is already described as the number of cyclic compositions of n into parts ≥ 2. For example, for the 12-TET and 5-TET tuning system, we have

$$\begin{aligned}
&W_{12}^{\mathcal{K}(\bullet)} = 30, \\
&W_{5}^{\mathcal{K}(\bullet)} = 2, \qquad \mathcal{W}_{5}^{\mathcal{K}(\bullet)} = \{\bullet\overset{(5)}{\circ\circ\circ\circ}, \bullet\overset{(3,2)}{\circ\circ\bullet\circ}\}.
\end{aligned}$$

Next, from (3.9) we get

$$a_n^{\mathcal{K}(\bullet)} \triangleq \sum_{\substack{k\notin\mathcal{K}(\bullet) \\ \exists k'\in\mathcal{K}(\bullet),\ k'>k}} C_{n-k}^{\mathcal{K}(\bullet)} = C_{n-1}^{\mathcal{K}(\bullet)}$$

and thus $a_1^{\mathcal{K}(\bullet)} = 1$, $a_2^{\mathcal{K}(\bullet)} = 0$ and $a_{n+3}^{\mathcal{K}(\bullet)} = F_n^{(1)}$ is the 1-Fibonacci sequence. Hence

$$\begin{aligned}
a^{\mathcal{K}(\bullet)}(z) &= \frac{z - z^2}{1 - z - z^2} \\
&= z + z^3 + z^4 + 2z^5 + 3z^6 + 5z^7 + 8z^8 + 13z^9 + 21z^{10} + 34z^{11} + 55z^{12} + \dots.
\end{aligned}$$

For instance, in 12-TET and 5-TET tuning systems, we have

$$\begin{aligned}
&a_{12}^{\mathcal{K}(\bullet)} = 55 \\
&a_{5}^{\mathcal{K}(\bullet)} = 2, \qquad \{\bullet\overset{(4,1)}{\circ\circ\circ\bullet}, \bullet\overset{(2,2,1)}{\circ\bullet\circ\bullet}\}.
\end{aligned} \tag{4.13}$$

Thus (1.2) in Theorem 1 yields the transversal generating function

$$\dim_{\mathsf{T}}^{\varphi(X,\bullet)}(z) = W^{\mathcal{K}(\bullet)}(z) + \frac{z - z^2}{1 - z - z^2}. \tag{4.14}$$

For example, in 12-TET and 5-TET tuning system, we have

$$\begin{aligned}
&\dim_{\mathsf{T}}\big(\varphi\big(\mathcal{L}_{12}(X,\bullet)\big)\big) = 30 + 55 = 85, \\
&\dim_{\mathsf{T}}\big(\varphi\big(\mathcal{L}_{5}(X,\bullet)\big)\big) = 2 + 2 = 4, \qquad T_5^{(X,\bullet)} = \{\bullet\overset{(5)}{\circ\circ\circ\circ}, \bullet\overset{(3,2)}{\circ\circ\bullet\circ}, \bullet\overset{(4,1)}{\circ\circ\circ\bullet}, \bullet\overset{(2,2,1)}{\circ\bullet\circ\bullet}\}.
\end{aligned} \tag{4.15}$$

For orbital dimensions, first we have

$$\begin{aligned} C^{\mathcal{K}(\bullet)}(z) &= \frac{1-z}{1-z-z^2} \qquad (4.16)\\ &= 1+z^2+z^3+2z^4+3z^5+5z^6+8z^7+13z^8+21z^9+34z^{10}+\dots. \end{aligned}$$

For example, in 12-TET and 5-TET tuning systems, we get

$$C_{12}^{\mathcal{K}(\bullet)} = 89, \qquad (4.17)$$

$$C_5^{\mathcal{K}(\bullet)} = 3, \qquad \mathcal{C}_5^{\mathcal{K}(\bullet)} = \{\overset{(5)}{\bullet\circ\circ\circ\circ}, \overset{(3,2)}{\bullet\circ\circ\bullet\circ}, \overset{(2,3)}{\bullet\circ\bullet\circ\circ}\}. \qquad (4.18)$$

Next, according to item 3 in Theorem 1, we need the cumulative generating function of the class represented by $a^{\mathcal{K}(\bullet)}(z)$, with respect to the number of notes. From (3.11),

$$a_{n,m}^{\mathcal{K}(\bullet)} = \sum_{\substack{k\notin\mathcal{K}(\bullet)\\ \exists k'\in\mathcal{K}(\bullet),\ k'>k}} C_{n-k,m}^{\mathcal{K}(\bullet)} = C_{n-1,m}^{\mathcal{K}(\bullet)},$$

thus, using (2.4), we get

$$a^{\mathcal{K}(\bullet)}(z,u) = uzC^{\mathcal{K}(\bullet)}(z,u) = \frac{uz-uz^2}{1-z-uz^2}.$$

Hence, from (3.13) we get

$$b^{\mathcal{K}(\bullet)}(z,u) = u\frac{\partial}{\partial u}a^{\mathcal{K}(\bullet)}(z,u) = \frac{uz(1-z)^2}{(1-z-uz^2)^2},$$

and from (3.12) we obtain

$$\begin{aligned} b^{\mathcal{K}(\bullet)}(z) &= \frac{z(1-z)^2}{(1-z-z^2)^2} \qquad (4.19)\\ &= z+2z^3+2z^4+5z^5+8z^6+15z^7+26z^8+46z^9+80z^{10}+\dots \end{aligned}$$

(the coefficients are A006367 in [39]). For example, in the 12-TET and 5-TET tuning system (for the later see (4.13)), we get

$$b_{12}^{\mathcal{K}(\bullet)} = 240, \qquad (4.20)$$

$$b_5^{\mathcal{K}(\bullet)} = 5, \qquad \overbrace{\{(4,1),(1,4)\}}^{\mathcal{O}_\alpha((4,1))} \cup \overbrace{\{(2,2,1),(2,1,2),(1,2,2)\}}^{\mathcal{O}_\alpha((2,2,1))}. \qquad (4.21)$$

Hence, (1.3) in Theorem 1, together with (4.16) and (4.19), yield

$$\begin{aligned} \dim_{\mathsf{O}}^{\varphi(X,\bullet)}(z) &= C^{\mathcal{K}(\bullet)}(z) + b^{\mathcal{K}(\bullet)}(z) = \frac{(1-z)(1-2z^2)}{(1-z-z^2)^2} \qquad (4.22)\\ &= 1+z+z^2+3z^3+4z^4+8z^5+13z^6+23z^7+39z^8+67z^9+\dots, \end{aligned}$$

which corresponds to A206268 in [39], described as the number of compositions with at most one 1. For example, in the 12-TET and 5-TET tuning system (see (4.17), (4.18), (4.20), and (4.21)),

$$\dim_{\mathsf{O}}\big(\varphi\big(\mathcal{L}_{12}(X,\bullet)\big)\big) = 89 + 240 = 329,$$

$$\dim_{\mathsf{O}}\big(\varphi\big(\mathcal{L}_{5}(X,\bullet)\big)\big) = 3 + 5 = 8, \qquad \mathcal{O}_\alpha\big(\varphi\big(\mathcal{L}_{5}(X,\bullet)\big)\big) = \left\{ \begin{array}{ccc} (5), & (3,2), & (2,3), \\ (4,1), & (1,4), & \\ (2,2,1), & (2,1,2), & (1,2,2) \end{array} \right\}.$$

For a global transversal, according to remark 4.8, with (4.10) and (4.14) we get

$$\begin{aligned}
\dim_{\mathsf{T}}^{\varphi(X)}(z) &= \dim_{\mathsf{T}}^{\varphi(X,\circ)}(z) + \dim_{\mathsf{T}}^{\varphi(X,\bullet)}(z) - \dim_{\mathsf{T}}^{\varphi(X,\circ)\cap\varphi(X,\bullet)}(z) \\
&= W^{\mathcal{K}(\circ)}(z) + W^{\mathcal{K}(\bullet)}(z) + \frac{z-z^2}{1-z-z^2} - \frac{z}{1-z} \\
&= z + 2z^2 + 3z^3 + 5z^4 + 6z^5 + 11z^6 + 13z^7 + 22z^8 + 31z^9 \ldots
\end{aligned}$$

(the corresponding sequence of coefficients has no record in [39]). For example, for the 12-TET and 5-TET tuning system, the transversal dimensions of the golden mean scales are (for the later see (4.12) and (4.15))

$$\dim_{\mathsf{T}}\big(\varphi\big(\mathcal{L}_{12}(X)\big)\big) = 115$$

$$\dim_{\mathsf{T}}\big(\varphi\big(\mathcal{L}_{5}(X)\big)\big) = 6, \qquad T_5^{(X,\circ)} \cup T_5^{(X,\bullet)} \triangleq \left\{ \begin{array}{l} \overset{(1,1,1,1,1)}{\circ\circ\circ\circ\circ}, \overset{(1,1,1,2)}{\circ\circ\circ\circ\bullet}, \overset{(1,2,2)}{\circ\circ\bullet\circ\bullet}, \\ \overset{(5)}{\bullet\circ\circ\circ\circ}, \overset{(3,2)}{\bullet\circ\circ\bullet\circ}, \overset{(4,1)}{\bullet\circ\circ\circ\bullet} \end{array} \right\}$$

($(1,2,2)$ and $(2,2,1)$ are equal as wheels).

Finally, for the global orbital, now we use (4.11) and (4.22), but first observe that, according to remark 4.8, in addition to the empty composition, there are two kinds of compositions in the intersection (see (4.9)): one kind is formed by compositions of even integers $n = 2d$ of length d and period 1, and the other kind is formed by compositions of odd $n = 2d + 1$ of length $d + 1$ that are aperiodic. We conclude that

$$\begin{aligned}
\dim_{\mathsf{O}}^{\varphi(X)}(z) &= \dim_{\mathsf{O}}^{\varphi(X,\circ)}(z) + \dim_{\mathsf{O}}^{\varphi(X,\bullet)}(z) - \dim_{\mathsf{O}}^{\varphi(X,\circ)\cap\varphi(X,\bullet)}(z) - 1 \\
&= \frac{1}{1-z-z^2} + \frac{(1-z)(1-2z^2)}{(1-z-z^2)^2} - \frac{z^2}{1-z^2} - \sum_{d=0}^{\infty}(d+1)z^{2d+1} - 1 \\
&= \frac{1 - z - 3z^2 + 3z^3 + 4z^4 - 5z^5 - 2z^6 + 2z^7}{(1-z^2)^2(1-z-z^2)^2} \\
&= 1 + z + 2z^2 + 4z^3 + 8z^4 + 13z^5 + 25z^6 + 40z^7 + 72z^8 + 117z^9 + \ldots.
\end{aligned}$$

For example, in the 12-TET and 5-TET tuning system, we get

$$\dim_{\mathsf{O}}\big(\varphi\big(\mathcal{L}_{12}(X)\big)\big) = 561$$

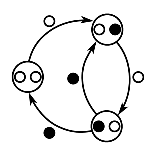
$$\dim_{\mathsf{O}}\big(\varphi\big(\mathcal{L}_5(X)\big)\big) = 13, \qquad \textsc{modes}\big(\mathcal{L}_5(X)\big) = \left\{ \begin{array}{l} (1,1,1,1,1),(1,1,1,2),(1,1,2,1), \\ (1,2,1,1),(2,1,1,1),(1,2,2), \\ (2,2,1),(2,1,2),(5),(3,2), \\ (2,3),(4,1),(1,4) \end{array} \right\}.$$

$\diamond$

Example 10. Let $X = \mathsf{X}_{\mathcal{F}}$ where $\mathcal{F} = \{\bullet\bullet, \circ\circ\circ\}$. Then X is a 2-step SFT. Hence, $X^{[2]}$ is the vertex shift represented by the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

The corresponding directed graph is



(the labels on the edges define a 1-block code that is a conjugacy from $X^{[2]}$ to X, with a 2-block code inverse). Then $\varphi\big(\mathcal{L}(X)\big)$ coincides with the scales defined by $X^{[2]}$ with the distinguished set of symbols rule for $\{\circ\circ, \circ\bullet\}$. The matrix (3.14) is

$$\begin{pmatrix} 0 & z \\ z^2 & z^2 \end{pmatrix}.$$

Then the $\circ\circ$-admisible scales are compositions with summands in $\{1,2\}$ such that the first summand equals to 1, and no adjacent 1s, except perhaps the last two summands. On the other hand, the $\circ\bullet$-admisible scales are compositions with summands in $\{1,2\}$ such that the first summand equals to 2, and no adjacent ones, except perhaps the last two summands. Further details are left to the reader.

$\diamond$

5 Related works and discussion

We have seen a general method to deduce transversal and orbital generating functions of classes of musical scales induced by vertex shift spaces through the distinguished symbol rule. We have found approaches that use generating functions at least in the texts [27, 8], and learned that enumeration problems in music go back at least to the works of [37, 36, 18]. Our methods can serve to complement and generalize several other works that address characterizations and classification of musical scales like [35, 28], also octave subdivisions [23], optimal spelling of pitches of musical scales [9], tuning systems other that 12-TET [23], scales and constraint programming [25], modular arithmetic sequences and scales [4], algebras of periodic rhythms and scales [6], formalisms to generate pure-tone systems that best approximate modulation/transposition properties of equal-tempered scales [29], tuning systems and modes [19], etc. Moreover, other combinatorial classes, such as non-crossing configurations [15] like dissections of polygons and RNA secondary structures [22], can be incorporated to complement works that address constructions of musical scales like [34].

There are many references that address the theory of musical scales that are relevant to our work, like the fundamentals [17, 40], from the point of view of mathematics inclusive [26, 32], several of which are related to combinatorics on words [2, 14, 13, 1].

In our arguments, a key ingredient has been the use of first return loop systems, which arise in the study of classification problems of Markov shifts [20, 12]. In fact, studying music theory in contexts of dynamical systems has been an active area of research, for example [5], see also [41]. Furthermore, the results presented here can serve as a basis to adapt other related areas of mathematics in music, such as thermodynamic formalism and random environments [7] (e.g. to compute (relative) partition functions).

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