arXiv:2009.04052v2 [math.NT] 11 Sep 2020

The abc Conjecture Implies That Only Finitely Many Cullen Numbers Are Repunits

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Abstract

Assuming the abc conjecture with $\epsilon = 1$, we use elementary methods to show that for any integer $s \geq 2$, there are only finitely many s-Cullen numbers that are repunits. More precisely, for fixed s, there are only finitely many positive integers n, b, and q with $n, b \geq 2$ and $q \geq 3$ such that

$$C_{s,n} = ns^n + 1 = \frac{b^q - 1}{b - 1}.$$

1 Introduction

Definition 1. A *Cullen number* is an number of the form $C_n = n2^n + 1$, where n is a positive integer. These are A002064 in the OEIS. An *s-Cullen number* is a number of the form $C_{n,s} = ns^n + 1$, where s, n are positive integers with $s \geq 2$. See A050914, for example, for the 3-Cullen numbers. Cullen and Dubner ([3] and [4]) introduced the two families, respectively.

The first significant result on Cullen numbers occurred in 1976, when Hooley [5] showed that almost all Cullen numbers are composite. (It is conjectured, but not proven, that infinitely many are prime.)

Luca and Stănică [6] showed that the intersection of the Cullen numbers with the Fibonacci sequence is finite. This result was generalized to s-Cullen numbers (for fixed s) by Marques [7]. Bilu, Marques, and Togbé [2] (among other things) generalized to the intersection of s-Cullen numbers with other recurrence sequences.

In this paper, we are considering the intersection of the s-Cullen numbers (for fixed s) with a two–parameter family, namely the repunits. We are able to prove, conditionally, that the intersection of the repunits and Cullen numbers (or s-Cullen numbers, for fixed s) is finite.

Definition 2. A *repunit* is a positive integer n that we can write as

$$\frac{b^q - 1}{b - 1} = \sum_{j=0}^{q-1} b^j = b^{q-1} + b^{q-2} + \cdots + b + 1 = (11 \cdots 1)_b$$

for some integer *base* $b > 1$ and some integer exponent q. Beiler [1] introduced the name in the 1960s.

For example, base-10 repunits are A002275 in the OEIS, base-15 repunits are A135518, and base-2 repunits are A000225. We impose the condition $q \geq 3$ to avoid the trivial representation of any number x as 11 in base $x - 1$.

Definition 3. The *radical* of a positive integer n is the product of all the primes that divide n, so if $n = \prod_{p_i|n} p_i^{a_i}$, then $\text{rad}(n) = \prod_{p_i|n} p_i$. For example, the radical of $90 = 2 \cdot 3^2 \cdot 5$ is $30 = 2 \cdot 3 \cdot 5$.

Conjecture 4. The *abc conjecture* of Oesterlé [9] and Masser [8] states that if a,b, and c are relatively prime integers such that $a + b = c$, then for any $\epsilon > 0$, there are only finitely many exceptions to the equation

$$c < \text{rad}(abc)^{1+\epsilon}.$$

In the next section, we use this conjecture with $\epsilon = 1$.

2 Main result

Theorem 5. *The abc conjecture with $\epsilon = 1$ implies that for any fixed integer s, $s \geq 2$, there are only finitely many s-Cullen numbers $C_{s,n}$ that are repunits.*

We divide our theorem into two cases, each of which we prove as a proposition. The first proposition shows that there are only finitely many s-Cullen numbers that can be written as repunits of length three, and the second shows that there are only finitely many s-Cullen numbers that can be written as repunits of length greater than three. As the union of two finite sets is finite itself, the two propositions prove the theorem.

Proposition 6. *The abc conjecture implies that for any fixed integer s, $s \geq 2$, there are only finitely many s-Cullen numbers that are repunits of length three.*

Proof. Suppose that $C_n = ns^n + 1$ is a repunit of length three, i.e., $C_{s,n} = ns^n + 1 = b^2 + b + 1$. Our assumption lets us infer that $ns^n = b(b+1)$. Let us first consider the case

$$b + 1 < \operatorname{rad}(b(b+1))^2 = \operatorname{rad}(ns^n)^2 < (sn)^2,$$

then

$$ns^n = b(b+1) < (b+1)^2 < (ns)^4 = n^4 s^4.$$

By taking logarithms, we see that

$$n - 4 < 3 \log_s(n).$$

The last equation holds for only finitely many values of n. In the case where the inequality $b + 1 < \operatorname{rad}(b(b+1))^2$ does not hold, the abc conjecture with $\epsilon = 1$ says that we only have examples for finitely many values of b. Therefore, there are only finitely many n such that $C_{s,n}$ is a repunit of length three.

□

Proposition 7. *The abc conjecture implies that for any fixed s, there are only finitely many s-Cullen numbers that are repunits of length greater than three.*

Proof. Suppose that $C_n = ns^n + 1$ is a repunit of length greater than three, i.e., that C_n can written as $\frac{b^{q-1}}{b-1} = b^{q-1}+b^{q-2}+\cdots+b+1$ for some $b \geq 2, q \geq 4$. This supposition leads us to the insight that

$$ns^n = b(b^{q-2} + \cdots + b + 1) = b\left(\frac{b^{q-1}-1}{b-1}\right),$$

which we rewrite as

$$(b-1)ns^n = b(b^{q-1}-1).$$

In the case where $b^{q-1} < \text{rad}(b(b^{q-1}-1))^2$, then

$$b^{q-1} < \text{rad}((b-1)ns^n)^2$$

and

$$b^{q-1} - 1 < s^2n^2(b-1)^2$$

which simplifies to

$$\frac{b(b^{q-1}-1)}{b-1} < s^2n^2b(b-1).$$

The previous equation shows us that

$$\frac{b(b^{q-1}-1)}{b-1} + 1 < s^2n^2b^2,$$

and thus

$$ns^n < s^2n^2b^2,$$

or

$$s^{n-2} < nb^2.$$

We know that $ns^n > b^{q-1}$, so $(ns^n)^{\frac{2}{q-1}} > b^2$. This then gives us a further upper bound on the equation above, as

$$s^{n-2} < nb^2 < n^{1+\frac{2}{q-1}}s^{\frac{2n}{q-1}}$$

so

$$s^{n\left(1-\frac{2}{q-1}\right)-2} < n^{\frac{q+1}{q-1}}.$$

If we take the log base s of both sides, we see that

$$n\frac{q-3}{q-1} - 2 < \frac{q+1}{q-1}\log_s n.$$

We assumed that $q \geq 4$, giving us

$$\frac{1}{3}n < \frac{5}{3}\log_s n + 2,$$

or

$$n < 5\log_s(n) + 6,$$

revealing that $C_{s,n}$ can only be a repunit for finitely many n.

As the abc conjecture, with $\epsilon = 1$, implies that there are at most finitely many pairs (x, m) such that $x^m > \text{rad}(x(x^m - 1))^2$, there are finitely many s-Cullen numbers for any fixed s that are also repunits of length greater than three. □

3 Known examples

We see that $C_{s,1}$ is a repunit exactly when $s + 1$ is.

When $C_{s,2}$ is a repunit of length 3, we have that $2s^2 + 1 = x^2 + x + 1$, or $s^2 = x(x + 1)/2$, i.e., s^2 is a square triangular number. See A001110. These numbers were characterized by Euler, and help us find the infinite family $C_{6,2}$, $C_{35,2}$, $C_{204,2}\ldots$ of Cullen numbers which are length-three repunits. The case $C_{s,2}$ being a repunit of length 4 is ruled out by looking at the corresponding elliptic curve. In general, if we fix n and q, we get a curve over $\mathbb{Q}$, and it may prove enjoyable to employ the techniques of arithmetic geometry. Such an investigation, however, is beyond the scope of this short note.

Other than the two families noted above, there are no s-Cullen numbers $C_{s,n}$ which are also a repunits with $s \leq 75$ and $n \leq 75$ or with $s \leq 10^6$ and $n \leq 10$. The PARI/GP [10] code for this brief computation is available at github.com/31and8191/Cullen.

4 Acknowledgments

The second author would like to thank Ms. Joemese Malloy and the Thurgood Marshall Child Development Center for their extraordinary measures that allowed her to do mathematics with peace of mind, knowing that her child was safe, loved, and well-cared for during the pandemic. As always, she would like to thank her husband, Loren LaLonde, whose support was all the more meaningful during this difficult time.

The first author would like to thank his wife, Christina Ruiz Grantham, for her patience during his unexpected arrival as mathematician-in-residence at the onset of the pandemic. He thanks his children, Salem and Jack, for their enthusiasm at his insertion of number theory into their elementary-school mathematics curriculum.

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2020 *Mathematics Subject Classification*: Primary 11B83.
Keywords: Generalized Cullen Number, Cullen Number, repunit, abc conjecture.

(Concerned with sequences A002064 and A002275.)