PROJECTIONS AND ANGLE SUMS OF PERMUTOHEDRA AND OTHER POLYTOPES

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Abstract. Let $(x_1, \dots, x_n) \in \mathbb{R}^n$. Permutohedra of types A and B are convex polytopes in $\mathbb{R}^n$ defined by

$$\mathcal{P}_n^A = \text{conv}\{(x_{\sigma(1)}, \dots, x_{\sigma(n)}) : \sigma \in \text{Sym}(n)\}$$

and

$$\mathcal{P}_n^B = \text{conv}\{(\varepsilon_1 x_{\sigma(1)}, \dots, \varepsilon_n x_{\sigma(n)}) : (\varepsilon_1, \dots, \varepsilon_n) \in \{\pm 1\}^n, \sigma \in \text{Sym}(n)\},$$

where $\text{Sym}(n)$ denotes the group of permutations of the set $\{1, \dots, n\}$. We derive a closed formula for the number of j-faces of $G\mathcal{P}_n^A$ and $G\mathcal{P}_n^B$ for a linear map $G : \mathbb{R}^n \to \mathbb{R}^d$ satisfying some minor general position assumptions. In particular, we will show that the face numbers of the projected permutohedra do not depend on the linear mapping G. Furthermore, we derive formulas for the sum of the conic intrinsic volumes of the tangent cones of $\mathcal{P}_n^A$ and $\mathcal{P}_n^B$ at all of their j-faces. The same is done for the Grassmann angles. We generalize all these results to polytopes whose normal fan is the fan of some hyperplane arrangement.

1. Introduction

In the work of Donoho and Tanner [8] the following interesting statement can be found: The number of j-faces of the image of the n-dimensional cube $[0,1]^n$ under a linear map $G : \mathbb{R}^n \to \mathbb{R}^d$ does not depend on the choice of G provided G is in "general position". That is to say, the cube is an equiprojective polytope as defined by Shephard [23]. More precisely, by [8, Eq. (1.6)] we have

$$f_j(G[0,1]^n) = 2\binom{n}{j} \sum_{l=n-d}^{n-j-1} \binom{n-j-1}{l}; \tag{1.1}$$

for all $0 \le j < d \le n$, where $f_j(P)$ denotes the number of j-faces of a polytope P, and $G[0,1]^n$ is the image of $[0,1]^n$ under G. In the present paper, we investigate similar questions for the *permutohedra* of types A and B. These are the polytopes $\mathcal{P}_n^A$ and $\mathcal{P}_n^B$ in $\mathbb{R}^n$ defined by

$$\mathcal{P}_n^A := \mathcal{P}_n^A(x_1, \dots, x_n) := \text{conv}\big\{(x_{\sigma(1)}, \dots, x_{\sigma(n)}) : \sigma \in \text{Sym}(n)\big\}$$

and

$$\mathcal{P}_n^B := \mathcal{P}_n^B(x_1, \dots, x_n) := \text{conv}\big\{(\varepsilon_1 x_{\sigma(1)}, \dots, \varepsilon_n x_{\sigma(n)}) : (\varepsilon_1, \dots, \varepsilon_n) \in \{\pm 1\}^n, \sigma \in \text{Sym}(n)\big\}$$

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for a point $(x_1,\dots,x_n)\in\mathbb{R}^n$. Here, $\mathrm{Sym}(n)$ denotes the group of all permutations of the set $\{1,\dots,n\}$. These polytopes have been studied starting with the work of Schoute [22] in 1911; see [6, 19, 12] as well as [29, Example 0.10], [28, Section 5.3], [4, pp. 58–60, 254–258] and [7, Example 2.2.5]. We prove that under certain minor general position assumptions on the linear map $G:\mathbb{R}^n\to\mathbb{R}^d$, the number of j-faces of the projected permutohedra $G\mathcal{P}_n^A$ and $G\mathcal{P}_n^B$ is constant and given by the formulas

$$f_j(G\mathcal{P}_n^A)=2\left\{ n \atop n-j \right\}\left(\left[n-j \atop n-d+1 \right]+\left[n-j \atop n-d+3 \right]+\dots\right) \tag{1.2}$$

for all $1\le j<d\le n-1$, provided that $x_1>\dots>x_n$, and

$$f_j(G\mathcal{P}_n^B)=2T(n,n-j)\big(B(n-j,n-d+1)+B(n-j,n-d+3)+\dots\big), \tag{1.3}$$

for all $1\le j<d\le n$, provided that $x_1>\dots>x_n>0$. Here, $\left[n \atop k \right]$ and $\left\{ n \atop k \right\}$ denote the Stirling numbers of the first and second kind, respectively. The Stirling number of the first kind $\left[n \atop k \right]$ can be defined as the number of permutations of the set $\{1,\dots,n\}$ having exactly k cycles, while the Stirling number of the second kind $\left\{ n \atop k \right\}$ is defined as the number of partitions of the set $\{1,\dots,n\}$ into k non-empty, disjoint subsets. The numbers $B(n,k)$ and $T(n,k)$ denote the B-analogues to the Stirling numbers of the first and second kind, respectively, defined by the following formulas:

$$(t+1)(t+3)\cdot\ldots\cdot(t+2n-1)=\sum_{k=0}^{n}B(n,k)t^k,\quad T(n,k)=\sum_{m=k}^{n}2^{m-k}\binom{n}{m}\left\{ m \atop k \right\}.$$

It turns out that these results can be generalized to any polytope $P\subset\mathbb{R}^n$ whose normal fan, that is the set of normal cones at all faces F of P, coincides with the fan of some linear hyperplane arrangement $\mathcal{A}$. A linear hyperplane arrangement is a finite collection of distinct linear hyperplanes in $\mathbb{R}^n$. The hyperplanes dissect the space into finitely many cones or chambers, and the fan of the arrangement is the set of all faces of these chambers. The number of j-faces of the projected polytope GP does not depend on the choice of the linear mapping G (under some minor general position assumptions on G) and can be expressed in terms of the coefficients of the $(n-j)$-th level characteristic polynomials of the hyperplane arrangement $\mathcal{A}$.

It is known [29, Theorem 7.16] that zonotopes, that is Minkowski sums of finitely many line segments, are special cases of the named class of polytopes. One simple special example is the cube $[0,1]^n$ appearing in the formula (1.1). We show that the permutohedra of types A and B are also special cases of this class of polytopes since their faces and normal fans can be characterized in terms of reflection arrangements of types A_{n-1} and B_n, respectively. On the other hand, it turns out that permutohedra are zonotopes only in some rare exceptional cases, namely if the numbers $x_1,\dots,x_n$ form an arithmetic sequence.

From the formulas (1.2) and (1.3), we derive results on generalized angle sums of permutohedra. In particular, we compute the sum of the d-th conic intrinsic volumes υ_d of the tangent cones T_F of $\mathcal{P}_n^A$ and $\mathcal{P}_n^B$ at their j-faces F as follows:

$$\sum_{F\in\mathcal{F}_j(\mathcal{P}_n^A)}\upsilon_d(T_F(\mathcal{P}_n^A))=\left\{ n \atop n-j \right\}\left[n-j \atop n-d \right],\quad \text{for all } 0\le j\le d\le n-1,$$

$$\sum_{F\in\mathcal{F}_j(\mathcal{P}_n^B)}\upsilon_d(T_F(\mathcal{P}_n^B))=T(n,n-j)B(n-j,n-d),\quad \text{for all } 0\le j\le d\le n.$$

The same is done for the sums of the d-th Grassmann angles γ_d of the tangent cones. The corresponding formulas are

$$\sum_{F\in\mathcal{F}_j(\mathcal{P}_n^A)}\gamma_d(T_F(\mathcal{P}_n^A))=2\begin{Bmatrix} n\\ n-j\end{Bmatrix}\sum_{l=0}^{\infty}\begin{bmatrix} n-j\\ n-d-2l-1\end{bmatrix},\quad\text{for all } 0\le j\le d\le n-1,$$

$$\sum_{F\in\mathcal{F}_j(\mathcal{P}_n^B)}\gamma_d(T_F(\mathcal{P}_n^B))=2T(n,n-j)\sum_{l=0}^{\infty}B(n-j,n-d-2l-1),\quad\text{for all } 0\le j\le d\le n.$$

Moreover, we compute the same angle sums for the above-mentioned more general class of polytopes whose normal fans are fans of hyperplane arrangements.

The paper is organized as follows. Section 2 introduces the necessary notation and some important but well-known definitions and results from convex and integral geometry. In Section 3, we state the main results of this paper. In its first part, Section 3.1, we recall various characterizations of the permutohedra, their faces, and normal fans. Section 3.2 contains some necessary results on general position. In Section 3.3, we state the formulas for the number of faces of the projected permutohedra and more general polytopes mentioned above. Finally, Section 3.4 deals with the angle sums of these polytopes. Section 4 is dedicated to the proofs of the results from Section 3.

2. Preliminaries

In this section, we are going to introduce necessary facts and notation from convex and integral geometry. These facts are well-known and can be skipped at first reading.

2.1. Facts from convex geometry.

For a set $M\subset\mathbb{R}^n$ denote by $\operatorname{lin} M$ and $\operatorname{aff} M$ the linear hull and the affine hull of M, respectively. They are defined as the minimal linear and the minimal affine subspace of $\mathbb{R}^n$ containing M, respectively. Equivalently, $\operatorname{lin} M$ can be defined as the set of all linear combinations of elements in M, while $\operatorname{aff} M$ can be defined as the set of all affine combinations of elements in M. Similarly, the convex hull of M is denoted by $\operatorname{conv} M$ and defined as the minimal convex set containing M, or, equivalently,

$$\operatorname{conv} M:=\big\{\lambda_1x_1+\ldots+\lambda_mx_m: m\in\mathbb{N}, x_1,\ldots,x_m\in M,\lambda_1+\ldots+\lambda_m\ge 0,\lambda_1+\ldots+\lambda_m=1\big\}.$$

The positive hull of a set M is denoted by $\operatorname{pos} M$ and defined as

$$\operatorname{pos} M:=\big\{\lambda_1x_1+\ldots+\lambda_mx_m: m\in\mathbb{N}, x_1,\ldots,x_m\in M,\lambda_1,\ldots,\lambda_m\ge 0\big\}.$$

The relative interior of a set M is the set of all interior points of M relative to its affine hull $\operatorname{aff} M$ and it is denoted by $\operatorname{relint} M$. The set of interior points of M is denoted by $\operatorname{int} M$.

A *polyhedral set* is an intersection of finitely many closed half-spaces (whose boundaries need not pass through the origin). A bounded polyhedral set is called *polytope*. Equivalently, a polytope can be defined as the convex hull of a finite set of points. A *polyhedral cone* (or just cone) is an intersection of finitely many closed half-spaces whose boundaries contain the origin and therefore a special case of the polyhedral sets. Equivalently, a polyhedral cone can be defined as the positive hull of a finite set of points. The dimension of a polyhedral set P is defined as the dimension of its affine hull $\operatorname{aff} P$.

A supporting hyperplane for a polyhedral set $P\subset\mathbb{R}^n$ is an affine hyperplane H with the property that $H\cap P\neq\emptyset$ and P lies entirely in one of the closed half-spaces bounded by H. A *face* of a polyhedral set P (of arbitrary dimension) is a set of the form $F=P\cap H$, for a supporting hyperplane H, or the set P itself. Equivalently, the faces of a polyhedral set P are obtained by replacing some of the half-spaces, whose intersection defines the polyhedral set, by their boundaries and taking the intersection. The set of all faces of P is denoted by $\mathcal{F}(C)$ and the set of all k-dimensional faces (or just k-faces) of P by $\mathcal{F}_k(P)$ for $k \in \{0, \dots, n\}$. The number of k-faces of P is denoted by $f_k(C) := \#\mathcal{F}_k(C)$. In general, the number of elements in a set M is denoted by $|M|$ or $\#M$. The 0-dimensional faces are called *vertices*. In the case of a cone, the only possible vertex is the origin.

The *dual cone* C° (or polar cone) of a cone $C \subset \mathbb{R}^n$ is defined as

$$C^\circ := \{v \in \mathbb{R}^n : \langle v, x \rangle \le 0 \ \forall x \in C\}.$$

The *tangent cone* $T_F(P)$ of a polyhedral set $P \subset \mathbb{R}^n$ at a face F of P is defined by

$$T_F(P) = \{x \in \mathbb{R}^n : f_0 + \varepsilon x \in P \text{ for some } \varepsilon > 0\},$$

where f_0 is any point in the relative interior of F. This definition does not depend on the choice of f_0. The *normal cone* of P at the face F is defined as the dual of the tangent cone, that is $N_F(P) = T_F(P)^\circ$.

2.2. **Grassmann angles and conic intrinsic volumes.** Now, we are going to introduce some important geometric functionals of cones. Let $C \subset \mathbb{R}^n$ be a cone and g be an n-dimensional standard Gaussian distributed vector. Then, the *k-th conic intrinsic volume* of C is defined as

$$\upsilon_k(C) := \sum_{F \in \mathcal{F}_k(C)} \mathbb{P}(\Pi_C(g) \in \operatorname{relint} F), \quad k = 0, \dots, n,$$

where Π_C denotes the orthogonal projection on C, that is, $\Pi_C(x)$ is the vector in C which minimizes the Euclidean distance to $x \in \mathbb{R}^n$.

The conic intrinsic volumes are the analogues of the usual intrinsic volumes in the setting of conical or spherical geometry. Equivalently, the conic intrinsic volumes can be defined using the spherical Steiner formula, as done in [21, Section 6.5]. For further properties of conic intrinsic volumes we refer to [2, Section 2.2] and [21, Section 6.5].

Following Grnbaum [11], we define the *Grassmann angles* $\gamma_k(C)$, $k \in \{0, \dots, n\}$, of cone C as follows. Let W_{n-k} be random linear subspace of $\mathbb{R}^n$ with uniform distribution on the Grassmannian of all $(n-k)$-dimensional subspaces. Then, the k-th Grassmann angle of C is defined as

$$\gamma_k(C) := \mathbb{P}(W_{n-k} \cap C \neq \{0\}), \quad k = 0, \dots, n.$$

If the lineality space $C \cap -C$ of a cone C, which is the maximal linear subspace contained in C, has dimension $j \in \{0, \dots, n-1\}$, the Grassmann angles satisfy

$$1 = \gamma_0(C) = \ldots = \gamma_j(C) \ge \gamma_{j+1}(C) \ge \ldots \ge \gamma_n(C) = 0.$$

As proved in [11, Eq. (2.5)], the Grassmann angles do not depend on the dimension of the ambient linear subspace. This means that if we embed C in $\mathbb{R}^N$ with $N \ge n$, we obtain the same Grassmann angles. Therefore, it is convenient to write $\gamma_N(C) := 0$ for all $N \ge \dim C$. If C is not a linear subspace, then $\frac{1}{2}\gamma_k(C)$ is also known as the *k-th conic quermassintegral*; see [21, Eqs. (1)-(4)] or [13].

The conic intrinsic volumes and Grassmann angles satisfy a linear relation, called the *conic Crofton formula*. More precisely, we have

$$\gamma_k(C) = 2 \sum_{i=1,3,5,\ldots} \upsilon_{k+i}(C) \tag{2.1}$$

for all $k \in \{0, \dots, n\}$ and for every cone $C \subset \mathbb{R}^n$ which is not a linear subspace, according to [21, p.261]. Consequently,

$$\upsilon_k(C) = \frac{1}{2}\gamma_{k-1}(C) - \frac{1}{2}\gamma_{k+1}(C), \tag{2.2}$$

for all $k \in \{0, \dots, n\}$, where in the cases $k = 0$ and $k = n$ we have to define $\gamma_{-1}(C) = 1$ and $\gamma_{n+1}(C) = 0$. Then, (2.2) follows from (2.1) and the identity $\upsilon_0(C) + \upsilon_2(C) + \ldots = 1/2$; see [3, Eq. (5.3)].

3. Main results

3.1. **Permutohedra of types A and B.** In this section, we introduce permutohedra of types A and B, give an exact characterization of their faces, and compute their normal fans.

Definitions of the permutohedra. For $x_1, \dots, x_n \in \mathbb{R}$ the *permutohedron of type* A is defined as the following polytope in $\mathbb{R}^n$:

$$\mathcal{P}_n^A = \mathcal{P}_n^A(x_1, \dots, x_n) := \operatorname{conv}\left\{(x_{\sigma(1)}, \dots, x_{\sigma(n)}) : \sigma \in \operatorname{Sym}(n)\right\},$$

where $\operatorname{Sym}(n)$ is the group of all permutations of the set $\{1, \dots, n\}$. The permutohedron lies in the hyperplane $\{t \in \mathbb{R}^n : t_1 + \ldots + t_n = x_1 + \ldots + x_n\}$ and therefore has at most dimension $n - 1$. Similarly, the *permutohedron of type* B is defined as the following polytope in $\mathbb{R}^n$:

$$\mathcal{P}_n^B = \mathcal{P}_n^B(x_1, \dots, x_n) := \operatorname{conv}\left\{(\varepsilon_1 x_{\sigma(1)}, \dots, \varepsilon_n x_{\sigma(n)}) : \varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \{\pm 1\}^n, \sigma \in \operatorname{Sym}(n)\right\}.$$

Note that $\mathcal{P}_n^A(x_1, \dots, x_n)$ does not change under permutations of $x_1, \dots, x_n$, whereas $\mathcal{P}_n^B(x_1, \dots, x_n)$ stays invariant under signed permutations. Therefore, it is not a restriction of generality to assume that $x_1 \geq \ldots \geq x_n$ in the A-case and $x_1 \geq \ldots \geq x_n \geq 0$ in the B-case.

The next lemma is due to Rado [20] (see also [28, Section 5.3], [4, p. 257] and [17, Corollary B.3]) and describes $\mathcal{P}_n^A$ as a set of solutions to a finite system of affine inequalities.

Lemma 3.1. *Assume that $x_1 \geq \ldots \geq x_n$. Then, a point $(t_1, \dots, t_n) \in \mathbb{R}^n$ belongs to the permutohedron $\mathcal{P}_n^A(x_1, \dots, x_n)$ of type A if and only if*

$$t_1 + \cdots + t_n = x_1 + \ldots + x_n$$

and, for every non-empty subset $M \subset \{1, \dots, n\}$, we have

$$\sum_{i \in M} t_i \leq x_1 + \ldots + x_{|M|}.$$

An analogous result for the permutohedron of type B, together with a proof and references to the original literature, can be found in [17, Corollary C.5.a].

Lemma 3.2. *Assume that $x_1 \geq \ldots \geq x_n \geq 0$. Then, a point $(t_1, \dots, t_n) \in \mathbb{R}^n$ belongs to the permutohedron $\mathcal{P}_n^B(x_1, \dots, x_n)$ of type B if and only if for every non-empty subset $M \subset \{1, \dots, n\}$, we have*

$$\sum_{i \in M} |t_i| \leq x_1 + \ldots + x_{|M|}. \tag{3.1}$$

Faces of the permutohedra. We now state an exact characterization of the faces of both types of permutohedra. To this end, we need to introduce some useful notation.

Let $\mathcal{R}_{n,j}$ be the set of all ordered partitions $(B_1,\dots,B_j)$ of the set $\{1,\dots,n\}$ into j non-empty, disjoint and distinguishable subsets $B_1,\dots,B_j$. Furthermore, let $\mathcal{T}_{n,j}$ be the set of all pairs $(\mathcal{B},\eta)$, where $\mathcal{B}=(B_1,\dots,B_{j+1})$ is an ordered partition of the set $\{1,\dots,n\}$ into $j+1$ disjoint distinguishable subsets such that $B_1,\dots,B_j$ are non-empty, whereas B_{j+1} may be empty or not, and $\eta: B_1\cup\ldots\cup B_j\to\{\pm1\}$. In what follows, we write $\eta_i:=\eta(i)$ for ease of notation.

Proposition 3.3. *Suppose that $x_1>\ldots>x_n$. Then, for $j\in\{0,\dots,n-1\}$, the j-dimensional faces of $\mathcal{P}_n^A(x_1,\dots,x_n)$ are in one-to-one correspondence with the ordered partitions $\mathcal{B}\in\mathcal{R}_{n,n-j}$. The j-face corresponding to the ordered partition $\mathcal{B}=(B_1,\dots,B_{n-j})\in\mathcal{R}_{n,n-j}$ is given by*

$$F_{\mathcal{B}}=\operatorname{conv}\{(x_{\sigma(1)},\dots,x_{\sigma(n)}):\sigma\in I_{\mathcal{B}}\}.$$

Here, $I_{\mathcal{B}}\subset Sym(n)$ is the set of all permutations $\sigma\in Sym(n)$ such that

$$\sigma(B_1)=\{1,\dots,|B_1|\},\;\;\sigma(B_2)=\{|B_1|+1,\dots,|B_1\cup B_2|\},\;\;\dots,\\ \sigma(B_{n-j})=\{|B_1\cup\ldots\cup B_{n-j-1}|+1,\dots,n\}.$$

Equivalently, the face $F_{\mathcal{B}}$ can be written as

$$F_{\mathcal{B}}=\Big\{(t_1,\dots,t_n)\in\mathcal{P}_n^A(x_1,\dots,x_n):\sum_{i\in B_1\cup\ldots\cup B_l}t_i=x_1+\ldots+x_{|B_1\cup\ldots\cup B_l|}\;\forall l=1,\dots,n-j-1\Big\}.$$

Proposition 3.4. *Suppose that $x_1>\ldots>x_n>0$. Then, for $j\in\{0,\dots,n\}$, the j-dimensional faces of $\mathcal{P}_n^B(x_1,\dots,x_n)$ are in one-to-one correspondence with the pairs $(\mathcal{B},\eta)\in\mathcal{T}_{n,n-j}$. The j-face corresponding to the pair $(\mathcal{B},\eta)$, where $\mathcal{B}=(B_1,\dots,B_{n-j+1})$, is given by*

$$F_{\mathcal{B},\eta}=\operatorname{conv}\{(\varepsilon_1x_{\sigma(1)},\dots,\varepsilon_nx_{\sigma(n)}):(\sigma,\varepsilon)\in I_{\mathcal{B},\eta}\}.$$

Here, $I_{\mathcal{B},\eta}\subset\mathrm{Sym}(n)\times\{\pm1\}^n$ is the set of all pairs $(\sigma,\varepsilon)\in\mathrm{Sym}(n)\times\{\pm1\}^n$ such that

$$\sigma(B_1)=\{1,\dots,|B_1|\},\;\;\sigma(B_2)=\{|B_1|+1,\dots,|B_1\cup B_2|\},\;\;\dots,\\ \sigma(B_{n-j+1})=\{|B_1\cup\ldots\cup B_{n-j}|+1,\dots,n\}$$

and $\varepsilon_i=\eta_i$ for all $i\in B_1\cup\ldots\cup B_{n-j}$, while the remaining ε_i's take arbitrary values in the set $\{\pm1\}$. Equivalently, the face $F_{\mathcal{B},\eta}$ can be written as

$$F_{\mathcal{B},\eta}=\Big\{(t_1,\dots,t_n)\in\mathcal{P}_n^B(x_1,\dots,x_n):\sum_{i\in B_1\cup\ldots\cup B_l}\eta_it_i=x_1+\ldots+x_{|B_1\cup\ldots\cup B_l|}\;\forall l=1,\dots,n-j\Big\}.$$

Proofs of Proposition 3.3 can be found in [4, pp. 254-256] or in [28, Section 5.3]. Without proof, versions of the same proposition are stated in [19, Proposition 2.6] and in Exercise 2.9 on p. 96 of [7]. For completeness, a proof of Proposition 3.4 (which may also be known) will be provided in Section 4.1.

Normal fans of permutohedra. Following [29, Chapter 7], a *fan* in $\mathbb{R}^n$ is defined as a family $\mathcal{F}$ of non-empty cones with the following two properties:

(i) Every non-empty face of a cone in $\mathcal{F}$ is also a cone in $\mathcal{F}$.
(ii) The intersection of any two cones in $\mathcal{F}$ is a face of both.

For a non-empty polytope $P \subset \mathbb{R}^n$ the *normal fan* of P is defined as the set of normal cones of P, that is

$$\mathcal{N}(P) := \{N_F(P) : F \in \mathcal{F}(P)\}.$$

For a hyperplane arrangement $\mathcal{A}$, which is a finite set of linear hyperplanes in $\mathbb{R}^n$, the complement $\mathbb{R}^n \backslash \bigcup_{H \in \mathcal{A}} H$ is a disjoint union of open convex sets. The set of closures of these "regions" is denoted by $\mathcal{R}(\mathcal{A})$ and called the *conical mosaic* generated by $\mathcal{A}$. Clearly, the set of all faces of these cones defines a fan denoted by $\mathcal{F}(\mathcal{A})$:

$$\mathcal{F}(\mathcal{A}) = \bigcup_{C \in \mathcal{R}(\mathcal{A})} \mathcal{F}(C).$$

Also, we denote the set of j-dimensional cones of this fan by $\mathcal{F}_j(\mathcal{A})$.

The following theorems exactly characterize the normal fans of the permutohedra of types A and B. Denote by $\mathcal{A}(A_{n-1})$ the hyperplane arrangement consisting of the hyperplanes

$$\{\beta \in \mathbb{R}^n : \beta_i = \beta_j\}, \quad 1 \le i < j \le n. \tag{3.2}$$

Similarly, let $\mathcal{A}(B_n)$ be the hyperplane arrangement that consists of the hyperplanes

$$\begin{aligned} &\{\beta \in \mathbb{R}^n : \beta_i = \beta_j\}, \quad 1 \le i < j \le n, \\ &\{\beta \in \mathbb{R}^n : \beta_i = -\beta_j\}, \quad 1 \le i < j \le n, \\ &\{\beta \in \mathbb{R}^n : \beta_i = 0\}, \quad 1 \le i \le n. \end{aligned} \tag{3.3}$$

These arrangements, also called reflection arrangements of types A_{n-1} and B_n, as well as the cones they generate (called the Weyl chambers), will be further discussed in Section 4.3.

Theorem 3.5. *For $x_1 > \ldots > x_n$ the normal fan $\mathcal{N}(\mathcal{P}_n^A(x_1, \ldots, x_n))$ of the permutohedron of type A coincides with the fan $\mathcal{F}(\mathcal{A}(A_{n-1}))$ generated by the hyperplane arrangement $\mathcal{A}(A_{n-1})$.*

Theorem 3.6. *For $x_1 > \ldots > x_n > 0$ the normal fan $\mathcal{N}(\mathcal{P}_n^B(x_1, \ldots, x_n))$ of the permutohedron of type B coincides with the fan $\mathcal{F}(\mathcal{A}(B_n))$ generated by the hyperplane arrangement $\mathcal{A}(B_n)$.*

Both theorems seem to be known, see, e.g., [12, Section 3.1], but for the sake of completeness we will give their proofs in Section 4.1. For example, the normal cones at the vertices of the permutohedra coincide with the Weyl chambers of types A_{n-1} and B_n, which was used to compute their statistical dimension in [3, Proposition 3.5].

3.2. **Results on general position.** Before stating our main results on the face numbers of projected permutohedra, we need to introduce the terminology of *general position* in the context of hyperplane arrangements and polyhedral sets. Moreover, we formulate general position assumptions for both types of permuthohedra that are necessary for our main results in Section 3.3.

Let M be an affine subspace of $\mathbb{R}^n$. Denote by $L \subset \mathbb{R}^n$ the unique linear subspace such that $M = t + L$ holds for some $t \in \mathbb{R}^n$, that is the translation of M passing through the origin. We say that M is in *general position with respect to a linear subspace* $L' \subset \mathbb{R}^n$ if

$$\dim(L \cap L') = \max(\dim L + \dim L' - n, 0).$$

A linear subspace L' is said to be in *general position with respect to a polyhedral set* P if the affine hull of each face F of P is in general position with respect to L'.

For a linear hyperplane arrangement $\mathcal{A}$ in $\mathbb{R}^n$, the *lattice* $\mathcal{L}(\mathcal{A})$ generated by $\mathcal{A}$ consists of all linear subspaces of $\mathbb{R}^n$ that can be represented as intersections of hyperplanes from $\mathcal{A}$. Denote by

$\mathcal{L}_j(\mathcal{A})$ the set of j-dimensional subspaces from the lattice $\mathcal{L}(\mathcal{A})$. A linear subspace $L' \subset \mathbb{R}^n$ is said to be in *general position with respect to the hyperplane arrangement* $\mathcal{A}$, if all $K \in \mathcal{L}(\mathcal{A})$ satisfy

$$\dim(K \cap L') = \max(\dim L' + \dim K - n, 0), \tag{3.4}$$

that is, if L' is in general position with respect to each $K \in \mathcal{L}(\mathcal{A})$.

Now, we are able to formulate two equivalent general position assumptions that we need to impose on a linear mapping $G \in \mathbb{R}^{d\times n}$ in the case of a permutohedron of type A.

Corollary 3.7. *Let $1 \le d \le n-1$ and $x_1 > \ldots > x_n$. For a matrix $G \in \mathbb{R}^{d\times n}$ with* $\operatorname{rank} G = d$, *the following two conditions are equivalent:*

- *(A1) The $(n-d)$-dimensional linear subspace* $\ker G$ *is in general position with respect to* $\mathcal{P}_n^A(x_1, \ldots, x_n)$.
- *(A2) The d-dimensional linear subspace* $(\ker G)^\perp$ *is in general position with respect to the reflection arrangement $\mathcal{A}(A_{n-1})$ defined in* (3.2).

An analogous result can be formulated for the permutohedron of type B.

Corollary 3.8. *Let $1 \le d \le n$ and $x_1 > \ldots > x_n > 0$. For a matrix $G \in \mathbb{R}^{d\times n}$ with* $\operatorname{rank} G = d$, *the following two conditions are equivalent:*

- *(B1) The $(n-d)$-dimensional linear subspace* $\ker G$ *is in general position with respect to* $\mathcal{P}_n^B(x_1, \ldots, x_n)$.
- *(B2) The d-dimensional linear subspace* $(\ker G)^\perp$ *is in general position with respect to the reflection arrangement $\mathcal{A}(B_n)$ defined in* (3.3).

Corollaries 3.7 and 3.8 follow from the more general Theorem 3.11, which we will state in Section 3.3. Therefore, their proofs will be postponed to Section 4.2.

3.3. **Face numbers of projected permutohedra and more general polytopes.** In this section, we state our main results on the number of faces of projected permutohedra of types A and B and, more generally, of polytopes whose normal fan coincides with the fan generated by a hyperplane arrangement. In case of the named polytopes, the face numbers of the projected polytopes are independent of the projection provided it satisfies some minor general position assumption.

Permutohedra of types A and B. The formulas will be stated in terms of Stirling numbers defined as follows. The (signless) *Stirling number of the first kind* $\left[{n \atop k}\right]$ is the number of permutations of the set $\{1, \ldots, n\}$ having exactly k cycles. Equivalently, these numbers can be defined as the coefficients of the polynomial

$$t(t+1)\cdot \ldots \cdot (t+n-1) = \sum_{k=0}^{n} \left[{n \atop k}\right] t^k \tag{3.5}$$

for $n \in \mathbb{N}_0$, with the convention that $\left[{n \atop k}\right] = 0$ for $n \in \mathbb{N}_0$, $k \notin \{0, \ldots, n\}$ and $\left[{0 \atop 0}\right] = 1$. The B-analogues to the Stirling numbers of the first kind, denoted by $B(n,k)$, are defined as the coefficients of the polynomial

$$(t+1)(t+3)\cdot \ldots \cdot (t+2n-1) = \sum_{k=0}^{n} B(n,k) t^k \tag{3.6}$$

for $n \in \mathbb{N}$ and, by convention, $B(n,k) = 0$ for $k \notin \{0, \ldots, n\}$. The triangular array of integers $B(n,k)$ appears as entry A028338 in [24].

The *Stirling number of the second kind* $\left\{{n \atop k}\right\}$ is defined as the number of partitions of the set $\{1, \ldots, n\}$ into k non-empty subsets. The B-analogues to the Stirling numbers of the second kind, denoted by $T(n,k)$, are defined as

$$T(n,k) = \sum_{m=k}^{n} 2^{m-k} \binom{n}{m} \left\{{m \atop k}\right\}. \tag{3.7}$$

They appear as entry A039755 in [24] and were studied by Suter [27].

Our main results for the permutohedra of types A and B are as follows.

Theorem 3.9. *Let $x_1 > \ldots > x_n$ be given. For a matrix $G \in \mathbb{R}^{d \times n}$ with $\operatorname{rank} G = d$ and satisfying one of the equivalent general position assumptions (A1) or (A2), we have*

$$f_j(G\mathcal{P}_n^A) = 2\left\{{n \atop n-j}\right\} \left(\left[{n-j \atop n-d+1}\right] + \left[{n-j \atop n-d+3}\right] + \ldots \right),$$

for all $0 \leq j < d \leq n-1$.

Theorem 3.10. *Let $x_1 > \ldots > x_n > 0$ be given. For a matrix $G \in \mathbb{R}^{d \times n}$ with $\operatorname{rank} G = d$ satisfying one of the equivalent general position assumptions (B1) or (B2), we have*

$$f_j(G\mathcal{P}_n^B) = 2T(n, n-j)\big(B(n-j, n-d+1) + B(n-j, n-d+3) + \ldots\big),$$

for all $0 \leq j < d \leq n$.

The proofs of Theorems 3.9 and 3.10 are postponed to Section 4.3.

A more general class of polytopes. We are able to formulate a more general result which is valid for all polytopes $P \subset \mathbb{R}^n$ whose normal fan $\mathcal{N}(P) := \{N_F(P) : F \in \mathcal{F}(P)\}$ coincides with the fan $\mathcal{F}(\mathcal{A}) := \bigcup_{C \in \mathcal{R}(\mathcal{A})} \mathcal{F}(C)$ generated by some hyperplane arrangement $\mathcal{A}$. Here, $\mathcal{R}(\mathcal{A})$ denotes the conical mosaic in $\mathbb{R}^n$ consisting of the n-dimensional cones generated by $\mathcal{A}$. In Theorems 3.5 and 3.6 we already observed that the permutohedra of types A and B are special cases of this class of polytopes. Before stating the result, we need to introduce the characteristic polynomial of a hyperplane arrangement.

The *rank* of a linear hyperplane arrangement $\mathcal{A}$ in $\mathbb{R}^n$ is defined by

$$\operatorname{rank}(\mathcal{A}) = n - \dim\left(\bigcap_{H \in \mathcal{A}} H\right), \quad \operatorname{rank}(\emptyset) = 0.$$

The *characteristic polynomial* $\chi_{\mathcal{A}}(t)$ of $\mathcal{A}$ can be defined by the following Whitney formula:

$$\chi_{\mathcal{A}}(t) = \sum_{\mathcal{C} \subset \mathcal{A}} (-1)^{\#\mathcal{C}} t^{n - \operatorname{rank}(\mathcal{C})}; \tag{3.8}$$

see, e.g., [18, Lemma 2.3.8] or [25, Theorem 2.4], as well as [25, Section 1.3] or [26, Section 3.11.2] for other definitions using the Möbius function on the intersection poset of $\mathcal{A}$.

Similar to the case of the permutohedra, we need to impose certain general position assumptions on the linear mapping G under consideration.

Theorem 3.11. *Let $P \subset \mathbb{R}^n$ be a polytope such that the normal fan $\mathcal{N}(P)$ coincides with the fan $\mathcal{F}(\mathcal{A})$ of some hyperplane arrangement $\mathcal{A}$. For $1 \leq d \leq \dim P$ and a matrix $G \in \mathbb{R}^{d \times n}$ with $\operatorname{rank} G = d$ the following two general position assumptions are equivalent:*

- *(G1) The $(n-d)$-dimensional linear subspace $\ker G$ is in general position with respect to P.*
- *(G2) The d-dimensional linear subspace $(\ker G)^\perp$ is in general position with respect to the hyperplane arrangement $\mathcal{A}$.*

The proof of this theorem is postponed to Section 4.2.

Theorem 3.12. *Let $P \subset \mathbb{R}^n$ be a polytope such that the normal fan $\mathcal{N}(P)$ coincides with the fan $\mathcal{F}(\mathcal{A})$ of a hyperplane arrangement $\mathcal{A}$. Moreover, let $G \in \mathbb{R}^{d\times n}$ be a matrix with $\operatorname{rank} G = d$ such that one of the equivalent general position assumptions (G1) or (G2) is satisfied. Then, the number of j-faces of the projected polytope GP is independent of the linear map G and given by*

$$f_j(GP) = 2 \sum_{M\in\mathcal{L}_{n-j}(\mathcal{A})} (a^M_{n-d+1} + a^M_{n-d+3} + \ldots),$$

for $0 \leq j < d \leq \dim P$, where the numbers a^M_k are (up to a sign) the coefficients of the characteristic polynomial of the induced hyperplane arrangement $\mathcal{A}|M := \{H \cap M : H \in \mathcal{A}, M \nsubseteq H\}$ in the ambient space M:

$$\chi_{A|M}(t) = \sum_{k=0}^{n-j} (-1)^{n-j-k} a^M_k t^k. \tag{3.9}$$

Also recall that $\mathcal{L}_{n-j}(\mathcal{A})$ is set of all $(n-j)$-dimensional linear subspaces that can be represented as intersections of hyperplanes from $\mathcal{A}$. By convention, we have $\mathcal{L}_n(\mathcal{A}) := \{\mathbb{R}^n\}$.

The proof of Theorem 3.12 is postponed to Section 4.3. As a consequence, the polytopes considered in Theorem 3.12 belong to the class of equiprojective polytopes as defined in [23].

Permutohedra and zonotopes as special cases. As mentioned above, the permutohedra are special cases of the above class of polytopes since their normal fans coincide with the fans of reflection arrangements. Thus, Theorems 3.9 and 3.10 can be derived from Theorem 3.12 using formulas for the coefficients of the characteristic polynomials of the reflection arrangements or rather their restrictions to linear subspaces $M \in \mathcal{L}_{n-j}(\mathcal{A}(A_{n-1}))$, respectively, $M \in \mathcal{L}_{n-j}(\mathcal{A}(B_n))$. These coefficients were already computed by Amelunxen and Lotz [2, Lemma 6.5]. In Section 4.3 however, we will prove Theorems 3.9 and 3.10 using an equivalent approach that includes computing the number of faces of Weyl chambers that are non-trivially intersected by a linear subspace.

Besides permutohedra, the zonotopes are also a special case of the above class of polytopes whose normal fan is the fan of a hyperplane arrangement. A *zonotope* $Z = Z(V) \subset \mathbb{R}^n$ is a Minkowski sum of a finite number of line segments, and therefore, can be written as

$$Z(V) = [-v_1, v_1] + \ldots + [-v_p, v_p] + z$$

for some $p \in \mathbb{N}$, a matrix $V = (v_1, \ldots, v_p) \in \mathbb{R}^{n\times p}$ and $z \in \mathbb{R}^n$. Following [29, Definition 7.13], a zonotope $Z = Z(V)$ can equivalently be defined as the image of a cube under an affine map, that is,

$$Z(V) := V[-1, +1]^p + z = \{Vy + z : y \in [-1, +1]^p\}.$$

In the book of Ziegler [29, Theorem 7.16] it is proved that for a zonotope $Z = Z(V) \subset \mathbb{R}^n$, the normal fan $\mathcal{N}(Z)$ of Z coincides with the fan $\mathcal{F}(\mathcal{A})$ of the hyperplane arrangement

$$\mathcal{A} = \mathcal{A}_V := \{H_1, \ldots, H_p\}$$

in $\mathbb{R}^n$, where $H_i := \{x \in \mathbb{R}^n : \langle x, v_i\rangle = 0\}$ for $i = 1, \ldots, p$. It is known [29, Example 7.15] that $\mathcal{P}^A_n(n, n-1, \ldots, 2, 1)$ is a zonotope and the natural question arises if the permutohedra of types A and B are zonotopes for all $(x_1, \ldots, x_n) \in \mathbb{R}^n$. The following proposition shows that this is not the case.

Proposition 3.13. *For $x_1 > \ldots > x_n$, the permutohedron $\mathcal{P}_n^A(x_1,\ldots,x_n)$ of type A is a zonotope if and only if $x_1,\ldots,x_n$ are in arithmetic progression, that is,*

$$x_1 = a + (n-1)b,\ x_2 = a + (n-2)b, \ldots, x_{n-1} = a + b,\ x_n = a \tag{3.10}$$

for some $a \in \mathbb{R}$ and $b > 0$.

For $x_1 > \ldots > x_n > 0$, the permutohedron $\mathcal{P}_n^B(x_1,\ldots,x_n)$ of type B is a zonotope if and only if $x_1,\ldots,x_n$ are in arithmetic progression, that is if (3.10) *holds for some $a > 0$ and $b > 0$.*

The proof is postponed to Section 4.1.

3.4. **Angle sums of permutohedra, zonotopes and other polytopes.** The next theorem follows from Theorem 3.9 using a well-known formula from Affentranger and Schneider [1] and will be proved in Section 4.4. Recall that the Grassmann angles γ_d and the conic intrinsic volumes υ_d were defined in Section 2.2.

Theorem 3.14. *Let $x_1 > \ldots > x_n$ be given. We have*

$$\sum_{F \in \mathcal{F}_j(\mathcal{P}_n^A)} \upsilon_d(T_F(\mathcal{P}_n^A)) = \left\{ {n \atop n-j} \right\} \left[{n-j \atop n-d} \right] \tag{3.11}$$

and

$$\sum_{F \in \mathcal{F}_j(\mathcal{P}_n^A)} \gamma_d(T_F(\mathcal{P}_n^A)) = 2 \left\{ {n \atop n-j} \right\} \sum_{l=0}^{\infty} \left[{n-j \atop n-d-2l-1} \right] \tag{3.12}$$

for $0 \le j \le d \le n-1$.

The following theorem is an analogue of Theorem 3.14 in the B-case.

Theorem 3.15. *Let $x_1 > \ldots > x_n > 0$ be given. We have*

$$\sum_{F \in \mathcal{F}_j(\mathcal{P}_n^B)} \upsilon_d(T_F(\mathcal{P}_n^B)) = T(n, n-j) B(n-j, n-d) \tag{3.13}$$

and

$$\sum_{F \in \mathcal{F}_j(\mathcal{P}_n^B)} \gamma_d(T_F(\mathcal{P}_n^B)) = 2 T(n, n-j) \sum_{l=0}^{\infty} B(n-j, n-d-2l-1) \tag{3.14}$$

for $0 \le j \le d \le n$.

Note that (3.11) and (3.13) recover results from Amelunxen and Lotz [2]. Among other things, they derived a formula for the j-th level characteristic polynomial of the reflection arrangements $\mathcal{A}(A_{n-1})$ and $\mathcal{A}(B_n)$, see [2, Lemma 6.5], which yields the sums of the $(n-d)$-th conic intrinsic volumes over all j-dimensional regions from $\mathcal{F}(\mathcal{A}(A_{n-1}))$, respectively $\mathcal{F}(\mathcal{A}(B_n))$, see [2, Theorem 6.1]. These formulas coincide with the sums computed in (3.11) and (3.13) (with j replaced by $n-j$). This does not come as a surprise since the normal fans of the permutohedra are the fans of the corresponding reflection arrangements, following Theorems 3.5 and 3.6. The proofs of Theorems 3.14 and 3.15 are postponed to Section 4.4.

For more general polytopes whose normal fans are generated by a hyperplane arrangement, the following result holds.

Theorem 3.16. *Let $P \subset \mathbb{R}^n$ be a polytope whose normal fan $\mathcal{N}(P)$ coincides with the fan $\mathcal{F}(\mathcal{A})$ of a hyperplane arrangement $\mathcal{A}$. Then, we have*

$$\sum_{F \in \mathcal{F}_j(P)} \upsilon_d(T_F(P)) = \sum_{M \in \mathcal{L}_{n-j}(\mathcal{A})} a_{n-d}^M$$

and

$$\sum_{F \in \mathcal{F}_j(P)} \gamma_d(T_F(P)) = 2 \sum_{M \in \mathcal{L}_{n-j}(\mathcal{A})} (a_{n-d-1}^M + a_{n-d-3}^M + \ldots)$$

for $0 \le j \le d \le \dim P$. Recall that the numbers a_k^M were defined in (3.9) *and $\mathcal{L}_j(\mathcal{A})$ denotes the set of j-dimensional subspaces from the lattice of $\mathcal{A}$. By convention, we put $\mathcal{L}_n(\mathcal{A}) := \{\mathbb{R}^n\}$.*

The proof of this theorem is also postponed to Section 4.4. Theorems 3.14 and 3.15 can be deduced from Theorem 3.16 since the normal fans of the permutohedra of types A and B coincide with the fans $\mathcal{F}(\mathcal{A}(A_{n-1}))$ and $\mathcal{F}(\mathcal{A}(B_n))$, respectively, following Theorems 3.5 and 3.6. Using the known formulas [2, Lemma 6.5] for the coefficients of the j-th level characteristic polynomials yields the results. In Section 4.4, we are going to prove Theorems 3.14 and 3.15 by using Theorems 3.9 and 3.10 on the face numbers of projected permutohedra of types A and B, respectively, and a formula of Affentranger and Schneider [1]. Let us also mention that applying Theorem 3.16 to a full-dimensional zonotope P with $d = n$ we recover a formula stated in [16, Theorem 12].

4. Proofs

This section is dedicated to proving the main results from Section 3. In Section 4.1, we are going to prove the characterization of the faces and the normal fans of permutohedra. Section 4.2 contains the proofs of the equivalences between the general position assumptions of Theorem 3.11 and its Corollaries 3.7 and 3.8 from Section 3.2. Moreover, in Section 4.3, we prove the main results of this paper on the number of j-faces of the projected permutohedra and the more general class of polytopes. Finally, we will also prove the results on the angle sums of the same polytopes in Section 4.4.

4.1. **Permutohedra: Proofs of Propositions 3.4 and 3.13, and Theorems 3.5 and 3.6.**

Before starting with the proofs, let us mention the well-known fact that the points $(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ are indeed vertices of $\mathcal{P}_n^A$ for all $\sigma \in \mathrm{Sym}(n)$. Similarly, the points $(\varepsilon_1 x_{\sigma(1)}, \ldots, \varepsilon_n x_{\sigma(n)})$ are vertices of $\mathcal{P}_n^B$ for all $\varepsilon \in \{\pm 1\}^n$, $\sigma \in \mathrm{Sym}(n)$. Let us explain this in the B-case. It suffices to prove the claim for the point $x = (x_1, \ldots, x_n)$ of $\mathcal{P}_n^B$, where $x_1 \ge \ldots \ge x_n \ge 0$. Suppose that there are points $y = (y_1, \ldots, y_n) \in \mathcal{P}_n^B$ and $z = (z_1, \ldots, z_n) \in \mathcal{P}_n^B$ such that $x = (y + z)/2$. By Lemma 3.2, we have $|y_1| \le x_1$ and $|z_1| \le x_1$. Thus, we have $y_1 = z_1 = x_1$. Given this, we can consider the second coordinate in the same way. Inductively, we obtain $y_i = z_i = x_i$ for all $i = 1, \ldots, n$, which means that $(x_1, \ldots, x_n)$ is indeed a vertex of $\mathcal{P}_n^B$.

In the case where $x_1 > \ldots > x_n$, the permutohedron $\mathcal{P}_n^A$ has dimension $n - 1$. Similarly, for $x_1 > \ldots > x_n > 0$, the permutohedron $\mathcal{P}_n^B$ of type B has dimension n.

Faces of permutohedra of type B. We are going to prove Proposition 3.4. Suppose that $x_1 > \ldots > x_n > 0$. Recall that $\mathcal{T}_{n,n-j}$ denotes the set of all pairs $(\mathcal{B}, \eta)$, where $\mathcal{B} = (B_1, \ldots, B_{n-j+1})$ is an ordered partition of the set $\{1, \ldots, n\}$ into $n - j + 1$ disjoint distinguishable subsets such that $B_1, \ldots, B_{n-j}$ are non-empty, whereas B_{n-j+1} may be empty or not, and $\eta : B_1 \cup \ldots \cup B_{n-j} \to \{\pm 1\}$. For Proposition 3.4, we want to prove that there is a one-to-one correspondence between the j-faces of $\mathcal{P}_n^B(x_1,\dots,x_n)$ and the pairs $(\mathcal{B},\eta)\in\mathcal{T}_{n,n-j}$, and that the j-face corresponding to the pair $(\mathcal{B},\eta)$ is given by

$$F_{\mathcal{B},\eta} = \operatorname{conv}\{(\varepsilon_1 x_{\sigma(1)},\dots,\varepsilon_n x_{\sigma(n)}) : (\sigma,\varepsilon)\in I_{\mathcal{B},\eta}\} \tag{4.1}$$

$$= \left\{ t\in\mathcal{P}_n^B(x_1,\dots,x_n) : \sum_{i\in B_1\cup\ldots\cup B_l} \eta_i t_i = x_1+\ldots+x_{|B_1\cup\ldots\cup B_l|}\ \forall l=1,\dots,n-j \right\}. \tag{4.2}$$

Here, $I_{\mathcal{B},\eta}\subset \operatorname{Sym}(n)\times\{\pm1\}^n$ is the set of all pairs $(\sigma,\varepsilon)\in\operatorname{Sym}(n)\times\{\pm1\}^n$ such that $\sigma(B_1)=\{1,\dots,|B_1|\},\sigma(B_2)=\{|B_1|+1,\dots,|B_1\cup B_2|\},\dots$ and $\varepsilon_i=\eta_i$ for $i\in B_1\cup\ldots\cup B_{n-j}$.

Proof of Proposition 3.4. Let $F\in\mathcal{F}(\mathcal{P}_n^B)$ be a face of $\mathcal{P}_n^B(x_1,\dots,x_n)$ with $x_1>\ldots>x_n>0$. Either, we have $F=\mathcal{P}_n^B$, which means there is nothing to prove, or there is a supporting hyperplane $H=\{t\in\mathbb{R}^n : \alpha_1t_1+\ldots+\alpha_nt_n=b\}$ for some $\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{R}^n\backslash\{0\}$ and $b\in\mathbb{R}$ such that

$$H\cap\mathcal{P}_n^B=F \quad\text{and}\quad \mathcal{P}_n^B\subset H^- := \{t\in\mathbb{R}^n : \alpha_1t_1+\ldots+\alpha_nt_n\le b\}. \tag{4.3}$$

Without loss of generality, we may assume that $\alpha_1\ge\ldots\ge\alpha_n\ge0$ (otherwise apply a signed permutation of $\{1,\dots,n\}$ to $(\alpha_1,\dots,\alpha_n)$ and all other objects). Then,

$$\underbrace{\alpha_1=\cdots=\alpha_{i_1}}_{\text{group }1} > \underbrace{\alpha_{i_1+1}=\ldots=\alpha_{i_2}}_{\text{group }2} > \ldots > \underbrace{\alpha_{i_{m-1}+1}=\ldots=\alpha_{i_m}}_{\text{group }m} > \underbrace{\alpha_{i_m+1}=\ldots=\alpha_n=0}_{\text{group }m+1}, \tag{4.4}$$

for some $m\in\{1,\dots,n\}$ and $1\le i_1<\ldots<i_m\le n$. Note that for $i_m=n$, no α_i's are required to be zero, which means that the last group is empty. Then, $\mathcal{P}_n^B\subset H^-$ implies that

$$\alpha_1\varepsilon_1x_{\sigma(1)}+\ldots+\alpha_n\varepsilon_nx_{\sigma(n)}\le b, \quad \text{for all } \varepsilon=(\varepsilon_1,\dots,\varepsilon_n)\in\{\pm1\}^n, \sigma\in\operatorname{Sym}(n).$$

The first equation of (4.3) implies that there is a pair $(\sigma',\varepsilon')\in\operatorname{Sym}(n)\times\{\pm1\}^n$ such that

$$\alpha_1\varepsilon_1'x_{\sigma'(1)}+\ldots+\alpha_{i_m}\varepsilon_{i_m}'x_{\sigma'(i_m)} = \alpha_1\varepsilon_1'x_{\sigma'(1)}+\ldots+\alpha_n\varepsilon_n'x_{\sigma'(n)} = b.$$

Since the α_i's and the x_i's are non-increasing and non-negative, the swapping lemma (see, e.g., [4, p. 254]) states that $\alpha_1\varepsilon_1x_{\sigma(1)}+\ldots+\alpha_n\varepsilon_nx_{\sigma(n)}$ attains its maximal value if we choose $\varepsilon_i=+1$ and $\sigma(i)=i$ for all $i\in\{1,\dots,n\}$. It follows that, in fact, we have

$$\alpha_1x_1+\ldots+\alpha_nx_n=b. \tag{4.5}$$

Denote the groups of indices appearing in (4.4) by

$$B_1=\{1,\dots,i_1\},\ \ldots,\ B_m=\{i_{m-1}+1,\dots,i_m\},\ B_{m+1}=\{i_m+1,\dots,n\}, \tag{4.6}$$

where B_{m+1} may or may not be empty. Defining in our case $\eta_i:=1$ for all $i\in\{1,\dots,i_m\}$ we obtain that, under (4.6), the set $I_{\mathcal{B},\eta}$ consists of all pairs $(\sigma,\varepsilon)\in\operatorname{Sym}(n)\times\{\pm1\}^n$ such that

$$\sigma(B_1)=B_1,\ \ldots,\ \sigma(B_m)=B_m,\ \sigma(B_{m+1})=B_{m+1}$$

and $\varepsilon_i=\eta_i$ for all $i\in\{1,\dots,i_m\}$. Consequently, from (4.5) and (4.4) it follows that

$$\alpha_1\varepsilon_1x_{\sigma(1)}+\ldots+\alpha_n\varepsilon_nx_{\sigma(n)}=b \quad \text{for all } (\sigma,\varepsilon)\in I_{\mathcal{B},\eta}. \tag{4.7}$$

Furthermore, it follows from (4.5), (4.4) and the swapping lemma that

$$\alpha_1\varepsilon_1x_{\sigma(1)}+\ldots+\alpha_n\varepsilon_nx_{\sigma(n)}<b \quad \text{for all } (\sigma,\varepsilon)\in(\operatorname{Sym}(n)\times\{\pm1\}^n)\backslash I_{\mathcal{B},\eta}. \tag{4.8}$$

Indeed, if $(\sigma,\varepsilon)\notin I_{\mathcal{B},\eta}$, then there is the possibility that we have a strictly negative term on the left-hand side of (4.8) which means that we could make it strictly larger by changing the sign of this term. Thus, we can assume all these terms to be non-negative. Then, $(\sigma,\varepsilon)\notin I_{\mathcal{B},\eta}$ implies that there is a pair of indices $1 \le i < j \le n$ such that $\alpha_i > \alpha_j$ und $x_{\sigma(i)} < x_{\sigma(j)}$ and we can apply the swapping lemma to strictly increase the left-hand side.

According to (4.7) and (4.8), the vertices $(\varepsilon_1 x_{\sigma(1)}, \dots, \varepsilon_n x_{\sigma(n)})$ with $(\sigma, \varepsilon) \in I_{\mathcal{B},\eta}$ are the only vertices of $\mathcal{P}_n^B$ that belong to the supporting hyperplane H. It follows from [29, Proposition 2.3] that F is the convex hull of these vertices, that is $F = F_{\mathcal{B},\eta}$, where

$$F_{\mathcal{B},\eta} := \operatorname{conv}\left\{(\varepsilon_1 x_{\sigma(1)}, \dots, \varepsilon_n x_{\sigma(n)}) : (\sigma, \varepsilon) \in I_{\mathcal{B},\eta}\right\}. \tag{4.9}$$

Essentially the same argument shows that, conversely, a set of the form $F_{\mathcal{B},\eta}$ is a face of $\mathcal{P}_n^B$. At the beginning, we applied a signed permutation to all objects including $(\alpha_1, \dots, \alpha_n)$ to achieve that the α_i's are non-increasing and non-negative. Applying the inverse signed permutation proves that the faces of $\mathcal{P}_n^B$ coincide with the sets of the form $F_{\mathcal{B},\eta}$ as defined in (4.9), for some pair $(\mathcal{B}, \eta) \in \mathcal{T}_{n,m}$. Furthermore, for two different pairs $(\mathcal{B}', \eta'), (\mathcal{B}'', \eta'')$ we have $I_{\mathcal{B}',\eta'} \neq I_{\mathcal{B}'',\eta''}$, which implies that the corresponding sets $F_{\mathcal{B}',\eta'}$ and $F_{\mathcal{B}'',\eta''}$ are different, since their sets of vertices are different. Finally, the polytope $F_{\mathcal{B},\eta}$, for $(\mathcal{B}, \eta) \in \mathcal{T}_{n,m}$, is isometric to the direct product $\mathcal{P}^A_{|B_1|} \times \dots \times \mathcal{P}^A_{|B_m|} \times \mathcal{P}^B_{|B_{m+1}|}$, which follows from the description of the vertices of $F_{\mathcal{B},\eta}$. It follows that $\dim F_{\mathcal{B},\eta} = n - m$.

Now, we prove the equivalence of the representations (4.1) and (4.2). To this end, we take some pair $(\mathcal{B}, \eta) \in \mathcal{T}_{n,m}$, assuming without restriction of generality that

$$B_1 := \{1, \dots, i_1\}, \;\; B_2 := \{i_1 + 1, \dots, i_2\}, \;\; \dots, \;\; B_{m+1} := \{i_m + 1, \dots, n\},$$

where $1 \le i_1 < \dots < i_m \le n$ for some $m \in \{1, \dots, n\}$, and $\eta_i = 1$ for $i \in \{1, \dots, i_m\}$. Our goal is to prove that $F_{\mathcal{B},\eta} = M$, where

$$M := \left\{t \in \mathcal{P}_n^B(x_1, \dots, x_n) : \sum_{i \in B_1 \cup \dots \cup B_l} t_i = x_1 + \dots + x_{|B_1 \cup \dots \cup B_l|} \; \forall l = 1, \dots, m\right\}.$$

The inclusion $F_{\mathcal{B},\eta} \subset M$ holds trivially and we only need to prove that $M \subset F_{\mathcal{B},\eta}$.

Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n \backslash \{0\}$ be such that condition (4.4) holds. The above arguments show that the hyperplane $H = \{t \in \mathbb{R}^n : \alpha_1 t_1 + \dots + \alpha_n t_n = b\}$ with $b := \alpha_1 x_1 + \dots + \alpha_n x_n$ is a supporting hyperplane of the face $F_{\mathcal{B},\eta}$, that is

$$H \cap \mathcal{P}_n^B = F_{\mathcal{B},\eta} \quad \text{and} \quad \mathcal{P}_n^B \subset H^- := \{t \in \mathbb{R}^n : \alpha_1 t_1 + \dots + \alpha_n t_n \le b\}.$$

Suppose now that there is some $y \notin F_{\mathcal{B},\eta}$ such that $y \in M \subset \mathcal{P}_n^B \subset H^-$. This already yields

$$\alpha_1 y_1 + \dots + \alpha_{i_m} y_{i_m} = \alpha_1 y_1 + \dots + \alpha_n y_n < t = \alpha_1 x_1 + \dots + \alpha_{i_m} x_{i_m},$$

since $y \in H^-$, but $y \notin H$. It follows that

$$\begin{aligned}
&\alpha_{i_m}(y_1 + \dots + y_{i_m}) + \sum_{l=1}^{m-1} (\alpha_{i_l} - \alpha_{i_{l+1}}) \sum_{i=1}^{i_l} y_i \\
&= \alpha_1 y_1 + \dots + \alpha_{i_m} y_{i_m} \\
&< \alpha_1 x_1 + \dots + \alpha_{i_m} x_{i_m} \\
&= \alpha_{i_m}(x_1 + \dots + x_{i_m}) + \sum_{l=1}^{m-1} (\alpha_{i_l} - \alpha_{i_{l+1}}) \sum_{i=1}^{i_l} x_i,
\end{aligned}$$

which is a contradiction to $y \in M$. This proves that both representations (4.1) and (4.2) are equivalent. □

Normal fans of permutohedra. Before we start with the proofs of Theorems 3.5 and 3.6, we need to prove a short lemma concerning the interior of a polytope.

Lemma 4.1. *Let $P \subset \mathbb{R}^n$ be a polytope with non-empty interior (i.e.* $\dim P = n$*) such that P is given by the following affine inequalities*

$$P = \{x \in \mathbb{R}^n : l_1(x) \leq 0, \dots, l_m(x) \leq 0\}$$

for some $m \in \mathbb{N}$ and affine-linear functions $l_i(x) = \langle x, y_i \rangle + b_i$, where $y_i \in \mathbb{R}^n \backslash \{0\}$ and $b_i \in \mathbb{R}$, $i = 1, \dots, m$. Then, we have

$$\operatorname{int} P = \{x \in \mathbb{R}^n : l_1(x) < 0, \dots, l_m(x) < 0\}.$$

Proof. Suppose $x \in \mathbb{R}^n$ satisfies the conditions $l_1(x) < 0, \dots, l_m(x) < 0$. Since the functions $l_1, \dots, l_m$ are continuous, we also have $l_1(y) < 0, \dots, l_m(y) < 0$ for all y in some small enough neighborhood of x. Thus, x lies in $\operatorname{int} P$.

Now let $x \in P$ satisfy $l_i(x) = 0$ for some $i \in \{1, \dots, m\}$. Then, in each neighborhood of x, we can find a point y with $l_i(y) > 0$. This means that $x \notin \operatorname{int} P$, thus completing the proof. □

Remark 4.2. For a j-face F of a polyhedral set $P \subset \mathbb{R}^n$, the normal cone $N_F(P)$ is $(n-j)$-dimensional. In order to prove this, assume that the linear hull $M := \operatorname{lin} N_F(P)$ is k-dimensional for some $k < n - j$. Then, we have $M^\perp = M^\circ \subset N_F(P)^\circ = T_F(P)$. But since $\dim M^\perp = n - k > j$, this is a contradiction to the fact that the maximal linear subspace L contained in $T_F(P)$ is j-dimensional. Also, since $L \subset T_F(P)$, we have $L^\perp \supset N_F(P)$ and therefore $\dim N_F(P) \leq n - j$.

Now, let $x_1 > \ldots > x_n$ be given. For Theorem 3.5, we want to prove that $\mathcal{N}(\mathcal{P}_n^A(x_1, \dots, x_n)) = \mathcal{F}(\mathcal{A}(A_{n-1}))$.

Proof of Theorem 3.5. From Proposition 3.3 we know that each j-face of $\mathcal{P}_n^A$, for a $j \in \{0, \dots, n-1\}$, is uniquely defined by an ordered partition $\mathcal{B} = (B_1, \dots, B_{n-j}) \in \mathcal{R}_{n,n-j}$ of the set $\{1, \dots, n\}$ and given by

$$F_\mathcal{B} = \left\{ (t_1, \dots, t_n) \in \mathcal{P}_n^A : \sum_{i \in B_1 \cup \ldots \cup B_l} t_i = x_1 + \ldots + x_{|B_1 \cup \ldots \cup B_l|} \text{ for all } l = 1, \dots, n-j \right\}.$$

Now, take a point $t \in \operatorname{relint} F_\mathcal{B}$. We claim that x satisfies the following conditions:

$$\sum_{i \in B_1 \cup \ldots \cup B_l} t_i = x_1 + \ldots + x_{|B_1 \cup \ldots \cup B_l|} \quad \forall l = 1, \dots, n-j, \tag{4.10}$$

and

$$\sum_{i \in M} t_i < x_1 + \ldots + x_{|M|} \quad \forall M \subset \{1, \dots, n\} : M \notin \{B_1, B_1 \cup B_2, \dots, B_1 \cup \ldots \cup B_{n-j}\}. \tag{4.11}$$

In order to prove this, consider the affine subspace

$$L_\mathcal{B} := \left\{ (t_1, \dots, t_n) \in \mathbb{R}^n : \sum_{i \in B_1 \cup \ldots \cup B_l} t_i = x_1 + \ldots + x_{|B_1 \cup \ldots \cup B_l|} \text{ for all } l = 1, \dots, n-j \right\},$$

which is of dimension j since the conditions are linearly independent. Then, following Lemma 3.1, we can represent $F_\mathcal{B}$ as the set of points $(t_1, \dots, t_n) \in L_\mathcal{B}$ such that

$$\sum_{i \in M} t_i \leq x_1 + \ldots + x_{|M|} \quad \forall M \subset \{1, \dots, n\} : M \notin \{B_1, B_1 \cup B_2, \dots, B_1 \cup \ldots \cup B_{n-j}\}.$$

Since $\dim F_{\mathcal{B}} = j$, the characterization of relint $F_{\mathcal{B}}$ in (4.11) follows from Lemma 4.1 applied to the ambient affine subspace $L_{\mathcal{B}}$ instead of $\mathbb{R}^n$.

Now, we want to determine the tangent cone $T_{F_{\mathcal{B}}}(\mathcal{P}_n^A)$. By definition, the tangent cone is given by

$$T_{F_{\mathcal{B}}}(\mathcal{P}_n^A) = \{v \in \mathbb{R}^n : t + \varepsilon v \in \mathcal{P}_n^A \text{ for some } \varepsilon > 0\},$$

where $t \in \operatorname{relint} F_{\mathcal{B}}$. Following Lemma 3.1, for a $v \in \mathbb{R}^n$, the condition $t + \varepsilon v \in \mathcal{P}_n^A$ holds for some $\varepsilon > 0$ if and only if

$$\sum_{i=1}^{n}(t_i + \varepsilon v_i) = x_1 + \ldots + x_n \quad \text{and} \quad \sum_{i \in M}(t_i + \varepsilon v_i) \le x_1 + \ldots + x_{|M|} \ \forall M \subset \{1, \ldots, n\}.$$

Since $t_1 + \ldots + t_n = x_1 + \ldots + x_n$, the first condition is satisfied if and only if $v_1 + \ldots + v_n = 0$. We observe that if we choose $\varepsilon > 0$ small enough, the second condition is satisfied for all sets $M \subset \{1, \ldots, n\}$ such that $M \notin \{B_1, B_1 \cup B_2, \ldots, B_1 \cup \ldots \cup B_{n-j}\}$, due to (4.11). For the sets $B_1, B_1 \cup B_2, \ldots, B_1 \cup \ldots \cup B_{n-j}$, we obtain that

$$\sum_{i \in B_1 \cup \ldots \cup B_l} v_i \le 0,$$

following (4.10). Therefore, the tangent cone is given by

$$T_{F_{\mathcal{B}}}(\mathcal{P}_n^A) = \Big\{v \in \mathbb{R}^n : v_1 + \ldots + v_n = 0, \sum_{i \in B_1 \cup \ldots \cup B_l} v_i \le 0 \ \forall l = 1, \ldots, n-j-1\Big\}.$$

Thus, the corresponding normal cone is given by

$$N_{F_{\mathcal{B}}}(\mathcal{P}_n^A) = T_{F_{\mathcal{B}}}(\mathcal{P}_n^A)^\circ = \{x \in \mathbb{R}^n : \forall 1 \le l_1 \le l_2 \le n-j \ \forall i_1 \in B_{l_1}, i_2 \in B_{l_2}, \text{ we have } x_{i_1} \ge x_{i_2}\}.$$

Note that the conditions of $N_{F_{\mathcal{B}}}(\mathcal{P}_n^A)$ imply $x_{i_1} = x_{i_2}$ for all $i_1, i_2 \in B_l$, $l = 1, \ldots, n-j$. The cone $N_{F_{\mathcal{B}}}(\mathcal{P}_n^A)$ is an $(n-j)$-dimensional cone in the fan $\mathcal{F}(\mathcal{A}(A_{n-1}))$ and it is easy to check that, going through all ordered partitions $\mathcal{B} \in \mathcal{R}_{n,n-j}$, we obtain all $(n-j)$-dimensional cones of the fan $\mathcal{N}(\mathcal{A}(A_{n-1}))$; see, e.g., [15, Section 2.7]. This completes the proof. □

Now, let $x_1 > \ldots > x_n > 0$ be given. For Theorem 3.6, we want to prove that the normal fan $\mathcal{N}(\mathcal{P}_n^B(x_1, \ldots, x_n))$ coincides with the fan $\mathcal{F}(\mathcal{A}(B_n))$ generated by the hyperplane arrangement $\mathcal{A}(B_n)$.

Proof of Theorem 3.6. From Proposition 3.4 we know that each j-face of $\mathcal{P}_n^B$, for a $j \in \{0, \ldots, n\}$, is uniquely defined by a pair $(\mathcal{B}, \eta) \in \mathcal{T}_{n,n-j}$, where $\mathcal{B} = (B_1, \ldots, B_{n-j+1})$, and given by

$$F_{\mathcal{B},\eta} = \Big\{(t_1, \ldots, t_n) \in \mathcal{P}_n^B : \sum_{i \in B_1 \cup \ldots \cup B_l} \eta_i t_i = x_1 + \ldots + x_{|B_1 \cup \ldots \cup B_l|} \ \forall l = 1, \ldots, n-j\Big\}.$$

Now, we claim that

$$T_{F_{\mathcal{B},\eta}}(\mathcal{P}_n^B) = \Big\{v \in \mathbb{R}^n : \sum_{i \in B_1 \cup \ldots \cup B_l} \eta_i v_i \le 0 \ \forall l = 1, \ldots, n-j\Big\}. \tag{4.12}$$

In order to prove this, take a point $t \in \operatorname{relint} F_{\mathcal{B},\eta}$. Then,

$$\sum_{i \in B_1 \cup \ldots \cup B_l} \eta_i t_i = x_1 + \ldots + x_{|B_1 \cup \ldots \cup B_l|} \quad \forall l = 1, \ldots, n-j \tag{4.13}$$

and

$$\sum_{i\in M}|t_i| < x_1+\ldots+x_{|M|} \quad \forall M\subset\{1,\ldots,n\}: M\notin\{B_1, B_1\cup B_2,\ldots,B_1\cup\ldots\cup B_{n-j}\}. \tag{4.14}$$

This can easily be justified in the same way as in the A-case using Lemmas 3.2 and 4.1. Note that (4.13) implies that $\operatorname{sgn} t_i = \eta_i$ for all $i\in B_1\cup\ldots\cup B_{n-j}$ such that $t_i\neq 0$. Otherwise, if $\eta_{i_0} = -\operatorname{sgn} t_{i_0}$ for some $i_0\in\{1,\ldots,n\}$ with $t_{i_0}\neq 0$, we would have

$$\sum_{B_1\cup\ldots\cup B_l}|t_i| > \sum_{B_1\cup\ldots\cup B_l}\eta_i t_i = x_1+\ldots+x_{|B_1\cup\ldots\cup B_l|}$$

for some $l\in\{1,\ldots,n-j\}$ in contradiction to $t\in\mathcal{P}_n^B$.

Now, recall that the tangent cone is defined by

$$T_{F_{\mathcal{B},\eta}}(\mathcal{P}_n^B) = \{v\in\mathbb{R}^n : t+\varepsilon v\in\mathcal{P}_n^B \text{ for some } \varepsilon>0\}$$

for $t\in\operatorname{relint} F_{\mathcal{B},\eta}$. In view of the characterization of points in $\mathcal{P}_n^B$ stated in Lemma 3.2, it follows that $v\in T_{F_{\mathcal{B},\eta}}(\mathcal{P}_n^B)$ if and only if there exists an $\varepsilon>0$ such that

$$\sum_{i\in M}|t_i+\varepsilon v_i| \le x_1+\ldots+x_{|M|} \quad \forall M\subset\{1,\ldots,n\}.$$

For all $M\subset\{1,\ldots,n\}$ with $M\notin\{B_1,B_1\cup B_2,\ldots,B_1\cup\ldots\cup B_{n-j}\}$ this condition is satisfied due to (4.14) provided $\varepsilon>0$ is small enough. If $t_i\neq 0$ for all $i\in B_1\cup\ldots\cup B_{n-j}$, the remaining conditions are equivalent to

$$\sum_{i\in B_1\cup\ldots\cup B_l}\eta_i(t_i+\varepsilon v_i)\le x_1+\ldots+x_{|B_1\cup\ldots\cup B_l|} \quad \forall l=1,\ldots,n-j. \tag{4.15}$$

This follows from the fact that $\operatorname{sgn}(t_i+\varepsilon v_i) = \operatorname{sgn} t_i = \eta_i$ for $\varepsilon>0$ chosen small enough. By (4.13), we obtain

$$\sum_{i\in B_1\cup\ldots\cup B_l}\eta_i v_i\le 0,$$

for all $l=1,\ldots,n-j$. This proves (4.12). At this point, it remains to prove that $t_i\neq 0$ for all $i\in B_1\cup\ldots\cup B_{n-j}$. In order to do this, assume $t_i=0$ for some $i\in B_l$ and some $l\in\{1,\ldots,n-j\}$. Defining $D_i := (B_1\cup\ldots\cup B_l)\backslash\{i\}$, we have

$$\sum_{j\in D_i}\eta_j t_j = \sum_{j\in B_1\cup\ldots\cup B_l}\eta_j t_j = x_1+\ldots+x_{|B_1\cup\ldots\cup B_l|},$$

due to (4.13). If $D_i = B_1\cup\ldots\cup B_m$ for some $m<l$, we obtain

$$x_1+\ldots+x_{|B_1\cup\ldots\cup B_m|} = \sum_{j\in D_i}\eta_j t_j = x_1+\ldots+x_{|B_1\cup\ldots\cup B_l|},$$

in contradiction to $x_i>0$ for all $i=1,\ldots,n$. If, on the other hand, $D_i\neq B_1\cup\ldots\cup B_m$ for all $m<l$, we have

$$x_1+\ldots+x_{|B_1\cup\ldots\cup B_l|} = \sum_{j\in D_i}\eta_j t_j < x_1+\ldots+x_{|D_i|},$$

following (4.14). This is a contradiction to $D_i\subset B_1\cup\ldots\cup B_l$ proving that $t_i\neq 0$ for all $i\in B_1\cup\ldots\cup B_{n-j}$.

Thus, the normal cone of $\mathcal{P}_n^B$ at $F_{\mathcal{B},\eta}$ is given by

$$\begin{aligned}N_{F_{\mathcal{B},\eta}}(\mathcal{P}_n^B) &= T_{F_{\mathcal{B},\eta}}(\mathcal{P}_n^B)^\circ \\ &= \big\{x\in\mathbb{R}^n : \forall 1\le l_1\le l_2\le n-j\ \forall i_1\in B_{l_1}, i_2\in B_{l_2},\ \text{we have } \eta_{i_1}x_{i_1}\ge \eta_{i_2}x_{i_2}\ge 0;\\ &\qquad\qquad \forall i\in B_{n-j+1} \text{ we have } x_i=0\big\}.\end{aligned}$$

The cone $N_{F_\mathcal{B}}(\mathcal{P}_n^B)$ is an $(n-j)$-face of a Weyl chamber of type B_n and we can observe that, going through all pairs $(\mathcal{B},\eta)\in\mathcal{T}_{n,n-j}$, we obtain all $(n-j)$-dimensional cones of the fan $\mathcal{N}(\mathcal{A}(B_n))$; see, e.g., [15, Section 2.4]. This completes the proof. □

Permutohedra and zonotopes. In order to prove Proposition 3.13, we need to verify that for $x_1 > \ldots > x_n$, the permutohedron $\mathcal{P}_n^A(x_1,\ldots,x_n)$ of type A (respectively, for $x_1>\ldots>x_n>0$, the permutohedron $\mathcal{P}_n^B(x_1,\ldots,x_n)$ of type B) is a zonotope if and only if $x_1,\ldots,x_n$ are in arithmetic progression, that is, $x_{j+1}-x_j = x_j - x_{j-1}$ for all admissible j.

Proof of Proposition 3.13. We will prove both the A- and the B-case together and assume that $x_1>\ldots>x_n$ and $x_1>\ldots>x_n>0$, respectively. In the book of Ziegler [29, Example 7.15], it is shown that $\mathcal{P}_n^A(n,n-1,\ldots,1)$ is a zonotope. By shifting and rescaling, we obtain that

$$\mathcal{P}_n^A\big(a+(n-1)b, a+(n-2)b,\ldots,a+b,a\big)$$

is also a zonotope for each $a\in\mathbb{R}$ and $b>0$. Similarly, we can also prove that $\mathcal{P}_n^B(n,n-1,\ldots,1)$ is a zonotope and therefore also $\mathcal{P}_n^A(a+(n-1)b,a+(n-2)b,\ldots,a+b,a)$ for each $a>0$ and $b>0$. This follows from the representation of $\mathcal{P}_n^B(n,n-1,\ldots,1)$ as the following Minkowski sum of line segments:

$$\mathcal{P}_n^B(n,n-1,\ldots,1) = \sum_{1\le i<j\le n}\Big[-\frac{e_i-e_j}{2},\frac{e_i-e_j}{2}\Big] + \sum_{1\le i<j\le n}\Big[-\frac{e_i+e_j}{2},\frac{e_i+e_j}{2}\Big] + \sum_{1\le i\le n}[-e_i,e_i].$$

In order to prove this, we observe that this Minkowski sum is invariant under signed permutations of the coordinates. Additionally, we can compute the vertices of this Minkowski sum, that is, the points of the Minkowski sum that maximize a linear function $v\mapsto\langle c,v\rangle$, $\mathbb{R}^n\to\mathbb{R}$, for a vector $c\in\mathbb{R}^n$, provided the maximizer is unique. Applying a signed permutation, we may assume that $c_1\ge c_2\ge\ldots\ge c_n\ge 0$. On the line segment $[-\frac{e_i-e_j}{2},\frac{e_i-e_j}{2}]$, the function $v\mapsto\langle c,v\rangle$ is uniquely maximized by the right-hand boundary $\frac{e_i-e_j}{2}$ provided $c_i>c_j$. For $c_i=c_j$, the maximizer is not unique. Therefore, we may assume that $c_1>c_2>\ldots>c_n>0$. Then, the unique maximizer of $v\mapsto\langle c,v\rangle$ is given by the sum of the right-hand boundaries of the line segments:

$$\begin{aligned}v &= \sum_{1\le i<j\le n}\frac{e_i-e_j}{2} + \sum_{1\le i<j\le n}\frac{e_i+e_j}{2} + \sum_{1\le i\le n} e_i\\ &= \sum_{1\le i<j\le n} e_i + \sum_{1\le i\le n} e_i\\ &= (n,n-1,\ldots,1)^\top.\end{aligned}$$

Hence, the vertices of the above Minkowski sum have the form $(\varepsilon_1\sigma(n),\varepsilon_2\sigma(n-1),\ldots,\varepsilon_n\sigma(1))$, for all $\varepsilon\in\{\pm1\}^n,\sigma\in\mathrm{Sym}(n)$. This proves the representation of $\mathcal{P}_n^B(a+(n-1)b,a+(n-2)b,\ldots,a+b,a)$ as the Minkowski sum of certain line segments. In particular, this polytope is a zonotope.

To prove the other direction, assume that $\mathcal{P}_n^A(x_1,\dots,x_n)$ with $n \geq 3$ is a zonotope and use that a polytope P is a zonotope if and only if every 2-dimensional face of P is centrally symmetric [29, p. 200]. Following Proposition 3.3, we know that the convex hull F of the six points

$$(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_4, x_5, \dots, x_n), \quad \sigma \in \mathrm{Sym}(3)$$

is a 2-face of $\mathcal{P}_n^A(x_1,\dots,x_n)$. This face is centrally symmetric around some $a = (a_1,\dots,a_n)$. This means that for each vertex z of F, also $2a - z$ is a vertex of F. Thus, we obtain the conditions

$$2a_1 - x_1, 2a_1 - x_2, 2a_1 - x_3 \in \{x_1, x_2, x_3\}.$$

From $x_1 > x_2 > x_3$, we obtain

$$2a_1 - x_1 = x_3 \quad \text{and} \quad 2a_1 - x_2 = x_2$$

and therefore also $x_1 + x_3 = 2x_2$. This yields $x_3 - x_2 = x_2 - x_1$. Analogously, by considering more general 2-faces of $\mathcal{P}_n^A(x_1,\dots,x_n)$, one proves that $x_{j+1} - x_j = x_j - x_{j-1}$ for all admissible j. Thus, $x_1,\dots,x_n$ are in arithmetic progression.

The proof that $x_1,\dots,x_n$ are in arithmetic progression if $\mathcal{P}_n^B(x_1,\dots,x_n)$ is a zonotope follows in the same way as in the A-case since the considered 2-faces of $\mathcal{P}_n^A$ are also 2-faces of $\mathcal{P}_n^B$, following Proposition 3.4. □

4.2. **General position: Proofs of Theorem 3.11 and Corollaries 3.7 and 3.8.** In this section, we prove the equivalences of the general position assumptions stated in Sections 3.2 and 3.3. In fact, Corollaries 3.7 and 3.8 follow from Theorem 3.11.

Let $P \subset \mathbb{R}^n$ be polytope such that the normal fan $\mathcal{N}(P)$ coincides with the fan of a hyperplane arrangement $\mathcal{A}$. For a $1 \leq d \leq \dim P$ and a matrix $G \in \mathbb{R}^{d\times n}$ with $\operatorname{rank} G = d$, we want to prove that the following two general position assumptions are equivalent:

- (G1) The $(n-d)$-dimensional linear subspace $\ker G$ is in general position with respect to P.
- (G2) The d-dimensional linear subspace $(\ker G)^\perp$ is in general position with respect to the hyperplane arrangement $\mathcal{A}$.

Proof of Theorem 3.11. Let $F \in \mathcal{F}_k(P)$ be a k-face of P for some $k \in \{0,\dots,\dim P\}$ and let L be the linear subspace parallel to $\operatorname{aff} F$ with the same dimension as $\operatorname{aff} F$, that is, $\operatorname{aff} F = t + L$ for some $t \in \mathbb{R}^n$. Then, the normal cone $N_F(P)$ is $(n-k)$-dimensional, due to Remark 4.2, and coincides with some $(n-k)$-dimensional cone from the fan of $\mathcal{A}$, that is, an $(n-k)$-face of the conical mosaic generated by $\mathcal{A}$. Thus, $\operatorname{lin} N_F(P)$ can be represented as an intersection of hyperplanes from $\mathcal{A}$ and therefore is an element of the lattice $\mathcal{L}(\mathcal{A})$.

On the other hand, by definition, $T_F(P)$ contains the linear subspace L. Thus, we have $(\operatorname{aff} F)^\perp = L^\perp \supset T_F(P)^\circ = N_F(P)$. Since both $N_F(P)$ and $(\operatorname{aff} F)^\perp$ are $(n-k)$-dimensional, we obtain

$$L^\perp = (\operatorname{aff} F)^\perp = \operatorname{lin} N_F(P) \in \mathcal{L}_{n-k}(\mathcal{A}).$$

The same argumentation applied backwards shows that, conversely, each $(n-k)$-dimensional subspace $K \in \mathcal{L}(\mathcal{A})$ coincides with $\operatorname{lin} N_F(P)$ for some k-face F of P. If we write $\operatorname{aff} F = t + L$ for some $t \in \mathbb{R}^n$, as above, then we obtain $K = (\operatorname{aff} F)^\perp = L^\perp$.

The equivalence of (G1) and (G2) follows easily from these observations. Condition (G1) is not satisfied, if and only if

$$\dim(L \cap \ker G) \neq \max\{k-d, 0\}$$

for some $k \in \{0, \dots, \dim P\}$ and some k-dimensional linear subspace L such that $\operatorname{aff} F = t + L$ for some k-face F of P and $t \in \mathbb{R}^n$. Following the above observation, $L^\perp \in \mathcal{L}(\mathcal{A})$ and

$$\begin{aligned}\dim\big((\ker G)^\perp \cap L^\perp\big) &= n - \dim(L + \ker G)\\ &= n - \big(\dim(\ker G) + \dim L - \dim(L \cap \ker G)\big)\\ &= d - k + \dim(L \cap \ker G)\end{aligned}$$

holds true, and we arrive at

$$\dim\big((\ker G)^\perp \cap L^\perp\big) \neq \max\{0, d-k\}.$$

Thus, $(\ker G)^\perp$ is not in general position with respect to $\mathcal{A}$ and therefore, (G2) is not satisfied. Since every $K \in \mathcal{L}(\mathcal{A})$ can be represented as $L^\perp$ as above, the same argument applies backwards. □

Now, let $x_1 > \ldots > x_n$ and $G \in \mathbb{R}^{d \times n}$ be a matrix with $\operatorname{rank} G = d$, $1 \leq d \leq n-1 = \dim \mathcal{P}_n^A$. We want to prove that the following conditions are equivalent:

(A1) The $(n-d)$-dimensional linear subspace $\ker G$ is in general position to $\mathcal{P}_n^A(x_1, \dots, x_n)$.
(A2) The d-dimensional linear subspace $(\ker G)^\perp$ is in general position with respect to the reflection arrangement $\mathcal{A}(A_{n-1})$,

where $\mathcal{A}(A_{n-1})$ is the hyperplane arrangement as defined in (3.2).

Proof of Corollary 3.7. Following Theorem 3.5, the normal fan $\mathcal{N}(\mathcal{P}_n^A)$ coincides with $\mathcal{F}(\mathcal{A}(A_{n-1}))$. Thus, the equivalence of (A1) and (A2) is a special case of Theorem 3.11. □

Now, let $x_1 > \ldots > x_n > 0$ and $G \in \mathbb{R}^{d \times n}$ be a matrix with $\operatorname{rank} G = d$, $1 \leq d \leq n = \dim \mathcal{P}_n^B$. For Corollary 3.8, we want to verify that the following conditions are equivalent:

(B1) The $(n-d)$-dimensional linear subspace $\ker G$ is in general position to $\mathcal{P}_n^B(x_1, \dots, x_n)$.
(B2) The d-dimensional linear subspace $(\ker G)^\perp$ is in general position with respect to the reflection arrangement $\mathcal{A}(B_n)$,

where $\mathcal{A}(B_n)$ is the hyperplane arrangement as defined in (3.3).

Proof of Corollary 3.8. Following Theorem 3.6, the normal fan $\mathcal{N}(\mathcal{P}_n^B)$ is given by the fan $\mathcal{F}(\mathcal{A}(B_n))$. Thus, the equivalence of (B1) and (B2) is a special case of Theorem 3.11. □

4.3. **Face numbers: Proofs of Theorems 3.9, 3.10 and 3.12.** In this section, we are going to prove our main results from Section 3.3 on the number of j-faces of projected permutohedra and more general polytopes. We start with the proof of Theorem 3.12 and proceed with the similar proofs of Theorems 3.9 and 3.10.

The following lemma, known as Farkas' Lemma, will be used in the proof of the named theorems. For the proof, we refer to [2, Lemma 2.4] and [13, Lemma 2.1].

Lemma 4.3 (Farkas)**.** *Let $C \subset \mathbb{R}^n$ be a full-dimensional cone and $L \subset \mathbb{R}^n$ a linear subspace. Then, we have*

$$\operatorname{int}(C) \cap L \neq \emptyset \Leftrightarrow C^\circ \cap L^\perp = \{0\}.$$

Furthermore, we need a formula for the number of regions generated by a hyperplane arrangement that are intersected by a linear subspace non-trivially. For the proof, we refer to [10, Theorem 3.1] or [14, Theorem 3.3] in combination with [14, Lemma 3.5].

Lemma 4.4. *Let $L_d \subset \mathbb{R}^n$ be a d-dimensional linear subspace that is in general position with respect to a hyperplane arrangement $\mathcal{A}$. Then, the number of regions in $\mathcal{R}(\mathcal{A})$ (which is the set of closed polyhedral cones of the conical mosaic generated by $\mathcal{A}$) intersected by L_d is given by*

$$\#\{R \in \mathcal{R}(\mathcal{A}) : \operatorname{int} R \cap L_d \neq \emptyset\} = \#\{R \in \mathcal{R}(\mathcal{A}) : R \cap L_d \neq \{0\}\} = 2(a_{n-d+1} + a_{n-d+3} + \ldots),$$

where the a_k's are defined by the characteristic polynomial $\chi_{\mathcal{A}}(t) = \sum_{k=0}^{n} (-1)^{n-k} a_k t^k$.

Now, let $P \subset \mathbb{R}^n$ be a polytope such that the normal fan $\mathcal{N}(P)$ coincides with the fan $\mathcal{F}(\mathcal{A})$ of a hyperplane arrangement $\mathcal{A}$. Moreover, let $G \in \mathbb{R}^{d\times n}$ be a matrix with $\operatorname{rank} G = d \leq \dim P$ such that one of the equivalent general position assumptions (G1) or (G2) is satisfied. For Theorem 3.12, we want to prove that the number of j-faces of the projected polytope GP is given by

$$f_j(GP) = 2 \sum_{M\in\mathcal{L}_{n-j}(\mathcal{A})} (a^M_{n-d+1} + a^M_{n-d+3} + \ldots)$$

for $0 \leq j < d \leq \dim P$, where the numbers a^M_k are (up to a sign) the coefficients of the characteristic polynomial of the hyperplane arrangement $\mathcal{A}|M := \{H \cap M : H \in \mathcal{A}, M \nsubseteq H\}$ in the ambient space M:

$$\chi_{A|M}(t) = \sum_{k=0}^{n-j} (-1)^{n-j-k} a^M_k t^k.$$

Proof of Theorem 3.12. Consider first the case when P is full-dimensional. Let F be a j-face of P and $0 \leq j < d \leq n$ be given. Then, following [1] or [9, Proposition 5.3], GF is a j-face of GP if and only if

$$\operatorname{int} T_F(P) \cap \ker G = \emptyset$$

since the general position assumption (G1) is satisfied. Using Farkas' Lemma 4.3, this is equivalent to

$$(\ker G)^\perp \cap N_F(P) \neq \{0\}.$$

Thus, using that $\mathcal{N}(P) = \mathcal{F}(\mathcal{A})$ and, in particular, $\{N_F(P) : F \in \mathcal{F}_j(P)\} = \mathcal{F}_{n-j}(\mathcal{A})$, we obtain

$$\begin{aligned} f_j(GP) &= \sum_{F\in\mathcal{F}_j(P)} \mathbb{1}_{\{GF\in\mathcal{F}_j(GP)\}} \\ &= \sum_{F\in\mathcal{F}_j(P)} \mathbb{1}_{\{(\ker G)^\perp\cap N_F(P)\neq\{0\}\}} \\ &= \sum_{D\in\mathcal{F}_{n-j}(\mathcal{A})} \mathbb{1}_{\{(\ker G)^\perp\cap D\neq\{0\}\}} \\ &= \sum_{M\in\mathcal{L}_{n-j}(\mathcal{A})} \sum_{D\in\mathcal{F}_{n-j}(\mathcal{A}):D\subset M} \mathbb{1}_{\{((\ker G)^\perp\cap M)\cap D\neq\{0\}\}}, \end{aligned} \tag{4.16}$$

since each $(n-j)$-dimensional cone D from the fan $\mathcal{F}(\mathcal{A})$ of the hyperplane arrangement $\mathcal{A}$ is contained in a unique $(n-j)$-dimensional subspace $M \in \mathcal{L}(\mathcal{A})$ that can be represented as an intersection of hyperplanes from $\mathcal{A}$. The $(n-j)$-dimensional cones $D \in \mathcal{F}_{n-j}(\mathcal{A})$ with $D \subset M$

are the closures of the $(n-j)$-dimensional regions generated by the induced arrangement $\mathcal{A}|M = \{H \cap M : H \in \mathcal{A}, M \not\subseteq H\}$ in M and therefore, we obtain

$$f_j(GP) = \sum_{M \in \mathcal{L}_{n-j}(\mathcal{A})} \sum_{R \in \mathcal{R}(\mathcal{A}|M)} \mathbb{1}_{\{((\ker G)^\perp \cap M) \cap R \neq \{0\}\}}.$$

Due to general position assumption (G2), the subspace $(\ker G)^\perp \cap M$ is of codimension $n-d$ in M and additionally in general position with respect to $\mathcal{A}|M$ in M. Thus, we can apply Lemma 4.4 to the ambient linear subspace M and arrive at

$$f_j(GP) = 2 \sum_{M \in \mathcal{L}_{n-j}(\mathcal{A})} (a^M_{n-d+1} + a^M_{n-d+3} + \ldots), \tag{4.17}$$

which completes the proof in the full-dimensional case.

Now, suppose $p := \dim P < n$. We want to restrict all arguments to the p-dimensional linear subspace L satisfying $\operatorname{aff} P = t + L$ for some $t \in \mathbb{R}^n$, and then apply the already known full-dimensional case in the ambient space L. At first, we observe that $\operatorname{rank}(G|_L) = d$, since $\dim \ker(G|_L) = \dim(L \cap \ker G) = p - d \geq 0$ because $\ker G$ is in general position with respect to P due to general position assumption (G1). Furthermore, we need to verify whether the conditions (G1) and (G2) also hold in the restricted case where n is replaced by p, G is replaced by the restriction $G|_L$ of G to L, and $\mathcal{A}$ is replaced by $\mathcal{A}|L = \{H \cap L : H \in \mathcal{A}, L \not\subseteq H\} = \{H \cap L : H \in \mathcal{A}\}$. The last equation is due to $L^\perp \subset \operatorname{lin} N_F(P)$ for all faces F of P, and therefore, $L^\perp \subset H$ for all $H \in \mathcal{A}$, since the linear hull $\operatorname{lin} N_F(P)$ coincides with an intersection of hyperplanes from $\mathcal{A}$. Thus, we also observe that the elements of $\mathcal{A}|L$ and $\mathcal{A}$ are in one-to-one correspondence via the mapping $H' \mapsto H' + L^\perp$ and, the inverse map is given by $H \cap L \leftarrow\!\shortmid H$.

Also, following (G1) for P in $\mathbb{R}^n$, $\ker(G|_L)$ is in general position with respect to K, for each linear subspace K such that $\operatorname{aff} F = t + K$ for some face F of P, since

$$\dim(K \cap \ker(G|_L)) = \dim(K \cap L \cap \ker G) = \dim(K \cap \ker G).$$

Thus, (G1) is also satisfied if we restrict all objects to L. Then, (G2) is also satisfied in the restricted version due to the equivalence of (G1) and (G2) proved in Theorem 3.11. Thus, we can apply (4.17) in the restricted case to obtain

$$f_j(GP) = 2 \sum_{M' \in \mathcal{L}_{p-j}(\mathcal{A}|L)} \big(a^{M'}_{p-d+1} + a^{M'}_{p-d+3} + \ldots\big),$$

since $(\mathcal{A}|L)|M' = \mathcal{A}|M'$ and therefore $\chi_{(\mathcal{A}|L)|M'}(t) = \chi_{\mathcal{A}|M'}(t)$. Next we observe that the linear subspaces $M' \in \mathcal{L}_{p-j}(\mathcal{A}|L)$ are in one-to-one correspondence to the linear subspaces $M \in \mathcal{L}_{n-j}(\mathcal{A})$ via $M' \mapsto M' + L^\perp =: M$. Following the Whitney formula for the characteristic polynomial (3.8) and the identity $\mathcal{A}|(M' + L^\perp) = (\mathcal{A}|M') + L^\perp$, we obtain the relation

$$\chi_{\mathcal{A}|(M'+L^\perp)}(t) = \chi_{(\mathcal{A}|M')+L^\perp}(t) = t^{n-p}\chi_{\mathcal{A}|M'}(t),$$

for all $M' \in \mathcal{L}_{p-j}(\mathcal{A}|L)$, and thus, $a^{M'}_k = a^{M'+L^\perp}_{k+n-p}$. Hence, we arrive at

$$f_j(GP) = 2 \sum_{M' \in \mathcal{L}_{p-j}(\mathcal{A}|L)} \big(a^{M'+L^\perp}_{n-d+1} + a^{M'+L^\perp}_{n-d+3} + \ldots\big) = 2 \sum_{M \in \mathcal{L}_{n-j}(\mathcal{A})} \big(a^M_{n-d+1} + a^M_{n-d+3} + \ldots\big),$$

which completes the proof. □

Permutohedron of type A. Now, we are going to prove the formulas for the number of faces of the projected permutohedra of types A and B. Before starting with the proof of Theorem 3.9, we need to introduce the Weyl chambers of type A_{n-1} and the corresponding reflection group and reflection arrangement. The *reflection group* $\mathcal{G}(A_{n-1})$ *of type* A_{n-1} acts on $\mathbb{R}^n$ by permuting the coordinates in an arbitrary way, that is, the $n!$ elements of $\mathcal{G}(A_{n-1})$ are the linear mappings

$$g_\sigma : \mathbb{R}^n \to \mathbb{R}^n, \quad (\beta_1, \dots, \beta_n) \mapsto (\beta_{\sigma(1)}, \dots, \beta_{\sigma(n)}),$$

where $\sigma \in \mathrm{Sym}(n)$. The *closed fundamental Weyl chamber of type* A_{n-1} is the cone

$$\mathcal{C}(A_{n-1}) = \{(\beta_1, \dots, \beta_n) \in \mathbb{R}^n : \beta_1 \geq \dots \geq \beta_n\}.$$

Then, the *closed Weyl chambers of type* A_{n-1} are the cones of the form $g\mathcal{C}(A_{n-1})$, where $g \in \mathcal{G}(A_{n-1})$, that is, the cones

$$C_\sigma^A := \{(\beta_1, \dots, \beta_n) \in \mathbb{R}^n : \beta_{\sigma(1)} \geq \beta_{\sigma(2)} \geq \dots \geq \beta_{\sigma(n)}\}, \quad \sigma \in \mathrm{Sym}(n).$$

Equivalently, the Weyl chambers can be defined as the conical mosaic generated by the reflection arrangement $\mathcal{A}(A_{n-1})$ consisting of the hyperplanes

$$\{\beta \in \mathbb{R}^n : \beta_i = \beta_j\}, \quad 1 \leq i < j \leq n.$$

Note that with this notation, the permutohedron $\mathcal{P}_n^A(x_1, \dots, x_n)$ for a point $(x_1, \dots, x_n) \in \mathbb{R}^n$ is just the convex hull of all points $g(x_1, \dots, x_n)$, where $g \in \mathcal{G}(A_{n-1})$.

In order to prove Theorem 3.9, we need to evaluate the number of j-faces $F \in \mathcal{F}_j(\mathcal{A}(A_{n-1}))$ that are intersected non-trivially by a d-dimensional linear subspace satisfying some general position assumption.

Lemma 4.5. *The number of j-faces of Weyl chambers of type A_{n-1} (where each face is counted exactly once) intersected non-trivially by a d-dimensional subspace L_d in general position to the reflection arrangement $\mathcal{A}(A_{n-1})$ is given by*

$$\sum_{F \in \mathcal{F}_j(\mathcal{A}(A_{n-1}))} \mathbb{1}_{\{F \cap L_d \neq \{0\}\}} = 2 \left\{ {n \atop j} \right\} \left(\left[{j \atop n-d+1} \right] + \left[{j \atop n-d+3} \right] + \dots \right),$$

for all $j \in \{1, \dots, n\}$.

Recall that the numbers $\left[{n \atop k} \right]$ and $\left\{ {n \atop k} \right\}$ denote the Stirling numbers of the first and second kind, respectively, as defined in Section 3.3.

Note that for $j = n$, this result is already known. That is, the number of Weyl chambers of type A_{n-1} intersected by a d-dimensional subspace L_d in general position with respect to $\mathcal{A}(A_{n-1})$ is given by

$$\sum_{F \in \mathcal{F}_n(\mathcal{A}(A_{n-1}))} \mathbb{1}_{\{F \cap L_d \neq \{0\}\}} = \sum_{\sigma \in \mathrm{Sym}(n)} \mathbb{1}_{\{C_\sigma^A \cap L_d \neq \{0\}\}} = 2 \left(\left[{n \atop n-d+1} \right] + \left[{n \atop n-d+3} \right] + \dots \right); \tag{4.18}$$

see [14, Theorem 3.4]. The proof of Lemma 4.5 is similar to that of [15, Theorem 2.8], where a related formula has been established in a setting where the faces are counted with certain non-trivial multiplicities.

Proof of Lemma 4.5. The j-dimensional faces of the Weyl chamber C_σ^A are enumerated by collections of indices $1 \leq i_1 < \dots < i_{j-1} \leq n-1$ as follows:

$$C_\sigma^A(i_1, \dots, i_{j-1}) := \{\beta \in \mathbb{R}^n : \beta_{\sigma(1)} = \dots = \beta_{\sigma(i_1)} \geq \dots \geq \beta_{\sigma(i_{j-1}+1)} = \dots = \beta_{\sigma(n)}\}$$

It is easy to see that $C_\sigma^A(i_1,\dots,i_{j-1})$ may also be a j-face of another Weyl chamber $C_{\sigma'}^A$ for some $\sigma'\neq\sigma$ (the permutations inside each group of equations of the defining conditions of $C_\sigma^A(i_1,\dots,i_{j-1})$ can be chosen arbitrary without changing the face itself).

Now, we will introduce a notation for all j-faces of all Weyl chambers of type A_{n-1} in the way that each face is counted exactly once. Recall that $\mathcal{R}_{n,j}$ denotes the set of all ordered partitions $\mathcal{B}=(B_1,\dots,B_j)$ of the set $\{1,\dots,n\}$ into j disjoint non-empty subsets. For such an ordered partition $\mathcal{B}$, we define the polyhedral cone

$$Q_\mathcal{B}=\big\{\beta\in\mathbb{R}^n:\text{ for all }1\leq l_1\leq l_2\leq j\text{ and }i_1\in B_{l_1},i_2\in B_{l_2}\text{ we have }\beta_{i_1}\geq\beta_{i_2}\big\}.$$

Note that these conditions imply $\beta_{i_1}=\beta_{i_2}$ for all $i_1,i_2\in B_l$ and all $1\leq l\leq j$. We can observe that each j-face $F\in\mathcal{F}_j(\mathcal{A}(A_{n-1}))$ coincides with $Q_\mathcal{B}$ for a unique ordered partition $\mathcal{B}\in\mathcal{R}_{n,j}$ and that, conversely, each cone $Q_\mathcal{B}$ is a j-face from $\mathcal{F}_j(\mathcal{A}(A_{n-1}))$. Thus, we have

$$\sum_{F\in\mathcal{F}_j(\mathcal{A}(A_{n-1}))}\mathbb{1}_{\{F\cap L_d\neq\{0\}\}}=\sum_{\mathcal{B}\in\mathcal{R}_{n,j}}\mathbb{1}_{\{Q_\mathcal{B}\cap L_d\neq\{0\}\}}.$$

Now, we want to evaluate the right hand-side of the above equation. At first, consider the case $j\leq n-d$. Since L_d is in general position with respect to $\mathcal{A}(A_{n-1})$, we know that for each $Q_\mathcal{B}$

$$\dim(L_d\cap\operatorname{lin}Q_\mathcal{B})=\max(j+d-n,0)=0,$$

since $\operatorname{lin}Q_\mathcal{B}\in\mathcal{L}(\mathcal{A}_{n-1})$. Therefore, Lemma 4.5 becomes trivial since both sides vanish. From now on, assume that $j\geq n-d+1$. The j-dimensional linear hull of $Q_\mathcal{B}$ is given by

$$W_\mathcal{B}=\big\{\beta\in\mathbb{R}^n:\text{for all }1\leq l\leq j\text{ and }i_1,i_2\in B_l\text{ we have }\beta_{i_1}=\beta_{i_2}\big\}.$$

Using $y_l:=\beta_i$, where $i\in B_l$ is arbitrary and $l=1,\dots,j$, as coordinates on $W_\mathcal{B}$, allows us to identify this linear hull with $\mathbb{R}^j$. This identification is linear (which is sufficient for what follows) but not isometric. The subspace $W_\mathcal{B}$ naturally decomposes into $j!$ Weyl chambers of type A_{j-1} of the form

$$W_\mathcal{B}(\pi)=\{(\beta_1,\dots,\beta_n)\in W_\mathcal{B}:y_{\pi(1)}\geq\dots\geq y_{\pi(j)}\},$$

where $\pi\in\operatorname{Sym}(j)$. Note that the Weyl chamber in $W_\mathcal{B}$ corresponding to the identity permutation $\pi(i)=i$ for all $1\leq i\leq j$ is $Q_\mathcal{B}$ itself. Since $L_d\subset\mathbb{R}^n$ has dimension d and is in general with respect to the reflection arrangement $\mathcal{A}(A_{n-1})$ and since $W_\mathcal{B}$ is an intersection of hyperplanes from $\mathcal{A}(A_{n-1})$, as mentioned above, it follows that the subspace $L_d\cap W_\mathcal{B}\subset W_\mathcal{B}$ has dimension $d-n+j$ and is in general position with respect to the reflection arrangement of type A_{j-1} in $W_\mathcal{B}$. This can be easily verified using the definition (3.4). Thus, we obtain

$$\sum_{\pi\in\operatorname{Sym}(j)}\mathbb{1}_{\{L_d\cap W_\mathcal{B}(\pi)\neq\{0\}\}}=\sum_{\pi\in\operatorname{Sym}(j)}\mathbb{1}_{\{(L_d\cap W_\mathcal{B})\cap W_\mathcal{B}(\pi)\neq\{0\}\}}=2\left(\begin{bmatrix}j\\n-d+1\end{bmatrix}+\begin{bmatrix}j\\n-d+3\end{bmatrix}+\dots\right),$$

following (4.18) applied to $L_d\cap W_\mathcal{B}$ in the ambient linear subspace $W_\mathcal{B}$. If we take the sum over all ordered partitions $\mathcal{B}\in\mathcal{R}_{n,j}$, we arrive at

$$\sum_{\mathcal{B}\in\mathcal{R}_{n,j}}\sum_{\pi\in\operatorname{Sym}(j)}\mathbb{1}_{\{L_d\cap W_\mathcal{B}(\pi)\neq\{0\}\}}=2\sum_{\mathcal{B}\in\mathcal{R}_{n,j}}\left(\begin{bmatrix}j\\n-d+1\end{bmatrix}+\begin{bmatrix}j\\n-d+3\end{bmatrix}+\dots\right).\tag{4.19}$$

The right-hand side of (4.19) can be easily computed using that the number of (unordered) partitions of $\{1,\dots,n\}$ into j non-empty subsets is given by the Stirling number $\left\{{n\atop j}\right\}$ of the second kind. Therefore, the number of ordered partitions is given by $j!\left\{{n\atop j}\right\}$, since the sets of each partition can be arranged in an arbitrary order. Moreover, since the j-face $Q_\mathcal{B}$ can be represented as $W_{\mathcal{B}'}(\pi')$ in

$j!$ ways and the sets $B_1,\dots,B_j$ are (up to their order) the same in all representations, the left-hand side of (4.19) can be rewritten as

$$j!\sum_{\mathcal{B}\in\mathcal{R}_{n,j}}\mathbb{1}_{\{Q_{\mathcal{B}}\cap L_d\neq\{0\}\}}.$$

Combining these results yields

$$\sum_{F\in\mathcal{F}_j(\mathcal{A}(A_{n-1}))}\mathbb{1}_{\{F\cap L_d\neq\{0\}\}}=\sum_{\mathcal{B}\in\mathcal{R}_{n,j}}\mathbb{1}_{\{Q_{\mathcal{B}}\cap L_d\neq\{0\}\}}=2\left\{{n\atop j}\right\}\left(\left[{j\atop n-d+1}\right]+\left[{j\atop n-d+3}\right]+\dots\right),$$

which completes the proof. □

Now, we turn to the proof of Theorem 3.9. Let $G\in\mathbb{R}^{d\times n}$ be a matrix with $\operatorname{rank}G=d$ satisfying one of the equivalent general position assumptions (A1) or (A2). Assume $x_1>\ldots>x_n$. We want to show that

$$f_j(G\mathcal{P}_n^A)=2\left\{{n\atop n-j}\right\}\left(\left[{n-j\atop n-d+1}\right]+\left[{n-j\atop n-d+3}\right]+\dots\right)$$

holds for all $0\leq j<d\leq n-1$.

Proof of Theorem 3.9. Since $\mathcal{P}_n^A$ is a polytope whose normal fan coincides with the fan generated by $\mathcal{A}(A_{n-1})$ and the equivalent general positions assumptions (A1) and (A2) are satisfied, we can apply (4.16) from the proof of Theorem 3.12 with P replaced by $\mathcal{P}_n^A$ and $\mathcal{A}$ replaced by $\mathcal{A}(A_{n-1})$, and obtain

$$f_j(G\mathcal{P}_n^A)=\sum_{D\in\mathcal{F}_{n-j}(\mathcal{A}(A_{n-1}))}\mathbb{1}_{\{(\ker G)^\perp\cap D\neq\{0\}\}}.$$

As was shown in the proof of Theorem 3.12, (4.16) was applicable although $\mathcal{P}_n^A$ is not full-dimensional. Following Lemma 4.5, we arrive at

$$f_j(G\mathcal{P}_n^A)=2\left\{{n\atop n-j}\right\}\left(\left[{n-j\atop n-d+1}\right]+\left[{n-j\atop n-d+3}\right]+\dots\right),$$

since $(\ker G)^\perp$ has dimension d and is in general position with respect to the reflection arrangement $\mathcal{A}(A_{n-1})$, due to (A1). This completes the proof. □

Permutohedron of type B. For the proof of Theorem 3.10, we need to introduce the Weyl chambers of type B_n. The *reflection group* $\mathcal{G}(B_n)$ acts on $\mathbb{R}^n$ by permuting the coordinates in an arbitrary way and multiplying an arbitrary number of coordinates by -1. Thus, the $2^n n!$ elements of $\mathcal{G}(B_n)$ are the linear mappings

$$g_{\varepsilon,\sigma}:\mathbb{R}^n\to\mathbb{R}^n,\quad(\beta_1,\dots,\beta_n)\mapsto(\varepsilon_1\beta_{\sigma(1)},\dots,\varepsilon_n\beta_{\sigma(n)}),$$

where $\sigma\in\operatorname{Sym}(n)$ and $\varepsilon=(\varepsilon_1,\dots,\varepsilon_n)\in\{\pm1\}^n$. The closed *fundamental Weyl chamber of type B_n* is the cone

$$\mathcal{C}(B_n)=\{(\beta_1,\dots,\beta_n)\in\mathbb{R}^n:\beta_1\geq\ldots\geq\beta_n\geq0\}.$$

Then, the *closed Weyl chambers of type B_n* are the cones $g\,\mathcal{C}(A_{n-1})$, where $g\in\mathcal{G}(B_n)$, that is, the cones

$$C^B_{\varepsilon,\sigma}:=\{(\beta_1,\dots,\beta_n)\in\mathbb{R}^n:\varepsilon_1\beta_{\sigma(1)}\geq\varepsilon_2\beta_{\sigma(2)}\geq\ldots\geq\varepsilon_n\beta_{\sigma(n)}\geq0\},\quad\varepsilon\in\{\pm1\}^n,\sigma\in\operatorname{Sym}(n).$$

Equivalently, the Weyl chambers of type B_n can be defined as the conical mosaic generated by the reflection arrangement $\mathcal{A}(B_n)$ consisting of the hyperplanes (3.3). Again, we observe that with this notation, the permutohedron $\mathcal{P}_n^B(x_1,\dots,x_n)$ for a point $(x_1,\dots,x_n)\in\mathbb{R}^n$ is just the convex hulls of all points $g(x_1,\dots,x_n)$, where $g\in\mathcal{G}(B_n)$.

In order to show Theorem 3.10, we need to prove the following lemma.

Lemma 4.6. *The number of j-faces of Weyl chambers of type B_n (where each face is counted exactly once) intersected non-trivially by a d-dimensional subspace L_d in general position to $\mathcal{A}(B_n)$ is given by*

$$\sum_{F\in\mathcal{F}_j(\mathcal{A}(B_n))}\mathbb{1}_{\{F\cap L_d\neq\{0\}\}}=2T(n,j)\big(B(j,n-d+1)+B(j,n-d+3)+\dots\big),$$

for all $j\in\{1,\dots,n\}$.

Recall that the numbers $B(n,k)$ and $T(n,k)$ are the B-analogues to the Stirling numbers of the first and second kind, respectively, as defined in (3.6) and (3.7). Also note that for $j=n$, this lemma is already known. That is, the number of Weyl chambers of type B_n intersected by a d-dimensional subspace L_d in general position with respect to $\mathcal{A}(B_n)$ is given by

$$\begin{aligned}\sum_{F\in\mathcal{F}_n(\mathcal{A}(B_n))}\mathbb{1}_{\{F\cap L_d\neq\{0\}\}}&=\sum_{(\varepsilon,\sigma)\in\{\pm1\}^n\times\mathrm{Sym}(n)}\mathbb{1}_{\{C^B_{\varepsilon,\sigma}\cap L_d\neq\{0\}\}}\\&=2\big(B(n,n-d+1)+B(n,n-d+3)+\dots\big);\end{aligned}\tag{4.20}$$

see [14, Theorem 3.4] or [15, Theorem 2.4]. The proof of Lemma 4.6 is similar to that of [15, Theorem 2.1].

Proof of Lemma 4.6. Since this is proven similarly to Lemma 4.5, we will not give the proof in full detail. At first, we will introduce the notation for all j-faces of all Weyl chambers of type B_n in the way that each face is counted exactly once. Recall that $\mathcal{T}_{n,j}$ denotes the set of all pairs $(\mathcal{B},\eta)$, where $\mathcal{B}=(B_1,\dots,B_{j+1})$ is an ordered partition of the set $\{1,\dots,n\}$ into $j+1$ disjoint distinguishable subsets such that $B_1,\dots,B_j$ are non-empty, whereas B_{j+1} may be empty or not, and $\eta:B_1\cup\ldots\cup B_j\to\{\pm1\}$. For ease of notation set $\eta(i)=\eta_i$, $i\in B_1\cup\ldots\cup B_j$. The j-face from $\mathcal{F}_j(\mathcal{A}(B_n))$ corresponding to $(\mathcal{B},\eta)$ is then given by

$$\begin{aligned}Q_{\mathcal{B},\eta}=\big\{\beta\in\mathbb{R}^n:&\text{ for all }1\le l_1\le l_2\le j\text{ and }i_1\in B_{l_1},i_2\in B_{l_2}\text{ we have }\eta_{i_1}\beta_{i_1}\ge\eta_{i_2}\beta_{i_2}\ge0;\\&\text{for all }i\in B_{j+1}\text{ we have }\beta_i=0\big\}.\end{aligned}$$

Note that these conditions imply $\eta_{i_1}\beta_{i_1}=\eta_{i_2}\beta_{i_2}$ for all $i_1,i_2\in B_l$ and all $1\le l\le j$. Similarly to the A-case, we obtain

$$\sum_{F\in\mathcal{F}_j(\mathcal{A}(B_n))}\mathbb{1}_{\{F\cap L_d\neq\{0\}\}}=\sum_{(\mathcal{B},\eta)\in\mathcal{T}_{n,j}}\mathbb{1}_{\{Q_{\mathcal{B},\eta}\cap L_d\neq\{0\}\}}.$$

Now, we want to evaluate the right hand-side of the above equation. Like in the A-case, Theorem 3.10 becomes trivial for $j\le n-d$. Thus, assume that $j\ge n-d+1$. The j-dimensional linear hull of $Q_{\mathcal{B},\eta}$ is given by

$$\begin{aligned}W_{\mathcal{B},\eta}=\big\{\beta\in\mathbb{R}^n:&\text{ for all }1\le l\le j\text{ and }i_1,i_2\in B_l\text{ we have }\eta_{i_1}\beta_{i_1}=\eta_{i_2}\beta_{i_2};\\&\text{for all }i\in B_{j+1}\text{ we have }\beta_i=0\big\}.\end{aligned}$$

We can use $y_l:=\eta_i\beta_i$ for $i\in B_l$ and $l=1,\dots,j$ as coordinates on $W_{\mathcal{B}}$. Then, we can identify this linear hull with $\mathbb{R}^j$ and naturally decompose $W_{\mathcal{B},\eta}$ into $2^jj!$ Weyl chambers of type B_n of the form

$$W_{\mathcal{B},\eta}(\pi,\delta)=\{(\beta_1,\dots,\beta_n)\in W_{\mathcal{B},\eta}:\delta_1y_{\pi(1)}\ge\ldots\ge\delta_jy_{\pi(j)}\ge0\},$$

where $\pi \in \mathrm{Sym}(j)$ and $\delta = (\delta_1, \dots, \delta_j) \in \{\pm 1\}^j$. Since $L_d \subset \mathbb{R}^n$ has dimension d and is in general with respect to the reflection arrangement $\mathcal{A}(B_n)$ it follows from the definition that the subspace $L_d \cap W_{\mathcal{B}} \subset W_{\mathcal{B}}$ has dimension $d-n+j$ and is in general position with respect to the reflection arrangement of type B_j in $W_{\mathcal{B},\eta}$. Thus, we obtain

$$\sum_{(\pi,\delta)\in \mathrm{Sym}(j)\times\{\pm 1\}^j} \mathbb{1}_{\{L_d \cap W_{\mathcal{B},\eta}(\pi,\delta)\neq\{0\}\}} = 2\big(B(j,n-d+1)+B(j,n-d+3)+\dots\big),$$

following (4.20) applied to $L_d \cap W_{\mathcal{B},\eta}$ in the ambient linear subspace $W_{\mathcal{B},\eta}$. We arrive at

$$\begin{aligned}
&\sum_{(\mathcal{B},\eta)\in\mathcal{T}_{n,j}} \sum_{(\pi,\delta)\in \mathrm{Sym}(j)\times\{\pm 1\}^j} \mathbb{1}_{\{L_d \cap W_{\mathcal{B},\eta}(\pi,\delta)\neq\{0\}\}} \\
&= 2\sum_{(\mathcal{B},\eta)\in\mathcal{T}_{n,j}} (B(j,n-d+1)+B(j,n-d+3)+\dots) \\
&= 2(B(j,n-d+1)+B(j,n-d+3)+\dots)\cdot \#\mathcal{T}_{n,j}.
\end{aligned} \tag{4.21}$$

The number of elements in $\mathcal{T}_{n,j}$ is given by

$$\#\mathcal{T}_{n,j} = \sum_{r=0}^{n-j} 2^{n-r} j! \binom{n}{r} \left\{ {n-r \atop j} \right\} = 2^j j! \sum_{r=j}^{n} 2^{r-j}\binom{n}{r}\left\{ {r \atop j} \right\} = 2^j j! T(n,j).$$

The last equation follows from (3.7) and the first equation can be proved as follows. There are $\binom{n}{r}$ possibilities to fix the elements of the set B_{j+1}, given it has cardinality $r \in \{0,\dots,n-j\}$. Then, there are 2^{n-r} possibilities for the choice of signs of the other $n-r$ elements in $B_1,\dots,B_j$. At last, the number of ordered partitions of $\{1,\dots,n\}\backslash B_{j+1}$ into j non-empty subsets is given by $j!\left\{ {n-r \atop j} \right\}$. Summing over all admissible values of r yields the first equation.

Moreover, since each j-face $Q_{\mathcal{B},\eta}$ can be represented as $W_{\mathcal{B}',\eta'}(\pi',\delta')$ in $2^j j!$ ways, the sum in (4.21) can be rewritten as

$$2^j j! \sum_{(\mathcal{B},\eta)\in\mathcal{T}_{n,j}} \mathbb{1}_{\{Q_{\mathcal{B},\eta}\cap L_d \neq\{0\}\}}.$$

Combining the above yields

$$\sum_{F\in\mathcal{F}_j(\mathcal{A}(B_n))} \mathbb{1}_{\{F\cap L_d\neq\{0\}\}} = 2T(n,j)\big(B(j,n-d+1)+B(j,n-d+3)+\dots\big),$$

which completes the proof. □

Now, we are finally able to present the proof of Theorem 3.10. Let $n \geq d$ and $G \in \mathbb{R}^{d\times n}$ be a matrix with $\operatorname{rank} G = d$ satisfying one of the equivalent general position assumptions (B1) or (B2). Assume $x_1 > \dots > x_n > 0$. We want to prove that

$$f_j(G\mathcal{P}_n^B) = 2T(n,n-j)\big(B(n-j,n-d+1)+B(n-j,n-d+3)+\dots\big)$$

holds for all $0 \leq j < d \leq n$.

Proof of Theorem 3.10. Since $\mathcal{P}_n^B$ is a polytope whose normal fan coincides with the fan generated by $\mathcal{A}(B_n)$ and the equivalent general positions assumptions (B1) and (B2) are satisfied, we can apply (4.16) with P replaced by $\mathcal{P}_n^B$ and $\mathcal{A}$ replaced by $\mathcal{A}(B_n)$. Thus, we obtain

$$f_j(G\mathcal{P}_n^B) = \sum_{D\in\mathcal{F}_{n-j}(\mathcal{A}(B_n))} \mathbb{1}_{\{D\cap(\ker G)^\perp\neq\{0\}\}}$$

$$= 2T(n, n-j)\big(B(n-j, n-d+1) + B(n-j, n-d+3) + \dots\big).$$

where we applied Lemma 4.6 in the last step. □

4.4. **Angle Sums: Proofs of Theorems 3.14, 3.15 and 3.16.** The main ingredient in the proofs of Theorems 3.14, 3.15 and 3.16 is the following formula:

$$\sum_{F\in\mathcal{F}_j(P)} \gamma_d(T_F(P)) = f_j(P) - \mathbb{E}\, f_j(GP) \tag{4.22}$$

for a polyhedral set $P \subset \mathbb{R}^n$ with non-empty interior, a Gaussian random matrix $G \in \mathbb{R}^{d\times n}$ and all $0 \le j < d \le n$. This formula (with the Gaussian projection G replaced by a projection on a d-dimensional uniform subspace) is often used throughout the existing literature, most prominently by Affentranger and Schneider [1], and a detailed proof can be found in [9, Theorem 4.8]. Results from Baryshnikov and Vitale [5] imply that the f-vector of the Gaussian projection of a polyhedral set P has the same distribution as the f-vector of a uniform projection of P.

Suppose $x_1 > \dots > x_n$. For Theorem 3.14, we want to show that the formulas

$$\sum_{F\in\mathcal{F}_j(\mathcal{P}_n^A)} \gamma_d(T_F(\mathcal{P}_n^A)) = 2\left\{ {n \atop n-j} \right\} \sum_{l=0}^{\infty} \left[{n-j \atop n-d-2l-1} \right], \qquad \sum_{F\in\mathcal{F}_j(\mathcal{P}_n^A)} \upsilon_d(T_F(\mathcal{P}_n^A)) = \left\{ {n \atop n-j} \right\} \left[{n-j \atop n-d} \right]$$

hold for all $1 \le j \le d \le n-1$.

Proof of Theorem 3.14. Let first $1 \le j < d \le n-1$ be given. Take a Gaussian random matrix $G \in \mathbb{R}^{d\times n}$ meaning that its entries are independent and standard Gaussian distributed random variables. This matrix has $\operatorname{rank} G = d$ a.s. since the rows of G are a.s. linearly independent; see, e.g., the proof of Theorem 4.17 in [9]. Also, the random matrix G satisfies general position assumption (A1) a.s. This can be justified by noting that $\ker G$ has a rotationally invariant distribution which implies that $\ker G$ is uniformly distributed in the Grassmannian of all $(n-d)$-dimensional subspaces of $\mathbb{R}^n$. This means that $\ker G$ is in general position with respect to each subspace $L \subset \mathbb{R}^n$ a.s., following [21, Lemma 13.2.1]. In particular, $\ker G$ is in general position with respect to $\mathcal{P}_n^A$. Thus we can apply (4.22) and Theorem 3.9 and obtain

$$\begin{aligned}
\sum_{F\in\mathcal{F}_j(\mathcal{P}_n^A)} \gamma_d(T_F(\mathcal{P}_n^A)) &= f_j(\mathcal{P}_n^A) - \mathbb{E}\, f_j(G\mathcal{P}_n^A) \\
&= f_j(\mathcal{P}_n^A) - 2\left\{ {n \atop n-j} \right\} \left(\left[{n-j \atop n-d+1} \right] + \left[{n-j \atop n-d+3} \right] + \dots \right) \\
&= (n-j)!\left\{ {n \atop n-j} \right\} - 2\left\{ {n \atop n-j} \right\} \left(\left[{n-j \atop n-d+1} \right] + \left[{n-j \atop n-d+3} \right] + \dots \right),
\end{aligned}$$

where we used Proposition 3.3 to determine the number of j-faces of $\mathcal{P}_n^A$, which coincides with the number of ordered partitions of the set $\{1, \dots, n\}$ into $n-j$ non-empty subsets. Note that for $x_1 > \dots > x_n$ the permutohedron $\mathcal{P}_n^A$ is only $(n-1)$-dimensional. The formula (4.22) was still applicable in the ambient linear subspace $\operatorname{lin} \mathcal{P}_n^A$ since the Grassmann angles do not depend on the dimension of ambient linear subspace.

We can simplify the above formula using that

$$\left[{n \atop 1} \right] + \left[{n \atop 3} \right] + \dots = \left[{n \atop 2} \right] + \left[{n \atop 4} \right] + \dots = \frac{n!}{2}.$$

Thus, we obtain

$$\sum_{F\in\mathcal{F}_j(\mathcal{P}_n^A)}\gamma_d(T_F(\mathcal{P}_n^A)) = 2\left\{{n \atop n-j}\right\}\left(\frac{(n-j)!}{2}-\left(\left[{n-j \atop n-d+1}\right]+\left[{n-j \atop n-d+3}\right]+\dots\right)\right)$$
$$= 2\left\{{n \atop n-j}\right\}\sum_{l=0}^{\infty}\left[{n-j \atop n-d-2l-1}\right],$$

which completes the proof of (3.12).

To deduce (3.11) we use the relation between the Grassmann angles and the conic intrinsic volumes (2.2), namely

$$\upsilon_d(C) = \frac{1}{2}\gamma_{d-1}(C) - \frac{1}{2}\gamma_{d+1}(C),$$

for all $d \in \{0,\dots,n\}$ provided C is not a linear subspace, where in the cases $d = 0$ and $d = n$ we have to define $\gamma_{-1}(C) = 1$ and $\gamma_{n+1}(C) = 0$. For $d \in \{j+1,\dots,n-1\}$, we have

$$\sum_{F\in\mathcal{F}_j(\mathcal{P}_n^A)}\upsilon_d(T_F(\mathcal{P}_n^A)) = \frac{1}{2}\sum_{F\in\mathcal{F}_j(\mathcal{P}_n^A)}\big(\gamma_{d-1}(T_F(\mathcal{P}_n^A)) - \gamma_{d+1}(T_F(\mathcal{P}_n^A))\big)$$
$$= \left\{{n \atop n-j}\right\}\left(\sum_{l=0}^{\infty}\left[{n-j \atop n-d-2l}\right] - \sum_{l=0}^{\infty}\left[{n-j \atop n-d-2l-2}\right]\right)$$
$$= \left\{{n \atop n-j}\right\}\left[{n-j \atop n-d}\right].$$

To complete the proof, note that in the case $j = d$ we have $\upsilon_j(T_F(\mathcal{P}_n^A)) = \upsilon_{n-j}(N_F(\mathcal{P}_n^A))$, which is the solid angle of $N_F(\mathcal{P}_n^A)$. By Theorem 3.5, the sum of these angles over all $F \in \mathcal{F}_j(\mathcal{P}_n^A)$ is the number of linear subspaces in $\mathcal{L}_{n-j}(\mathcal{A}(A_{n-1}))$, which is given by $\left\{{n \atop n-j}\right\}$. □

Now, suppose $x_1 > \dots > x_n > 0$. In order to prove Theorem 3.15, we need to verify the formulas

$$\sum_{F\in\mathcal{F}_j(\mathcal{P}_n^B)}\upsilon_d(T_F(\mathcal{P}_n^B)) = T(n,n-j)B(n-j,n-d), \tag{4.23}$$

$$\sum_{F\in\mathcal{F}_j(\mathcal{P}_n^B)}\gamma_d(T_F(\mathcal{P}_n^B)) = 2T(n,n-j)\sum_{l=0}^{\infty}B(n-j,n-d-2l-1) \tag{4.24}$$

for $0 \le j \le d \le n$, where the numbers $B(n,k)$ and $T(n,k)$ are the B-analogues to the Stirling numbers of first and second kind, respectively, as defined in (3.6) and (3.7).

Proof of Theorem 3.15. Similarly to the proof of Theorem 3.14, we use the formula (4.22). Let $G \in \mathbb{R}^{d\times n}$ be a Gaussian random matrix and let $0 \le j < d \le n$. As seen in the proof of Theorem 3.14, $\ker G = d$ a.s. and $\ker G$ is in general position with respect to each linear subspace $L \subset \mathbb{R}^n$ a.s. Thus, $\ker G$ is also in general position with respect to $\mathcal{P}_n^B$ a.s. and therefore, general position assumption (B1) is a.s. satisfied. Applying (4.22) and Theorem 3.10, we obtain

$$\sum_{F\in\mathcal{F}_j(\mathcal{P}_n^B)}\gamma_d(T_F(\mathcal{P}_n^B)) = f_j(\mathcal{P}_n^B) - \mathbb{E}\,f_j(G\mathcal{P}_n^B)$$
$$= f_j(\mathcal{P}_n^B) - 2T(n,n-j)\big(B(n-j,n-d+1) + B(n-j,n-d+3) + \dots\big).$$

By Proposition 3.4, the number of j-faces of $\mathcal{P}_n^B$ is given by the number of pairs $(\mathcal{B},\eta)\in\mathcal{T}_{n,n-j}$. This number was already computed in the proof of Lemma 4.6 and is given by

$$f_j(\mathcal{P}_n^B)=\#\mathcal{T}_{n,n-j}=2^{n-j}(n-j)!T(n,n-j).$$

Using the equation

$$B(n,1)+B(n,3)+\ldots=B(n,0)+B(n,2)+\ldots=2^{n-1}n!,$$

we obtain

$$\begin{aligned}&\sum_{F\in\mathcal{F}_j(\mathcal{P}_n^B)}\gamma_d(T_F(\mathcal{P}_n^B))\\&\quad=2T(n,n-j)\big(2^{n-j-1}(n-j)!-(B(n-j,n-d+1)+B(n-j,n-d+3)+\ldots)\big)\\&\quad=2T(n,n-j)\sum_{l=0}^{\infty}B(n-j,n-d-2l-1)\end{aligned}$$

which completes the proof of (4.24).

In order to prove (4.23), we use the relation (2.2), and obtain for $d\in\{j+1,\ldots,n\}$

$$\begin{aligned}\sum_{F\in\mathcal{F}_j(\mathcal{P}_n^B)}\upsilon_d(T_F(\mathcal{P}_n^B))&=\frac{1}{2}\sum_{F\in\mathcal{F}_j(\mathcal{P}_n^B)}\big(\gamma_{d-1}(T_F(\mathcal{P}_n^B))-\gamma_{d+1}(T_F(\mathcal{P}_n^B))\big)\\&=T(n,n-j)B(n-j,n-d).\end{aligned}$$

The case $j=d$ is treated similarly to the A-case. □

At last, we want to prove Theorem 3.16. Let $P\subset\mathbb{R}^n$ be a polytope whose normal fan $\mathcal{N}(P)$ coincides with the fan $\mathcal{F}(\mathcal{A})$ of a hyperplane arrangement $\mathcal{A}$ in $\mathbb{R}^n$. Then, we want to prove the formulas

$$\begin{aligned}\sum_{F\in\mathcal{F}_j(P)}\upsilon_d(T_F(P))&=\sum_{M\in\mathcal{L}_{n-j}(\mathcal{A})}a_{n-d}^M,\\\sum_{F\in\mathcal{F}_j(P)}\gamma_d(T_F(P))&=2\sum_{M\in\mathcal{L}_{n-j}(\mathcal{A})}(a_{n-d-1}^M+a_{n-d-3}^M+\ldots)\end{aligned}$$

for $0\le j\le d\le\dim P$, where the numbers a_k^M are defined by $\chi_{A|M}(t)=\sum_{k=0}^{j}(-1)^{j-k}a_k^Mt^k$ and $\mathcal{L}_j(\mathcal{A})$ denotes the set of j-dimensional subspaces from the lattice of $\mathcal{A}$.

Proof of Theorem 3.16. Let $G\in\mathbb{R}^{d\times n}$ be a Gaussian random matrix. In the same way as in the proof of Theorem 3.14, we can show that G satisfies the equivalent general position assumptions (G1) and (G2) a.s. Again, we use the formula (4.22) applied to the ambient affine subspace aff P. Together with Theorem 3.12, we obtain

$$\begin{aligned}\sum_{F\in\mathcal{F}_j(P)}\gamma_d(T_F(P))&=f_j(P)-\mathbb{E}\,f_j(GP)\\&=f_j(P)-2\sum_{M\in\mathcal{L}_{n-j}(\mathcal{A})}(a_{n-d+1}^M+a_{n-d+3}^M+\ldots)\end{aligned}\tag{4.25}$$

for all $0\le j<d\le\dim P$. For a j-face F of P, the corresponding normal cone $N_F(P)$ is $(n-j)$-dimensional and since the normal fan of P is given by the fan $\mathcal{F}(\mathcal{A})$, we obtain $f_j(P)=\#\mathcal{F}_{n-j}(\mathcal{A})$.

The number of regions of the arrangements $\mathcal{A}$ can be expressed through the characteristic polynomial by means of the Zaslavsky formula

$$\#\mathcal{R}(\mathcal{A}) = (-1)^n \chi_{\mathcal{A}}(-1);$$

see [25, Theorem 2.5]. Since each $(n-j)$-dimensional cone of the fan $\mathcal{F}(\mathcal{A})$ is uniquely contained in an $(n-j)$-dimensional subspace $M \in \mathcal{L}_{n-j}(\mathcal{A})$, and since the $(n-j)$-dimensional cones of $\mathcal{F}(\mathcal{A})$ contained in M are the cones of the conical mosaic in M generated by the induced arrangement $\mathcal{A}|M := \{H \cap M : H \in \mathcal{A}, M \nsubseteq H\}$, we obtain

$$\begin{aligned}\#\mathcal{F}_{n-j}(\mathcal{A}) &= \sum_{M \in \mathcal{L}_{n-j}(\mathcal{A})} \#\{F \in \mathcal{F}_{n-j}(\mathcal{A}) : F \subset M\} \\ &= \sum_{M \in \mathcal{L}_{n-j}(\mathcal{A})} \#\mathcal{R}(\mathcal{A}|M) \\ &= \sum_{M \in \mathcal{L}_{n-j}(\mathcal{A})} (-1)^{n-j} \chi_{\mathcal{A}|M}(-1).\end{aligned}$$

Furthermore, we have

$$\chi_{\mathcal{A}|M}(-1) = \sum_{k=0}^{n-j} (-1)^{n-j-k} a_k^M (-1)^k = (-1)^{n-j} \sum_{k=0}^{n-j} a_k^M.$$

Combining these results, we arrive at

$$f_j(P) = \sum_{M \in \mathcal{L}_{n-j}(\mathcal{A})} \sum_{k=0}^{n-j} a_k^M.$$

Now, we can insert this formula for $f_j(P)$ in (4.25) together with the identity

$$a_0^M + a_2^M + \ldots = a_1^M + a_3^M + \ldots,$$

see [14, Remark 3.2], and obtain

$$\begin{aligned}\sum_{F \in \mathcal{F}_j(P)} \gamma_d(T_F(P)) &= \sum_{M \in \mathcal{L}_{n-j}(\mathcal{A})} \sum_{k=0}^{n-j} a_k^M - 2 \sum_{M \in \mathcal{L}_{n-j}(\mathcal{A})} (a_{n-d+1}^M + a_{n-d+3}^M + \ldots) \\ &= 2 \sum_{M \in \mathcal{L}_{n-j}(\mathcal{A})} (a_{n-d-1}^M + a_{n-d-3}^M + \ldots)\end{aligned}$$

for $0 \le j < d \le \dim P$.

The formula for the sums of the conic intrinsic volumes follows from relation (2.2):

$$\begin{aligned}\sum_{F \in \mathcal{F}_j(P)} \upsilon_d(T_F(P)) &= \frac{1}{2} \sum_{F \in \mathcal{F}_j(P)} \big(\gamma_{d-1}(T_F(P)) - \gamma_{d+1}(T_F(P))\big) \\ &= \sum_{M \in \mathcal{L}_{n-j}(\mathcal{A})} a_{n-d}^M\end{aligned}$$

for $d \in \{j+1, \ldots, n\}$. In the case $j = d$, the same argument as used at the end of the proof of Theorem 3.14 shows that $\sum_{F \in \mathcal{F}_j(P)} \upsilon_j(T_F(P)) = \#\mathcal{L}_{n-j}(\mathcal{A}) = \sum_{M \in \mathcal{L}_{n-j}(\mathcal{A})} a_{n-j}^M$ since $a_{n-j}^M = 1$. The proof is complete. $\square$

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