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SOME NEW SERIES FOR $1/\pi$ MOTIVATED BY CONGRUENCES

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Abstract. In this paper, via symbolic computation we deduce a family of six new series for $1/\pi$, for example,

$$\sum_{n=0}^{\infty}\frac{41673840n+4777111}{5780^k}W_n\left(\frac{1444}{1445}\right)=\frac{147758475}{\sqrt{95}\,\pi}$$

where $W_n(x)=\sum_{k=0}^n\binom nk\binom{n+k}k\binom{2k}k\binom{2(n-k)}{n-k}x^k$. In addition, we pose 17 new series for $1/\pi$ motivated by congruences; for example, we conjecture that

$$\sum_{k=0}^{\infty}\frac{4290k+367}{3136^k}\binom{2k}{k}T_k(14,1)T_k(17,16)=\frac{5390}{\pi},$$

where $T_k(b,c)$ is the coefficient of x^k in the expansion of $(x^2+bx+c)^k$.

1. Introduction

Let $n\in\mathbb{N}=\{0,1,2,\ldots\}$. In 1894 J. Franel [8] introduced the usual Franel numbers $f_n=\sum_{k=0}^n\binom ni^3$ $(n\in\mathbb{N})$ and the Franel numbers $f_n^{(4)}=\sum_{k=0}^n\binom nk^4$ $(n\in\mathbb{N})$ of order four. By the Zeilberger algorithm (cf. [9]), the sequence $(f_n^{(4)})_{n\geq0}$ satisfies the following recurrence first claimed by Franel:

$$(n+2)^3f_{n+2}^{(4)}=4(1+n)(3+4n)(5+4n)f_n^{(4)}+2(3+2n)(7+9n+3n^2)f_{n+1}^{(4)}.$$

M. Rogers and A. Straub [11] confirmed the suthor's conjectural series for $1/\pi$ involving Franel polynomials.

In 2005 Y. Yang used mofular forms of level 10 to discover the following curious identity relating Franel numbers of order four to Ramanujan-type series for $1/\pi$:

$$\sum_{k=0}^{\infty}\frac{4k+1}{36^k}f_k^{(4)}=\frac{18}{\sqrt{15}\,\pi}.$$

More this kind of identities were deduced by S. Cooper [4] in 2012 via modular forms. For the classical Ramanujan-type series for $1/\pi$, one may consult [1, 2, 10] and the nice survey given by Cooper [5, Chapter 14].

Key words and phrases. Ramanujan-type series for $1/\pi$, congruences, binomial coefficients, symbolic computation.

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For $n\in\mathbb{N}$ the polynomial

$$W_n(x)=\sum_{k=0}^n\binom{n}{k}\binom{n+k}{k}\binom{2k}{k}\binom{2(n-k)}{n-k}x^k$$
$$=\sum_{k=0}^n\binom{n+k}{2k}\binom{2k}{k}^2\binom{2(n-k)}{n-k}x^k$$

at $x=-1$ coincides with $(-1)^nf_n^{(4)}$, this can be easily verified since the sequence $((-1)^nW_n(-1))_{n\geq0}$ satisfies the same recurrence as $(f_n^{(4)})_{n\geq0}$. In 2011 the author [12, (3.1)-(3.10)] proposed ten identities of the form

$$\sum_{k=0}^\infty\frac{ak+b}{m^k}W_k\left(\frac{1}{m}\right)=\frac{C}{\pi},$$

where a,b,m are integers with $am\neq0$, and C^2 is rational. They were later confirmed in [6].

In this paper we establish six new series for $1/\pi$ involving $W_n(x)$.

Theorem 1.1. *We have the following identities:*

$$\sum_{k=0}^\infty\frac{45k+8}{40^k}W_k\left(\frac{9}{10}\right)=\frac{215\sqrt{15}}{12\pi},\tag{1.1}$$
$$\sum_{k=0}^\infty\frac{1360k+389}{(-60)^k}W_k\left(\frac{16}{15}\right)=\frac{205\sqrt{15}}{\pi},\tag{1.2}$$
$$\sum_{k=0}^\infty\frac{735k+124}{200^k}W_k\left(\frac{49}{50}\right)=\frac{10125\sqrt{7}}{56\pi},\tag{1.3}$$
$$\sum_{k=0}^\infty\frac{376380k+69727}{(-320)^k}W_k\left(\frac{81}{80}\right)=\frac{260480\sqrt{5}}{3\pi},\tag{1.4}$$
$$\sum_{k=0}^\infty\frac{348840k+47461}{1300^k}W_k\left(\frac{324}{325}\right)=\frac{1314625\sqrt{2}}{12\pi},\tag{1.5}$$
$$\sum_{k=0}^\infty\frac{41673840k+4777111}{5780^k}W_k\left(\frac{1444}{1445}\right)=\frac{147758475}{\sqrt{95}\,\pi}.\tag{1.6}$$

We also have 9 conjectural series for $1/\pi$ involving $W_n(x)$ as listed in the following conjecture.

Conjecture 1.1. *We have the following identities:*

$$\sum_{k=0}^{\infty}\frac{4k+1}{6^k}W_k\left(-\frac{1}{8}\right)=\frac{\sqrt{72+42\sqrt{3}}}{\pi},\tag{1.7}$$

$$\sum_{k=0}^{\infty}\frac{392k+65}{(-108)^k}W_k\left(-\frac{49}{12}\right)=\frac{387\sqrt{3}}{\pi},\tag{1.8}$$

$$\sum_{k=0}^{\infty}\frac{168k+23}{112^k}W_k\left(\frac{63}{16}\right)=\frac{1652\sqrt{3}}{9\pi},\tag{1.9}$$

$$\sum_{k=0}^{\infty}\frac{1512k+257}{(-320)^k}W_k\left(-\frac{405}{64}\right)=\frac{1184\sqrt{35}}{5\pi},\tag{1.10}$$

$$\sum_{k=0}^{\infty}\frac{56k+9}{324^k}W_k\left(\frac{25}{4}\right)=\frac{1134\sqrt{35}}{125\pi},\tag{1.11}$$

$$\sum_{k=0}^{\infty}\frac{13000k-1811}{(-1296)^k}W_k\left(-\frac{625}{9}\right)=\frac{49356\sqrt{39}}{5\pi},\tag{1.12}$$

$$\sum_{k=0}^{\infty}\frac{9360k-1343}{1300^k}W_k\left(\frac{900}{13}\right)=\frac{21515\sqrt{39}}{3\pi},\tag{1.13}$$

$$\sum_{k=0}^{\infty}\frac{56355k+2443}{(-5776)^k}W_k\left(-\frac{83521}{361}\right)=\frac{4669535\sqrt{2}}{68\pi},\tag{1.14}$$

$$\sum_{k=0}^{\infty}\frac{5928k+253}{5780^k}W_k\left(\frac{1156}{5}\right)=\frac{28951\sqrt{2}}{4\pi}.\tag{1.15}$$

Remark 1.1. Note that the left-hand sides of (1.1)-(1.15) have the form $\sum_{k=0}^{\infty}(ak+b)W_k(x)/m^k$ with mx an integer square. Motivated by congruences, the author found (1.1)-(1.15) during August 23-29, 2020.

van Hamme [20] thought that classical Ramanujan-type series for $1/\pi$ should have their p-adic analogues involving the p-adic Gamma function. This does not hold in general for generalized Ramanujan-type series, for example, the author [13, Conjecture 1.5] discovered the identity

$$\sum_{n=0}^{\infty}\frac{6n-1}{256^n}\binom{2n}{n}\sum_{k=0}^{n}\binom{2k}{k}^2\binom{2(n-k)}{n-k}12^{n-k}=\frac{8\sqrt{3}}{\pi}$$

(which was later confirmed in [6]) and conjectured its related p-adic congruence

$$\sum_{n=0}^{p-1}\frac{6n-1}{256^n}\binom{2n}{n}\sum_{k=0}^{n}\binom{2k}{k}^2\binom{2(n-k)}{n-k}12^{n-k}\equiv -p \pmod{p^2}$$

(with p any prime greater than 3) which has noting to do with the Legendre symbol $(\frac{-3}{p})$.

For the author's philosophy to generate series for $1/\pi$ via congruences, one may consult the survey [13] and the recent paper [18, Section 1].

We will prove Theorem 1.1 in the next section, and present related conjectural congruences in Section 3. In Section 4-6, we will pose 8 other new conjectural series for $1/\pi$ motivated by congruences.

2. Proof of Theorem 1.1

Lemma 2.1. *For $|z|\le 1/30$, we have*

$$\sum_{k=0}^{\infty}\frac{z^k}{(1+4z)^{k+1}}W_k\left(\frac{1}{1+4z}\right)=\sum_{n=0}^{\infty}f_n^{(4)}z^n \tag{2.1}$$

and

$$\sum_{k=0}^{\infty}\frac{kz^k}{(1+4z)^{k+1}}W_k\left(\frac{1}{1+4z}\right)=\sum_{n=0}^{\infty}n(f_n^{(4)}+4s_n)z^n, \tag{2.2}$$

where

$$s_n:=\sum_{0\le j<n}(-1)^{n-1-j}\binom{n-1}{j}\binom{n+j}{j}\binom{2j}{j}\binom{2(n-1-j)}{n-1-j}. \tag{2.3}$$

Proof. Let N be any nonnegative integer. Then

$$\begin{aligned}
&\sum_{k=0}^{N}\frac{z^k}{(1+4z)^{k+1}}W_k\left(\frac{1}{4z+1}\right)\\
=&\sum_{k=0}^{N}z^k\sum_{j=0}^{k}\binom{k+j}{2j}\binom{2j}{j}^2\binom{2(k-j)}{k-j}(1+4z)^{-j-k-1}\\
=&\sum_{k=0}^{N}z^k\sum_{j=0}^{k}\binom{k+j}{2j}\binom{2j}{j}^2\binom{2(k-j)}{k-j}\sum_{r=0}^{\infty}\binom{-j-k-1}{r}(4z)^r\\
=&\sum_{n=0}^{\infty}z^n\sum_{k=0}^{\min\{n,N\}}\sum_{j=0}^{k}\binom{k+j}{2j}\binom{2j}{j}^2\binom{2(k-j)}{k-j}\binom{-j-k-1}{n-k}4^{n-k}\\
=&\sum_{n=0}^{\infty}z^n\sum_{j=0}^{\min\{n,N\}}\binom{2j}{j}^2\sum_{k=j}^{\min\{n,N\}}\binom{k+j}{2j}\binom{2(k-j)}{k-j}\binom{n+j}{k+j}(-4)^{n-k}\\
=&\sum_{n=0}^{\infty}z^n\sum_{j=0}^{\min\{n,N\}}(-4)^{n-j}\binom{2j}{j}^2\binom{n+j}{2j}\sum_{k=j}^{\min\{n,N\}}\binom{n-j}{k-j}\frac{\binom{2(k-j)}{k-j}}{(-4)^{k-j}}.
\end{aligned}$$

Similarly,

$$\sum_{k=0}^{N}\frac{kz^k}{(1+4z)^{k+1}}W_k\left(\frac{1}{4z+1}\right)$$
$$=\sum_{n=0}^{\infty}z^n\sum_{j=0}^{\min\{n,N\}}(-4)^{n-j}\binom{2j}{j}^2\binom{n+j}{2j}\sum_{k=j}^{\min\{n,N\}}k\binom{n-j}{n-k}\frac{\binom{2(k-j)}{k-j}}{(-4)^{k-j}}.$$

Clearly $\binom{2m}{m}\le(1+1)^{2m}=4^m$ for all $m\in\mathbb{N}$. Thus

$$\Big|\sum_{k=j}^{\min\{n,N\}}\binom{n-j}{k-j}\frac{\binom{2(k-j)}{k-j}}{(-4)^{k-j}}\Big|\le\sum_{k\ge j}\binom{n-j}{k-j}=2^{n-j}$$

and hence

$$\Big|\sum_{j=0}^{\min\{n,N\}}(-4)^{n-j}\binom{2j}{j}^2\binom{n+j}{2j}\sum_{k=j}^{\min\{n,N\}}\binom{n-j}{k-j}\frac{\binom{2(k-j)}{k-j}}{(-4)^{k-j}}\Big|$$
$$\le\sum_{j=0}^{\min\{n,N\}}4^n\binom{n+j}{2j}\binom{2j}{j}2^{n-j}\le 8^n\sum_{j=0}^{n}\binom{n}{j}\binom{n+j}{j}\left(\frac{2-1}{2}\right)^j=8^nP_n(2),$$

where

$$P_n(x)=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\left(\frac{x-1}{2}\right)^k$$

is the Legendre polyomial of degree n. Similarly,

$$\Big|\sum_{j=0}^{\min\{n,N\}}(-4)^{n-j}\binom{2j}{j}^2\binom{n+j}{2j}\sum_{k=j}^{\min\{n,N\}}k\binom{n-j}{n-k}\frac{\binom{2(k-j)}{k-j}}{(-4)^{k-j}}\Big|$$
$$\le\sum_{j=0}^{\min\{n,N\}}4^n\binom{n+j}{2j}\binom{2j}{j}\min\{n,N\}2^{n-j}\le n8^nP_n(2).$$

By the Laplace-Heine formula (cf. [19, p. 194]),

$$P_n(2)\sim\frac{(2+\sqrt{3})^{n+1/2}}{\sqrt{2n\pi}\sqrt[4]{3}}\quad\text{as } n\to+\infty.$$

As $8(2+\sqrt{3})<29.86$, we have $n8^nP_n(2)<30^n$ if n is sufficiently. Recall that $|z|<1/30$.

In view of the above,

$$\begin{aligned}&\lim_{N\to+\infty}\sum_{k=0}^N\frac{z^k}{(1+4z)^k}W_k\left(\frac{1}{1+4z}\right)\\=&\lim_{N\to+\infty}\sum_{n=0}^N z^n\sum_{j=0}^n(-4)^{n-j}\binom{2j}{j}^2\binom{n+j}{2j}\sum_{k=j}^n\binom{n-j}{n-k}\binom{-1/2}{k-j}\\=&\sum_{n=0}^\infty z^n\sum_{j=0}^n(-4)^{n-j}\binom{2j}{j}^2\binom{n+j}{2j}\binom{n-j-1/2}{n-j}\\=&\sum_{n=0}^\infty z^n\sum_{j=0}^n\binom{n+j}{2j}\binom{2j}{j}^2\binom{2(n-j)}{n-j}(-1)^{n-j}\\=&\sum_{n=0}^\infty z^n(-1)^nW_n(-1)=\sum_{n=0}^\infty f_n^{(4)}z^n.\end{aligned}$$

Similarly,

$$\begin{aligned}&\lim_{N\to+\infty}\sum_{k=0}^N\frac{kz^k}{(1+4z)^k}W_k\left(\frac{1}{1+4z}\right)-\sum_{n=0}^\infty nf_n^{(4)}z^n\\=&\lim_{N\to+\infty}\sum_{n=0}^N z^n\sum_{j=0}^n(-4)^{n-j}\binom{2j}{j}^2\binom{n+j}{2j}\sum_{k=j}^n(k-n)\binom{n-j}{n-k}\binom{-1/2}{k-j}\\=&-\sum_{n=0}^\infty z^n\sum_{j=0}^n(-4)^{n-j}\binom{2j}{j}^2\binom{n+j}{2j}(n-j)\sum_{j\le k<n}\binom{n-j-1}{n-k-1}\binom{-1/2}{k-j}\\=&-\sum_{n=0}^\infty z^n\sum_{j=0}^n(-4)^{n-j}(n-j)\binom{n}{j}\binom{n+j}{j}\binom{2j}{j}\binom{n-j-3/2}{n-j-1}\\=&\sum_{n=0}^\infty nz^n\sum_{0\le j<n}4^{n-j}\binom{n-1}{j}\binom{n+j}{j}\binom{2j}{j}\binom{-1/2}{n-j-1}\\=&\sum_{n=0}^\infty nz^n\sum_{0\le j<n}(-1)^{n-j-1}4\binom{n-1}{j}\binom{n+j}{j}\binom{2j}{j}\binom{2(n-j-1)}{n-j-1}.\end{aligned}$$

So we have the desired result. □

Lemma 2.2. *For any $n\in\mathbb{N}$ we have*

$$\begin{aligned}&5n(4n+1)((n+2)s_{n+2}-16ns_n)\\=&(30n^3+54n^2+7n-2)f_{n+1}^{(4)}+2(60n^3+58n^2+17n+2)f_n^{(4)}.\end{aligned}\tag{2.4}$$

Proof. Let u_n denote the left-hand side or the right-hand side of (2.4). Via the Zeilberger algorithm, we find that

$$\begin{aligned}&(1+n)(3+n)^3(5+4n)u_{n+2}\\&\times(344+2572n+8198n^2+13329n^3+10875n^4+4190n^5+600n^6)\\=&2(2+n)(9+4n)P(n)u_{n+1}+4(1+n)(2+n)(3+4n)(5+4n)(9+4n)Q(n)u_n\end{aligned}$$

for all $n=0,1,2,\ldots$, where

$$\begin{aligned}P(n)=&62208+506208n+1799416n^2+3578972n^3+4250502n^4\\&+3104119n^5+1401609n^6+380700n^7+56940n^8+3600n^9\end{aligned}$$

and

$$Q(n)=40108+127005n+164335n^2+110729n^3+40825n^4+7790n^5+600n^6.$$

Note also that $u_0=0$, $u_1=2150$ and $u_2=103680$. As both sides of (2.4) give the same integer sequence $(u_n)_{n\geq0}$, we have (2.4) as desired. □

Now we are able to present an auxiliary theorem.

Theorem 2.3. *Let a,b and x be complex numbers with $|x-1|\geq7.5$. Then*

$$\begin{aligned}&\frac{10}{x}(x-1)^2(x-2)\sum_{n=0}^\infty\frac{an+b}{(4x)^n}W_n\left(1-\frac1x\right)\\=&\sum_{k=0}^\infty(2ax(5x-7)k+a(10x-13)+10b(x-1)(x-2))\frac{f_k^{(4)}}{(4x-4)^k}.\end{aligned}\tag{2.5}$$

Proof. Note that $|1/(4x-4)|\leq1/30$. Applying (2.1) with $z=1/(4x-4)$, we get

$$\sum_{n=0}^\infty\frac{1}{(4x)^n}W_n\left(1-\frac1x\right)=\frac{x}{x-1}\sum_{k=0}^\infty\frac{f_k^{(4)}}{(4x-4)^k}.\tag{2.6}$$

If we have

$$\sum_{n=0}^\infty\frac{n}{(4x)^n}W_n\left(1-\frac1x\right)=\frac{x}{10(x-1)^2(x-2)}\sum_{k=0}^\infty\frac{(10x-14)(kx+1)+1}{(4x-4)^k}f_k,\tag{2.7}$$

then combining (2.6) with (2.7) we immediately get (2.5). The identity (2.7) is equivalent to the following one with $z=1/(4x-4)$:

$$\begin{aligned}&5(1-4z)\sum_{k=0}^\infty\frac{kz^k}{(1+4z)^{k+1}}W_k\left(\frac{1}{1+4z}\right)\\=&\sum_{k=0}^\infty((5-8z)(1+4z)k+4z(5-6z))f_kz^k.\end{aligned}\tag{2.8}$$

Below we prove (2.8) for $|z|\leq1/30$. For convenience, we write $[z^m]f(z)$ with $m\in\mathbb{N}$ to denote the coefficient of z^m in the power series expansion of $f(z)$.

By Lemmas 2.1, for any $n\in\mathbb{N}$ we have

$$\begin{aligned}&[z^{n+1}](1-16z^2)\sum_{k=1}^{\infty}\frac{kz^{k-1}}{(1+4z)^{k+1}}W_k\left(\frac{1}{1+4z}\right)\\=&[z^{n+2}](1-16z^2)\sum_{m=0}^{\infty}m(f_m^{(4)}+4s_m)z^m\\=&(n+2)(f_{n+2}^{(4)}+4s_{n+2})-16n(f_n^{(4)}+4s_n)\\=&(n+2)f_{n+2}^{(4)}-16nf_n^{(4)}+4((n+2)s_{n+2}-ns_n).\end{aligned}$$

Now let $n\in\mathbb{Z}^+$. By the recurrence of $(f_m^{(4)})_{m\geq0}$, we have

$$4n(4n+1)(4n-1)f_{n-1}^{(4)}=(n+1)^3f_{n+1}^{(4)}-2(2n+1)(3n^2+3n+1)f_n^{(4)}$$

and hence

$$\begin{aligned}&n(4n+1)((32n+52)f_{n+1}^{(4)}+(96n+56)f_n^{(4)}-32(4n-1)f_{n-1}^{(4)})\\=&4n(4n+1)(8n+13)f_{n+1}^{(4)}+8n(4n+1)(12n+7)f_n^{(4)}\\&-8(n+1)^3f_{n+1}^{(4)}+16(2n+1)(3n^2+3n+1)f_n^{(4)}\\=&4(30n^3+54n^2+7n-2)f_{n+1}^{(4)}+8(60n^3+58n^2+17n+2)f_n^{(4)}\\=&20n(4n+1)((n+2)s_{n+2}-ns_n)\end{aligned}$$

with the aid of Lemma 2.2. Combining this with the last paragraph, we get

$$\begin{aligned}&[z^{n+1}]5(16z^2-1)\sum_{k=1}^{\infty}\frac{kz^{k-1}}{(1+4z)^{k+1}}W_k\left(\frac{1}{1+4z}\right)\\=&-5(n+2)f_{n+2}^{(4)}+80nf_{n+1}^{(4)}-20((n+2)s_{n+2}-ns_n)\\=&-5(n+2)f_{n+2}^{(4)}+80nf_{n+1}^{(4)}-(32n+52)f_{n+1}^{(4)}\\&-(96n+56)f_n^{(4)}+32(4n-1)f_{n-1}^{(4)}\\=&[z^{n+1}](32z^2-12z-5)\left(4\sum_{k=0}^{\infty}(k+1)f_kz^k+\sum_{k=1}^{\infty}kf_kz^{k-1}\right)\\&-[z^{n+1}](32z^2+8z)\sum_{k=0}^{\infty}f_kz^k\end{aligned}$$

In view of (2.2),

$$\begin{aligned}&5(16z^2-1)\sum_{k=1}^{\infty}\frac{kz^{k-1}}{(1+4z)^{k+1}}W_k\left(\frac{1}{1+4z}\right)\\=&5(16z^2-1)\sum_{m=1}^{\infty}m(f_m^{(4)}+4s_m)z^{m-1}\\=&5(16z^2-1)(6+68z+900z^2+\ldots)=-30-340z-4020z^2-\ldots\end{aligned}$$

Combining this with the final result in the last paragraph, we find that

$$5(16z^2-1)\sum_{k=1}^{\infty}\frac{kz^{k-1}}{(1+4z)^{k+1}}W_k\left(\frac{1}{1+4z}\right)$$
$$=(4z+1)(8z-5)\left(4\sum_{k=0}^{\infty}(k+1)f_kz^k++\sum_{k=1}^{\infty}kf_kz^{k-1}\right)-8z(4z+1)\sum_{k=0}^{\infty}f_kz^k$$

and hence

$$5(4z-1)\sum_{k=1}^{\infty}\frac{kz^{k-1}}{(1+4z)^{k+1}}W_k\left(\frac{1}{1+4z}\right)$$
$$=(8z-5)\left(4\sum_{k=0}^{\infty}(k+1)f_kz^k+\sum_{k=1}^{\infty}kf_kz^{k-1}\right)-8z\sum_{k=0}^{\infty}f_kz^k.$$

This yields the desired (2.8).

The proof of Theorem 2.3 is now complete. □

Proof of Theorem 1.1. In light of Theorem 2.3, we have

$$\sum_{k=0}^{\infty}\frac{45k+8}{40^k}W_k\left(\frac{9}{10}\right)=\frac{1075}{72}\sum_{k=0}^{\infty}\frac{4k+1}{36^k}f_k^{(4)},$$
$$\sum_{k=0}^{\infty}\frac{1360k+389}{(-60)^k}W_k\left(\frac{16}{15}\right)=\frac{9225}{32}\sum_{k=0}^{\infty}\frac{4k+1}{(-64)^k}f_k^{(4)},$$
$$\sum_{k=0}^{\infty}\frac{735k+124}{200^k}W_k\left(\frac{49}{50}\right)=\frac{10125}{784}\sum_{k=0}^{\infty}\frac{60k+11}{196^k}f_k^{(4)},$$
$$\sum_{k=0}^{\infty}\frac{376380k+69727}{(-320)^k}W_k\left(\frac{81}{80}\right)=\frac{5209600}{243}\sum_{k=0}^{\infty}\frac{17k+3}{(-324)^k}f_k,$$
$$\sum_{k=0}^{\infty}\frac{348840k+47461}{1300^k}W_k\left(\frac{324}{325}\right)=\frac{1314625}{243}\sum_{k=0}^{\infty}\frac{65k+9}{1296^k}f_k^{(4)},$$
$$\sum_{k=0}^{\infty}\frac{41673840k+4777111}{5780^k}W_k\left(\frac{1444}{1445}\right)=\frac{147758475}{1444}\sum_{k=0}^{\infty}\frac{408k+47}{5776^k}f_k^{(4)}.$$

By S. Cooper [4],

$$\sum_{k=0}^{\infty}\frac{4k+1}{36^k}f_k^{(4)}=\frac{6\sqrt{15}}{5\pi},\quad \sum_{k=0}^{\infty}\frac{4k+1}{(-64)^k}f_k^{(4)}=\frac{32\sqrt{15}}{45\pi},$$
$$\sum_{k=0}^{\infty}\frac{60k+11}{196^k}f_k^{(4)}=\frac{14\sqrt{7}}{\pi},\quad \sum_{k=0}^{\infty}\frac{17k+3}{(-324)^k}f_k=\frac{81\sqrt{5}}{20\pi},$$
$$\sum_{k=0}^{\infty}\frac{65k+9}{1296^k}f_k^{(4)}=\frac{81\sqrt{2}}{4\pi},\quad \sum_{k=0}^{\infty}\frac{408k+47}{5776^k}f_k^{(4)}=\frac{76\sqrt{95}}{5\pi}.$$

So we have the desired (1.1)-(1.6). This concludes the proof. □

3. Congruences related to the identities (1.1)-(1.15)

In [13, Section 3] the author introduced the polynomials

$$S_n(x)=\sum_{k=0}^n\binom nk^4x^k\quad(n=0,1,2,\ldots)\tag{3.1}$$

and made conjectures on $\sum_{k=0}^{p-1}S_k(x)$ modulo p^2 (with p an odd prime) for each integer x among the numbers

$$1,\ -2,\ \pm4,\ -9,\ 12,\ 16,\ -20,\ 36,\ -64,\ 196,\ -324,\ 1296,\ 5776.$$

See also [17, Conjectures 49-51].

Theorem 1.1 and its proof are actually motivated by our following conjecture.

Conjecture 3.1. *Let p be an odd prime. Then, for any p-adic integer $x\not\equiv0\pmod p$ we have*

$$\sum_{k=0}^{p-1}\frac{1}{(4x)^k}W_k\left(1-\frac1x\right)\equiv\sum_{k=0}^{p-1}S_k(4x-4)\pmod p.\tag{3.2}$$

When

$$x\in\left\{2,\ \pm\frac54,\ \pm4,\ 5,\ 10,\ -15,\ 50,\ -80,\ 325,\ 1445\right\},$$

we have the further congruence

$$\sum_{k=0}^{p-1}\frac{1}{(4x)^k}W_k\left(1-\frac1x\right)\equiv\sum_{k=0}^{p-1}S_k(4x-4)\pmod{p^2}.\tag{3.3}$$

The identity (1.1) is motivated by the following conjecture on related congruences.

Conjecture 3.2. (i) *For any $n\in\mathbb{Z}^+$ we have*

$$\frac{10^{n-1}}{4n}\sum_{k=0}^{n-1}(45k+8)40^{n-1-k}W_k\left(\frac{9}{10}\right)\in\mathbb{Z}^+.$$

(ii) *Let $p\not=2,5$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{45k+8}{40^k}W_k\left(\frac{9}{10}\right)\equiv\frac{p}{16}\left(129\left(\frac{-15}{p}\right)-1\right)\pmod{p^2}.$$

When $(\frac{-15}{p})=1$, for any $n\in\mathbb{Z}^+$ the number

$$\sum_{k=0}^{pn-1}\frac{45k+8}{40^k}W_k\left(\frac{9}{10}\right)-p\sum_{k=0}^{p-1}\frac{45k+8}{40^k}W_k\left(\frac{9}{10}\right)$$

divided by $(pn)^2$ is a p-adic integer.

The identity (1.2) is motivated by the following conjecture on related congruences.

Conjecture 3.3. (i) *For any $n\in\mathbb{Z}^+$ we have*

$$\frac{15^{n-1}}{n}\sum_{k=0}^{n-1}(1360k+389)(-60)^{n-1-k}W_k\left(\frac{16}{15}\right)\in\mathbb{Z}^+.$$

(ii) *Let $p>5$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{1360k+389}{(-60)^k}W_k\left(\frac{16}{15}\right)\equiv\frac{p}{2}\left(779\left(\frac{-15}{p}\right)-1\right)\pmod{p^2}.$$

When $(\frac{-15}{p})=1$, for any $n\in\mathbb{Z}^+$ the number

$$\sum_{k=0}^{pn-1}\frac{1360k+389}{(-60)^k}W_k\left(\frac{9}{10}\right)-p\sum_{k=0}^{p-1}\frac{1360k+389}{(-60)^k}W_k\left(\frac{9}{10}\right)$$

divided by $(pn)^2$ is a p-adic integer.

The identity (1.3) is motivated by the following conjecture on related congruences.

Conjecture 3.4. (i) *For any $n\in\mathbb{Z}^+$ we have*

$$\frac{50^{n-1}}{4n}\sum_{k=0}^{n-1}(735k+124)200^{n-1-k}W_k\left(\frac{49}{50}\right)\in\mathbb{Z}^+.$$

(ii) *Let $p\neq 2,5$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{735k+124}{200^k}W_k\left(\frac{49}{50}\right)\equiv\frac{p}{32}\left(3969\left(\frac{-7}{p}\right)-1\right)\pmod{p^2}.$$

When $(\frac{p}{7})=1$, for any $n\in\mathbb{Z}^+$ the number

$$\sum_{k=0}^{pn-1}\frac{735k+124}{200^k}W_k\left(\frac{49}{50}\right)-p\sum_{k=0}^{p-1}\frac{735k+124}{200^k}W_k\left(\frac{49}{50}\right)$$

divided by $(pn)^2$ is a p-adic integer.

The identity (1.4) is motivated by the following conjecture on related congruences.

Conjecture 3.5. (i) *For any $n\in\mathbb{Z}^+$ we have*

$$\frac{80^{n-1}}{n}\sum_{k=0}^{n-1}(376380k+69727)(-1)^k320^{n-1-k}W_k\left(\frac{81}{80}\right)\in\mathbb{Z}^+.$$

(ii) *Let $p\neq 2,5$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{376380k+69727}{(-320)^k}W_k\left(\frac{81}{80}\right)\equiv\frac{p}{3}\left(209198\left(\frac{-5}{p}\right)-17\right)\pmod{p^2}.$$

When $(\frac{-5}{p})=1$, *for any* $n\in\mathbb{Z}^+$ *the number*

$$\sum_{k=0}^{pn-1}\frac{376380k+69727}{(-320)^k}W_k\left(\frac{81}{80}\right)-p\sum_{k=0}^{p-1}\frac{376380k+69727}{(-320)^k}W_k\left(\frac{81}{80}\right)$$

divided by $(pn)^2$ *is a* p*-adic integer.*

The identity (1.5) is motivated by the following conjecture on related congruences.

Conjecture 3.6. (i) *For any* $n\in\mathbb{Z}^+$ *we have*

$$\frac{325^{n-1}}{n}\sum_{k=0}^{n-1}(348840k+47461)1300^{n-1-k}W_k\left(\frac{324}{325}\right)\in\mathbb{Z}^+,$$

and this number is odd if and only if $n\in\{2^a:\ a\in\mathbb{N}\}$.

(ii) *Let* $p\neq 2,5,13$ *be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{348840k+47461}{1300^k}W_k\left(\frac{324}{325}\right)\equiv\frac{p}{3}\left(142384\left(\frac{-2}{p}\right)-1\right)\pmod{p^2}.$$

When $p\equiv 1,3\pmod 8$, *for any* $n\in\mathbb{Z}^+$ *the number*

$$\sum_{k=0}^{pn-1}\frac{348840k+47461}{1300^k}W_k\left(\frac{324}{325}\right)-p\sum_{k=0}^{p-1}\frac{348840k+47461}{1300^k}W_k\left(\frac{324}{325}\right)$$

divided by $(pn)^2$ *is a* p*-adic integer.*

The identity (1.6) is motivated by the following conjecture on related congruences.

Conjecture 3.7. (i) *For any* $n\in\mathbb{Z}^+$ *we have*

$$\frac{1445^{n-1}}{n}\sum_{k=0}^{n-1}(41673840k+4777111)5780^{n-1-k}W_k\left(\frac{1444}{1445}\right)\in\mathbb{Z}^+,$$

and this number is odd if and only if $n\in\{2^a:\ a\in\mathbb{N}\}$.

(ii) *Let* $p\neq 2,5,17$ *be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{41673840k+4777111}{5780^k}W_k\left(\frac{1444}{1445}\right)\equiv p\left(4777113\left(\frac{-95}{p}\right)-2\right)\pmod{p^2}.$$

When $(\frac{-95}{p})=1$, *for any* $n\in\mathbb{Z}^+$ *the number*

$$\sum_{k=0}^{pn-1}\frac{5928k+253}{5780^k}W_k\left(\frac{1156}{5}\right)-p\sum_{k=0}^{p-1}\frac{5928k+253}{5780^k}W_k\left(\frac{1156}{5}\right)$$

divided by $(pn)^2$ *is a* p*-adic integer.*

The identity (1.7) is motivated by our following conjecture on related congruences (see also [16, A337332]).

Conjecture 3.8. (i) *For any integer $n>1$, we have*

$$\frac{8^{n-1}}{6n}\sum_{k=0}^{n-1}(4k+1)6^{n-1-k}W_k\left(-\frac{1}{8}\right)\in\mathbb{Z}^+.$$

(ii) *For any prime $p>5$, we have*

$$\frac{1}{p}\sum_{k=0}^{p-1}\frac{4k+1}{6^k}W_k\left(-\frac{1}{8}\right)\equiv\begin{cases}(-3)^{(p-1)/4}\pmod{p} & \text{if } p\equiv 1\pmod{12},\\ -5(-3)^{(p-1)/4}\pmod{p} & \text{if } p\equiv 5\pmod{12},\\ -(-3)^{(p+5)/4}\pmod{p} & \text{if } p\equiv 7\pmod{12},\\ -(-3)^{(p+1)/4}\pmod{p} & \text{if } p\equiv 11\pmod{12}.\end{cases}$$

(iii) *For any prime $p>3$, we have*

$$\sum_{k=0}^{p-1}\frac{1}{6^k}W_k\left(-\frac{1}{8}\right)$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } p\equiv 1\pmod{4}\ \&\ p=x^2+4y^2\ (x,y\in\mathbb{Z}),\\ 0\pmod{p^2} & \text{if } p\equiv 3\pmod{4}.\end{cases}$$

Remark 3.1. Sun [18, Identity (I5)] asserts that

$$\sum_{k=0}^{\infty}\frac{6k+1}{256^k}\binom{2k}{k}^2T_k(8,-2)=\frac{2}{\pi}\sqrt{8+6\sqrt{2}},$$

which is similar to (3.1). Similar to Conjecture 3.8, we conjecture that for any integer $n>1$ we have

$$\frac{1}{4n\binom{2n}{n}}\sum_{k=0}^{n-1}(6k+1)256^{n-1-k}\binom{2k}{k}^2T_k(8,-2)\in\mathbb{Z}^+,$$

and that for any odd prime p we have

$$\sum_{k=0}^{p-1}\frac{6k+1}{256^k}\binom{2k}{k}^2T_k(8,-2)\equiv\begin{cases}2^{(p-1)/4}p\pmod{p^2} & \text{if } p\equiv 1\pmod{4},\\ -2^{(p+1)/4}p\pmod{p^2} & \text{if } p\equiv 3\pmod{4}.\end{cases}$$

The identity (1.8) is motivated by the following conjecture on related congruences.

Conjecture 3.9. (i) *For any $n\in\mathbb{Z}^+$ we have*

$$\frac{12^{n-1}}{n}\sum_{k=0}^{n-1}(392k+65)(-1)^k108^{n-1-k}W_k\left(-\frac{49}{12}\right)\in\mathbb{Z}^+.$$

(ii) *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{392k+65}{(-108)^k}W_k\left(-\frac{49}{12}\right)\equiv p\left(86\left(\frac{-3}{p}\right)-21\left(\frac{21}{p}\right)\right)\pmod{p^2}.$$

When $(\frac{p}{7})=1$, *for any* $n\in\mathbb{Z}^+$ *the number*

$$\sum_{k=0}^{pn-1}\frac{392k+65}{(-108)^k}W_k\left(-\frac{49}{12}\right)-p\left(\frac{p}{3}\right)\sum_{k=0}^{p-1}\frac{392k+65}{(-108)^k}W_k\left(-\frac{49}{12}\right)$$

divided by $(pn)^2$ *is a* p*-adic integer.*

(iii) *For any prime* $p>5$, *we have*

$$\sum_{k=0}^{p-1}\frac{1}{(-108)^k}W_k\left(-\frac{49}{12}\right)$$
$$\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } (\frac{-2}{p})=(\frac{p}{3})=(\frac{p}{7})=1\ \&\ p=x^2+42y^2,\\ 2p-8x^2 \pmod{p^2} & \text{if } (\frac{p}{7})=1,\ (\frac{-2}{p})=(\frac{p}{3})=-1\ \&\ p=2x^2+21y^2,\\ 12x^2-2p \pmod{p^2} & \text{if } (\frac{-2}{p})=1,\ (\frac{p}{3})=(\frac{p}{7})=-1\ \&\ p=3x^2+14y^2,\\ 2p-24x^2 \pmod{p^2} & \text{if } (\frac{p}{3})=1,\ (\frac{-2}{p})=(\frac{p}{7})=-1\ \&\ p=6x^2+7y^2,\\ 0 \pmod{p^2} & \text{if } (\frac{-42}{p})=-1,\end{cases}$$

where x *and* y *are integers.*

Remark 3.2. Note that the imaginary quadratic field $\mathbb{Q}(\sqrt{-42})$ has class number four. For primes in the form x^2+dy^2 with $x,y\in\mathbb{Z}$, one my consult the book [7].

The identity (1.9) is motivated by the following conjecture on related congruences.

Conjecture 3.10. (i) *For any* $n\in\mathbb{Z}^+$ *we have*

$$\frac{16^{n-1}}{n}\sum_{k=0}^{n-1}(168k+23)112^{n-1-k}W_k\left(\frac{63}{16}\right)\in\mathbb{Z}^+.$$

(ii) *Let* $p\neq 2,7$ *be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{168k+23}{112^k}W_k\left(\frac{63}{16}\right)\equiv\frac{p}{2}\left(59\left(\frac{-3}{p}\right)-13\left(\frac{21}{p}\right)\right)\pmod{p^2}.$$

When $(\frac{p}{7})=1$, *for any* $n\in\mathbb{Z}^+$ *the number*

$$\sum_{k=0}^{pn-1}\frac{168k+23}{112^k}W_k\left(\frac{63}{16}\right)-p\left(\frac{p}{3}\right)\sum_{k=0}^{p-1}\frac{168k+23}{112^k}W_k\left(\frac{63}{16}\right)$$

divided by $(pn)^2$ *is a* p*-adic integer.*

(iii) *For any prime $p>3$ with $p\neq 7$, we have*

$$\sum_{k=0}^{p-1}\frac{1}{112^k}W_k\left(\frac{63}{16}\right)$$
$$\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } (\frac{-2}{p})=(\frac{p}{3})=(\frac{p}{7})=1\ \&\ p=x^2+42y^2,\\ 2p-8x^2 \pmod{p^2} & \text{if } (\frac{p}{7})=1,\ (\frac{-2}{p})=(\frac{p}{3})=-1\ \&\ p=2x^2+21y^2,\\ 12x^2-2p \pmod{p^2} & \text{if } (\frac{-2}{p})=1,\ (\frac{p}{3})=(\frac{p}{7})=-1\ \&\ p=3x^2+14y^2,\\ 2p-24x^2 \pmod{p^2} & \text{if } (\frac{p}{3})=1,\ (\frac{-2}{p})=(\frac{p}{7})=-1\ \&\ p=6x^2+7y^2,\\ 0 \pmod{p^2} & \text{if } (\frac{-42}{p})=-1,\end{cases}$$

where x and y are integers.

The identity (1.10) is motivated by the following conjecture on related congruences.

Conjecture 3.11. (i) *For any $n\in\mathbb{Z}^+$ we have*

$$\frac{64^{n-1}}{n}\sum_{k=0}^{n-1}(1512k+257)(-1)^k320^{n-1-k}W_k\left(-\frac{405}{64}\right)\in\mathbb{Z}^+.$$

(ii) *Let $p\neq 2,5$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{1512k+257}{(-320)^k}W_k\left(-\frac{405}{64}\right)\equiv\frac{p}{10}\left(2849\left(\frac{-35}{p}\right)-279\left(\frac{5}{p}\right)\right) \pmod{p^2}.$$

When $(\frac{p}{7})=1$, for any $n\in\mathbb{Z}^+$ the number

$$\sum_{k=0}^{pn-1}\frac{1512k+257}{(-320)^k}W_k\left(-\frac{405}{64}\right)-p\left(\frac{p}{5}\right)\sum_{k=0}^{p-1}\frac{1512k+257}{(-320)^k}W_k\left(-\frac{405}{64}\right)$$

divided by $(pn)^2$ is a p-adic integer.

(iii) *For any prime $p>5$, we have*

$$\sum_{k=0}^{p-1}\frac{1}{(-320)^k}W_k\left(-\frac{405}{64}\right)$$
$$\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } (\frac{2}{p})=(\frac{p}{5})=(\frac{p}{7})=1\ \&\ p=x^2+70y^2,\\ 2p-8x^2 \pmod{p^2} & \text{if } (\frac{p}{7})=1,\ (\frac{2}{p})=(\frac{p}{5})=-1\ \&\ p=2x^2+35y^2,\\ 20x^2-2p \pmod{p^2} & \text{if } (\frac{p}{5})=1,\ (\frac{2}{p})=(\frac{p}{7})=-1\ \&\ p=5x^2+14y^2,\\ 28x^2-2p \pmod{p^2} & \text{if } (\frac{2}{p})=1,\ (\frac{p}{5})=(\frac{p}{7})=-1\ \&\ p=7x^2+10y^2,\\ 0 \pmod{p^2} & \text{if } (\frac{-70}{p})=-1,\end{cases}$$

where x and y are integers.

Remark 3.3. Note that the imaginary quadratic field $\mathbb{Q}(\sqrt{-70})$ has class number four.

The identity (1.11) is motivated by the following conjecture on related congruences.

Conjecture 3.12. (i) *For any $n\in\mathbb{Z}^+$ we have*

$$\frac{4^{n-1}}{n}\sum_{k=0}^{n-1}(56k+9)324^{n-1-k}W_k\left(\frac{25}{4}\right)\in\mathbb{Z}^+.$$

(ii) *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{56k+9}{324^k}W_k\left(\frac{25}{4}\right)\equiv\frac{p}{5}\left(49\left(\frac{-35}{p}\right)-4\left(\frac{5}{p}\right)\right)\pmod{p^2}.$$

When $(\frac{p}{7})=1$, for any $n\in\mathbb{Z}^+$ the number

$$\sum_{k=0}^{pn-1}\frac{56k+9}{324^k}W_k\left(\frac{25}{4}\right)-p\left(\frac{p}{5}\right)\sum_{k=0}^{p-1}\frac{56k+9}{324^k}W_k\left(\frac{25}{4}\right)$$

divided by $(pn)^2$ is a p-adic integer.

(iii) *For any prime $p>5$, we have*

$$\sum_{k=0}^{p-1}\frac{1}{324^k}W_k\left(\frac{25}{4}\right)$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2}&\text{if }(\frac{2}{p})=(\frac{p}{5})=(\frac{p}{7})=1\ \&\ p=x^2+70y^2,\\ 2p-8x^2\pmod{p^2}&\text{if }(\frac{p}{7})=1,\ (\frac{2}{p})=(\frac{p}{5})=-1\ \&\ p=2x^2+35y^2,\\ 20x^2-2p\pmod{p^2}&\text{if }(\frac{p}{5})=1,\ (\frac{2}{p})=(\frac{p}{7})=-1\ \&\ p=5x^2+14y^2,\\ 28x^2-2p\pmod{p^2}&\text{if }(\frac{2}{p})=1,\ (\frac{p}{5})=(\frac{p}{7})=-1\ \&\ p=7x^2+10y^2,\\ 0\pmod{p^2}&\text{if }(\frac{-70}{p})=-1,\end{cases}$$

where x and y are integers.

The identity (1.12) is motivated by the following conjecture on related congruences.

Conjecture 3.13. (i) *For any $n>1$ we have*

$$\frac{9^{n-1}}{n}\sum_{k=0}^{n-1}(13000k-1811)(-1)^k1296^{n-1-k}W_k\left(-\frac{625}{9}\right)\in\mathbb{Z}^+,$$

and this number is odd if and only if n is a power of two.

(ii) *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{13000k-1811}{(-1296)^k}W_k\left(-\frac{625}{9}\right)\equiv\frac{p}{5}\left(11882\left(\frac{-39}{p}\right)-20937\right)\pmod{p^2}.$$

When $(\frac{-39}{p})=1$, for any $n\in\mathbb{Z}^+$ the number

$$\sum_{k=0}^{pn-1}\frac{13000k-1811}{(-1296)^k}W_k\left(-\frac{625}{9}\right)-p\sum_{k=0}^{p-1}\frac{13000k-1811}{(-1296)^k}W_k\left(-\frac{625}{9}\right)$$

divided by $(pn)^2$ is a p-adic integer.

(iii) *For any prime $p>5$, we have*

$$\sum_{k=0}^{p-1}\frac{1}{(-1296)^k}W_k\left(-\frac{625}{9}\right)$$
$$\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } (\frac{2}{p})=(\frac{p}{3})=(\frac{p}{13})=1\ \&\ p=x^2+78y^2,\\ 8x^2-2p \pmod{p^2} & \text{if } (\frac{2}{p})=1,\ (\frac{p}{3})=(\frac{p}{13})=-1\ \&\ p=2x^2+39y^2,\\ 12x^2-2p \pmod{p^2} & \text{if } (\frac{p}{13})=1,\ (\frac{2}{p})=(\frac{p}{3})=-1\ \&\ p=3x^2+26y^2,\\ 24x^2-2p \pmod{p^2} & \text{if } (\frac{p}{3})=1,\ (\frac{2}{p})=(\frac{p}{13})=-1\ \&\ p=6x^2+13y^2,\\ 0 \pmod{p^2} & \text{if } (\frac{-78}{p})=-1,\end{cases}$$

where x and y are integers.

Remark 3.4. Note that the imaginary quadratic field $\mathbb{Q}(\sqrt{-78})$ has class number four.

The identity (1.13) is motivated by the following conjecture on related congruences.

Conjecture 3.14. (i) *For any $n>1$ we have*

$$\frac{13^{n-1}}{n}\sum_{k=0}^{n-1}(9360k-1343)1300^{n-1-k}W_k\left(\frac{900}{13}\right)\in\mathbb{Z}^+,$$

and this number is odd if and only if n is a power of two.

(ii) *Let $p\neq 2,5,13$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{9360k-1343}{1300^k}W_k\left(\frac{900}{13}\right)\equiv\frac{p}{5}\left(7944\left(\frac{-39}{p}\right)-14659\right)\pmod{p^2}.$$

When $(\frac{-39}{p})=1$, for any $n\in\mathbb{Z}^+$ the number

$$\sum_{k=0}^{pn-1}\frac{9360k-1343}{1300^k}W_k\left(\frac{900}{13}\right)-p\sum_{k=0}^{p-1}\frac{9360k-1343}{1300^k}W_k\left(\frac{900}{13}\right)$$

divided by $(pn)^2$ is a p-adic integer.

(iii) *For any prime $p>5$, we have*

$$\sum_{k=0}^{p-1}\frac{1}{1300^k}W_k\left(\frac{900}{13}\right)$$
$$\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } (\frac{2}{p})=(\frac{p}{3})=(\frac{p}{13})=1\ \&\ p=x^2+78y^2,\\ 8x^2-2p \pmod{p^2} & \text{if } (\frac{2}{p})=1,\ (\frac{p}{3})=(\frac{p}{13})=-1\ \&\ p=2x^2+39y^2,\\ 12x^2-2p \pmod{p^2} & \text{if } (\frac{p}{13})=1,\ (\frac{2}{p})=(\frac{p}{3})=-1\ \&\ p=3x^2+26y^2,\\ 24x^2-2p \pmod{p^2} & \text{if } (\frac{p}{3})=1,\ (\frac{2}{p})=(\frac{p}{13})=-1\ \&\ p=6x^2+13y^2,\\ 0 \pmod{p^2} & \text{if } (\frac{-78}{p})=-1,\end{cases}$$

where x and y are integers.

The identity (1.14) is motivated by the following conjecture on related congruences.

Conjecture 3.15. (i) *For any $n\in\mathbb{Z}^+$ we have*

$$\frac{361^{n-1}}{n}\sum_{k=0}^{n-1}(56355k+2443)(-1)^k5776^{n-1-k}W_k\left(-\frac{83521}{361}\right)\in\mathbb{Z}^+.$$

(ii) *Let $p\neq 2,19$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{56355k+2443}{(-5776)^k}W_k\left(-\frac{83521}{361}\right)\equiv\frac{7p}{323}\left(426855\left(\frac{-2}{p}\right)-314128\right)\pmod{p^2}.$$

When $p\equiv 1,3\pmod 8$, for any $n\in\mathbb{Z}^+$ the number

$$\sum_{k=0}^{pn-1}\frac{56355k+2443}{(-5776)^k}W_k\left(-\frac{83521}{361}\right)-p\sum_{k=0}^{p-1}\frac{56355k+2443}{(-5776)^k}W_k\left(-\frac{83521}{361}\right)$$

divided by $(pn)^2$ is a p-adic integer.

(iii) *For any prime $p\neq 2,19$, we have*

$$\sum_{k=0}^{p-1}\frac{1}{(-5776)^k}W_k\left(-\frac{83521}{361}\right)$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } (\frac{-2}{p})=(\frac{p}{5})=(\frac{p}{13})=1\ \&\ p=x^2+130y^2,\\ 8x^2-2p\pmod{p^2} & \text{if } (\frac{-2}{p})=1,\ (\frac{p}{5})=(\frac{p}{13})=-1\ \&\ p=2x^2+65y^2,\\ 20x^2-2p\pmod{p^2} & \text{if } (\frac{p}{5})=1,\ (\frac{-2}{p})=(\frac{p}{13})=-1\ \&\ p=5x^2+26y^2,\\ 40x^2-2p\pmod{p^2} & \text{if } (\frac{p}{13})=1,\ (\frac{-2}{p})=(\frac{p}{5})=-1\ \&\ p=10x^2+13y^2,\\ p\delta_{p,17}\pmod{p^2} & \text{if } (\frac{-130}{p})=-1,\end{cases}$$

where x and y are integers.

Remark 3.5. Note that the imaginary quadratic field $\mathbb{Q}(\sqrt{-130})$ has class number four.

The identity (1.15) is motivated by the following conjecture on related congruences.

Conjecture 3.16. (i) *For any $n\in\mathbb{Z}^+$ we have*

$$\frac{5^{n-1}}{n}\sum_{k=0}^{n-1}(5928k+253)5780^{n-1-k}W_k\left(\frac{1156}{5}\right)\in\mathbb{Z}^+,$$

and this number is odd if and only if $n\in\{2^a:\ a\in\mathbb{N}\}$.

(ii) *Let $p\neq 2,5,17$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{5928k+253}{5780^k}W_k\left(\frac{1156}{5}\right)\equiv\frac{p}{85}\left(81744\left(\frac{-2}{p}\right)-60239\right)\pmod{p^2}.$$

When $p\equiv 1,3 \pmod{8}$, for any $n\in\mathbb{Z}^+$ the number

$$\sum_{k=0}^{pn-1}\frac{5928k+253}{5780^k}W_k\left(\frac{1156}{5}\right)-p\sum_{k=0}^{p-1}\frac{5928k+253}{5780^k}W_k\left(\frac{1156}{5}\right)$$

divided by $(pn)^2$ is a p-adic integer.

(iii) *For any prime $p\neq 2,5,17$, we have*

$$\sum_{k=0}^{p-1}\frac{1}{5780^k}W_k\left(\frac{1156}{5}\right)$$

$$\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } (\frac{-2}{p})=(\frac{p}{5})=(\frac{p}{13})=1\ \&\ p=x^2+130y^2,\\ 8x^2-2p \pmod{p^2} & \text{if } (\frac{-2}{p})=1,\ (\frac{p}{5})=(\frac{p}{13})=-1\ \&\ p=2x^2+65y^2,\\ 20x^2-2p \pmod{p^2} & \text{if } (\frac{p}{5})=1,\ (\frac{-2}{p})=(\frac{p}{13})=-1\ \&\ p=5x^2+26y^2,\\ 40x^2-2p \pmod{p^2} & \text{if } (\frac{p}{13})=1,\ (\frac{-2}{p})=(\frac{p}{5})=-1\ \&\ p=10x^2+13y^2,\\ 0 \pmod{p^2} & \text{if } (\frac{-130}{p})=-1,\end{cases}$$

where x and y are integers.

4. A new type series for $1/\pi$ involving generalized central trinomial coefficients

For $b,c\in\mathbb{Z}$ and $n\in\mathbb{N}$ the generalized trinomial coefficient $T_n(b,c)$ denotes the coefficient of x^n in the expansion of $(x^2+bx+c)^n$.

In 2011, the author [12, 14] posed over 60 conjectural series for $1/\pi$ of the following seven types with a,b,c,d,m integers and $mbcd(b^2-4c)$ nonzero.

Type I. $\sum_{k=0}^\infty\frac{a+dk}{m^k}\binom{2k}{k}^2T_k(b,c)$.
Type II. $\sum_{k=0}^\infty\frac{a+dk}{m^k}\binom{2k}{k}\binom{3k}{k}T_k(b,c)$.
Type III. $\sum_{k=0}^\infty\frac{a+dk}{m^k}\binom{4k}{2k}\binom{2k}{k}T_k(b,c)$.
Type IV. $\sum_{k=0}^\infty\frac{a+dk}{m^k}\binom{2k}{k}^2T_{2k}(b,c)$.
Type V. $\sum_{k=0}^\infty\frac{a+dk}{m^k}\binom{2k}{k}\binom{3k}{k}T_{3k}(b,c)$.
Type VI. $\sum_{k=0}^\infty\frac{a+dk}{m^k}T_k(b,c)^3$,
Type VII. $\sum_{k=0}^\infty\frac{a+dk}{m^k}\binom{2k}{k}T_k(b,c)^2$,

Though some of these new families of conjectural series for $1/\pi$ have been proved (see, e.g., [3]), the three conjectual series for $1/\pi$ of type VI and two of type VII remain open.

In a recent published paper [18] the author proposed four conjectural series for $1/\pi$ of a new type:

Type VIII. $\sum_{k=0}^\infty\frac{a+dk}{m^k}T_k(b,c)T_k(b_*,c_*)^2$,

where a,b,b_*,c,c_*,d,m are integers with $mbb_*cc_*d(b^2-4c)(b_*^2-4c_*)(b^2c_*-b_*^2c)\neq 0$.

Here we introduce series for $1/\pi$ involving generalized central trinomial coefficients of the following novel type:

Type IX. $\sum_{k=0}^\infty\frac{a+dk}{m^k}\binom{2k}{k}T_k(b,c)T_k(b_*,c_*)$, where a,b,b_*,c,c_*,d,m are integers with $mbb_*cc_*d(b^2-4c)(b_*^2-4c_*)(b^2c_*-b_*^2c)\neq 0$.

Conjecture 4.1. *We have the following identities:*

$$\sum_{k=0}^\infty\frac{4290k+367}{3136^k}\binom{2k}{k}T_k(14,1)T_k(17,16)=\frac{5390}{\pi}\tag{IX1}$$

and

$$\sum_{k=0}^\infty\frac{540k+137}{3136^k}\binom{2k}{k}T_k(2,81)T_k(14,81)=\frac{98}{3\pi}(10+7\sqrt5).\tag{IX2}$$

The conjectural identity (IX1) is motivated by the author's following conjecture on congruences.

Conjecture 4.2. (i) *For any integer $n>1$, we have*

$$n\binom{2n}{n}\ \Big|\ \sum_{k=0}^{n-1}(4290k+367)3136^{n-1-k}\binom{2k}{k}T_k(14,1)T_k(17,16).$$

(ii) *Let p be an odd prime with $p\neq 7$. Then*

$$\begin{aligned}&\sum_{k=0}^{p-1}\frac{4290k+367}{3136^k}\binom{2k}{k}T_k(14,1)T_k(17,16)\\ \equiv&\frac p2\left(1430\left(\frac{-1}{p}\right)+39\left(\frac{3}{p}\right)-735\right)\pmod{p^2}.\end{aligned}$$

Moreover, when $p\equiv 1\pmod{12}$, for any $n\in\mathbb{Z}^+$ the number

$$\begin{aligned}&\sum_{k=0}^{pn-1}\frac{4290k+367}{3136^k}\binom{2k}{k}T_k(14,1)T_k(17,16)\\ &-p\sum_{k=0}^{n-1}\frac{4290k+367}{3136^k}\binom{2k}{k}T_k(14,1)T_k(17,16)\end{aligned}$$

divided by $(pn)^2\binom{2n}{n}$ is a p-adic integer.

(iii) *For any prime $p>7$, we have*

$$\begin{aligned}&\left(\frac{-1}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{3136^k}T_k(14,1)T_k(17,16)\\ \equiv&\begin{cases}4x^2-2p\pmod{p^2}&\text{if }p\equiv1,4\pmod{15}\ \&\ p=x^2+15y^2\ (x,y\in\mathbb{Z}),\\ 2p-12x^2\pmod{p^2}&\text{if }p\equiv2,8\pmod{15}\ \&\ p=3x^2+5y^2\ (x,y\in\mathbb{Z}),\\ 0\pmod{p^2}&\text{if }(\frac{-15}{p})=-1.\end{cases}\end{aligned}$$

Remark 4.1. Note that the imaginary quadratic field $\mathbb{Q}(\sqrt{-15})$ has class number two.

The conjectural identity (IX2) is motivated by the following conjecture on congruences.

Conjecture 4.3. (i) *For any integer $n>1$, we have*

$$2n\binom{2n}{n}\ \Big|\ \sum_{k=0}^{n-1}(540k+137)3136^{n-1-k}\binom{2k}{k}T_k(2,81)T_k(14,81).$$

(ii) *Let p be an odd prime with $p\neq 7$. Then*

$$\sum_{k=0}^{p-1}\frac{540k+137}{3136^k}\binom{2k}{k}T_k(2,81)T_k(14,81)$$
$$\equiv\frac{p}{3}\left(270\left(\frac{-1}{p}\right)-104\left(\frac{-2}{p}\right)+245\left(\frac{-5}{p}\right)\right)\pmod{p^2}.$$

Moreover, when $p\equiv\pm1,\pm9\pmod{40}$, for any $n\in\mathbb{Z}^+$ the number

$$\sum_{k=0}^{pn-1}\frac{540k+137}{3136^k}\binom{2k}{k}T_k(2,81)T_k(14,81)$$
$$-p\left(\frac{-1}{p}\right)\sum_{k=0}^{n-1}\frac{540k+137}{3136^k}\binom{2k}{k}T_k(2,81)T_k(14,81)$$

divided by $(pn)^2\binom{2n}{n}$ is a p-adic integer.

(iii) *For any prime $p>7$, we have*

$$\left(\frac{-1}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{3136^k}T_k(2,81)T_k(14,81)$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } (\frac{2}{p})=(\frac{p}{3})=(\frac{p}{5})=1\ \&\ p=x^2+15y^2,\\ 8x^2-2p\pmod{p^2} & \text{if } (\frac{2}{p})=1,\ (\frac{p}{3})=(\frac{p}{5})=-1\ \&\ p=2x^2+15y^2,\\ 20x^2-2p\pmod{p^2} & \text{if } (\frac{p}{5})=1,\ (\frac{2}{p})=(\frac{p}{3})=-1\ \&\ p=5x^2+6y^2,\\ 2p-12x^2\pmod{p^2} & \text{if } (\frac{p}{3})=1,\ (\frac{2}{p})=(\frac{p}{5})=-1\ \&\ p=3x^2+10y^2,\\ 0\pmod{p^2} & \text{if } (\frac{-30}{p})=-1,\end{cases}$$

where x and y are integers.

5. Series for $1/\pi$ involving $F_n(x):=\sum_{k=0}^n\binom{n}{k}\binom{n+2k}{2k}\binom{2k}{k}x^{n-k}$

As mentioned in [15, Remark 4.4], an identity of MacMahon implies that the polynomial

$$F_n(x)=\sum_{k=0}^n\binom{n}{k}\binom{n+2k}{2k}\binom{2k}{k}x^{n-k}$$

at $x=-4$ coincides with the Franel number $f_n=\sum_{k=0}^n\binom{n}{k}^3$. Conjecture 4.4 of Sun [15] lists ten conjectural series for $1/\pi$ involving $F_n(x)$ with $x\neq-4$;

eight of them were later confirmed in [6], but the following two remain open:

$$\sum_{k=0}^{\infty}\frac{357k+103}{2160^k}\binom{2k}{k}F_k(-324)=\frac{90}{\pi},\tag{5.1}$$

$$\sum_{k=0}^{\infty}\frac{k}{3645^k}\binom{2k}{k}F_k(486)=\frac{10}{3\pi}.\tag{5.2}$$

Here we pose the following new conjecture.

Conjecture 5.1. *We have the following identities:*

$$\sum_{k=0}^{\infty}\frac{6k+1}{(-1728)^k}\binom{2k}{k}F_k(-324)=\frac{24}{25\pi}\sqrt{375+120\sqrt{10}},\tag{5.3}$$

$$\sum_{k=0}^{\infty}\frac{4k+1}{(-160)^k}\binom{2k}{k}F_k(-20)=\frac{\sqrt{30}}{5\pi}\cdot\frac{5+\sqrt[3]{145+30\sqrt{6}}}{\sqrt[6]{145+30\sqrt{6}}},\tag{5.4}$$

$$\sum_{k=0}^{\infty}\frac{1290k+289}{27648^k}\binom{2k}{k}F_k(-2160)=\frac{96\sqrt{15}}{\pi},\tag{5.5}$$

$$\sum_{k=0}^{\infty}\frac{804k+49}{276480^k}\binom{2k}{k}F_k(12096)=\frac{120\sqrt{15}}{\pi},\tag{5.6}$$

$$\sum_{k=0}^{\infty}(24k+5)\left(\frac{2}{135}\right)^kF_k\left(-\frac{27}{8}\right)=\frac{3}{2\pi}(5\sqrt{6}+4\sqrt{15}).\tag{5.7}$$

Remark 5.1. The author found (5.3)-(5.7) during August 19-27, 2020. As all of them converge quickly, one can easily check them via `Mathematica` or `Maple`.

The identity (5.3) is motivated by [15, Conjecture 4.6] and the following conjecture.

Conjecture 5.2. *Let $n>1$ be an integer. Then*

$$\frac{1}{n\binom{2n}{n}}\sum_{k=0}^{n-1}(-1)^k(6k+1)1728^{n-1-k}\binom{2k}{k}F_k(-324)\in\mathbb{Z}^+,$$

and this number is odd if and only if $n\in\{2^a+1:\ a\in\mathbb{N}\}$.

Remark 5.2. The reader might wonder how we found the right-hand side of the identity (5.3). We thought that the left-hand side of (5.3) times π is an algebraic number and found the form of this algebraic number via calculating its first 100 digits and using the Maple command `identify`.

The identity (5.4) is motivated by the following conjecture on related congruences.

Conjecture 5.3. (i) *Let $n>1$ be an integer. Then*

$$\frac{1}{n\binom{2n}{n}}\sum_{k=0}^{n-1}(-1)^k(4k+1)160^{n-1-k}\binom{2k}{k}F_k(-20)\in\mathbb{Z}^+,$$

and this number is odd if and only if $n\in\{2^a+1:\ a\in\mathbb{N}\}$.

(ii) *For any odd prime p, we have*

$$\left(\frac{p}{5}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{(-160)^k}F_k(-20)$$

$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } p\equiv 1,3\pmod{8}\ \&\ p=x^2+2y^2\ (x,y\in\mathbb{Z}),\\ 0\pmod{p^2} & \text{if } p\equiv 5,7\pmod{8}.\end{cases}$$

Remark 5.3. See [16, A337247] for a sequence related to the first part. Part (i) of this conjecture implies that for any odd prime $p\neq 5$ we have

$$\sum_{k=0}^{p-1}\frac{4k+1}{(-160)^k}\binom{2k}{k}F_k(-20)\equiv 0\pmod{p},$$

which was observed by the author on Jan. 18, 2012.

The identity (5.5) is motivated by the following conjecture on congruences.

Conjecture 5.4. (i) *Let $n>1$ be an integer. Then*

$$\frac{1}{n\binom{2n}{n}}\sum_{k=0}^{n-1}(1290k+289)27648^{n-1-k}\binom{2k}{k}F_k(-2160)\in\mathbb{Z}^+,$$

and this number is odd if and only if $n\in\{2^a+1:\ a\in\mathbb{N}\}$.

(ii) *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{1290k+289}{27648^k}\binom{2k}{k}F_k(-2160)\equiv p\left(104\left(\frac{3}{p}\right)+185\left(\frac{-15}{p}\right)\right)\pmod{p^2}.$$

Moreover, if $(\frac{-5}{p})=1$ then for any $n\in\mathbb{Z}^+$ the number

$$\sum_{k=0}^{pn-1}\frac{1290k+289}{27648^k}\binom{2k}{k}F_k(-2160)-p\left(\frac{3}{p}\right)\sum_{k=0}^{n-1}\frac{1290k+289}{27648^k}\binom{2k}{k}F_k(-2160)$$

divided by $(pn)^2\binom{2n}{n}$ is a p-adic integer.

(iii) *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{27648^k}F_k(-2160)$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2}&\text{if }(\frac{-1}{p})=(\frac{p}{3})=(\frac{p}{5})=(\frac{p}{11})=1,\ p=x^2+165y^2,\\ 2x^2-2p\pmod{p^2}&\text{if }(\frac{-1}{p})=(\frac{p}{3})=(\frac{p}{5})=(\frac{p}{11})=-1,\ 2p=x^2+165y^2,\\ 2p-12x^2\pmod{p^2}&\text{if }(\frac{-1}{p})=(\frac{p}{5})=-1,\ (\frac{p}{3})=(\frac{p}{11})=1,\ p=3x^2+55y^2,\\ 2p-6x^2\pmod{p^2}&\text{if }(\frac{-1}{p})=(\frac{p}{5})=1,\ (\frac{p}{3})=(\frac{p}{11})=-1,\ 2p=3x^2+55y^2,\\ 20x^2-2p\pmod{p^2}&\text{if }(\frac{-1}{p})=(\frac{p}{11})=1,\ (\frac{p}{3})=(\frac{p}{5})=-1,\ p=5x^2+33y^2,\\ 10x^2-2p\pmod{p^2}&\text{if }(\frac{-1}{p})=(\frac{p}{11})=-1,\ (\frac{p}{3})=(\frac{p}{5})=1,\ 2p=5x^2+33y^2,\\ 44x^2-2p\pmod{p^2}&\text{if }(\frac{-1}{p})=(\frac{p}{3})=-1,\ (\frac{p}{5})=(\frac{p}{11})=1,\ p=11x^2+15y^2,\\ 22x^2-2p\pmod{p^2}&\text{if }(\frac{-1}{p})=(\frac{p}{3})=1,\ (\frac{p}{5})=(\frac{p}{11})=-1,\ 2p=11x^2+15y^2,\\ 0\pmod{p^2}&\text{if }(\frac{-165}{p})=-1,\end{cases}$$

where x and y are integers.

Remark 5.4. Note that the imaginary quadratic field $\mathbb{Q}(\sqrt{-165})$ has class number eight.

The identity (5.6) is motivated by the following conjecture on related congruences.

Conjecture 5.5. (i) *Let $n>1$ be an integer. Then*

$$\frac{1}{n\binom{2n}{n}}\sum_{k=0}^{n-1}(804k+49)276480^{n-1-k}\binom{2k}{k}F_k(12096)\in\mathbb{Z}^+,$$

and this number is odd if and only if $n\in\{2^a+1:\ a\in\mathbb{N}\}$.

(ii) *Let $p>5$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{804k+49}{276480^k}\binom{2k}{k}F_k(12096)\equiv p\left(95\left(\frac{-15}{p}\right)-46\left(\frac{30}{p}\right)\right)\pmod{p^2}.$$

Moreover, if $p\equiv 1,3\pmod 8$ then for any $n\in\mathbb{Z}^+$ the number

$$\sum_{k=0}^{pn-1}\frac{804k+49}{276480^k}\binom{2k}{k}F_k(12096)-p\left(\frac{-15}{p}\right)\sum_{k=0}^{n-1}\frac{804k+49}{276480^k}\binom{2k}{k}F_k(12096)$$

divided by $(pn)^2\binom{2n}{n}$ is a p-adic integer.

(iii) *Let $p>5$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{276480^k}F_k(12096)$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2}&\text{if }(\frac{-2}{p})=(\frac{p}{3})=(\frac{p}{5})=(\frac{p}{7})=1\ \&\ p=x^2+210y^2,\\ 8x^2-2p\pmod{p^2}&\text{if }(\frac{-2}{p})=(\frac{p}{7})=1,\ (\frac{p}{3})=(\frac{p}{5})=-1\ \&\ p=2x^2+105y^2,\\ 2p-12x^2\pmod{p^2}&\text{if }(\frac{-2}{p})=(\frac{p}{3})=1,\ (\frac{p}{5})=(\frac{p}{7})=-1\ \&\ p=3x^2+70y^2,\\ 20x^2-2p\pmod{p^2}&\text{if }(\frac{-2}{p})=(\frac{p}{3})=(\frac{p}{5})=(\frac{p}{7})=-1\ \&\ p=5x^2+42y^2,\\ 2p-24x^2\pmod{p^2}&\text{if }(\frac{-2}{p})=(\frac{p}{5})=1,\ (\frac{p}{3})=(\frac{p}{7})=-1\ \&\ p=6x^2+35y^2,\\ 28x^2-2p\pmod{p^2}&\text{if }(\frac{-2}{p})=(\frac{p}{5})=-1,\ (\frac{p}{3})=(\frac{p}{7})=1\ \&\ p=7x^2+30y^2,\\ 40x^2-2p\pmod{p^2}&\text{if }(\frac{-2}{p})=(\frac{p}{7})=-1,\ (\frac{p}{3})=(\frac{p}{5})=1\ \&\ p=10x^2+21y^2,\\ 56x^2-2p\pmod{p^2}&\text{if }(\frac{-2}{p})=(\frac{p}{3})=-1,\ (\frac{p}{5})=(\frac{p}{7})=1\ \&\ p=14x^2+15y^2,\\ 0\pmod{p^2}&\text{if }(\frac{-210}{p})=-1,\end{cases}$$

where x and y are integers.

Remark 5.5. Note that the imaginary quadratic field $\mathbb{Q}(\sqrt{-210})$ has class number eight.

The identity (5.7) is motivated by the following conjecture on related congruences.

Conjecture 5.6. (i) *Let n be any positive integer. Then*

$$\frac{4^{n-1}}{n\binom{2n-1}{n-1}}\sum_{k=0}^{n-1}(24k+5)135^{n-1-k}2^k\binom{2k}{k}F_k\left(-\frac{27}{8}\right)\in\mathbb{Z}^+,$$

and this number is congruent to 5 modulo 8.

(ii) *Let $p>5$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{(24k+5)2^k}{135^k}\binom{2k}{k}F_k\left(-\frac{27}{8}\right)\equiv p\left(95\left(4\frac{-6}{p}\right)+\left(\frac{-15}{p}\right)\right)\pmod{p^2}.$$

Moreover, if $(\frac{10}{p})=1$ then for any $n\in\mathbb{Z}^+$ the number

$$\sum_{k=0}^{pn-1}\frac{(24k+5)2^k}{135^k}\binom{2k}{k}F_k\left(-\frac{27}{8}\right)-p\left(\frac{-6}{p}\right)\sum_{k=0}^{n-1}\frac{(24k+5)2^k}{135^k}\binom{2k}{k}F_k\left(-\frac{27}{8}\right)$$

divided by $(pn)^2\binom{2n}{n}$ is a p-adic integer.

(iii) *Let $p>5$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{2^k\binom{2k}{k}}{135^k}F_k\left(-\frac{27}{8}\right)$$
$$\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } (\frac{2}{p})=(\frac{p}{3})=(\frac{p}{5})=1 \ \&\ p=x^2+30y^2,\\ 8x^2-2p \pmod{p^2} & \text{if } (\frac{2}{p})=1,\ (\frac{p}{3})=(\frac{p}{5})=-1 \ \&\ p=2x^2+15y^2,\\ 2p-12x^2 \pmod{p^2} & \text{if } (\frac{p}{3})=1,\ (\frac{2}{p})=(\frac{p}{5})=-1 \ \&\ p=3x^2+10y^2,\\ 20x^2-2p \pmod{p^2} & \text{if } (\frac{p}{5})=1,\ (\frac{2}{p})=(\frac{p}{3})=-1 \ \&\ p=5x^2+6y^2,\\ 0 \pmod{p^2} & \text{if } (\frac{-30}{p})=-1,\end{cases}$$

where x and y are integers.

Remark 5.6. Note that the imaginary quadratic field $\mathbb{Q}(\sqrt{-30})$ has class number four.

6. One more conjectural series for $1/\pi$ and related congruences

In Jan. 2012 the author (cf. [14, (8)]) conjectured that

$$\sum_{n=0}^{\infty}\frac{28n+5}{576^n}\binom{2n}{n}\sum_{k=0}^{n}\frac{5^k\binom{2k}{k}^2\binom{2(n-k)}{n-k}^2}{\binom{n}{k}}=\frac{9}{\pi}(2+\sqrt{2}), \tag{6.1}$$

which remains open up to now. Here we pose a similar conjecture.

Conjecture 6.1. *We have the following identity:*

$$\sum_{n=0}^{\infty}\frac{182n+31}{576^n}\binom{2n}{n}\sum_{k=0}^{n}\frac{\binom{2k}{k}^2\binom{2(n-k)}{n-k}^2}{\binom{n}{k}}\left(-\frac{25}{16}\right)^k=\frac{189}{2\pi}. \tag{6.2}$$

This is motivated by the author's following conjecture on related congruences.

Conjecture 6.2. *Let $p>3$ be a prime. Then*

$$\sum_{n=0}^{p-1}\frac{182n+31}{576^n}\binom{2n}{n}\sum_{k=0}^{n}\frac{\binom{2k}{k}^2\binom{2n-2k}{n-k}^2}{\binom{n}{k}}\left(-\frac{25}{16}\right)^k$$
$$\equiv\frac{p}{2}\left(63\left(\frac{-1}{p}\right)-1\right) \pmod{p^2}.$$

Also,

$$\sum_{n=0}^{p-1}\frac{\binom{2n}{n}}{576^n}\sum_{k=0}^{n}\frac{\binom{2k}{k}^2\binom{2n-2k}{n-k}^2}{\binom{n}{k}}\left(-\frac{25}{16}\right)^k$$
$$\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } (\frac{p}{7})=1 \ \&\ p=x^2+7y^2\ (x,y\in\mathbb{Z}),\\ 0 \pmod{p^2} & \text{if } (\frac{p}{7})=-1,\ \text{i.e., } p\equiv 3,5,6 \pmod{7}.\end{cases}$$

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