# New Combinatorial Interpretations of the Fibonacci Numbers Squared, Golden Rectangle Numbers, and Jacobsthal Numbers Using Two Types of Tile 

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#### Abstract

We consider the tiling of an $n$-board (a board of size $n \times 1$ ) with squares of unit width and (1,1)-fence tiles. A (1,1)-fence tile is composed of two unit-width square subtiles separated by a gap of unit width. We show that the number of ways to tile an $n$-board using unit-width squares and $(1,1)$-fence tiles is equal to a Fibonacci number squared when $n$ is even and a golden rectangle number (the product of two consecutive Fibonacci numbers) when $n$ is odd. We also show that the number of tilings of boards using $n$ such square and fence tiles is a Jacobsthal number. Using combinatorial techniques we prove identities involving sums of Fibonacci and Jacobsthal numbers in a straightforward way. Some of these identities appear to be new. We also construct and obtain identities for a known Pascal-like triangle (which has alternating ones and zeros along one side) whose ( $n, k$ )th entry is the number of tilings using $n$ tiles of which $k$ are fence tiles. There is a simple relation between this triangle and the analogous one for tilings of an $n$-board. Connections between the triangles and Riordan arrays are also demonstrated. With the help of the triangles, we express the Fibonacci numbers squared, golden rectangle numbers, and Jacobsthal numbers as double sums of products of two binomial coefficients.


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## 1 Introduction

The $(n+1)$ th Fibonacci number ( $\underline{\text { A000045 }}$ ), defined by $F_{n+1}=\delta_{n, 1}+F_{n}+F_{n-1}, F_{n<1}=0$, where $\delta_{i, j}$ is 1 if $i=j$ and zero otherwise, can be interpreted as the number of ways to tile an $n$-board (a board of size $n \times 1$ composed of $1 \times 1$ cells) with $1 \times 1$ squares (henceforth referred to simply as squares) and $2 \times 1$ dominoes [ 4,3$]$. More generally, the number of ways to tile an $n$-board with all the $r \times 1 r$-ominoes from $r=1$ up to $r=k$ is the $k$-step (or $k$-generalized) Fibonacci number $F_{n+1}^{(k)}=\delta_{n, 1}+F_{n}^{(k)}+F_{n-1}^{(k)}+\cdots+F_{n-k+1}^{(k)}$, with $F_{n<1}^{(k)}=0$ [3]. Edwards [5] showed that it is possible to obtain a combinatorial interpretation of the tribonacci numbers (the 3-step Fibonacci numbers, $\mathbf{A 0 0 0 0 7 3}^{\text {) }}$ as the number of tilings of an $n$-board using just two types of tiles, namely, squares and $\left(\frac{1}{2}, 1\right)$-fence tiles. A $(w, g)$-fence tile is composed of two subtiles (called posts) of size $w \times 1$ separated by a gap of size $g \times 1$. We presented a bijection between the Fibonacci numbers squared (A007598) and the tilings of an $n$-board with half-squares (i.e., $\frac{1}{2} \times 1$ tiles always oriented so that the shorter side is horizontal) and ( $\frac{1}{2}, \frac{1}{2}$ )-fence tiles [7] and this was used to formulate combinatorial proofs of various identities $[7,8]$. We also identified a bijection between tiling an $n$-board with $\left(\frac{1}{2}, g\right)$-fence tiles where $g \in\{0,1,2, \ldots\}$ and strongly restricted permutations and then used it to obtain results concerning the permutations [6].

Here we show that the number of ways to tile an $n$-board using square and ( 1,1 )-fence tiles is a Fibonacci number squared if $n$ is even and a golden rectangle number (the product of two successive Fibonacci numbers, $\mathbf{A 0 0 1 6 5 4}^{\text {(f }} n$ is odd. We also consider the number of ways to tile boards using a total of $n$ of these tiles and refer to this as an $n$-tiling. We show that enumerating $n$-tilings yields the Jacobsthal numbers $J_{n \geq 0}=0,1,1,3,5,11,21,43,85,171, \ldots$ (A001045) where the $n$th Jacobsthal number is defined via

$$
\begin{equation*}
J_{n}=\delta_{n, 1}+J_{n-1}+2 J_{n-2}, \quad J_{n<1}=0 \tag{1}
\end{equation*}
$$

We use both types of tiling to formulate straightforward combinatorial proofs of identities involving the Fibonacci numbers squared, golden rectangle numbers, and Jacobsthal numbers. We also obtain two Pascal-like triangles (one for $n$-tilings, the other for tilings of an $n$-board) whose entries are the number of tilings with squares and (1,1)-fences which use a given number of fences. A number of properties of the triangles are derived including their relation to Riordan arrays. Finally, the triangles are used to express the Fibonacci numbers squared, golden rectangle numbers, and Jacobsthal numbers as double sums of a product of two binomial coefficients.

## 2 Tiling boards with squares and fences

When tiling a board with fences it is helpful to first determine the types of metatile since any tiling of the board can be expressed as a tiling using metatiles [5]. A metatile is an arrangement of tiles that exactly covers an integral number of adjacent cells and cannot be split into smaller metatiles $[5,6]$. When tiling with squares $(S)$ and $(1,1)$-fence tiles


Figure 1: An 8-board tiled with the three possible metatiles: a free square (cell 1), a filled fence (cells 2-4), and a bifence (cells 5-8). The symbolic representation of this tiling is SFSFF.
(henceforth referred to simply as fences or $F$ ), the simplest metatile is the square. To tile adjacent cells by starting with a single fence we must fill the gap with either a square or the post of another fence. These generate what we will refer to as the filled fence $(F S)$ and bifence ( $F F$ ) metatiles, respectively (Fig. 1). The bifence clearly has a length of 4. The filled fence has length 3 and the square inside will be called a captured square. A square which is not captured (and is therefore a metatile) is called a free square.

Theorem 1. Let $A_{n}$ be the number of ways to tile an n-board using squares and fences. Then

$$
\begin{equation*}
A_{n}=\delta_{n, 0}+A_{n-1}+A_{n-3}+A_{n-4}, \quad A_{n<0}=0 \tag{2}
\end{equation*}
$$

Proof. We condition on the last metatile $[2,6]$. If the last metatile is of length $l$ there will be $A_{n-l}$ ways to tile the remaining $n-l$ cells. The result (2) follows from the fact that there are three possible metatiles and these have lengths of 1,3 , and 4 . If $n=l$ there is exactly one tiling (which corresponds to that metatile filling the entire board) so we make $A_{0}=1$. There is no way to tile an $n$-board if $n<l$ and so $A_{n<0}=0$.

$$
A_{n \geq 0}=1,1,1,2,4,6,9,15,25,40,64,104,169,273,441,714,1156, \ldots \text { is } \underline{A 006498} \text {. As we }
$$ will shortly prove combinatorially, the even (odd) terms of this sequence are the Fibonacci numbers squared A007598 (golden rectangle numbers A001654).

Lemma 1. There is a bijection between the fence-square tilings of a $2 n$-board (a $(2 n+1)$ board) and the square-domino tilings of an ordered pair of $n$-boards (an ( $n+1$ )-board and an $n$-board).

Proof. Tile an $n$-board (an $(n+1)$-board) with the contents of the odd-numbered cells of the given $2 n$-board (( $2 n+1$ )-board) fence-square tiling and tile a second $n$-board (an $n$-board) with the contents of the even-numbered cells. The posts of any fence (which always lie on two consecutive odd or even cells) get mapped to a domino. The procedure is reversed by splicing the two square-domino tilings.

Theorem 2. For $n \geq 0$,

$$
\begin{align*}
A_{2 n} & =f_{n}^{2}  \tag{3a}\\
A_{2 n+1} & =f_{n} f_{n+1}, \tag{3b}
\end{align*}
$$

where $f_{n}=F_{n+1}$.

Proof. There are $f_{n}$ ways to tile an $n$-board using squares and dominoes [3]. From Lemma 1, $A_{2 n}$ is the same as the number of ways to tile an ordered pair of $n$-boards using squares and dominoes which is $f_{n}^{2}$, and $A_{2 n+1}$ is the same as the number of ways to tile an $n$-board and $(n+1)$-board using squares and dominoes which is $f_{n} f_{n+1}$.

As it is easily done and the result will be used in future work, in the following theorem we generalize Theorem 2 to the case of tiling an $n$-board with squares and ( $1, m-1$ )-fences for some fixed $m \in\{2,3, \ldots\}$.

Theorem 3. If $A_{n}^{(m)}$ is the number of ways to tile an $n$-board using squares and (1, $m-1$ )fences then for $n \geq 0$,

$$
A_{m n+r}^{(m)}=f_{n}^{m-r} f_{n+1}^{r}, \quad r=0, \ldots, m-1,
$$

where $f_{n}=F_{n+1}$.
Proof. We identify the following bijection between the tilings of a $(m n+r)$-board using squares and $(1, m-1)$-fences and the square-domino tilings of an ordered $m$-tuple of $r$ $(n+1)$-boards followed by $m-r n$-boards. For convenience we number the boards in this $m$-tuple from 0 to $m-1$ and the cells in the $(m n+r)$-board from 0 to $m n+r-1$. Tile board $j$ in the $m$-tuple with the contents (taken in order) of the cells of the given $(m n+r)$-board fence-square tiling whose cell number modulo $m$ is $j$. The posts of any ( $1, m-1$ )-fence (which will always lie on two consecutive cells with the same cell number modulo $m$ ) get mapped to a domino in board $j$. The procedure is reversed by splicing the square-domino tilings of the $m$-tuple of boards, hence establishing the bijection. The number of square-domino tilings of the $m$-tuple of boards is $f_{n+1}^{r} f_{n}^{m-r}$ and the result follows.

Theorem 4. If $B_{n}$ is the number of $n$-tilings using squares and fences then

$$
\begin{equation*}
B_{n}=J_{n+1} \tag{4}
\end{equation*}
$$

Proof. As in the proof of Theorem 1, we condition on the last metatile. If the last metatile contains $m$ tiles, there are $B_{n-m}$ possible $(n-m)$-tilings. As the three possible metatiles contain 1,2 , and 2 tiles we have

$$
\begin{equation*}
B_{n}=\delta_{n, 0}+B_{n-1}+2 B_{n-2}, \quad B_{n<0}=0 . \tag{5}
\end{equation*}
$$

where the $\delta_{n, 0}$ is to ensure that $B_{0}=1$ so that when an $n$-tiling is just one metatile we count precisely one tiling. Comparing (5) with (1) gives the result.

## 3 Combinatorial proofs of identities involving the Fibonacci squares and golden rectangle numbers

The proofs of Identities 1,2 , and 3 (and of Identities 6, 7, 8, and 9 in the next section) follow the techniques of Benjamin and Quinn [3].

Identity 1. For $n \geq 0$,

$$
\sum_{j=0}^{n} f_{j}^{2}=f_{n} f_{n+1}
$$

Proof. How many tilings of a $(2 n+1)$-board are there? Answer 1: $A_{2 n+1}=f_{n} f_{n+1}$. Answer 2: condition on the position of the last metatile which is not a bifence. The tiles after this metatile must be bifences and thus occupy $4 m$ cells ( $0 \leq m \leq\lfloor n / 2\rfloor$ ). If the last non-bifence metatile is a square there are $A_{2 n+1-4 m-1}=f_{n-2 m}^{2}$ tilings. If it is a filled fence there are $A_{2 n+1-4 m-3}=f_{n-1-2 m}^{2}$ tilings. Summing over all possible values of $m$ for both cases gives the result.

Identity 2. For $n \geq 0$,

$$
\sum_{j=0}^{2 n-1} f_{j} f_{j+1}=f_{2 n}^{2}-1
$$

Proof. How many tilings of a $4 n$-board use at least 1 square? Answer 1: $A_{4 n}-1=$ $f_{2 n}^{2}-1$ since the only way to tile a $4 n$-board without using squares is to use only bifences. Answer 2: condition on the position of the last metatile which is not a bifence. This gives $A_{4 n-4 m-1}=f_{2 n-2 m} f_{2 n-2 m-1}$ tilings if the last non-bifence metatile is a square and $A_{4 n-4 m-3}=f_{2 n-2 m-1} f_{2 n-2 m-2}$ if it is a filled fence for $0 \leq m<n$. Summing over $m$ for both cases gives the result.

## Identity 3.

$$
\begin{align*}
2 \sum_{j=0}^{n-2} f_{j} f_{j+2}-f_{n-2} f_{n-1} & =f_{n}^{2}-1, & & n>1  \tag{6a}\\
2 \sum_{j=0}^{n-2} f_{j} f_{j+2}+f_{n-1}^{2} & =f_{n} f_{n+1}-1, & & n>0 \tag{6b}
\end{align*}
$$

Proof. How many tilings of an $m$-board use at least 1 fence? Answer 1: $A_{m}-1$ since only the tiling using just squares does not use any fences. Answer 2: condition on the location of the last fence. If the last fence is in a metatile of length $l$ starting at cell $k+1$ (where $0 \leq k \leq m-l)$ there will be $A_{k}$ possible tilings since the cells after the metatile must all be occupied by squares. The two metatiles containing fences are of length 3 and 4 . Summing numbers of tilings over all positions and types of the last fence-containing metatile and equating the two answers gives

$$
2 \sum_{k=0}^{m-4} A_{k}+A_{m-3}=A_{m}-1
$$

Replacing $m$ in this by $2 n$ and $2 n+1$ leads to (6a) and (6b), respectively, after using (3) and $f_{j}^{2}+f_{j} f_{j+1}=f_{j} f_{j+2}$.

The proofs of Identities 4 and 5 (and of Identities 10, 11, and 12 in the next section) follow the techniques used in $[1,8]$. As far as we know, all these identities are new.

Identity 4. For $n \geq 0$,

$$
f_{n} f_{n+1}=1+\lfloor n / 2\rfloor+\sum_{j=1}^{n} j f_{n-j} f_{n-j+1}
$$

Proof. How many tilings of a $(2 n+1)$-board contain at least two squares? Answer 1: $A_{2 n+1}-$ $\frac{1}{2} n-1\left(A_{2 n+1}-\frac{1}{2}(n+1)\right)$ if $n$ is even (odd) since the only possible tilings with less than 2 squares when $n$ is even (odd) is one free square (filled fence) among $n / 2\left(\frac{1}{2}(n-1)\right)$ bifences and there are $\frac{1}{2} n-1\left(\frac{1}{2}(n+1)\right)$ such tilings. Answer 2: condition on the location of the second square. The metatile containing this must end on an even cell, $2 j$. Written in terms of symbols (see the caption to Fig. 1), the tiling of the first $2 j$ cells must end in $S$. This leaves one $S$ that may be placed anywhere among the $F$ symbols which number $j-1$. The number of ways to tile the cells to the right of the $2 j$ th cell is $A_{2 n+1-2 j}$. Summing over all possible $j$ gives $\sum_{j=1}^{n} j A_{2(n-j)+1}$. Equating this to Answer 1 and simplifying gives

$$
A_{2 n+1}-\lfloor n / 2\rfloor-1=\sum_{j=1}^{n} j A_{2(n-j)+1} .
$$

The identity follows from (3b).
Note that if we consider the tilings of a $2 n$-board that contain at least two squares we obtain Identity 2.1 of [8].

To generalize Identity 4 we first define $C_{n}^{(r)}$ as the number of ways to tile a $(2 n+1)$-board using $2 r+1$ squares (and $n-r$ fences).

Lemma 2. For $n \geq r \geq 0$,

$$
\begin{equation*}
C_{n}^{(r)}=C_{n-2}^{(r)}+\binom{n+r}{2 r}, \quad C_{n<0}^{(r)}=0 \tag{7}
\end{equation*}
$$

Proof. In symbolic form, a tiling can end in either $S$ or $F F$. If $S$, the number of ways to place the remaining $2 r$ squares and $n-r$ fences is $\binom{n+r}{2 r}$. If $F F$, there are $C_{n-2}^{(r)}$ ways to place the remaining tiles. There are no tilings if $n<0$.

As will be shown in Theorem $6, C_{n}^{(r)}$ is the $(n, r)$ th element of the $(1 /[(1-x)(1-$ $\left.\left.\left.x^{2}\right)\right], x /(1-x)^{2}\right)$ Riordan array (A158909).

Identity 5. For $p>0$,

$$
f_{n} f_{n+1}=\sum_{r=0}^{p-1} C_{n}^{(r)}+\sum_{j=p}^{n}\binom{j+p-1}{2 p-1} f_{n-j} f_{n+1-j}
$$

Proof. How many tilings of a $(2 n+1)$-board have at least $2 p$ squares? Answer 1: the total number of tilings minus the tilings that contain less than $2 p$ squares, i.e.,

$$
A_{2 n+1}-\sum_{r=0}^{p-1} C_{n}^{(r)}
$$

Answer 2: we condition on the location of the $2 p$ th square. If the metatile containing this lies on the $2 j$ th cell, in the symbolic representation, there are $2 p-1 S$ and $j-p F$ that precede the $2 p$ th $S$ and hence $\binom{j+p-1}{2 p-1}$ ways to arrange them. There are $A_{2 n+1-2 j}$ ways to place the remaining tiles after the $2 j$ th cell. Summing over all possible $j$ and equating the result to Answer 1 gives

$$
A_{2 n+1}-\sum_{r=0}^{p-1} C_{n}^{(r)}=\sum_{j=p}^{n}\binom{j+p-1}{2 p-1} A_{2(n-j)+1}
$$

and the identity follows from (3b).

## 4 Combinatorial proofs of identities involving the Jacobsthal numbers

Identity 6. For $n \geq 0$,

$$
2 \sum_{r=1}^{n} J_{r}=J_{n+2}-1
$$

Proof. How many $(n+1)$-tilings use at least 1 fence? Answer 1: $B_{n+1}-1$. Answer 2 : condition on the last metatile containing a fence. If this last metatile contains the $(j+1)$ th and $(j+2)$ th tiles $(0 \leq j \leq n-1)$ then there remain $j$ unspecified tiles. As there are two types of metatile containing a fence there are a total of $2 B_{j}$ tilings for each $j$. Summing over $j$ we have

$$
\sum_{j=0}^{n-1} 2 B_{j}=B_{n+1}-1
$$

and the identity follows from Theorem 4.
Identity 7. For $n \geq 0$,

$$
\sum_{r=1}^{2 n} J_{r}=J_{2 n+1}-1
$$

Proof. How many $2 n$-tilings use at least 1 square? Answer 1: $B_{2 n}-1$ since there is only one $2 n$-tiling without a square (the all-bifence tiling). Answer 2: condition on the last square which must be the $2(n-m)$ th tile $(0 \leq m \leq n-1)$ since any metatiles after the last square
are bifences. If this last square is a free square there are $2 n-2 m-1$ remaining unspecified tiles. There are $2 n-2 m-2$ if it is inside a filled fence. Summing over all possible $m$ and equating to Answer 1 gives

$$
B_{2 n}-1=\sum_{r=0}^{2 n-1} B_{r}
$$

The identity follows from (3b).
Identity 8. For $n \geq 0$,

$$
\sum_{r=1}^{2 n-1} J_{r}=J_{2 n}
$$

Proof. The proof follows that for Identity 7 but we count the number of $(2 n-1)$-tilings by conditioning on the last square.

Identity 9. For $m, n \geq 0$,

$$
J_{m+n+1}=J_{m+1} J_{n+1}+2 J_{m} J_{n}
$$

Proof. The number of $(m+n)$-tilings is $B_{m+n}$. Of these there are $B_{m} B_{n}$ tilings where the $m$ th tile is the last tile in a metatile. If the $m$ th tile is the first tile in a metatile containing two tiles, there are $m-1$ unspecified tiles before it and $n-1$ unspecified tiles after the metatile. As there are two kinds of two-tile metatiles we have $B_{m+n}=B_{m} B_{n}+2 B_{m-1} B_{n-1}$. The identity then follows from (3b).

Identity 10. For $n \geq 0$,

$$
J_{n+1}=\left\lceil\frac{1}{2}(n+1)\right\rceil+\sum_{j=1}^{n-1} j J_{n-j} .
$$

Proof. How many $n$-tilings have at least two squares? Answer 1: $B_{n}-\frac{1}{2}(n+1)\left(B_{n}-\frac{1}{2} n-1\right)$ when $n$ is odd (even) since the possible tilings with one square when $n$ is odd (even) are one filled fence (free square) placed among $\frac{1}{2}(n-1)\left(\frac{1}{2}(n-2)\right)$ bifences and there are $\frac{1}{2}(n+1)$ ( $n / 2$ ) such tilings, and the only possible tiling with no squares is the all-bifence tiling which only occurs when $n$ is even. Answer 2: condition on the second metatile containing an $S$. The symbolic representation of the tiling up to and including this must end in an $S$. If this $S$ is the $j$ th tile, there are $j-1$ ways to order the symbols preceding it and thus $(j-1) B_{n-j}$ $n$-tilings. Summing over all possible $j$, equating to Answer 1, and simplifying gives

$$
B_{n}-\left\lceil\frac{1}{2}(n+1)\right\rceil=\sum_{j=2}^{n}(j-1) B_{n-j}
$$

The identity is obtained on replacing $j$ by $j+1$ and using Theorem 4.

As before, we can generalize Identity 10 . We need the following definition and lemma. Let $D_{n}^{(r)}$ be the number of $n$-tilings that contain exactly $r$ squares. As the only tilings with no squares are the all-bifence tilings, for $n>0, D_{n}^{(0)}$ is $1(0)$ when $n$ is even (odd). For convenience we make $D^{(0)}(0)=1$.

Lemma 3. For $n \geq r>0$,

$$
\begin{equation*}
D_{n}^{(r)}=D_{n-2}^{(r)}+\binom{n-1}{r-1} \tag{8}
\end{equation*}
$$

Proof. The symbolic representation of a tiling must end in either $S$ or $F F$. If $S$, we are free to place the remaining $n-1$ tiles (of which $r-1$ are squares) in any order; this gives $\binom{n-1}{r-1}$ possibilities. If $F F$, there are $D_{n-2}^{(r)}$ ways to place the remaining tiles.

As will be shown in Theorem $5, D_{n}^{(r)}$ is the $\left(1 /\left(1-x^{2}\right), x /(1-x)\right)$ Riordan array ( $\underline{\text { A } 059260)}$ ).
Identity 11. For $p>0$,

$$
J_{n+1}=\sum_{r=0}^{p-1} D_{n}^{(r)}+\sum_{k=p}^{n}\binom{k-1}{p-1} J_{n+1-k}
$$

Proof. How many $n$-tilings have at least $p$ squares? Answer 1: the total number of tilings minus the tilings that contain less than $p$ squares, i.e.,

$$
B_{n}-\sum_{r=0}^{p-1} D_{n}^{(r)}
$$

Answer 2: we condition on the location of the $p$ th square. If it is the $k$ th tile, there are $\binom{k-1}{p-1}$ ways to place the first $k$ tiles and $B_{n-k}$ ways to place the remaining tiles. Summing over all possible $k$ and equating the result to Answer 1 gives

$$
B_{n}-\sum_{r=0}^{p-1} D_{n}^{(r)}=\sum_{k=p}^{n}\binom{k-1}{p-1} B_{n-k}
$$

and the identity follows from Theorem 4.
Identity 12. For $n>0$,

$$
J_{n+1}=n+J_{n-1}+\sum_{k=3}^{n}(2 k-5) J_{n+1-k}
$$

Proof. For $n>0$, how many $n$-tilings have at least two fences? Answer 1: $B_{n}-1-(n-2+1)$ since only the all-square tiling and tilings with 1 filled fence among $n-2$ squares have less than two fences. Answer 2: condition on the location of the second fence. If it is the $k$ th tile
( $k=3, \ldots, n-1$ ) and part of a filled fence or the first tile in a bifence, the first fence is part of a filled fence among $k-3$ squares and hence there are $2(k-2) B_{n-(k+1)}$ tilings for these cases. If the second fence is the end of bifence and is the $k$ th tile $(k=2, \ldots, n)$, the tiles before the bifence are all squares and hence there are $B_{n-k}$ tilings in this case. Summing over all possible $k$, changing $k$ to $k-1$ in the first sum, and equating to Answer 1 gives

$$
B_{n}-n=2 \sum_{k=4}^{n}(k-3) B_{n-k}+\sum_{k=2}^{n} B_{n-k}=B_{n-2}+\sum_{k=3}^{n}(2 k-5) B_{n-k}
$$

The identity then follows from Theorem 4.
Identity 13. For $n \geq 0$,

$$
J_{n+1}=F_{n+1}+\sum_{j=2}^{n} J_{j-1} F_{n+1-j}
$$

Proof. First note that the number of $n$-tilings with no bifences is given by $S_{n}=\delta_{0, n}+$ $S_{n-1}+S_{n-2}$ and hence $S_{n}=F_{n+1}$. How many $n$-tilings have at least one bifence? Answer 1 : $B_{n}-S_{n}$. Answer 2: condition on the last bifence. When the second fence it contains is the $j$ th tile $(j=2, \ldots, n)$ then the number of tilings is $B_{j-2} S_{n-j}$. Summing over all possible $j$ and equating this to Answer 1 gives

$$
B_{n}-S_{n}=\sum_{j=2}^{n} B_{j-2} S_{n-j}
$$

The identity follows from applying $S_{n}=F_{n+1}$ and Theorem 4 .

## 5 A Pascal-like triangle giving the number of $n$-tilings using $k$ fences

We define $\left\langle\begin{array}{l}n \\ k\end{array}\right\rangle$ as the number of $n$-tilings which contain exactly $k$ fences. We define $\left\langle\begin{array}{l}0 \\ 0\end{array}\right\rangle=1$ so that the result

$$
B_{n}=\sum_{k=0}^{n}\left\langle\begin{array}{l}
n  \tag{9}\\
k
\end{array}\right\rangle
$$

is valid for $n \geq 0$. The first 12 rows of the triangle whose entries are $\left\langle\begin{array}{l}n \\ k\end{array}\right\rangle$ are shown in Figure 2. As will be shown later via its connection with a Riordan array, the triangle is sequence A059259.

Identity 14. For $n \geq 0$,

$$
\left\langle\begin{array}{l}
n \\
0
\end{array}\right\rangle=1
$$

Proof. There is only one way to tile without using any fences.

| $n \backslash k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| :---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| 0 | 1 |  |  |  |  |  |  |  |  |  |  |  |  |
| 1 | 1 | 0 |  |  |  |  |  |  |  |  |  |  |  |
| 2 | 1 | 1 | 1 |  |  |  |  |  |  |  |  |  |  |
| 3 | 1 | 2 | 2 | 0 |  |  |  |  |  |  |  |  |  |
| 4 | 1 | 3 | 4 | 2 | 1 |  |  |  |  |  |  |  |  |
| 5 | 1 | 4 | 7 | 6 | 3 | 0 |  |  |  |  |  |  |  |
| 6 | 1 | 5 | 11 | 13 | 9 | 3 | 1 |  |  |  |  |  |  |
| 7 | 1 | 6 | 16 | 24 | 22 | 12 | 4 | 0 |  |  |  |  |  |
| 8 | 1 | 7 | 22 | 40 | 46 | 34 | 16 | 4 | 1 |  |  |  |  |
| 9 | 1 | 8 | 29 | 62 | 86 | 80 | 50 | 20 | 5 | 0 |  |  |  |
| 10 | 1 | 9 | 37 | 91 | 148 | 166 | 130 | 70 | 25 | 5 | 1 |  |  |
| 11 | 1 | 10 | 46 | 128 | 239 | 314 | 296 | 200 | 95 | 30 | 6 | 0 |  |
| 12 | 1 | 11 | 56 | 174 | 367 | 553 | 610 | 496 | 295 | 125 | 36 | 6 | 1 |

Figure 2: A Pascal-like triangle with entries $\left\langle\begin{array}{l}n \\ k\end{array}\right\rangle$ (A059259).

Identity 15. For $n \geq 1$,

$$
\left\langle\begin{array}{c}
n \\
1
\end{array}\right\rangle=n-1 .
$$

Proof. If only one of the $n$ tiles is a fence, there must be 1 filled fence and $n-2$ free squares making a total of $n-1$ metatile positions. The filled fence can be placed in any of these.

The following two identities describe, respectively, the entries in the first and second diagonal of the triangle.

Identity 16. For $n \geq 0$,

$$
\left\langle\begin{array}{l}
n \\
n
\end{array}\right\rangle= \begin{cases}1, & n \text { even } \\
0, & n \text { odd }\end{cases}
$$

Proof. An all-fence tiling must be composed of just bifences. This can only occur if the number of tiles is even.

Identity 17. For $m>0$,

$$
\left\langle\begin{array}{l}
2 m-1 \\
2 m-2
\end{array}\right\rangle=\left\langle\begin{array}{c}
2 m \\
2 m-1
\end{array}\right\rangle=m
$$

Proof. If there are $2 m-1$ or $2 m$ fences and 1 square, there must be $m-1$ bifences. The remaining metatile is then, respectively, a free square or a filled fence. There are $m$ possible positions for this remaining metatile.

The following identity shows that the third diagonal of the triangle is $\mathbf{A 0 0 2 6 2 0}$.

Identity 18. For $m>0$,

$$
\left\langle\begin{array}{c}
2 m \\
2 m-2
\end{array}\right\rangle=m^{2} ; \quad\left\langle\begin{array}{l}
2 m+1 \\
2 m-1
\end{array}\right\rangle=m(m+1)
$$

Proof. When 2 out of $2 m$ tiles are squares there must be either $m-1$ bifences and 2 free squares (totalling $m+1$ metatile positions) or $m-2$ bifences and 2 filled fences (giving $m$ metatile positions). There are $\binom{m+1}{2}$ places to put the squares in the first case and $\binom{m}{2}$ ways to place the filled fences in the second. The total number of tilings is thus $\binom{m}{2}+\binom{m+1}{2}=m^{2}$. When 2 out of $2 m+1$ tiles are squares, there must be $m-1$ bifences, 1 filled fence, and 1 free square, and thus $m+1$ metatile positions. There are therefore $2\binom{c+1}{2}=m(m+1)$ ways to place the free square and filled fence.

The following two identities show that the third and fourth columns of the triangle are $\underline{\mathrm{A} 000124}$ and $\underline{\text { A003600 }}$, respectively.

Identity 19. For $n \geq 2$,

$$
\left\langle\begin{array}{l}
n \\
2
\end{array}\right\rangle=\binom{n-2}{2}+n-1 .
$$

Proof. If there are 2 fences, there are either 2 filled fences or 1 bifence. In the first case there are $n-4$ free squares and hence a total of $n-2$ metatile positions in which to place the filled fences. There are thus $\binom{n-2}{2}$ ways to place the filled fences. In the second case there are $n-2$ free squares and thus $n-1$ metatile positions in which the bifence can be placed.

Identity 20. For $n \geq 3$,

$$
\left\langle\begin{array}{l}
n \\
3
\end{array}\right\rangle=\binom{n-3}{3}+2\binom{n-2}{2} .
$$

Proof. If there are 3 fences, there are either 3 filled fences or 1 bifence and 1 filled fence. In the first case there are $n-6$ free squares and 3 filled fences giving a total $n-3$ metatile positions to place the filled fences. In the second case there are $n-4$ free squares and thus $n-2$ metatile positions to place the filled fence and bifence.

Identity 21. For $n \geq k>0$,

$$
\binom{n}{k}=\left\langle\begin{array}{l}
n  \tag{10}\\
k
\end{array}\right\rangle+\left\langle\begin{array}{l}
n-1 \\
k-1
\end{array}\right\rangle .
$$

Proof. Interpret $\binom{n}{k}$ as the tilings of an $(n+k)$-board with $k$ dominoes $(D)$ and $n-k$ squares $(S)$. Proceeding from left to right along the board, replace $D D$ by a bifence, $D S$ by a filled fence, and then leave any of the remaining $S$ as they are. Except for the case of a 'left over' single $D$ at the right end of the board, this generates all possible $n$-tilings using $k$ fences. If the $(n+k)$-board ends in an isolated $D$, ignore it and hence obtain a $(n-1)$-tiling with $k-1$ fences. In both cases the scheme is reversible.

Identity 22. For $n>k>0$,

$$
\left\langle\begin{array}{l}
n  \tag{11}\\
k
\end{array}\right\rangle=\left\langle\begin{array}{c}
n-1 \\
k
\end{array}\right\rangle+\left\langle\begin{array}{l}
n-1 \\
k-1
\end{array}\right\rangle .
$$

Proof. An $n$-tiling such that $n>k$ must contain a free square or filled fence. Construct a bijection between $n$-tilings using $k$ fences and ( $n-1$ )-tilings using $k$ or $k-1$ fences as follows. In the $n$-tiling find the final square. If it is free, remove it to obtain an $(n-1)$-tiling with $k$ fences. If the square is part of a filled fence, remove the fence to obtain an ( $n-1$ )-tiling with $k-1$ fences.

Identity 23. For $n \geq r \geq 0,\left\langle{ }_{n}{ }^{n}-r\right\rangle=D_{n}^{(r)}$.
Proof. The result follows from the definition of $D_{n}^{(r)}$ since $\left\langle{ }_{n-r}^{n}\right\rangle$ is also the number of $n$-tilings containing $r$ squares.

Identity 24.

$$
\left\langle\begin{array}{l}
n  \tag{12}\\
k
\end{array}\right\rangle=\delta_{n, 0} \delta_{k, 0}+\left\langle\begin{array}{c}
n-1 \\
k
\end{array}\right\rangle+\left\langle\begin{array}{l}
n-2 \\
k-1
\end{array}\right\rangle+\left\langle\begin{array}{c}
n-2 \\
k-2
\end{array}\right\rangle .
$$

Proof. We count $\left\langle\begin{array}{l}n \\ k\end{array}\right\rangle$ by conditioning on the last metatile on the board. If the metatile contains $m$ tiles of which $j$ are fences, for the remaining tiles the number of $(n-m)$-tilings is $\left\langle\begin{array}{c}n-m \\ k-j\end{array}\right\rangle$. Summing these for the three types of metatile gives the result.

A $(p(x), q(x))$ Riordan array is a lower triangular matrix whose $(n, k)$ th entry is the coefficient of $x^{n}$ in the series for $p(x)\{q(x)\}^{k}[9]$.

Theorem 5. If $R(n, k)$ is the $(n, k)$ th entry of the $\left(1 /\left(1-x^{2}\right), x /(1-x)\right)$ Riordan array then

$$
\left\langle\begin{array}{l}
n  \tag{13}\\
k
\end{array}\right\rangle=R(n, n-k)
$$

Proof. Let $p=1 /\left(1-x^{2}\right), q=x /(1-x)$. Then $R(n-l, k-j)$ is the coefficient of $x^{n}$ in the expansion of $x^{l} p q^{k-j}$. Multiplying the identity $q=x+x^{2}+x^{2} q$ by $p q^{k-1}$ and taking the coefficient of $x^{n}$ gives $R(n, k)=R(n-1, k-1)+R(n-1, k-1)+R(n-2, k-1)+R(n-2, k)$ for $n>2, k>0$. Taking $R(n<0, k)=R(n<k, k)=0$ and including terms to arrive at a relation that is also compatible with the values of $R(k, n)$ for $0 \leq n \leq 2$ and $k=0$ gives

$$
\begin{equation*}
R(n, k)=\delta_{n, 0} \delta_{k, 0}+R(n-1, k-1)+R(n-1, k-1)+R(n-2, k-1)+R(n-2, k) \tag{14}
\end{equation*}
$$

which is then valid for all $n$ and $k$. Substituting (13) into (12), replacing $k$ by $n-k$, and noting that $\delta_{n, 0} \delta_{n-k, 0}$ can be rewritten as $\delta_{n, 0} \delta_{k, 0}$, gives (14).

From Identity 23, $R(n, k)=D_{n}^{(k)}$. In other words, a combinatorial interpretation of $R(n, k)$ is the number of $n$-tilings that use $k$ squares (and $n-k(1,1)$-fences). Then from Lemma 3 we have for $n \geq k \geq 0$,

$$
\begin{equation*}
R(n, k)=R(n-2, k)+\binom{n-1}{k-1} \tag{15}
\end{equation*}
$$

This allows us to prove a conjecture given in the OEIS entry for A059259 concerning A071921 which is the square array $a(n, m)$ given by $a(0, m \geq 0)=1$,

$$
\begin{equation*}
a(n, m)=\sum_{r=0}^{m-1}\binom{n-1+2 r}{n-1} . \tag{16}
\end{equation*}
$$

Using our notation, the conjecture is as follows.

## Identity 25.

$$
\left\langle\begin{array}{c}
n+2 m \\
2 m
\end{array}\right\rangle=a(n, m+1)
$$

Proof. From Theorem 5, $\left\langle\begin{array}{c}n+2 m \\ 2 m\end{array}\right\rangle=R(n+2 m, n)$. Repeatedly applying (8) gives

$$
R(n+2 m, m)=\binom{n-1+2 m}{n-1}+\binom{n-1+2(m-1)}{n-1}+\cdots+\binom{n-1+2}{n-1}+R(n, n)
$$

Using the fact that $R(n, n)=1$ the result follows from (16).
If $b, f$, and $s$ are, respectively, the numbers of bifences, filled fences, and free squares in an $n$-tiling using $k$ fences then it is easily seen that

$$
\begin{align*}
n & =2 b+2 f+s  \tag{17a}\\
k & =2 b+f \tag{17b}
\end{align*}
$$

Identity 26.

$$
\left\langle\begin{array}{l}
n  \tag{18}\\
k
\end{array}\right\rangle= \begin{cases}\sum_{b=b_{\min }}^{b_{\max }}\binom{n-k+b}{k-b}\binom{k-b}{b}, & b_{\min } \leq b_{\max } \\
0, & b_{\min }>b_{\max }\end{cases}
$$

where $b_{\text {min }}=\max (0,\lceil k-n / 2\rceil)$ and $b_{\max }=\lfloor k / 2\rfloor$.
Proof. For given values of $n$ and $k$ we sum the number of tilings for all possible values of $b$. The maximum number of bifences $b_{\text {max }}$ is obtained from (17b) when $f$ is 0 or 1 depending on whether $k$ is even or odd, respectively. Eliminating $f$ from (17) gives

$$
b=\frac{1}{2}(2 k-n+s) .
$$

If $2 k-n$ is negative, the minimum possible value of $b$ is zero. Otherwise $b_{\min }$ is obtained when $s$ is 0 or 1 when $2 k-n$ is even or odd, respectively. From (25) we have that the total number of metatiles, $b+f+s=n-k+b$. The number of ways of tiling using $b$ bifences, $f$ filled fences, and $s$ free squares is the multinomial coefficient $\binom{b+f+s}{b, f, s}$ which may be re-expressed as a product of binomial coefficients written in terms of $b$, $n$, and $k$. There will be no possible values of $b$ and therefore no tilings if $b_{\min }>b_{\max }$.

## Corollary 1.

$$
J_{n+1}=\sum_{k=0}^{n} \sum_{b=\max (0,\lceil k-n / 2\rceil)}^{\lfloor k / 2\rfloor}\binom{n-k+b}{k-b}\binom{k-b}{b} .
$$

Proof. The result follows from (9), Theorem 4, and Identity 26.
The ( $n, k$ ) th entry, which we will denote here by $\left[\begin{array}{c}n \\ k\end{array}\right]_{1 / 2}$, of the Pascal-like triangle A123521 is the number of ways to tile an $n$-board using $k\left(\frac{1}{2}, \frac{1}{2}\right)$-fences and $2(n-k)$ half-squares (with the shorter sides always horizontal) [8]. We now show that the $\left[\begin{array}{c}n \\ k\end{array}\right]_{1 / 2}$ triangle can be obtained from the $\left\langle\begin{array}{l}n \\ k\end{array}\right\rangle$ triangle by removing the odd downward diagonals of the latter which is equivalent to the following identity.

Identity 27. For $n \geq k \geq 0$,

$$
\left[\begin{array}{l}
n \\
k
\end{array}\right]_{1 / 2}=\left\langle\begin{array}{c}
2 n-k \\
k
\end{array}\right\rangle .
$$

Proof. The total post length of a $\left(\frac{1}{2}, \frac{1}{2}\right)$-fence is 1 . The entry $\left[\begin{array}{c}n \\ k\end{array}\right]_{1 / 2}$ can also be viewed as counting the number of tilings that use $k\left(\frac{1}{2}, \frac{1}{2}\right)$-fences and $2(n-k)$ half-squares since the total length occupied by the $n$ tiles is $k+2(n-k) \frac{1}{2}=n$. The entry $\left\langle\begin{array}{c}2 n-k \\ k\end{array}\right\rangle$ counts the number of tilings using $k(1,1)$-fences and $2(n-k)$ squares. This latter tiling differs from the former only in that the tiles are twice the length.

## 6 A Pascal-like triangle giving the number of tilings of an $n$-board using $k$ fences

We define $\left[\begin{array}{c}n \\ k\end{array}\right]$ as the number of tilings of an $n$-board which contain exactly $k$ fences (Fig. 3). We define $\left[\begin{array}{l}0 \\ 0\end{array}\right]=1$ so that the result

$$
A_{n}=\sum_{k=0}^{n}\left[\begin{array}{l}
n  \tag{19}\\
k
\end{array}\right]
$$

is valid for $n \geq 0$.
As a result of the following identity, the upward diagonals of the $\left\langle\begin{array}{l}n \\ k\end{array}\right\rangle$ triangle are the rows of the $\left[\begin{array}{l}n \\ k\end{array}\right]$ triangle. Equivalently, column $k$ of the $\left[\begin{array}{c}n \\ k\end{array}\right]$ triangle is obtained by displacing

| $n \backslash k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| :---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| 0 | 1 |  |  |  |  |  |  |  |  |  |  |  |  |
| 1 | 1 | 0 |  |  |  |  |  |  |  |  |  |  |  |
| 2 | 1 | 0 | 0 |  |  |  |  |  |  |  |  |  |  |
| 3 | 1 | 1 | 0 | 0 |  |  |  |  |  |  |  |  |  |
| 4 | 1 | 2 | 1 | 0 | 0 |  |  |  |  |  |  |  |  |
| 5 | 1 | 3 | 2 | 0 | 0 | 0 |  |  |  |  |  |  |  |
| 6 | 1 | 4 | 4 | 0 | 0 | 0 | 0 |  |  |  |  |  |  |
| 7 | 1 | 5 | 7 | 2 | 0 | 0 | 0 | 0 |  |  |  |  |  |
| 8 | 1 | 6 | 11 | 6 | 1 | 0 | 0 | 0 | 0 |  |  |  |  |
| 9 | 1 | 7 | 16 | 13 | 3 | 0 | 0 | 0 | 0 | 0 |  |  |  |
| 10 | 1 | 8 | 22 | 24 | 9 | 0 | 0 | 0 | 0 | 0 | 0 |  |  |
| 11 | 1 | 9 | 29 | 40 | 22 | 3 | 0 | 0 | 0 | 0 | 0 | 0 |  |
| 12 | 1 | 10 | 37 | 62 | 46 | 12 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |

Figure 3: A Pascal-like triangle with entries $\left[\begin{array}{c}n \\ k\end{array}\right]$ (A335964).
column $k$ of the $\left\langle\begin{array}{l}n \\ k\end{array}\right\rangle$ triangle downwards by $k$ (and filling the entries above with zeros). Thus we again obtain sequences $\underline{\mathrm{A} 000124}$ and $\underline{\mathrm{A} 003600}$ for the $k=2$ and $k=3$ columns, respectively (Identities 32 and 33).

Identity 28.

$$
\left[\begin{array}{l}
n \\
k
\end{array}\right]=\left\langle\begin{array}{c}
n-k \\
k
\end{array}\right\rangle .
$$

Proof. If a tiling contains $n-k$ tiles of which $k$ are fences, the total length is $n$.
The even rows of the triangle $\left[\begin{array}{l}n \\ k\end{array}\right]$ give the triangle $\left[\begin{array}{l}n \\ k\end{array}\right]_{1 / 2}$ (defined just before Identity 27).
Identity 29.

$$
\left[\begin{array}{c}
2 n \\
k
\end{array}\right]=\left[\begin{array}{l}
n \\
k
\end{array}\right]_{1 / 2}
$$

Proof. The number of tilings of a $2 n$-board with squares and ( 1,1 )-fences is the same as the number of tilings of an $n$-board with tiles of half the length.

Identity 30. For $n>0$,

$$
\left[\begin{array}{c}
n \\
0
\end{array}\right]=1 .
$$

Proof. There is only one way to tile a board without using any fences.
Identity 31. For $n>2$,

$$
\left[\begin{array}{c}
n \\
1
\end{array}\right]=n-2
$$

Proof. If there is only one fence, there must be 1 filled fence (and $n-3$ free squares). The filled fence can be placed in any of the $n-2$ metatile positions.

Identity 32. For $n>3$,

$$
\left[\begin{array}{l}
n \\
2
\end{array}\right]=\binom{n-4}{2}+n-3 .
$$

Proof. If there are 2 fences, there are either 2 filled fences or 1 bifence. In the first case there are $n-6$ free squares and thus a total of $n-4$ metatile positions to place the filled fences. There are thus $\binom{n-4}{2}$ ways to place the filled fences. In the second case there are $n-4$ free squares and thus $n-3$ metatile positions in which the bifence can be placed.

Identity 33. For $n>5$,

$$
\left[\begin{array}{l}
n \\
3
\end{array}\right]=\binom{n-6}{3}+2\binom{n-5}{2} .
$$

Proof. If there are 3 fences, there are either 3 filled fences or 1 bifence and a filled fence. In the first case there are $n-9$ free squares and 3 filled fences giving a total $n-6$ metatile positions for the filled fences. In the second case there are $n-7$ free squares and thus $n-5$ metatile positions for the filled fence and bifence.

Identity 34. For $n \geq r \geq 0,\left[\begin{array}{c}2 n+1 \\ n-r\end{array}\right]=C_{n}^{(r)}$.
Proof. The result follows from the definition of $C_{n}^{(r)}$ since $\left[\begin{array}{c}2 n+1 \\ n-r\end{array}\right]$ is also the number of ways to tile a $(2 n+1)$-board using $2 r+1$ squares.

Identity 35.

$$
\left[\begin{array}{c}
n  \tag{20}\\
k
\end{array}\right]=\left[\begin{array}{c}
n-1 \\
k
\end{array}\right]+\left[\begin{array}{l}
n-3 \\
k-1
\end{array}\right]+\left[\begin{array}{l}
n-4 \\
k-2
\end{array}\right]+\delta_{0, k} \delta_{0, n}
$$

Proof. We count $\left[\begin{array}{l}n \\ k\end{array}\right]$ by conditioning on the last metatile on the board. If the metatile is of length $l$ and contains $j$ fences, the number of ways to tile the remaining $n-l$ cells with $k-j$ fences is $\left[\begin{array}{c}n-l \\ k-j\end{array}\right]$. Summing these for the three types of metatile gives the result.

To show that $C_{n}^{(r)}$ is a Riordan array we first need a recursion relation that involves only the odd rows of the triangle.

## Identity 36.

$$
\left[\begin{array}{c}
2 n+1  \tag{21}\\
k
\end{array}\right]=\left[\begin{array}{c}
2 n-1 \\
k
\end{array}\right]+\left[\begin{array}{c}
2 n-1 \\
k-1
\end{array}\right]+\left[\begin{array}{c}
2 n-3 \\
k-1
\end{array}\right]+\left[\begin{array}{c}
2 n-3 \\
k-2
\end{array}\right]-\left[\begin{array}{c}
2 n-5 \\
k-3
\end{array}\right]+\delta_{0, k} \delta_{0, n}
$$

Proof. Let $E(n, k)$ denote (20). Then $E(2 n+1, k)+E(2 n, k)-E(2 n-1, k-1)$ gives the identity.

Theorem 6. If $\bar{R}(n, k)$ is the $(n, k)$ th entry of the $\left(1 /\left[(1-x)\left(1-x^{2}\right)\right], x /(1-x)^{2}\right)$ Riordan array then

$$
\left[\begin{array}{c}
2 n+1  \tag{22}\\
k
\end{array}\right]=\bar{R}(n, n-k)
$$

Proof. Let $p=1 /\left[(1-x)\left(1-x^{2}\right)\right], q=x /(1-x)^{2}$. The recursion relation without incorporating boundary conditions for a Riordan array with this particular $q$ is $\bar{R}(n, k)=$ $\bar{R}(n-1, k)+\bar{R}(n-1, k-1)+\bar{R}(n-2, k)+\bar{R}(n-2, k-1)-\bar{R}(n-3, k)$ for $n>2, k>0[8]$. Including terms to obtain a relation valid for all $n$ and $k$ (and taking $R(n, k)=0$ if $n<0$ or $n<k$ ) we arrive at

$$
\begin{equation*}
\bar{R}(n, k)=\delta_{n, 0} \delta_{k, 0}+\bar{R}(n-1, k)+\bar{R}(n-1, k-1)+\bar{R}(n-2, k)+\bar{R}(n-2, k-1)-\bar{R}(n-3, k) \tag{23}
\end{equation*}
$$

Substituting (22) into (21), replacing $k$ by $n-k$, and noting that $\delta_{n, 0} \delta_{n-k, 0}$ can be rewritten as $\delta_{n, 0} \delta_{k, 0}$, gives (23).

From Identity 34, $\bar{R}(n, k)=C_{n}^{(k)}$. In other words, a combinatorial interpretation of $\bar{R}(n, k)$ is the number of tilings of a $(2 n+1)$-board that use $2 k+1$ squares (and $2(n-k)$ ( 1,1 )-fences). Then from Lemma 2 we have for $n \geq k \geq 0$,

$$
\begin{equation*}
\bar{R}(n, k)=\bar{R}(n-2, k)+\binom{n+k}{2 k} \tag{24}
\end{equation*}
$$

If $b, f$, and $s$ are, respectively, the numbers of bifences, filled fences, and free squares in a tiling of an $n$-board using $k$ fences then it is easily seen that

$$
\begin{align*}
n & =4 b+3 f+s,  \tag{25a}\\
k & =2 b+f . \tag{25b}
\end{align*}
$$

Identity 37. For $m \geq 0$,

$$
\left[\begin{array}{ll}
4 m+p \\
2 m+q
\end{array}\right]= \begin{cases}1, & p=q=0 \\
m+1, & p=1, q=0 \text { or } p=3, q=1 \\
(m+1)^{2}, & p=2, q=0\end{cases}
$$

and for $m>0$,

$$
\left[\begin{array}{l}
4 m+p \\
2 m+q
\end{array}\right]= \begin{cases}m, & p=-1, q=-1 \\
m(m+1), & p=0, q=-1\end{cases}
$$

Proof. Substituting $n=4 m+p$ and $k=2 m+q$ into (25) and eliminating $m$ gives

$$
p-2 q=f+s
$$

in which the only possible values of $f$ and $s$ are non-negative integers. The number of bifences may then be expressed as

$$
\begin{equation*}
b=m+\frac{1}{2}(q-f) \tag{26}
\end{equation*}
$$

Thus when $p=q=0$ we must have $f=s=0$ which corresponds to a tiling using $m$ bifences only. When $p=1, q=0$ then we must have $f=0, s=1$ since (26) would imply a non-integral number of bifences if $f=1, s=0$. With the allowed case there are $m+1$ metatile positions in which to place the free square. When $p=3, q=1$ we must have $f=1$, $s=0$ and again there are $m+1$ places for the filled fence. For $p=2, q=0$, either $s=2$, $f=0$ or $s=0, f=2$. In the first case there are $\binom{m+2}{2}$ ways of placing the 2 free squares. In the second there are $m-1$ bifences and hence $\binom{m+1}{2}$ ways of placing the 2 filled fences. Adding gives the required result. With $p=q=-1$ the only possibility is $s=0, f=1$. There are then $m-1$ bifences and hence $m$ ways to place the filled fence. Finally, if $p=0$, $q=-1$ we must have $f=s=1$. With $m+1$ metatile positions, there are a total of $2\binom{m+1}{2}$ ways to place the filled fence and free square.

Identity 38.

$$
\left[\begin{array}{c}
n  \tag{27}\\
k
\end{array}\right]= \begin{cases}\sum_{b=b_{\min }}^{b_{\max }}\binom{n-2 k+b}{k-b}\binom{k-b}{b}, & b_{\min } \leq b_{\max } \\
0, & b_{\min }>b_{\max }\end{cases}
$$

where $b_{\text {min }}=\max \left(0,\left\lceil\frac{1}{2}(3 k-n)\right\rceil\right)$ and $b_{\max }=\lfloor k / 2\rfloor$.
Proof. For given values of $n$ and $k$ we sum the number of tilings for all possible values of $b$, the number of bifences. The maximum number of bifences is obtained from (25b) when $f$ is 0 or 1 depending on whether $k$ is even or odd, respectively. Eliminating $f$ from (25) gives

$$
b=\frac{1}{2}(3 k-n+s) .
$$

If $3 k-n$ is negative, the minimum possible value of $b$ is zero. Otherwise $b_{\min }$ is obtained when $s$ is 0 or 1 when $3 k-n$ is even or odd, respectively. From (25) we have that the total number of metatiles, $b+f+s=n-2 k+b$. The proof is then the same as for Identity 26 .

## Corollary 2.

$$
\begin{aligned}
f_{n}^{2} & =\sum_{k=0}^{2 n} \sum_{b=\max \left(0,\left\lceil\frac{1}{2}(3 k-2 n)\right\rceil\right)}^{\lfloor k / 2\rfloor}\binom{2 n-2 k+b}{k-b}\binom{k-b}{b}, \\
f_{n} f_{n+1} & =\sum_{k=0}^{2 n+1} \sum_{b=\max \left(0,\left\lceil\frac{1}{2}(3 k-2 n-1)\right\rceil\right)}^{\lfloor k / 2\rfloor}\binom{2 n+1-2 k+b}{k-b}\binom{k-b}{b} .
\end{aligned}
$$

Proof. The result follows from (19), Theorem 2, and Identity 38.

Identity 39. For $n \geq k \geq 0$,

$$
\left[\begin{array}{c}
2 n+1 \\
k
\end{array}\right]=\sum_{j=k-m}^{m}\binom{n+1-j}{j}\binom{n-(k-j)}{k-j}
$$

where $m=\min (\lfloor(n+1) / 2\rfloor, k)$.
Proof. From Lemma 1, $\left[\begin{array}{c}2 n+1 \\ k\end{array}\right]$ is also the number of square-domino tilings of an $(n+1)$-board and an $n$-board using $k$ dominoes in total. The number of ways to tile an $(n+1)$-board with $j$ dominoes (and $n+1-2 j$ squares) is $\binom{n+1-j}{j}$. If the $(n+1)$-board has $j$ dominoes then the $n$-board will have $k-j$ dominoes (and $n-2(k-j)$ squares). Hence there are $\binom{n+1-j}{j}\binom{n-(k-j)}{k-j}$ ways to tile the boards if the $(n+1)$-board has $j$ dominoes. Evidently $j$ cannot exceed $k$ or $\lfloor(n+1) / 2\rfloor$ and so $m \geq j \geq k-m$. We then sum over all possible values of $j$.

Identity 40. For $n \geq k \geq 0$,

$$
\left[\begin{array}{c}
2 n \\
k
\end{array}\right]=\sum_{j=k-m}^{m}\binom{n-j}{j}\binom{n-(k-j)}{k-j}
$$

where $m=\min (\lfloor n / 2\rfloor, k)$.
Proof. The proof is analogous to that of Identity 39.
Identity 40 is equivalent to Identity 3.2 in [8]. Summing Identities 39 and 40 over all possible $k$ will, respectively, give alternative ways of expressing $f_{n} f_{n+1}$ and $f_{n}^{2}$ as double sums of products of two binomial coefficients.

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