

Counting generalized Schröder paths

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Abstract

A Schröder path is a lattice path from $(0, 0)$ to $(2n, 0)$ with steps $(1, 1)$, $(1, -1)$ and $(2, 0)$ that never goes below the x -axis. A small Schröder path is a Schröder path with no $(2, 0)$ steps on the x -axis. In this paper, a 3-variable generating function $R_L(x, y, z)$ is given for Schröder paths and small Schröder paths respectively. As corollaries, we obtain the generating functions for several kinds of generalized Schröder paths counted according to the order in a unified way.

Keywords: Schröder path, Narayana number, generating function

1 Introduction

In this paper, we will consider the following sets of steps for lattice paths:

$$\begin{aligned} S_1 &= \{(1, 1), (1, -1)\}, \\ S_2 &= \{(r, r), (r, -r) | r \in \mathbb{N}^+\}, \\ S_3 &= \{(1, 1), (1, -1), (2, 0)\}, \\ S_4 &= \{(r, r), (r, -r), (2r, 0) | r \in \mathbb{N}^+\}, \end{aligned}$$

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$$S_5 = \{(1, 1), (1, -1), (2r, 0) | r \in \mathbb{N}^+\},$$

$$S_6 = \{(r, r), (r, -r), (2, 0) | r \in \mathbb{N}^+\},$$

where (r, r) , $(r, -r)$, $(2r, 0)$ are called *up steps*, *down steps* and *horizontal steps* respectively.

For a given set S of steps, let $L_S(n)$ denote the set of lattice paths from $(0, 0)$ to $(2n, 0)$ with steps in S , and never go below the x -axis. Let $A_S(n)$ denote the subset of $L_S(n)$ whose member paths have no horizontal steps on the x -axis. We denote by $L_S = \bigcup_{n \geq 1} L_S(n)$ and $A_S = \bigcup_{n \geq 1} A_S(n)$. Then $L_{S_1}(n)$, $L_{S_3}(n)$ and $A_{S_3}(n)$ are the sets of *Dyck paths*, *Schröder paths* and *small Schröder paths* of order n respectively.

It is well known that $|L_{S_1}(n)|$ is the n th *Catalan number* (A000108 in [8]), $|L_{S_3}(n)|$ is the n th *large Schröder number* (A006318), and $|A_{S_3}(n)|$ is the n th *small Schröder number* (A001003). Define a *peak* in a Dyck path to be a vertex between an up step and a down step. Then the number of Dyck paths of order n with k peaks is the well known *Narayana number* (A001263)

$$N(n, k) = \frac{1}{n} \binom{n}{k} \binom{n}{k-1}.$$

The n th *Narayana polynomial* is defined as $N_n(y) = \sum_{1 \leq k \leq n} N(n, k)y^k$ for $n \geq 1$ with $N_0(y) = 1$. In [11], Sulanke gave the generating function for the Narayana polynomial as

$$\sum_{n \geq 0} N_n(y)x^n = (1 + (1 - y)x - \sqrt{1 - 2(1 + y)x + (1 - y)^2x^2})/(2x). \quad (1.1)$$

Let

$$P_{S_i}(x) = 1 + \sum_{n \geq 1} |L_{S_i}(n)|x^n$$

and

$$Q_{S_i}(x) = 1 + \sum_{n \geq 1} |A_{S_i}(n)|x^n$$

denote the generating functions for $|L_{S_i}(n)|$ and $|A_{S_i}(n)|$ respectively. As one type of generalization of Dyck paths, $L_{S_2}(n)$ has been studied by several authors. The generating function $P_{S_2}(x)$ is given in [7] and [1] with different methods as

$$P_{S_2}(x) = (1 + 3x - \sqrt{1 - 10x + 9x^2})/(8x). \quad (1.2)$$

Moreover, Coker [1] and Sulanke [11] expressed $|L_{S_2}(n)|$ as a combination of Narayana numbers, and Woan [12] gave a three-term recurrence for $|L_{S_2}(n)|$. For other types of generalization of Dyck paths, readers can refer to [6] and [9].

Comparing to the above results about generalization of Dyck paths, generalization of Schröder paths has been rarely studied until Kung and Miler [7] gave the generating functions $P_{S_i}(x)$ ($4 \leq i \leq 6$). Later, Huh and Park [5] expressed $|A_{S_4}(n)|$ as a combination of Narayana numbers.

Note that we can also obtain Equation (1.2) by considering the number of runs of Dyck paths. Here a *run* in a lattice path is defined to be a vertex between two consecutive steps of the same kind. Let $R(n, k, S_1)$ denote the number of lattice paths in $L_{S_1}(n)$ with k runs. Since a Dyck path of order n with k peaks has $2n - 2k$ runs, we obtain from Equation (1.1) that

$$\begin{aligned} 1 + \sum_{n,k \geq 1} R(n, k, S_1) x^n y^k &= 1 + \sum_{n,k \geq 1} N(n, k) x^n y^{2n-2k} \\ &= (1 + (y^2 - 1)x - \sqrt{1 - 2(1 + y^2)x + (1 - y^2)^2 x^2}) / (2xy^2). \end{aligned} \quad (1.3)$$

Then Equation (1.2) is derived from Equation (1.3) by setting $y = 2$.

Motivated by the above observation, we study the number of runs for Schröder paths according to the following two types: a run is *diagonal* if it is the joint of two up steps or two down steps, and a run is *horizontal* if it is the joint of two horizontal steps.

For a Schröder path P , let $\text{dr}(P)$, $\text{hr}(P)$ and $\text{order}(P)$ denote the number of diagonal runs, the number of horizontal runs and the order of P respectively. Then the generating function $R_L(x, y, z)$ is defined for $L \subseteq L_{S_3}$ as

$$R_L(x, y, z) = 1 + \sum_{P \in L} x^{\text{order}(P)} y^{\text{dr}(P)} z^{\text{hr}(P)}.$$

In this paper, we give $R_L(x, y, z)$ for $L = L_{S_3}$ and $L = A_{S_3}$. As corollaries, we obtain the generating functions $P_{S_i}(x)$ and $Q_{S_i}(x)$ for $4 \leq i \leq 6$ in a unified way.

2 The case for Schröder paths

In the following, we use U , D and H to denote the steps $(1, 1)$, $(1, -1)$ and $(2, 0)$ respectively. For a lattice P and a step s , the *insertion* of s at a vertex

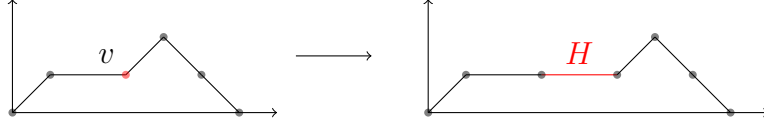


Figure 1: An example of the insertion of an H step.

v of P is defined as following: decompose P into two parts at v as $P = P_1P_2$, where P_i maybe empty. Then we connect the initial vertex of s to the end vertex of P_1 , and connect the end vertex of s to the initial vertex of P_2 . See Figure 1 for an example.

Given $P \in L_{S_1}(n)$ with k peaks, let V denote the set of vertices of P other than runs. We then insert m H steps to P as following:

- (1) We firstly choose i vertices from V , and insert an H step at each chosen vertex. In this step, we have $\binom{2k+1}{i}$ choices, and each insertion has no effect to the number of runs.
- (2) For the lattice path obtained after step (1), we choose j vertices from its runs, and insert an H step at each chosen vertex. In this step, we have $\binom{2n-2k}{j}$ choices, and the number of diagonal runs will decrease by j after insertion.
- (3) For the lattice path obtained after step (2), we insert the remaining $m - i - j$ H steps immediately after the $i + j$ H steps that have been inserted. In this step, we have $\binom{m-i-j}{i+j}$ choices, and the number of horizontal runs will increase by $m - i - j$ after insertion.

Let $\text{Ins}_m(P)$ denote the set of all Schröder paths obtained from P by the above insertion. Then we have

$$|\text{Ins}_m(P)| = \binom{2k+1}{i} \binom{2n-2k}{j} \binom{m-i-j}{i+j}. \quad (2.1)$$

Moreover, we have $\text{order}(P') = m+n$, $\text{dr}(P') = 2n-2k-j$, $\text{hr}(P') = m-i-j$ for each $P' \in \text{Ins}_m(P)$.

On the other hand, let HL_{S_3} denote the subset of L_{S_3} whose member paths consisting of H steps only. Let UL_{S_3} denote the subset of L_{S_3} whose member paths have at least one U step. It is obvious that each path of UL_{S_3}

can be obtained uniquely from a Dyck path by inserting some H steps as above. Thus we have

$$L_{S_3} = HL_{S_3} \cup UL_{S_3} = HL_{S_3} \cup \bigcup_{P \in L_{S_1}, m \geq 0} \text{Ins}_m(P).$$

Summarizing the above discussion, we then obtain the following result.

Theorem 2.1.

$$\begin{aligned} & R_{L_{S_3}}(x, y, z) \\ &= \frac{1 - xz + x}{1 - xz} (1 + (1 - w)u - \sqrt{1 - 2(1 + w)u + (1 - w)^2 u^2}) / (2u), \end{aligned}$$

where $u = x \left(\frac{x+y(1-xz)}{1-xz} \right)^2$ and $w = \left(\frac{1-xz+x}{x+y(1-xz)} \right)^2$.

Proof. By Equation (2.1), we have

$$\begin{aligned} & R_{L_{S_3}}(x, y, z) \\ &= 1 + \sum_{P \in HL_{S_3}} x^{\text{order}(P)} y^{\text{dr}(P)} z^{\text{hr}(P)} + \sum_{P \in UL_{S_3}} x^{\text{order}(P)} y^{\text{dr}(P)} z^{\text{hr}(P)} \\ &= 1 + \frac{x}{1 - xz} \\ &\quad + \sum_{\substack{n, k \geq 1 \\ m, i, j \geq 0}} N(n, k) \binom{2k+1}{i} \binom{2n-2k}{j} \binom{m-i-j}{i+j} x^{m+n} y^{2n-2k-j} z^{m-i-j} \\ &= \frac{1 - xz + x}{1 - xz} + \sum_{n, k \geq 1} N(n, k) x^n y^{2n-2k} \\ &\quad \cdot \sum_{\substack{0 \leq i \leq 2k+1 \\ 0 \leq j \leq 2n-2k}} \binom{2k+1}{i} \binom{2n-2k}{j} x^{i+j} y^{-j} \sum_{m \geq i+j} \binom{-i-j}{m-i-j} (-xz)^{m-i-j} \\ &= \frac{1 - xz + x}{1 - xz} \\ &\quad + \sum_{n, k \geq 1} N(n, k) x^n y^{2n-2k} \sum_{\substack{0 \leq i \leq 2k+1 \\ 0 \leq j \leq 2n-2k}} \binom{2k+1}{i} \binom{2n-2k}{j} x^{i+j} y^{-j} (1 - xz)^{-i-j} \\ &= \frac{1 - xz + x}{1 - xz} + \sum_{n, k \geq 1} N(n, k) x^n y^{2n-2k} \left(1 + \frac{x}{y(1-xz)} \right)^{2n-2k} \left(1 + \frac{x}{1-xz} \right)^{2k+1} \end{aligned}$$

$$= \frac{1 - xz + x}{1 - xz} \sum_{n \geq 0} N_n(w) u^n,$$

then Theorem 2.1 is derived from Equation (1.1). □

The generating functions $P_{S_i}(x)$ for $4 \leq i \leq 6$ were derived by Kung and Mier [7]. Here we can obtain them as a direct corollary of the above result.

Corollary 2.2. [7]

$$P_{S_4}(x) = \frac{(1-x)(1+x-4x^2) - (1-x)^2 \sqrt{1-12x+16x^2}}{2x(2-3x)^2},$$

$$P_{S_5}(x) = \frac{2x-1 + \sqrt{1-8x+12x^2-4x^3}}{2x(x-1)},$$

$$P_{S_6}(x) = \frac{1+2x-x^2 - \sqrt{(1-x)(1-11x+7x^2-x^3)}}{2x(x-2)^2}.$$

Proof. We use a bijection given by Huh and Park [5]. Let $\bar{L}_{S_3}(n)$ denote the set of Schröder paths of order n whose runs are colored in either black or white, and other vertices are colored in black only. For $P \in \bar{L}_{S_3}(n)$. Let $\phi(P)$ denote the lattice path obtained from P as following: delete all white vertices of P , and then connect adjacent black vertices with line segments. See [5, Figure 8] for an example. It is obvious that ϕ is a bijection from $\bar{L}_{S_3}(n)$ to $L_{S_4}(n)$, which implies that

$$P_{S_4}(x) = R_{L_{S_3}}(x, 2, 2) = \frac{(1-x)(1+x-4x^2) - (1-x)^2 \sqrt{1-12x+16x^2}}{2x(2-3x)^2}.$$

Similarly, we can obtain $P_{S_5}(x)$ and $P_{S_6}(x)$ by setting the pair (y, z) to be $(1, 2)$ and $(2, 1)$ in $R_{L_{S_3}}(x, y, z)$ respectively. □

Using the techniques in [4] ([Chapter VI]), Kung and Miler [7] gave the asymptotic formula for $|L_{S_4}(n)|$. The asymptotic formulas for $|L_{S_5}(n)|$ and $|L_{S_6}(n)|$ can be obtained from Corollary 2.2 in a similar way:

$$|L_{S_4}(n)| \sim \frac{\beta_1}{\alpha_1^n \sqrt{\pi n^3}},$$

$$|L_{S_5}(n)| \sim \frac{\beta_2}{\alpha_2^n \sqrt{\pi n^3}},$$

$$|L_{S_6}(n)| \sim \frac{\beta_3}{\alpha_3^n \sqrt{\pi n^3}},$$

where α_i and β_i are defined as following:

(1) $\alpha_1 = \frac{3-\sqrt{5}}{8}$ is the root of equation $f_1(x) = 1 - 12x + 16x^2 = 0$, and

$$\beta_1 = \frac{(1-\alpha_1)^2 \sqrt{-\alpha_1 f_1'(\alpha_1)}}{4\alpha_1(2-3\alpha_1)^2} = \frac{(35-15\sqrt{5})(\sqrt{6\sqrt{5}-10})}{4};$$

(2) $\alpha_2 = 0.16243 \dots$ is the root of equation $f_2(x) = 1 - 8x + 12x^2 - 4x^3 = 0$,

$$\text{and } \beta_2 = \frac{\sqrt{-\alpha_2 f_2'(\alpha_2)}}{4\alpha_2(1-\alpha_2)} = 1.55669 \dots;$$

(3) $\alpha_3 = 0.09678 \dots$ is the root of equation $f_3(x) = (1-x)(1-11x+7x^2-x^3)$,

$$\text{and } \beta_3 = \frac{\sqrt{-\alpha_3 f_3'(\alpha_3)}}{4\alpha_3(2-\alpha_3)^2} = 0.68998 \dots.$$

Theorem 2.1 can also be used to study colored Schröder paths. For instance, let $a(n)$ denote the number of Schröder paths of order n with their horizontal runs colored in one of three given colors. Then we obtain from Theorem 2.1 that

$$1 + \sum_{n \geq 1} a(n)x^n = R_{L_{S_3}}(x, 1, 3) = \frac{3x - 1 + \sqrt{1 - 10x + 25x^2 - 16x^3}}{2x(2x - 1)}. \quad (2.2)$$

The coefficients of the above function appear as sequence A186338 in OEIS, and is related to sequence A091866.

For the definition of pyramid and pyramid weight, see [2, Definition 2.1]. Let $T(n, k)$ denote the number of Dyck paths of order n that have pyramid weight k . Combing Equation 2.2 with a result of Denise and Simion ([2, Theorem 2.3]), we then obtain the following result.

Corollary 2.3.

$$a(n) = \sum_{k=0}^n T(n, k)2^k.$$

3 The case for small Schröder paths

A lattice path in A_{S_3} is said to be *primitive* if it does not intersect the x -axis except at $(0, 0)$ and $(2n, 0)$. Let PA_{S_3} denote the set of all primitive paths

in A_{S_3} . Since every path in A_{S_3} can be decomposed uniquely into a sequence of paths in PA_{S_3} , we have

$$R_{A_{S_3}}(x, y, z) = \frac{1}{1 - \bar{R}_{PA_{S_3}}(x, y, z)}, \quad (3.1)$$

where we use $\bar{R}_L(x, y, z)$ to denote the function $R_L(x, y, z) - 1$ for a given set L of lattice paths.

We now consider the generating function $\bar{R}_{PA_{S_3}}(x, y, z)$. Note that the set UL_{S_3} can be partitioned as $UL_{S_3} = \bigcup_{i=1}^4 U_i$, where

- (1) $U_1 = \{P \mid P \text{ starts with } U \text{ and ends with } D\}$;
- (2) $U_2 = \{P \mid P \text{ starts with } H \text{ and ends with } D\}$;
- (3) $U_3 = \{P \mid P \text{ starts with } U \text{ and ends with } H\}$;
- (4) $U_4 = \{P \mid P \text{ starts and ends with } H\}$.

As shown in Section 2, each path $P \in UL_{S_3}$ can be obtained uniquely from a Dyck path P' by inserting some H steps, and we have the following fact:

- (1) if it is not allowed to insert at either the initial vertex or the end vertex of P' , then $P \in U_1$;
- (2) if it is required to insert at the initial vertex of P' , and not allowed to insert at the end vertex, then $P \in U_2$;
- (3) if it is required to insert at the end vertex of P' , and not allowed to insert at the initial vertex, then $P \in U_3$;
- (4) if it is required to insert at both the initial vertex and the end vertex of P' , then $P \in U_4$.

Based on the above observation, we can obtain the following result after some calculation.

Lemma 3.1.

$$\bar{R}_{U_1}(x, y, z) = \frac{(1 - xz)^2}{(1 - xz + x)^2} \bar{R}_{UL_{S_3}}(x, y, z),$$

$$\begin{aligned}\bar{R}_{U_2}(x, y, z) &= \bar{R}_{U_3}(x, y, z) = \frac{x(1-xz)}{(1-xz+x)^2} \bar{R}_{ULS_3}(x, y, z), \\ \bar{R}_{U_4}(x, y, z) &= \frac{x^2}{(1-xz+x)^2} \bar{R}_{ULS_3}(x, y, z).\end{aligned}$$

Proof. The proof of the above result is almost the same as that of Theorem 2.1. Here we take $\bar{R}_{U_2}(x, y, z)$ as an example. By the definition of U_2 , we have

$$\begin{aligned}\bar{R}_{U_2}(x, y, z) &= \\ &= \sum_{\substack{n, k \geq 1 \\ i, j \geq 0 \\ m \geq 1}} N(n, k) \binom{2k-1}{i} \binom{2n-2k}{j} \binom{m-i-j-1}{i+j+1} x^{m+n} y^{2n-2k-j} z^{m-i-j-1} \\ &= \sum_{n, k \geq 1} N(n, k) x^n y^{2n-2k} \left(1 + \frac{x}{y(1-xz)}\right)^{2n-2k} \left(1 + \frac{x}{1-xz}\right)^{2k-1} \frac{x}{1-xz} \\ &= \frac{x(1-xz)}{(1-xz+x)^2} \bar{R}_{ULS_3}(x, y, z).\end{aligned}$$

□

Now we can obtain $R_{A_3}(x, y, z)$ as a direct corollary of the above result.

Theorem 3.2.

$$\begin{aligned}R_{A_{S_3}}(x, y, z) &= \\ &= \frac{1}{2} + \frac{-1 + zx + (1-z)x^2 + (1-xz)\sqrt{1-2(1+w)u + (1-w)^2u^2}}{2(y^2z - 2y + 1)x^2 - 2y^2x - 2x},\end{aligned}$$

where $u = x \left(\frac{x+y(1-xz)}{1-xz}\right)^2$ and $w = \left(\frac{1-xz+x}{x+y(1-xz)}\right)^2$.

Proof. Since

$$PA_{S_3} = \{UPD \mid P \text{ is empty or } P \in L_{S_3}\},$$

we obtain from Lemma 3.1 that

$$\begin{aligned}\bar{R}_{PA_{S_3}}(x, y, z) &= \\ &= x + x\bar{R}_{HL_{S_3}} + xy^2\bar{R}_{U_1} + xy\bar{R}_{U_2} + xy\bar{R}_{U_3} + x\bar{R}_{U_4} \\ &= \frac{x(1-xz+x)}{1-xz} + \frac{x}{w} \bar{R}_{ULS_3}(x, y, z) \\ &= \frac{x(1-xz+x)}{1-xz} + \frac{x}{w} (R_{LS_3}(x, y, z) - \frac{1-xz+x}{1-xz}).\end{aligned}\tag{3.2}$$

Combining Equation (3.1), Equation (3.2) and Theorem 2.1 together, we then obtain Theorem 3.2. \square

Setting the pair (y, z) to be $(2, 2)$, $(1, 2)$ and $(2, 1)$ respectively in Theorem 3.2, we then obtain the generating functions $Q_{S_i}(x)$ for $4 \leq i \leq 6$.

Corollary 3.3.

$$\begin{aligned} Q_{S_4}(x) &= \frac{1 + 4x - \sqrt{1 - 12x + 16x^2}}{10x}, \\ Q_{S_5}(x) &= \frac{-1 + \sqrt{1 - 8x + 12x^2 - 4x^3}}{2x(x - 2)}, \\ Q_{S_6}(x) &= \frac{-1 - 4x + x^2 + \sqrt{(1 - x)(1 - 11x + 7x^2 - x^3)}}{2x(x - 5)}. \end{aligned}$$

Expanding the above functions, we have

$$\begin{aligned} Q_{S_4}(x) &= 1 + x + 6x^2 + 41x^3 + 306x^4 + 2426x^5 + 20076x^6 + \cdots, \\ Q_{S_5}(x) &= 1 + x + 3x^2 + 12x^3 + 53x^4 + 248x^5 + 1209x^6 + \cdots, \\ Q_{S_6}(x) &= 1 + x + 6x^2 + 40x^3 + 293x^4 + 2286x^5 + 18637x^6 + \cdots. \end{aligned}$$

The coefficients of $Q_{S_4}(x)$ appear as sequence A078009 in OEIS. The generating functions $Q_{S_5}(x)$ and $Q_{S_6}(x)$, to our knowledge, have not been studied before. From Corollary 3.3, we can obtain the following asymptotic formulas:

$$\begin{aligned} |A_{S_4}(n)| &\sim \frac{\gamma_1}{\alpha_1^n \sqrt{\pi n^3}}, \\ |A_{S_5}(n)| &\sim \frac{\gamma_2}{\alpha_2^n \sqrt{\pi n^3}}, \\ |A_{S_6}(n)| &\sim \frac{\gamma_3}{\alpha_3^n \sqrt{\pi n^3}}, \end{aligned}$$

where α_i and f_i are the same as those in Section 2, and γ_i is defined as following:

$$\begin{aligned} \gamma_1 &= \frac{\sqrt{-\alpha_1 f_1'(\alpha_1)}}{20\alpha_1} = \frac{\sqrt{10 + 6\sqrt{5}}}{10}, \\ \gamma_2 &= \frac{\sqrt{-\alpha_2 f_2'(\alpha_2)}}{4\alpha_2(2 - \alpha_2)} = 0.70954 \cdots, \end{aligned}$$

$$\gamma_3 = \frac{\sqrt{-\alpha_3 f_3'(\alpha_3)}}{4\alpha_3(5 - \alpha_3)} = 0.50971 \dots .$$

It is well known (see, for example, [3, 10]) that $|L_{S_3}(n)| = 2|A_{S_3}(n)|$. Comparing the asymptotic formulas of $|L_{S_i}(n)|$ and $|A_{S_i}(n)|$ for $4 \leq i \leq 6$, we have the following analogue.

Corollary 3.4.

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{|L_{S_4}(n)|}{|A_{S_4}(n)|} &= \frac{5(1 - \alpha_1)^2}{(2 - 3\alpha_1)^2} = 1.39320 \dots , \\ \lim_{n \rightarrow \infty} \frac{|L_{S_5}(n)|}{|A_{S_5}(n)|} &= \frac{2 - \alpha_2}{1 - \alpha_2} = 2.19393 \dots , \\ \lim_{n \rightarrow \infty} \frac{|L_{S_6}(n)|}{|A_{S_6}(n)|} &= \frac{5 - \alpha_3}{(2 - \alpha_3)^2} = 1.35364 \dots . \end{aligned}$$

By giving a bijection between 5-colored Dyck paths and $|A_{S_4}(n)|$, Huh and Park [5] gave the following expression for $|A_{S_4}(n)|$, which we can also prove here with generating function.

Corollary 3.5. [5]

$$|A_{S_4}(n)| = \sum_{k=1}^n N(n, k) 5^{n-k}.$$

Proof. By Equation (1.1), we have

$$1 + \sum_{n, k \geq 1} N(n, k) 5^{n-k} x^n = \sum_{n \geq 0} N_n\left(\frac{1}{5}\right) (5x)^n = \frac{1 + 4x - \sqrt{1 - 12x + 16x^2}}{10x}.$$

Then Corollary 3.5 is derived from Corollary 3.3. \square

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