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THE POLYTOPE ALGEBRA OF GENERALIZED PERMUTAHEDRA

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Abstract. The polytope algebra of a zonotope can be endowed with the structure of a module over the Tits algebra of the corresponding hyperplane arrangement. We explore this structure and find relations between statistics on (signed) permutations and the composition series of the module associated with the (signed) braid arrangement. We prove that the relations defining the polytope algebra are compatible with the Hopf monoid structure of generalized permutahedra.

1. Introduction

In recent years, the family of generalized permutahedra has been a central object of study for many combinatorialists. They serve as a geometric model for many *classical* (type A) combinatorial objects. They were first introduced as *polymatroids* by Edmonds in [Edm70], where their relation to submodular functions and optimization was studied. More recently, Aguiar and Ardila [AA17] endowed this family with the structure of a Hopf monoid GP in the category of species. In doing so, they gave a unified framework to study similar algebraic structures over many different families of combinatorial objects.

Over three decades ago, McMullen [McM89] pioneered in endowing polytopes with an algebraic structure. At first glance, the Hopf monoid of Aguiar and Ardila seems to be very disconnected from McMullen's polytope algebra. However, there is a striking relationship between the notion of inversion on both structures, as already pointed out in [AA17]. Understanding the relation between both algebraic structures is the starting point of this work.

Let V be a real vector space. The polytope algebra $\Pi(V)$ is generated by the classes $[\mathfrak{p}]$ of polytopes $\mathfrak{p} \subseteq V$. These generators satisfy the *valuation* and *translation invariance* relations. We show that the relations defining McMullen's algebra are compatible with the Hopf monoid structure of generalized permutahedra. As a consequence, we construct a new Hopf monoid obtained as a quotient of GP by these relations.

Theorem 7.1. *The species* Π *of generalized permutahedra modulo the valuation and translation invariance relations is a Hopf monoid.*

The polytope algebra relative to a fixed polytope $\mathfrak{p} \subseteq V$, denoted $\Pi(\mathfrak{p})$, was considered by McMullen in [McM93a]. It is the subalgebra of $\Pi(V)$ generated by the classes of Minkowski summands of $\mathfrak{p}$. $\Pi(\mathfrak{p})$ is a graded algebra, with graded components $\Xi_r(\mathfrak{p})$ for $r = 0, 1, \dots, \dim(\mathfrak{p})$. For the case of zonotopes $\mathfrak{z}$, we endow McMullen's construction with an additional algebraic structure. $\Pi(\mathfrak{z})$ is a right-module over the Tits algebra $\Sigma[\mathcal{A}]$ of the corresponding hyperplane arrangement $\mathcal{A}$. McMullen showed that the algebra $\Pi(\mathfrak{p})$ has very nice properties when the polytope $\mathfrak{p}$ is simple. One of them is that the morphisms $\Pi(\mathfrak{p}) \to \Pi(\mathfrak{f})$, which are defined for every face $\mathfrak{f} \leq \mathfrak{p}$, are surjective. We prove that this also hold s for arbitrary zonotopes.

Proposition 5.2. *Let* $\mathfrak{z}$ *be a zonotope, not necessarily simple, and* $\mathfrak{f}$ *a face of* $\mathfrak{z}$*. Then, the morphism* $\psi_{\mathfrak{f}} : \Pi(\mathfrak{p}) \to \Pi(\mathfrak{f})$ *is surjective.*

The simple modules over $\Sigma[\mathcal{A}]$ are one dimensional and indexed by the flats of the arrangement [AM17, Chapter 9]. Given a module h over $\Sigma[\mathcal{A}]$, the number of copies of the simple module associated with the flat X that appear as a composition factor of h is $\eta_{\mathrm{X}}(\mathsf{h})$. We investigate these numbers for the module $\Pi(\mathfrak{z})$ in the particular case of the permutahedron π and the type B permutahedron π^B. They are closely related to the Eulerian numbers and the Eulerian numbers of type B, respectively. The main results in this direction are the following.

Theorem 6.1. *For any flat* X *of the braid arrangement in* $\mathbb{R}^d$ *and* $r = 0, 1, \ldots, d-1$,

$$\eta_{\mathrm{X}}(\Xi_r(\pi)) = \left|\{\sigma \in \mathfrak{S}_d \,:\, \mathrm{s}(\sigma) = \mathrm{X},\ \mathrm{exc}(\sigma) = r\}\right|.$$

Theorem 6.7. *For any flat* X *of signed braid arrangement in* $\mathbb{R}^d$ *and* $r = 0, 1, \ldots, d$,

$$\eta_{\mathrm{X}}(\Xi_r(\pi^B)) = \left|\{\sigma \in \mathfrak{B}_d \,:\, \mathrm{s}(\sigma) = \mathrm{X},\ \mathrm{exc}_B(\sigma) = r\}\right|.$$

This document is organized as follows. We review McMullen's construction in Section 2. The Tits algebra of a hyperplane arrangement and characteristic elements are the subject of Section 3. There, we also determine the dimensions of the eigenspaces of the action of a characteristic element over any module. Section 4 reviews the (signed) braid arrangement and its relation with (resp. type B) set partitions and (resp. signed) permutations. In Section 5 we introduce the McMullen module of a hyperplane arrangement. Section 6 contains the main results: we explore in-depth the module structure for the Coxeter arrangements of type A and B. In particular, we provide a conjectural eigenbasis for the action of the *Adams element* on the module $\Pi(\pi)$. In Section 7 we prove that the valuation and translation invariance relations are compatible with the Hopf monoid structure of GP. We conclude with some final remarks and questions in Section 8.

2. The polytope algebra

We briefly review the definition of the polytope algebra of McMullen [McM89] and its main properties. The subalgebra relative to a fixed polytope [McM93a] is studied at the end of this section. We start by establishing some notation and recalling some definitions.

2.1. **Preliminaries.** Let V be a real vector space of dimension d endowed with an inner product $\langle\cdot,\cdot\rangle$. For a polytope $\mathfrak{p} \subseteq V$ and a vector $v \in V$, let $\mathfrak{p}_v$ denote the face of $\mathfrak{p}$ maximized in the direction v. That is,

$$\mathfrak{p}_v = \{p \in \mathfrak{p} \,:\, \langle p, v\rangle \geq \langle q, v\rangle \text{ for all } q \in \mathfrak{p}\}.$$

The (outer) normal cone of a face $\mathfrak{f}$ of $\mathfrak{p}$ is the polyhedral cone

$$N(\mathfrak{f}, \mathfrak{p}) = \{v \in V \mid \mathfrak{f} \leq \mathfrak{p}_v\}.$$

The normal fan of $\mathfrak{p}$ is

$$\Sigma_{\mathfrak{p}} = \{N(\mathfrak{f}, \mathfrak{p}) \,:\, \mathfrak{f} \leq \mathfrak{p}\}.$$

There is a natural order-reversing correspondence between faces of $\mathfrak{p}$ and cones in $\Sigma_{\mathfrak{p}}$. For $F \in \Sigma_{\mathfrak{p}}$, we let $\mathfrak{p}_F \leq \mathfrak{p}$ denote the face whose normal cone is F.

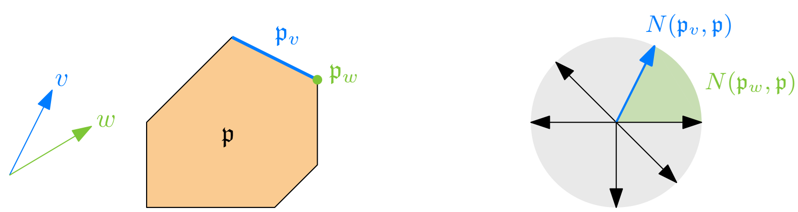


Figure 2.1. A 2-dimensional polytope $\mathfrak{p}$ and two of its faces $\mathfrak{p}_v, \mathfrak{p}_w$ maximized in directions v, w, respectively. On the right, the normal fan $\Sigma_{\mathfrak{p}}$.

Two polytopes $\mathfrak{p}$ and $\mathfrak{q}$ are said to be *normally equivalent* if $\Sigma_{\mathfrak{p}} = \Sigma_{\mathfrak{q}}$. If, on the other hand, $\Sigma_{\mathfrak{p}}$ *refines* $\Sigma_{\mathfrak{q}}$, we say that $\mathfrak{q}$ is a *deformation* of $\mathfrak{p}$. Recall that a fan Σ refines Σ' if every cone in Σ' is a union of cones in Σ.

The *Minkowski sum* of two subsets $A, B \subseteq V$ is

$$A + B = \{v + w \,:\, v \in A,\, w \in B\}.$$

We say that a set A is a *Minkowski summand* of S if $S = A + B$ for some set B. The Minkowski sum of two polytopes is a polytope, and every Minkowski summand of a polytope is necessarily a polytope. Moreover, the normal fan of $\mathfrak{p} + \mathfrak{q}$ is the common refinement of $\Sigma_\mathfrak{p}$ and $\Sigma_\mathfrak{q}$. Hence, $\Sigma_\mathfrak{p}$ refines the normal fan of any of its Minkowski summands.

The f-polynomial of a d-dimensional polytope $\mathfrak{p}$ is

$$f(\mathfrak{p}, z) = \sum_{i=0}^{d} f_i(\mathfrak{p}) z^i,$$

where $f_i(\mathfrak{p})$ is the number of i-dimensional faces of $\mathfrak{p}$. The h-polynomial of $\mathfrak{p}$ is defined by

$$h(\mathfrak{p}, z) = \sum_{i=0}^{d} h_i(\mathfrak{p}) z^i = f(\mathfrak{p}, z - 1).$$

The sequences $(f_0(\mathfrak{p}), \dots, f_d(\mathfrak{p}))$ and $(h_0(\mathfrak{p}), \dots, h_d(\mathfrak{p}))$ are the f-vector and h-vector of $\mathfrak{p}$, respectively. These polynomials behave nicely with respect to Cartesian products:

$$f(\mathfrak{p} \times \mathfrak{q}, z) = f(\mathfrak{p}, z) f(\mathfrak{q}, z) \qquad h(\mathfrak{p} \times \mathfrak{q}, z) = h(\mathfrak{p}, z) h(\mathfrak{q}, z).$$

2.2. **Definitions and structure theorem.** As an abelian group, the *polytope algebra* $\Pi(V)$ is generated by elements $[\mathfrak{p}]$, one for each polytope $\mathfrak{p} \subseteq V$. These generators satisfy the relations

$$[\mathfrak{p} \cup \mathfrak{q}] + [\mathfrak{p} \cap \mathfrak{q}] = [\mathfrak{p}] + [\mathfrak{q}], \tag{2.1}$$

whenever $\mathfrak{p}$, $\mathfrak{q}$ and $\mathfrak{p} \cup \mathfrak{q}$ are polytopes; and

$$[\mathfrak{p} + \{t\}] = [\mathfrak{p}] \tag{2.2}$$

for any polytope $\mathfrak{p}$ and translation vector $t \in V$. These relations are referred as the *valuation property* and the *translation invariance property*, respectively.

The group $\Pi(V)$ is endowed with a commutative product. The product of two generators is defined by means of the Minkowski sum

$$[\mathfrak{p}] \cdot [\mathfrak{q}] = [\mathfrak{p} + \mathfrak{q}], \tag{2.3}$$

and it is extended linearly to all $\Pi(V)$. Observe that the class of a point $1 := [\{o\}]$ is the unit of this algebra.

A fundamental operation on $\Pi(V)$ is given by the *dilations* δ_λ, defined for each scalar $\lambda \in \mathbb{R}$. For any subset $S \subseteq V$ and scalar λ, the dilate of S by λ is the set $\lambda S = \{\lambda v \,:\, v \in S\}$. The dilation $\delta_\lambda : \Pi(V) \to \Pi(V)$ is defined on the generators by

$$\delta_\lambda[\mathfrak{p}] = [\lambda \mathfrak{p}].$$

One can easily verify that δ_λ preserves relations (2.1) and (2.2). Further, $\delta_0[\mathfrak{p}] = 1$ for any polytope $\mathfrak{p}$.

Let $\Xi_0(V)$ be the subring of $\Pi(V)$ generated by 1, and Z_1 be the subgroup of $\Pi(V)$ generated by all elements of the form $[\mathfrak{p}] - 1$.

Lemma 2.1 ([McM89, Lemma 8]). *As an abelian group,* $\Pi(V)$ *has a direct sum decomposition*

$$\Pi(V) = \Xi_0(V) \oplus Z_1.$$

The dilation δ_0 *is the projection from* $\Pi(V)$ *to* $\Xi_0(V)$ *with kernel* Z_1.

It follows from the previous lemma that Z_1 is an ideal of Π. The next result shows that Z_1 is in fact nilpotent.

Lemma 2.2 ([McM89, Lemma 13])**.** *Let $\mathfrak{p}$ be a k-dimensional polytope. Then,*

$$([\mathfrak{p}]-1)^r = 0 \quad \textit{for } r > k.$$

Therefore, we can define maps

$$\begin{tikzcd} 1+Z_1 \arrow[r, bend left, "\log"] & Z_1 \arrow[l, bend left, "\exp"] \end{tikzcd} \tag{2.4}$$

by means of their usual power series

$$\log(1+x) = \sum_{k\geq 1} \frac{(-1)^{k-1}}{k} x^k \qquad \exp(x) = \sum_{k\geq 0} \frac{1}{k!} x^k$$

for $x \in Z_1$.

Lemma 2.3 ([McM89, Lemma 18])**.** *The maps (2.4) are inverse of each other. Moreover*

$$\log(x_1x_2) = \log(x_1) + \log(x_2) \ \textit{whenever } \delta_0 x_1 = \delta_0 x_2 = 1, \tag{2.5a}$$

$$\exp(x_1+x_2) = \exp(x_1)\exp(x_2) \ \textit{whenever } x_1, x_2 \in Z_1. \tag{2.5b}$$

In particular, $\log[\mathfrak{p}]$ is defined for any polytope $\mathfrak{p} \subseteq V$. If $\mathfrak{p}$ has dimension k, then

$$\log[\mathfrak{p}] = \sum_{r=0}^{k} \frac{(-1)^{k-1}}{k}([\mathfrak{p}]-1)^r \quad \text{and} \quad [\mathfrak{p}] = \sum_{r=0}^{k} \frac{1}{r!}(\log[\mathfrak{p}])^r.$$

It actually follows from the work of McMullen that $([\mathfrak{p}]-1)^k \neq 0$ for a k-dimensional polytope $\mathfrak{p}$. Therefore, all the terms appearing in the sums above are nonzero.

For $r \geq 1$ let $\Xi_r(V)$ be the subspace of $\Pi(V)$ generated by elements of the form $(\log[\mathfrak{p}])^r$.

Theorem 2.4 ([McM89, Theorem 1])**.** *The commutative ring $\Pi(V)$ is almost a graded $\mathbb{R}$-algebra, in the following sense:*

i. as an abelian group, $\Pi(V)$ admits a direct sum decomposition

$$\Pi(V) = \bigoplus_{r=0}^{d} \Xi_r(V);$$

ii. under multiplication,

$$\Xi_r(V)\cdot\Xi_s(V) = \Xi_{r+s},$$

with $\Xi_r = 0$ for $r > d$;

iii. $\Xi_0(V) \cong \mathbb{Z}$, and for $r = 1,\dots,d$, Ξ_r is a real vector space;

iv. the product of elements in $Z_1 = \bigoplus_{r\geq 1}\Xi_r(V)$ is bilinear;

v. the dilations δ_λ are algebra endomorphisms, and for $r = 0,1,\dots,d$, if $x \in \Xi_r(V)$ and $\lambda \geq 0$, then $\delta_\lambda x = \lambda^r x$.

Convention 2.5. As in later work of McMullen [McM93b, McM93a], we replace $\Xi_0(V) \cong \mathbb{Z}$ with the tensor product $\Xi_0(V)_\mathbb{R} := \mathbb{R}\otimes\Xi_0(V)$. We similarly let $\Pi(V)_\mathbb{R} := \Xi_0(V)_\mathbb{R} \oplus Z_1$. Hence, $\Pi(V)_\mathbb{R}$ is a graded $\mathbb{R}$-algebra. To simplify notation, we write Ξ_0 instead of $\Xi_0(V)_\mathbb{R}$ and Π instead of $\Pi(V)_\mathbb{R}$.

The last property in the theorem completely characterizes the graded components of Π, as the following result shows.

Lemma 2.6 ([McM89, Lemma 20])**.** *Let $x \in \Pi$ and $\lambda > 0$, with $\lambda \neq 1$. Then,*

$$x \in \Xi_r \qquad \textit{if and only if} \qquad \delta_\lambda x = \lambda^r x. \tag{2.6}$$

Example 2.7. Let $\mathfrak{l}$ be a line segment of arbitrary length. Lemma 2.2 shows that $([\mathfrak{l}]-1)^2 = 0$. Therefore,

$$\log[\mathfrak{l}] = [\mathfrak{l}] - 1.$$

It follows that $[\mathfrak{l}] - 1 \in \Xi_1$ and, by (2.6), $[\lambda\mathfrak{l}] - 1 = \delta_\lambda([\mathfrak{l}]-1) = \lambda[\mathfrak{l}] - \lambda$.

Lemma 2.8. *Let $v_1, \ldots, v_k \in V$ be nonzero vectors and let $\mathfrak{l}_i$ denote the line segment* $\mathrm{Conv}(o, v_i)$. *Then,*

$$\prod_{i=1}^{k} \log[\mathfrak{l}_i] \neq 0 \quad \iff \quad \{v_1, \ldots, v_k\} \text{ is linearly independent.}$$

Proof. Consider the polytope $\mathfrak{z} = \sum_i \mathfrak{l}_i$. Using (2.5a), we get

$$(\log[\mathfrak{z}])^k = \Big(\sum_{i=1}^{k} \log[\mathfrak{l}_i]\Big)^k = k! \prod_{i=1}^{k} \log[\mathfrak{l}_i].$$

The last equality follows since $k! \prod \log[\mathfrak{l}_i]$ is the only square-free term in the expansion of $(\sum \log[\mathfrak{l}_i])^k$, and $(\log[\mathfrak{l}])^2 = 0$ for any line segment $\mathfrak{l}$. Finally, $(\log[\mathfrak{z}])^k \neq 0$ if and only if $k \leq \dim(\mathfrak{z})$, and $\mathfrak{z}$ being the sum of k line segments has dimension at most k, with equality precisely when the vectors $v_1, \ldots, v_k$ are linearly independent. □

2.3. **Euler map and maximization operators.** The *Euler map* $x \mapsto x^*$ is the linear operator defined on generators by

$$[\mathfrak{p}]^* = \sum_{\mathfrak{q} \leq \mathfrak{p}} (-1)^{\dim(\mathfrak{q})} [\mathfrak{q}]. \tag{2.7}$$

The sum run over all nonempty faces $\mathfrak{q}$ of $\mathfrak{p}$. Up to a sign, the element $[\mathfrak{p}]^*$ corresponds to *the class of the interior* of $\mathfrak{p}$.

Theorem 2.9 ([McM89, Theorem 2])**.** *The Euler map is an involutory automorphism of* Π. *Moreover, for $x \in \Xi_r$ and $\lambda < 0$,*

$$\delta_\lambda x = \lambda^r x^*.$$

Theorem 2.10 ([McM89, Theorem 12])**.** *For any polytope* $\mathfrak{p}$*,*

$$[\mathfrak{p}] \cdot [-\mathfrak{p}]^* = 1. \tag{2.8}$$

That is, the class of a polytope is always invertible, with $[\mathfrak{p}]^{-1} = [-\mathfrak{p}]^*$.

Take a vector $v \in V$. We can define a *maximization operator* $\mathfrak{p} \mapsto \mathfrak{p}_v$ on the space of all polytopes $\mathfrak{p} \subseteq V$. The next result shows that it induces a well-defined map on Π.

Theorem 2.11 ([McM89, Theorem 7])**.** *The map $\mathfrak{p} \mapsto \mathfrak{p}_v$ induces an endomorphism $x \mapsto x_v$ on* Π*, defined on generators by*

$$[\mathfrak{p}] \mapsto [\mathfrak{p}]_v := [\mathfrak{p}_v].$$

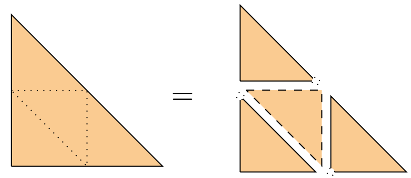
This endomorphism commutes with nonnegative dilations.

In particular, the morphism $x \mapsto x_v$ restricts to each graded component Ξ_r.

Example 2.12. Let $\mathfrak{p}$ be a 2-simplex. Expanding the power series for log, we get

$$2\log[\mathfrak{p}] = 4[\mathfrak{p}] - [2\mathfrak{p}] - 3.$$

Consider the decomposition of $2\mathfrak{p}$ shown below.



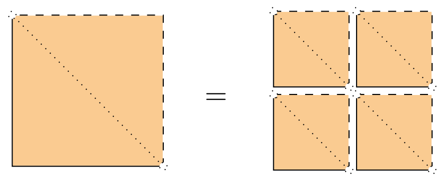
Note that the three copies of $\mathfrak{p}$ have a missing vertex, and only the interior of $-\mathfrak{p}$ appears. It shows that $[2\mathfrak{p}] = 3([\mathfrak{p}] - 1) + [-\mathfrak{p}]^*$. Therefore,

$$\log[\mathfrak{p}] = \frac{1}{2}\big([\mathfrak{p}] - [-\mathfrak{p}]^*\big).$$

Furthermore, using (2.8) one gets

$$(\log[\mathfrak{p}])^2 = \frac{1}{4}([2\mathfrak{p}] + [-2\mathfrak{p}]^* - 2) = [\mathfrak{p}] + [-\mathfrak{p}]^* - 2.$$

The last equality follows from the picture below. It also reflects the fact that $[\mathfrak{p}] + [-\mathfrak{p}]^* - 2 \in \Xi_2$, since it is an eigenvector of δ_2 of eigenvalue $4 = 2^2$.



2.4. **Subalgebra relative to a fixed polytope.** Fix a d-dimensional polytope $\mathfrak{p} \subseteq V$. Let $\Pi(\mathfrak{p})$ be the subalgebra of $\Pi(V)$ generated by the classes of Minkowski summands of $\mathfrak{p}$. A result of Shephard [Grü03, Section 15.2.7] implies that a polytope $\mathfrak{q}$ is a deformation of $\mathfrak{p}$ if and only if some small enough positive dilation of $\mathfrak{q}$ is a Minkowski summand of $\mathfrak{p}$. Consequently, $\Pi(\mathfrak{p})$ is the subalgebra generated by the classes of all deformations of $\mathfrak{p}$. In particular, the algebra $\Pi(\mathfrak{p})$ depends only on $\Sigma_{\mathfrak{p}}$.

The grading of $\Pi(V)$ induces a grading of $\Pi(\mathfrak{p})$. We let $\Xi_r(\mathfrak{p}) = \Pi(\mathfrak{p}) \cap \Xi_r(V)$. The dimension of these spaces was described by McMullen in the case of simple polytopes.

Theorem 2.13 ([McM93a, Theorem 6.1])**.** *Let $\mathfrak{p}$ be a d-dimensional simple polytope. Then,*

$$\dim(\Xi_r(\mathfrak{p})) = h_r(\mathfrak{p})$$

for $r = 0, 1, \ldots, d$.

Let $\mathfrak{f}$ be a face of $\mathfrak{p}$ and $v \in \operatorname{relint}\big(N(\mathfrak{f}, \mathfrak{p})\big)$. The maximization operator $x \mapsto x_v$ defines a morphism

$$\psi_{\mathfrak{f}} : \Pi(\mathfrak{p}) \to \Pi(\mathfrak{f}) \tag{2.9}$$

that only depends on the face $\mathfrak{f}$.

First observe that this map is well defined; that is, $[\mathfrak{q}_v] \in \Pi(\mathfrak{f})$ for every generator $[\mathfrak{q}]$ of $\Pi(\mathfrak{p})$. Indeed, if $\mathfrak{q}$ is a summand of $\mathfrak{p}$, say $\mathfrak{p} = \mathfrak{q} + \mathfrak{q}'$, then

$$\mathfrak{f} = \mathfrak{p}_v = \mathfrak{q}_v + \mathfrak{q}'_v,$$

so $\mathfrak{q}_v$ is a Minkowski summand of $\mathfrak{f}$. Moreover, since the normal fan of $\mathfrak{p}$ refines that of $\mathfrak{q}$, $\mathfrak{q}_w = \mathfrak{q}_v$ for any other $w \in \operatorname{relint}\big(N(\mathfrak{f}, \mathfrak{p})\big)$. Therefore the morphism (2.9) only depends on $\mathfrak{f}$ and not on the particular choice of v.

Theorem 2.14 ([McM93a, Theorem 2.4])**.** *Let $\mathfrak{p}$ be a simple polytope and $\mathfrak{f}$ a face of $\mathfrak{p}$. Then, the morphism $\psi_{\mathfrak{f}}$ is surjective.*

It is worth noting that $\psi_{\mathfrak{f}}$ is a morphism of graded algebras. This is a consequence of Theorem 2.11.

3. The Tits algebra of a linear hyperplane arrangement

3.1. **Basic definitions.** Let $\mathcal{A}$ be a (finite) linear hyperplane arrangement in V. A subspace of V obtained as the intersection of some hyperplanes in $\mathcal{A}$ is called a *flat*. The set of flats of $\mathcal{A}$ is denoted by $\mathcal{L}[\mathcal{A}]$, and it forms a graded lattice ordered by inclusion. The ambient space is the top element of $\mathcal{L}[\mathcal{A}]$, we denote it by $\top$. The intersection of all the hyperplanes in $\mathcal{A}$ is the minimal element of $\mathcal{L}[\mathcal{A}]$, we denote it by $\perp$.

The *characteristic polynomial* of $\mathcal{A}$ is

$$\chi(\mathcal{A}, t) := \sum_{\mathrm{X} \in \mathcal{L}[\mathcal{A}]} \mu(\mathrm{X}, \top)\, t^{\dim(\mathrm{X})},$$

where $\mu(\mathrm{X}, \top)$ denotes the Möbius function of $\mathcal{L}[\mathcal{A}]$. It is a monic polynomial of degree $\dim(V)$.

The *arrangement under a flat* X is the following collection of hyperplanes in ambient space X

$$\mathcal{A}^{\mathrm{X}} = \{\mathrm{H} \cap \mathrm{X} \,:\, \mathrm{H} \in \mathcal{A},\, \mathrm{X} \not\subseteq \mathrm{H}\}.$$

The hyperplanes in $\mathcal{A}$ split V into a collection $\Sigma[\mathcal{A}]$ of polyhedral cones called *faces*. Explicitly, the complement in V of the union of hyperplanes in $\mathcal{A}$ is the disjoint union of open subsets of V; and $\Sigma[\mathcal{A}]$ is the collection of the closures of these regions together with all their faces. $\Sigma[\mathcal{A}]$ is a poset under containment, its maximal elements are called *chambers*. It has a minimum element O, it coincides with the minimal flat $\perp$ of the arrangement. See Figure 3.1 for an example.

The support of a face F is the smallest flat $\mathrm{s}(F)$ containing it. It coincides with the linear span of F. The support map

$$\mathrm{s} : \Sigma[\mathcal{A}] \to \mathcal{L}[\mathcal{A}] \tag{3.1}$$

is surjective and order preserving.

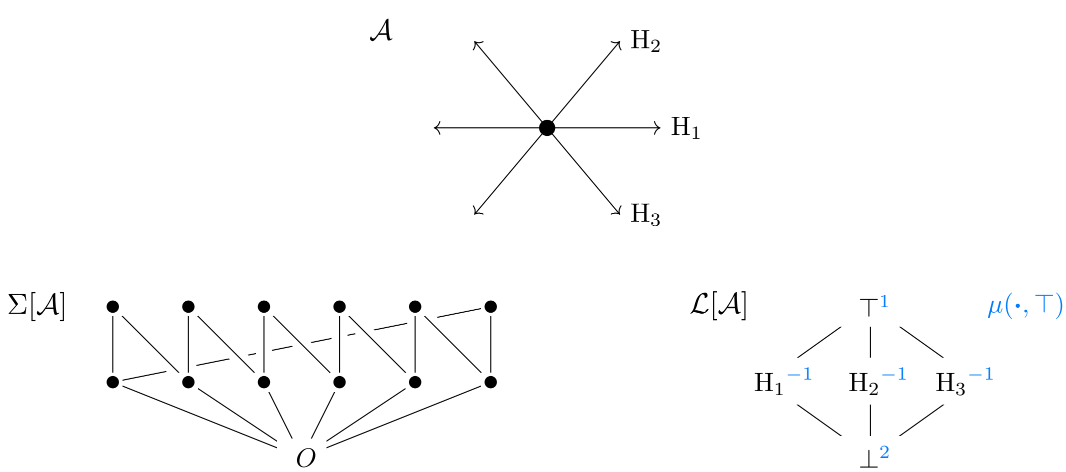


Figure 3.1. A 2-dimensional arrangement $\mathcal{A}$ (top) together with its poset of faces (left) and lattice of flats (right). The Möbius function of the lattice of flats is shown in blue. The characteristic polynomial of $\mathcal{A}$ is $\chi(\mathcal{A}; t) = t^2 - 3t + 2$.

3.2. **The Tits algebra.** The collection of faces of a hyperplane arrangement has the structure of a monoid. The product of two faces F and G, denoted FG, is the first face you encounter after moving a small positive distance from an interior point of F to an interior point of G, as illustrated in Figure 3.2. This product turns $\Sigma[\mathcal{A}]$ into a monoid, with unit O. One can easily verify the following properties. The first says that $\Sigma[\mathcal{A}]$ is a *left regular band* (LRB). For any faces F, G of $\mathcal{A}$:

$$FGF = FG \tag{3.2a}$$

$$FG = F \iff \mathrm{s}(G) \leq \mathrm{s}(F) \tag{3.2b}$$

$$FG = G \iff F \leq G \tag{3.2c}$$

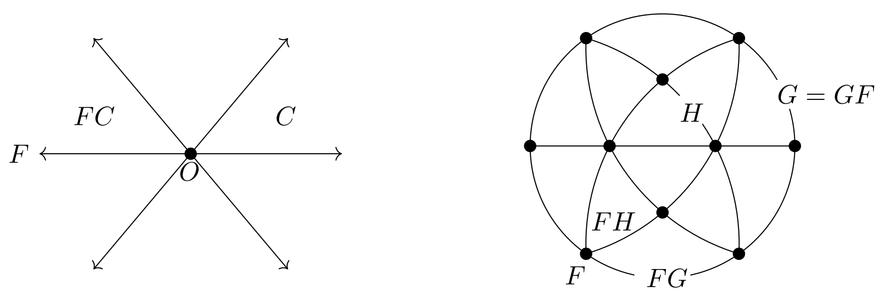


FIGURE 3.2. Product of faces in arrangements of rank 2 (left) and 3 (right). The arrangement of rank 3 is intersected with a sphere around the origin. F is a ray, G, H and FG are walls, and FH is a chamber.

The linearization $\Sigma[\mathcal{A}] := \mathbb{R}\Sigma[\mathcal{A}]$ of this monoid is the *Tits algebra* of $\mathcal{A}$. See [AM17, Chapters 1 and 9] for more details. We let $\mathtt{H}_F$ denote the basis element of $\Sigma[\mathcal{A}]$ associated with the face F of $\mathcal{A}$.

We view $\mathcal{L}[\mathcal{A}]$ as a commutative monoid with the join operation for the product. This makes the support map (3.1) a morphism of monoids. We let $\mathtt{H}_{\mathrm{X}}$ denote the basis element of $\mathbb{R}\mathcal{L}[\mathcal{A}]$ associated with the flat X of $\mathcal{A}$. $\mathbb{R}\mathcal{L}[\mathcal{A}]$ is the monoid algebra of $\mathcal{L}[\mathcal{A}]$, and the product between basis elements is given by

$$\mathtt{H}_{\mathrm{X}} \cdot \mathtt{H}_{\mathrm{Y}} = \mathtt{H}_{\mathrm{X} \vee \mathrm{Y}}.$$

A result of Solomon [Sol67, Theorem 1] shows that $\mathbb{R}\mathcal{L}[\mathcal{A}]$ is a split-semisimple algebra. This rests on the fact that the unique complete system of orthogonal idempotents for $\mathbb{R}\mathcal{L}[\mathcal{A}]$ consists of elements $\mathtt{Q}_{\mathrm{X}}$ uniquely determined by

$$\mathtt{Q}_{\mathrm{X}} = \sum_{\mathrm{Y}\colon \mathrm{Y} \geq \mathrm{X}} \mu(\mathrm{X}, \mathrm{Y}) \mathtt{H}_{\mathrm{Y}} \quad \text{or equivalently} \quad \mathtt{H}_{\mathrm{X}} = \sum_{\mathrm{Y}\colon \mathrm{Y} \geq \mathrm{X}} \mathtt{Q}_{\mathrm{Y}}.$$

3.3. **Modules over the Tits algebra.** The simple modules over a split-semisimple algebra are 1-dimensional and are in correspondence to its complete system of orthogonal idempotents, see for instance [AM17, Appendix D.3]. The module of $\mathbb{R}\mathcal{L}[\mathcal{A}]$ associated with the flat X is $\langle \mathtt{Q}_{\mathrm{X}} \rangle \subseteq \mathbb{R}\mathcal{L}[\mathcal{A}]$. Hence, the character χ_{X} associated with X is determined by

$$\chi_{\mathrm{X}}(\mathtt{Q}_{\mathrm{Y}}) = \begin{cases} 1 & \text{if } \mathrm{Y} = \mathrm{X}, \\ 0 & \text{otherwise.} \end{cases} \qquad \text{equivalently} \qquad \chi_{\mathrm{X}}(\mathtt{H}_{\mathrm{Y}}) = \begin{cases} 1 & \text{if } \mathrm{Y} \leq \mathrm{X}, \\ 0 & \text{otherwise.} \end{cases}$$

Let h be an arbitrary module over $\mathbb{R}\mathcal{L}[\mathcal{A}]$ and χ_{h} the corresponding character. For every flat X define

$$\xi_{\mathrm{X}}(\mathsf{h}) = \chi_{\mathsf{h}}(\mathtt{H}_{\mathrm{X}}) \quad \text{and} \quad \eta_{\mathrm{X}}(\mathsf{h}) = \chi_{\mathsf{h}}(\mathtt{Q}_{\mathrm{X}}). \tag{3.3}$$

The relation between the $\mathtt{H}$-basis and the $\mathtt{Q}$-basis imply that

$$\eta_{\mathrm{X}}(\mathsf{h}) = \sum_{\mathrm{Y}\,:\,\mathrm{Y}\geq\mathrm{X}} \mu(\mathrm{X},\mathrm{Y})\xi_{\mathrm{Y}}(\mathsf{h}). \tag{3.4}$$

Moreover, since the elements $\mathtt{H}_{\mathrm{X}}$ and $\mathtt{Q}_{\mathrm{X}}$ are idempotent,

$$\chi_{\mathsf{h}}(\mathtt{H}_{\mathrm{X}}) = \dim(\mathsf{h}\cdot\mathtt{H}_{\mathrm{X}}) \quad \text{and} \quad \chi_{\mathsf{h}}(\mathtt{Q}_{\mathrm{X}}) = \dim(\mathsf{h}\cdot\mathtt{Q}_{\mathrm{X}})$$

It follows that $\eta_{\mathrm{X}}(\mathsf{h})$ is the number of copies in h of the simple representation associated with $\mathtt{Q}_{\mathrm{X}}$. That is,

$$\mathsf{h} \cong \bigoplus_{\mathrm{X}} \langle \mathtt{Q}_{\mathrm{X}} \rangle^{\oplus \eta_{\mathrm{X}}(\mathsf{h})}$$

The algebra $\mathbb{R}\mathcal{L}[\mathcal{A}]$ is the maximal split-semisimple quotient of $\Sigma[\mathcal{A}]$ via the support map. It follows that the simple modules over $\Sigma[\mathcal{A}]$ are obtained from those of $\mathbb{R}\mathcal{L}[\mathcal{A}]$ factoring the action through the support map. That is, the simple modules over $\Sigma[\mathcal{A}]$ are 1-dimensional and indexed by flats. The character χ_{X} associated with the flat X evaluated on an element

$$w = \sum_{F} w^{F}\mathtt{H}_{F}$$

of $\Sigma[\mathcal{A}]$ yields

$$\chi_{\mathrm{X}}(w) = \sum_{F\,:\,\mathrm{s}(F)\leq\mathrm{X}} w^{F}. \tag{3.5}$$

Now, let h be an arbitrary right module over the Tits algebra $\Sigma[\mathcal{A}]$, and χ_{h} the corresponding character. Then, as with simple modules, χ_{h} factors through the support map. For a flat X, define $\xi_{\mathrm{X}}(\mathsf{h})$ and $\eta_{\mathrm{X}}(\mathsf{h})$ as in (3.3). It follows that

$$\xi_{\mathrm{X}}(\mathsf{h}) = \chi_{\mathsf{h}}(\mathtt{H}_{F}) = \dim(\mathsf{h}\cdot\mathtt{H}_{F})$$

where F is any face of support X. This is well defined since the character factors through the support map, but it can also be checked directly: $\mathsf{h}\cdot\mathtt{H}_{F} = \mathsf{h}\cdot\mathtt{H}_{G}$ whenever $\mathrm{s}(F) = \mathrm{s}(G)$. This is a consequence of (3.2a) and (3.2b).

The module h does not necessarily decompose as the sum of simple modules, so we consider composition series of h instead. The integer $\eta_{\mathrm{X}}(\mathsf{h})$ is the number of times the simple module with multiplicative character χ_{X} appears as a composition factor in a composition series of h.

3.4. **Characteristic elements and diagonalization.** Let t be a fixed scalar. An element w of the Tits algebra is *characteristic of parameter* t if for each flat X

$$\chi_{\mathrm{X}}(w) = t^{\dim(\mathrm{X})},$$

with $\chi_{\mathrm{X}}(w)$ as in (3.5). Characteristic elements determine the characteristic polynomial of the arrangement and also determine the characteristic polynomial of the arrangements under each flat. See [AM17, Section 12.4] and [ABM19] for more information.

We say that a scalar t is *non-critical* if t is not a root of $\chi(\mathcal{A}^{\mathrm{X}};t)$ for any flat X. A characteristic element w of non-critical parameter t uniquely determines a family of *Eulerian idempotents* $\mathtt{E} = \{\mathtt{E}_{\mathrm{X}}\}_{\mathrm{X}}$, which satisfies

$$w = \sum_{\mathrm{X}} t^{\dim(\mathrm{X})}\mathrm{E}_{\mathrm{X}}. \tag{3.6}$$

This is a consequence of [AM17, Propositions 11.9, 12.59]. A complete system of orthogonal idempotents $\mathtt{E} = \{\mathtt{E}_{\mathrm{X}}\}_{\mathrm{X}}$ of $\Sigma[\mathcal{A}]$ is an *Eulerian family* if each $\mathtt{E}_{\mathrm{X}}$ is of the form

$$\mathtt{E}_{\mathrm{X}} = \sum_{F\,:\,\mathrm{s}(F)\geq \mathrm{X}} a^F \mathtt{H}_F,$$

with $a^F \neq 0$ for at least one F with $\mathrm{s}(F) = \mathrm{X}$. By definition, they satisfy

$$\mathtt{E}_{\mathrm{X}}\mathtt{E}_{\mathrm{Y}} = \begin{cases} \mathtt{E}_{\mathrm{X}} & \text{if } \mathrm{X} = \mathrm{Y}, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{and} \qquad \sum_{\mathrm{X}} \mathtt{E}_{\mathrm{X}} = \mathtt{H}_O. \tag{3.7}$$

It follows that $\mathrm{s}(\mathtt{E}_{\mathrm{X}}) = \mathtt{Q}_{\mathrm{X}}$, and that $\mathtt{Q}_{\mathrm{X}} \mapsto \mathtt{E}_{\mathrm{X}}$ defines an algebra section $\mathbb{R}\mathcal{L}[\mathcal{A}] \to \Sigma[\mathcal{A}]$ of the support map. Since the corresponding characteristic element w is on the image of this section, its action on any $\Sigma[\mathcal{A}]$-module is diagonalizable. Let us make this explicit.

Let h be a right $\Sigma[\mathcal{A}]$-module. It follows from (3.7) that h has the following decomposition as a vector space:

$$\mathsf{h} = \bigoplus_{\mathrm{X}} \mathsf{h} \cdot \mathtt{E}_{\mathrm{X}}.$$

Moreover, the presentation of w as a weighted sum of Eulerian idempotents (3.6) shows that it acts on $\mathsf{h} \cdot \mathtt{E}_{\mathrm{X}}$ by multiplication by $t^{\dim(\mathrm{X})}$. Hence, the dimension of the eigenspace of the action of w with eigenvalue t^k is

$$\sum_{\dim(\mathrm{X})=k} \dim(\mathsf{h} \cdot \mathtt{E}_{\mathrm{X}}) = \sum_{\dim(\mathrm{X})=k} \chi_{\mathsf{h}}(\mathtt{E}_{\mathrm{X}}) = \sum_{\dim(\mathrm{X})=k} \chi_{\mathsf{h}}(\mathtt{Q}_{\mathrm{X}}) = \sum_{\dim(\mathrm{X})=k} \eta_{\mathrm{X}}(\mathsf{h}).$$

All these results have an identical counterpart for left modules. This is summed up in the following proposition.

Proposition 3.1. *Let h be a (left or right) module over $\Sigma[\mathcal{A}]$ and $w \in \Sigma[\mathcal{A}]$ a characteristic element of non-critical parameter t. Then, the action of w on h is diagonalizable. Furthermore, the eigenvalues of the action of w on h are*

$$t^k \qquad \textit{with multiplicity} \qquad \sum_{\dim(\mathrm{X})=k} \eta_{\mathrm{X}}(\mathsf{h}),$$

for $k = \dim(\perp), \dots, \dim(\mathcal{A})$.

4. Coxeter arrangements, permutation statistics and Eulerian polynomials

We will review some combinatorial aspects of the braid (type A) and signed braid (type B) arrangements. These are the reflection arrangements of the symmetric group $\mathfrak{S}_d$ and the hyperoctahedral group $\mathfrak{B}_d$, respectively. We make an explicit identification between flats of these arrangements and the corresponding notion of set partition. See [AM17, Sections 6.6 and 6.7] for further details. Some relevant statistics on elements of $\mathfrak{S}_d$ and $\mathfrak{B}_d$ are reviewed.

4.1. **Braid arrangement and symmetric group.** The *braid arrangement* $\mathcal{A}_d$ in $\mathbb{R}^d$ consists of the diagonal hyperplanes $x_i = x_j$ for $1 \leq i < j \leq d$. The central face is the line $x_1 = \cdots = x_d$. Intersecting with a sphere around the origin in the hyperplane $x_1 + \cdots + x_d = 0$ we obtain the *Coxeter complex of type A_{d-1}*. The pictures below show the cases $d = 3$ and 4. Points with the same color correspond to rays (1-dimensional faces) of $\mathcal{A}_d$ in the same $\mathfrak{S}_d$-orbit.

A *weak set partition* of a finite set I is a collection $\mathrm{X} = \{S_1, \dots, S_k\}$ of disjoint subsets whose union is I. The subsets S_i are the *blocks* of X. A *set partition* is a weak set partition with no empty blocks. We write $\mathrm{X} \vdash I$ to denote that X is a partition of I. If T is a union of blocks of X, we let $\mathrm{X}|_S \vdash S$ denote the corresponding partition of S.

Given a partition $\mathrm{X} \vdash [d]$, the corresponding flat of the braid arrangement is the intersection of the hyperplanes $x_i = x_j$ for i, j in the same block of X. Its dimension equals the number of blocks of X. We also use X to denote the corresponding flat. Hence, $\mathrm{X} \leq \mathrm{Y}$ if and only if the blocks of X are union of blocks of Y.

Let $\mathrm{X} = \{S_1, \dots, S_k\}$. The Möbius function of $\mathcal{L}[\mathcal{A}]$ is determined by

$$\mu(\perp, \mathrm{X}) = (-1)^{k-1}(k-1)!.$$

Using that a partition $\mathrm{Y} \geq \mathrm{X}$ corresponds to a partition $\mathrm{Y}|_{S_i}$ of each block S_i of X, we also have

$$\mu(\mathrm{X}, \mathrm{Y}) = \mu(\perp, \mathrm{Y}|_{S_1}) \dots \mu(\perp, \mathrm{Y}|_{S_k}). \tag{4.1}$$

A *set composition* of I is an ordered set partition $F = (S_1, \dots, S_k)$. We write $F \vDash I$ to denote that F is a composition of I, and let $\mathrm{s}(F) \vdash I$ be the underlying (unordered) set partition. Given a set composition $F \vDash [d]$, the corresponding face of the braid arrangement consists of points $x \in \mathbb{R}^d$ satisfying:

- $x_i = x_j$ whenever i, j belong to the same block of F,
- $x_i > x_j$ whenever the block containing i precedes the block containing j.

We also use F to denote the corresponding face. Under these identifications, the two possible definitions of $\mathrm{s}(F)$, as a flat or as a partition, agree.

4.1.1. *The symmetric group and Eulerian polynomial.* The symmetric group $\mathfrak{S}_d$ is the group of *permutations* $\sigma : [d] \to [d]$ under composition. It acts on $\mathbb{R}^d$ by permuting coordinates:

$$\sigma(x_1, x_2, \dots, x_d) = (x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(d)}).$$

We let $\mathrm{s}(\sigma)$ denote the subspace of fixed points by the action of σ; it is a flat of $\mathcal{A}_d$. Under the identification above, $\mathrm{s}(\sigma)$ is the partition of $[d]$ underlying the cycle decomposition of σ. Given a finite set S, we let $\mathfrak{C}(S)$ denote the collection of cyclic permutations on S. For a block $S \in \mathrm{s}(\sigma)$, we let $\sigma|_S \in \mathfrak{C}(S)$ be the restriction of σ to S.

We present some statistics on elements of $\mathfrak{S}_d$ that we will use in subsequent sections. For $\sigma \in \mathfrak{S}_d$, define

$$\mathrm{Des}(\sigma) = \{i \in [d-1] \,:\, \sigma(i) > \sigma(i+1)\} \qquad \mathrm{des}(\sigma) = \big|\mathrm{Des}(\sigma)\big|$$
$$\mathrm{Exc}(\sigma) = \{i \in [d-1] \,:\, \sigma(i) > i\} \qquad \mathrm{exc}(\sigma) = \big|\mathrm{Exc}(\sigma)\big|$$

Elements in the sets above are the *descents* and *excedances* of σ, respectively. It is a classical result that descents and excedances are *equidistributed* in $\mathfrak{S}_d$. That is,

$$\big|\{\sigma \in \mathfrak{S}_d \,:\, \mathrm{des}(\sigma) = k\}\big| = \big|\{\sigma \in \mathfrak{S}_d \,:\, \mathrm{exc}(\sigma) = k\}\big|,$$

for all possible values of k. Foata's *fundamental transformation* provides a simple proof of this result.

The *Eulerian polynomial* $A_d(z)$ keeps track of the distribution of descents, or excedances, in $\mathfrak{S}_d$:

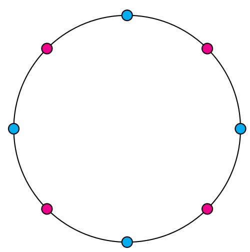
$$A_d(z) = \sum_{k=0}^{d-1} A_{d,k} z^k = \sum_{\sigma \in \mathfrak{S}_d} z^{\mathrm{exc}(\sigma)}.$$

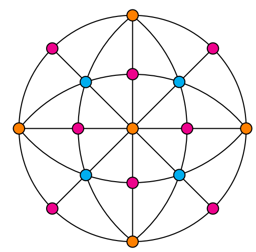
That is, the coefficient $A_{d,k}$ counts the number of permutations of $[d]$ with exactly k excedances. These coefficients are called *Eulerian numbers* (OEIS: A008292). The exponential generating function for these polynomials was originally given by Euler himself:

$$A(z,x) = \sum_{d \geq 0} A_d(z) \frac{x^d}{d!} = \frac{z-1}{z - e^{x(z-1)}}. \tag{4.2}$$

See [Foa10, Section 3] for a derivation of this formula.

4.2. **Signed braid arrangement and hyperoctahedral group.** The Coxeter arrangement of type B, or *signed braid arrangement* $\mathcal{A}_d^{\pm}$ in $\mathbb{R}^d$ consists of the hyperplanes $x_i = x_j$, $x_i = -x_j$ for $1 \leq i < j \leq d$ and $x_k = 0$ for $1 \leq k \leq d$. The *Coxeter complex of type B_d*, obtained by intersecting with the sphere in $\mathbb{R}^d$, is shown below for $d = 2$ and 3. Points with the same color correspond to rays of $\mathcal{A}_d^{\pm}$ in the same $\mathfrak{B}_d$-orbit.





Let I be a finite set with an fixed point free involution $x \mapsto \overline{x}$. For instance,

$$\pm[d] := \{-d, -d+1, \ldots, -1, 1, \ldots, d-1, d\} \quad \text{with involution } \overline{x} = -x.$$

A type B set partition X of I is a weak set partition satisfying two additional properties:

(1) There is one distinguished block $S_0 \in \mathrm{X}$ satisfying $\overline{S_0} = S_0$. It is called the *zero block* of X, and it might be empty.
(2) All the other blocks $S \in \mathrm{X} \setminus \{S_0\}$ are non-empty, satisfy $S \cap \overline{S} = \emptyset$ and $\overline{S} \in \mathrm{X}$. They are called the *nonzero blocks* of X

We write $\mathrm{X} \vdash^B I$ to denote that X is a type B partition of I. Note that the involution of I restricts to S_0, and that nonzero blocks come in pairs $S, \overline{S}$.

Convention 4.1. To simplify notation, we let $\mathrm{X} \vdash^B [d]$ denote that X is a type B partition of $\pm[d]$. When we write $\{S_0, S_1, \overline{S_1}, \ldots, S_k, \overline{S_k}\} \vdash^B [d]$, we assume that S_i contains the element $\max(S_i \sqcup \overline{S_i})$ for $1 \leq i \leq k$.

Given $\mathrm{X} \vdash^B [d]$, the corresponding flat of the braid arrangement is the intersection of the following hyperplanes:

$$x_i = 0 \text{ for each } i \in S_0$$
$$x_i = x_j \text{ for each } i, j \in [d] \text{ in the same block of X}$$
$$x_i = -x_j \text{ for each } i, j \in [d] \text{ with } i, -j \text{ in the same block of X}$$

For instance, the partition $\{1\bar{1}3\bar{3}, 2\bar{4}5, \bar{2}4\bar{5}, 67, \bar{6}\bar{7}\}$ corresponds to the flat of $\mathcal{A}_d^{\pm}$ consisting of points $x \in \mathbb{R}^d$ with:

$$x_1 = x_3 = 0, \qquad x_2 = -x_4 = x_5, \qquad x_6 = x_7.$$

If X has $2k+1$ blocks, then the dimension of the corresponding flat is k. We also write X for the corresponding flat. Hence, $\mathrm{X} \leq \mathrm{Y}$ if Y refines X.

Let $\mathrm{X} = \{S_0, S_1, \overline{S_1}, \ldots, S_k, \overline{S_k}\}$. The Möbius function of $\mathcal{L}[\mathcal{A}_d^{\pm}]$ is determined by

$$\mu(\perp, \mathrm{X}) = (-1)^k (2k-1)!!,$$

where $(2k-1)!!$ denotes the double factorial $(2k-1)!! = (2k-1)(2k-3)\ldots 1$. Using that a type B partition $\mathrm{Y} \geq \mathrm{X}$ is equivalent to a type B partition $\mathrm{Y}|_{S_0} \vdash^B S_0$ and partitions $\mathrm{Y}|_{S_i} \vdash S_i$ for $i = 1, \ldots, k$, we also have

$$\mu(\mathrm{X}, \mathrm{Y}) = \mu(\perp, \mathrm{Y}|_{S_0})\mu(\perp, \mathrm{Y}|_{S_1}) \ldots \mu(\perp, \mathrm{Y}|_{S_k}). \tag{4.3}$$

Observe that the first factor corresponds to the Möbius function of the type B arrangement and the remaining factors correspond to the Möbius function of the type A arrangement.

4.2.1. *The hyperoctahedral group.* The hyperoctahedral group $\mathfrak{B}_d$ is the group of *signed permutations* under composition. A signed permutation is a bijection $\sigma : \pm[d] \to \pm[d]$ satisfying $\sigma(i) = j$ if and only if $\sigma(\bar{i}) = \bar{j}$. Hence, the values $\sigma(1), \sigma(2), \ldots, \sigma(d)$ completely determine a signed permutation σ. The group $\mathfrak{B}_d$ acts on $\mathbb{R}^d$ by permutations and negation of coordinates:

$$\sigma(x_1, x_2, \ldots, x_d) = (x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(d)}),$$

where for $i \in [d]$, we let $x_{\bar{i}} = -x_i$. We let $\mathrm{s}(\sigma)$ denote the maximal point-fixed subspace by the action of σ; it is a flat of $\mathcal{A}_d^{\pm}$. Under the identification above, $\mathrm{s}(\sigma)$ is the type B partition of $[d]$ obtained from the underlying the cycle decomposition of σ by merging all the blocks that contain an element $i \in [d]$ and its negative $\bar{i}$.

Example 4.2. Let $\sigma \in \mathfrak{B}_6$ be defined by

$$\sigma(1) = 1 \quad \sigma(2) = \bar{2} \quad \sigma(3) = 4 \quad \sigma(4) = \bar{3} \quad \sigma(5) = \bar{6} \quad \sigma(6) = \bar{5}.$$

In cycle notation, we can write

$$\sigma = (1)(\bar{1})(2\bar{2})(34\bar{3}\bar{4})(5\bar{6})(\bar{5}6).$$

Then,

$$\mathrm{s}(\sigma) = \{2\bar{2}3\bar{3}4\bar{4}, 1, \bar{1}, 5\bar{6}, \bar{5}6\}.$$

Let $\sigma \in \mathfrak{B}_n$. The restriction $\sigma|_{S_0}$ to the zero block $S_0 \in \mathrm{s}(\sigma)$ is a signed permutation of S_0. Its action on $\mathbb{R}^{|S_0|/2}$ does not fix any nonzero vector, so $\mathrm{s}(\sigma|_{S_0}) = \perp$. For a nonzero block $S \in \mathrm{s}(\sigma)$, $\sigma|_S \in \mathfrak{C}(S)$ is a cyclic permutation of the elements in S. The restriction $\sigma|_{S \sqcup \overline{S}}$ is again a signed permutation, and it is completely determined by either $\sigma|_S$ or $\sigma|_{\overline{S}}$.

We present some statistics on signed permutations. For $\sigma \in \mathfrak{B}_d$, let

$$\begin{aligned}
\mathrm{Des}(\sigma) &= \{i \in [d-1] \cup \{0\} : \sigma(i) > \sigma(i+1)\} \quad \text{where } \sigma(0) := 0 & \mathrm{des}(\sigma) &= \big|\mathrm{Des}(\sigma)\big| \\
\mathrm{Exc}(\sigma) &= \{i \in [d-1] : \sigma(i) > i\} & \mathrm{exc}(\sigma) &= \big|\mathrm{Exc}(\sigma)\big| \\
\mathrm{Neg}(\sigma) &= \{i \in [d] : \sigma(i) < 0\} & \mathrm{neg}(\sigma) &= \big|\mathrm{Neg}(\sigma)\big| \\
& & \mathrm{fexc}(\sigma) &= 2\,\mathrm{exc}(\sigma) + \mathrm{neg}(\sigma)
\end{aligned}$$

Elements in the sets above are *descents*, *excedances* and *negations* of σ, respectively. The last statistic is called the *flag-excedance* of a signed permutation. We define one last statistic, the B-excedance of σ:

$$\mathrm{exc}_B(\sigma) = \lfloor \tfrac{\mathrm{fexc}(\sigma)+1}{2} \rfloor = \mathrm{exc}(\sigma) + \lfloor \tfrac{\mathrm{neg}(\sigma)+1}{2} \rfloor. \tag{4.4}$$

Foata and Han [FH09, Section 9] show that descents and B-excedances are equidistributed. That is,

$$\big|\{\sigma \in \mathfrak{B}_d : \mathrm{des}(\sigma) = k\}\big| = \big|\{\sigma \in \mathfrak{B}_d : \mathrm{exc}_B(\sigma) = k\}\big|,$$

for all possible values of k. The *type B Eulerian polynomial* $B_d(z)$ keeps track of the distribution of descents, or B-excedances, in $\mathfrak{B}_d$:

$$B_d(z) = \sum_{k=0}^{d} B_{d,k} z^k = \sum_{\sigma \in \mathfrak{B}_d} z^{\mathrm{exc}_B(\sigma)}.$$

The coefficients $B_{d,k}$ are the *Eulerian numbers of type B* (OEIS: A060187). The exponential generating function of these polynomials is first due to Brenti [Bre94, Theorem 3.4]. We will be interested in the *type B exponential generating function* of these polynomials:

$$B(z,x) = \sum_{d \geq 0} B_d(z) \frac{x^d}{(2d)!!} = \frac{(1-z)e^{x(1-z)/2}}{1 - ze^{x(1-z)}}, \tag{4.5}$$

where $(2d)!!$ is the double factorial $(2d)!! = (2d)(2d-2)\dots 2 = 2^d d!$. Substituting x by $2x$ we obtains Brenti's original formula.

5. McMullen module of the Tits algebra

Fix a hyperplane arrangement $\mathcal{A}$ in V. A *zonotope* of $\mathcal{A}$ is a polytope obtained as the Minkowski sum of segments $\mathfrak{l}_{\mathrm{H}}$ orthogonal to each hyperplane $\mathrm{H} \in \mathcal{A}$:

$$\mathfrak{z} = \sum_{\mathrm{H}} \mathfrak{l}_{\mathrm{H}}. \tag{5.1}$$

Consequently, the normal fan $\Sigma_{\mathfrak{z}}$ coincides with the collection of faces $\Sigma[\mathcal{A}]$ of the arrangement. We say that a polytope $\mathfrak{p}$ is a *generalized zonotope* of $\mathcal{A}$ it is a deformation of $\mathfrak{z}$.

It follows from (5.1) that the face $\mathfrak{z}_F$ of $\mathfrak{z}$ is a translate of

$$\mathfrak{z}_{\mathrm{X}} := \sum_{\mathrm{H}\,:\,\mathrm{H} \supseteq \mathrm{X}} \mathfrak{l}_{\mathrm{H}},$$

where $\mathrm{X} = \mathrm{s}(F)$. Hence, it is a Minkowski summand of $\mathfrak{z}$. In fact, this property characterizes zonotopes.

Lemma 5.1. *For a polytope $\mathfrak{p} \subseteq V$, the following are equivalent.*

i. *$\mathfrak{p}$ is a zonotope.*
ii. *Every face $\mathfrak{f} \leq \mathfrak{p}$ is a Minkowski summand of $\mathfrak{p}$.*
iii. *Every 1-dimensional face $\mathfrak{f} \leq \mathfrak{p}$ is a Minkowski summand of $\mathfrak{p}$.*

We now consider the algebra $\Pi(\mathfrak{z})$ introduced in Section 2.4. It is generated by the classes of generalized zonotopes of $\mathcal{A}$. Therefore, it only depends on the arrangement $\mathcal{A}$ and not on the particular choice of zonotope $\mathfrak{z}$.

Being a Minkowski summand is a transitive relation. Hence, the generators of $\Pi(\mathfrak{f})$ are also in $\Pi(\mathfrak{z})$ for any face $\mathfrak{f} \leq \mathfrak{z}$. That is, $\Pi(\mathfrak{f})$ is a subalgebra of $\Pi(\mathfrak{z})$. Moreover, if $\mathfrak{f} = \mathfrak{z}_v$ for some $v \in V$ and $\mathfrak{q}$ is a summand of $\mathfrak{f}$, then $\mathfrak{q}_v = \mathfrak{q}$. Therefore, the composition

$$\Pi(\mathfrak{f}) \hookrightarrow \Pi(\mathfrak{z}) \xrightarrow{\psi_{\mathfrak{f}}} \Pi(\mathfrak{f}),$$

where $\psi_{\mathfrak{f}}$ is the morphism (2.9), is the identity map. We have proved the following.

Proposition 5.2. *Let $\mathfrak{z}$ be a zonotope, not necessarily simple, and $\mathfrak{f}$ a face of $\mathfrak{z}$. Then, the morphism $\psi_{\mathfrak{f}}$ is surjective.*

Note that there is no natural morphism $\Pi(\mathfrak{f}) \to \Pi(\mathfrak{z})$ for arbitrary polytopes $\mathfrak{f} \leq \mathfrak{z}$.

Let F be a face of $\mathcal{A}$ and $\mathfrak{f}$ the corresponding face of $\mathfrak{z}$. We define the right multiplication of the basis element $\mathtt{H}_F \in \Sigma[\mathcal{A}]$ on $\Pi(\mathfrak{z})$ by means of the following composition:

$$\begin{tikzcd} \Pi(\mathfrak{z}) \arrow[r, two heads, "\psi_{\mathfrak{f}}"] \arrow[rr, bend right, "\cdot \mathtt{H}_F"'] & \Pi(\mathfrak{f}) \arrow[r, hook] & \Pi(\mathfrak{z}) \end{tikzcd}$$

That is, $[\mathfrak{q}] \cdot \mathtt{H}_F = [\mathfrak{q}_v]$ where $v \in \operatorname{relint}(F)$.

Proposition 5.3. *With this product, the algebra $\Pi(\mathfrak{z})$ is a right $\Sigma[\mathcal{A}]$-module. Moreover, each graded component $\Xi_r(\mathfrak{z})$ is a submodule of $\Pi(\mathfrak{z})$.*

Proof. The zero vector belongs to the central face O, so the action is clearly unital. Associativity follows from the following fact about polytopes [Grü03, Section 3.1.5]. If $\mathfrak{q} \subseteq V$ is a polytope and $v, w \in V$, then $(\mathfrak{q}_v)_w = \mathfrak{q}_{v+\lambda w}$ for any small enough $\lambda > 0$. Similarly, the definition of the Tits product is such that if $v \in \operatorname{relint}(F)$ and $w \in \operatorname{relint}(G)$, then $v + \lambda w \in \operatorname{relint}(FG)$ for any small enough $\lambda > 0$. Hence,

$$([\mathfrak{q}] \cdot \mathtt{H}_F) \cdot \mathtt{H}_G = (\mathfrak{q}_v)_w = \mathfrak{q}_{v+\lambda w} = [\mathfrak{q}] \cdot \mathtt{H}_{FG}.$$

It follows that this product gives $\Pi(\mathfrak{z})$ the structure of a right $\Sigma[\mathcal{A}]$-module.

Theorem 2.11 implies that nonnegative dilations are morphisms of modules. Moreover, for any $x \in \Xi_r(\mathfrak{z})$,

$$\delta_\lambda(x \cdot \mathtt{H}_F) = \delta_\lambda(x) \cdot \mathtt{H}_F = \lambda^r x \cdot \mathtt{H}_F = \lambda^r (x \cdot \mathtt{H}_F).$$

The characterization of the graded components Ξ_r in (2.6) then implies that $x \cdot \mathtt{H}_F \in \Xi_r(\mathfrak{z})$. Therefore, each graded component $\Xi_r(\mathfrak{z})$ is a right $\Sigma[\mathcal{A}]$-module. ☐

Remark 5.4. The characterization (2.6) not only shows that nonnegative dilations are morphisms of modules, but also that the action of basis elements $\mathtt{H}_F$ is by algebra morphisms.

Given the direct sum decomposition of $\Pi(\mathfrak{z})$ into proper modules $\Pi(\mathfrak{z}) = \bigoplus_r \Xi_r(\mathfrak{z})$, it is natural to ask about the module structure of each $\Xi_r(\mathfrak{z})$. Proposition 5.2 implies that

$$\xi_{\mathrm{X}}(\Xi_r(\mathfrak{z})) = \dim(\Xi_r(\mathfrak{z}) \cdot \mathtt{H}_F) = \dim(\Xi_r(\mathfrak{z}_{\mathrm{X}})),$$

where $F \in \Sigma[\mathcal{A}]$ is any face of support X. Hence,

$$\eta_{\mathrm{X}}(\Xi_r(\mathfrak{z})) = \sum_{\mathrm{Y} \geq \mathrm{X}} \mu(\mathrm{X}, \mathrm{Y}) \dim(\Xi_r(\mathfrak{z}_{\mathrm{Y}})).$$

If in addition $\mathcal{A}$ is a simplicial arrangement, Theorem 2.13 yields

$$\eta_{\mathrm{X}}(\Xi_r(\mathfrak{z})) = \sum_{\mathrm{Y} \geq \mathrm{X}} \mu(\mathrm{X}, \mathrm{Y}) h_r(\mathfrak{z}_{\mathrm{Y}}). \tag{5.2}$$

Let $w \in \Sigma[\mathcal{A}]$ be a characteristic element of non-critical parameter t, and let $\{\mathtt{E}_{\mathrm{X}}\}_{\mathrm{X}}$ be the corresponding Eulerian family. We say that an element $x \in \Pi(\mathfrak{z})$ is a double-eigenvector if it lies in one of the spaces

$$\bigoplus_{\dim(\mathrm{X}) = k} \Xi_r(\mathfrak{z}) \cdot \mathtt{E}_{\mathrm{X}},$$

for some r and k. Recall that en element is in this subspace if and only if

$$\delta_\lambda x = \lambda^r x \quad \text{and} \quad x \cdot w = t^k w$$

for any $\lambda > 0$, $\lambda \neq 1$.

5.1. **First example: the cube and the coordinate arrangement.** Let $\mathcal{C}_d$ be the coordinate arrangement in $\mathbb{R}^d$. It consists of the d coordinate hyperplanes $x_i = 0$ for $i = 1, \dots, d$. We identify the lattice of flats $\mathcal{L}[\mathcal{C}_d]$ with the (opposite) boolean algebra $2^{[d]}$ in the following manner:

$$S \subseteq [d] \longleftrightarrow \mathrm{X}_S := \bigcap_{i \in S} \{x \,:\, x_i = 0\}.$$

The flat X_S has codimension $|S|$. Note that $\mathrm{X}_S \leq \mathrm{X}_T$ if and only if $T \subseteq S$.

The d-cube $\mathfrak{c} = \mathfrak{c}_d = [0,1]^d$ a zonotope of $\mathcal{C}_d$. It is the Minkowski sum of the d line segments $\mathfrak{l}_i := \mathrm{Conv}(o, e_i)$ for $i = 1, \dots, d$. It is a simple polytope with h-vector $h(\mathfrak{c}_d, z) = (1+z)^d$. Furthermore, for any $S \subseteq [d]$ we have

$$\mathfrak{c}_{\mathrm{X}_S} = \sum_{i \in S} \mathfrak{l}_i \cong \mathfrak{c}_{|S|}.$$

Let us consider right $\Sigma[\mathcal{C}_d]$-module $\Pi(\mathfrak{c})$. For a flat X_S, formula (5.2) yields

$$\sum_r \eta_{\mathrm{X}_S}(\Xi_r(\mathfrak{c})) z^r = \sum_r \big(\sum_{T \subseteq S} \mu(T,S) h_r(\mathfrak{c}_{|T|}) \big) z^r = \sum_{T \subseteq S} (-1)^{|S|-|T|} (1+z)^{|T|} = z^{|S|}.$$

Hence,

$$\eta_{\mathrm{X}_S}(\Xi_r(\mathfrak{c})) = \begin{cases} 1 & \text{if } r = |S|, \\ 0 & \text{otherwise.} \end{cases}$$

That is, a series decomposition of $\Xi_r(\mathfrak{c})$ contains exactly one copy of the simple module indexed by X_S for every $S \in \binom{[d]}{r}$. We can go a step further.

For $t \neq 1$, consider the characteristic element $\gamma_t \in \Sigma[\mathcal{C}_d]$ introduced in [ABM19, Section 5.3]. It is defined by

$$\gamma_t^F = \begin{cases} (t-1)^{\dim(F)} & \text{if } F \text{ lies in the first orthant,} \\ 0 & \text{if not.} \end{cases}$$

For each flat X_S, there is exactly one face F_S in the first orthant whose support is X_S. Namely,

$$F_S = \big(\bigcap_{i \in S} \{x \,:\, x_i = 0\} \big) \cap \big(\bigcap_{i \notin S} \{x \,:\, x_i \geq 0\} \big).$$

We have $\mathrm{s}(F_S) = \mathrm{X}_S$ and $T \subseteq S$ if and only if $F_S \leq F_T$. A simple computation shows that the corresponding Eulerian family is determined by

$$\mathtt{E}_{\mathrm{X}_S} = \sum_{T \subseteq S} (-1)^{|S \setminus T|} \mathtt{H}_{F_T}.$$

For each $S \subseteq [d]$, define

$$y_S = \prod_{i \in S} \log[\mathfrak{l}_i] \in \Pi(\mathfrak{c}).$$

Lemma 2.8 shows that y_S is a nonzero element of $\Pi(\mathfrak{c})$.

We claim that $\{y_S\}_{S \subseteq [d]}$ is a basis of double-eigenvectors of $\Pi(\mathfrak{c})$. Explicitly, y_S is an eigenvector for the action of γ_t of eigenvalue $t^{d-|S|}$, and for the action of δ_λ of eigenvalue $\lambda^{|S|}$ ($\lambda > 0$). The second statement is clear, since $\log[\mathfrak{l}_i] \in \Xi_1(\mathfrak{c})$. Moreover, using that $\log[\mathfrak{l}_i] = [\mathfrak{l}_i] - 1$, we have

$$y_S = \prod_{i \in S} ([\mathfrak{l}_i] - 1) = \sum_{T \subseteq S} (-1)^{|S \setminus T|} [\mathfrak{c}_{\mathrm{X}_T}].$$

On the other hand, observe that

$$[\mathfrak{c}_{\mathrm{X}_S}] \cdot \mathtt{E}_{\mathrm{X}_S} = \sum_{T \subseteq S} (-1)^{|S \setminus T|} [\mathfrak{c}_{\mathrm{X}_S}] \cdot \mathtt{H}_{F_T} = \sum_{T \subseteq S} (-1)^{|S \setminus T|} [\mathfrak{c}_{\mathrm{X}_T}] = y_S.$$

Therefore,

$$y_S \in \Xi_r(\mathfrak{c}) \cap (\Pi(\mathfrak{c}) \cdot \mathtt{E}_{\mathrm{X}_S}) = \Xi_r(\mathfrak{c}) \cdot \mathtt{E}_{\mathrm{X}_S}.$$

The claim follows since $\dim(\mathrm{X}_S) = d - |S|$.

5.2. **Product of arrangements.** The Cartesian product of two arrangements $\mathcal{A}$ in V and $\mathcal{A}'$ in W is the following collection of hyperplanes in $V \oplus W$:

$$\mathcal{A} \times \mathcal{A}' = \{\mathrm{H} \oplus W \,:\, \mathrm{H} \in \mathcal{A}\} \cup \{V \oplus \mathrm{H} \,:\, \mathrm{H} \in \mathcal{A}'\}.$$

One can easily verify that $\Sigma[\mathcal{A} \times \mathcal{A}'] \cong \Sigma[\mathcal{A}] \times \Sigma[\mathcal{A}']$ as monoids. Hence, $\boldsymbol{\Sigma}[\mathcal{A} \times \mathcal{A}'] \cong \boldsymbol{\Sigma}[\mathcal{A}] \otimes \boldsymbol{\Sigma}[\mathcal{A}']$. In fact, it is also true that

$$\Pi(\mathfrak{z} \times \mathfrak{z}') \cong \Pi(\mathfrak{z}) \otimes \Pi(\mathfrak{z}'),$$

where $\mathfrak{z}$ and $\mathfrak{z}'$ are zonotopes of $\mathcal{A}$ and $\mathcal{A}'$, respectively, and therefore $\mathfrak{z} \times \mathfrak{z}'$ is a zonotope of $\mathcal{A} \times \mathcal{A}'$. Indeed, every generalized zonotope of $\mathcal{A} \times \mathcal{A}'$ is the Cartesian product of generalized zonotopes of $\mathcal{A}$ and $\mathcal{A}'$. The corresponding isomorphism is induced by

$$\begin{array}{ccc} \Pi(\mathfrak{z}) \otimes \Pi(\mathfrak{z}') & \to & \Pi(\mathfrak{z} \times \mathfrak{z}') \\ [\mathfrak{p}] \otimes [\mathfrak{q}] & \mapsto & [\mathfrak{p} \times \mathfrak{q}] \end{array}$$

The fact that this map is well-defined and a morphism of $\boldsymbol{\Sigma}[\mathcal{A} \times \mathcal{A}']$-modules follows from the ideas in Section 7.2.

6. Double diagonalization and Eulerian numbers

We study the double diagonalization problem for Coxeter arrangements of type A and B. Building on top of work by Björner [Bjö84], Brenti noticed that, for any Coxeter group W, the h-polynomial of the W-permutahedron (a zonotope of $\mathcal{A}$) is the corresponding W-Eulerian polynomial [Bre94, Theorem 2.3]. For the permutahedron and the type B permutahedron, these are the polynomials considered in 4.1 and 4.2, respectively.

6.1. **Type A.** Let $\mathcal{A} = \mathcal{A}_d$ be the braid arrangement in $\mathbb{R}^d$. The permutahedron $\pi = \pi_d \subseteq \mathbb{R}^d$ is the convex hull of the $\mathfrak{S}_d$-orbit the point $(1, 2, \dots, d)$. It is a zonotope of $\mathcal{A}$ and has dimension $d-1$. Deformations of π are called *generalized permutahedra*. For a flat/partition $\mathrm{X} = \{S_1, \dots, S_k\}$ of $\mathcal{A}$,

$$\pi_{\mathrm{X}} \cong \pi_{|S_1|} \times \cdots \times \pi_{|S_k|} \tag{6.1}$$

is a product of lower-dimensional permutahedra.

We consider the module $\Pi(\pi)$ as in Section 5. The main goal of this section will be to prove the following result.

Theorem 6.1. *For any flat $\mathrm{X} \in \mathcal{L}[\mathcal{A}_d]$ and $r = 0, 1, \dots, d-1$,*

$$\eta_{\mathrm{X}}(\Xi_r(\pi_d)) = \big|\{\sigma \in \mathfrak{S}_d \,:\, \mathrm{s}(\sigma) = \mathrm{X},\ \operatorname{exc}(\sigma) = r\}\big|.$$

The next lemma is an essential ingredient in the proof of Theorem 6.1.

Lemma 6.2.

$$\sum_{\{S_1,\dots,S_k\} \vdash [d]} \mu(\perp, \mathrm{X}) A_{|S_1|}(z) \cdot \ldots \cdot A_{|S_k|}(z) = \sum_{\sigma \in \mathfrak{C}(d)} z^{\operatorname{exc}(\sigma)}. \tag{6.2}$$

Proof. We will show that the exponential generating functions of both sides of (6.2) agree with the logarithm of $A(z,x)$ defined in (4.2). Note that both sums are empty in the case $d = 0$.

Recall that $\mu(\perp, \mathrm{X}) = (-1)^{k-1}(k-1)!$, where $k = |\mathrm{X}|$. Thus, a direct application of the Compositional Formula [Sta99, Theorem 5.1.4] shows that the exponential generating function of the LHS of (6.2) is the composition of

$$\sum_{d\geq 1}(-1)^{d-1}(d-1)!\frac{x^d}{d!} = \sum_{d\geq 1}(-1)^{d-1}\frac{x^d}{d} = \log(1+x)$$

with

$$\sum_{d\geq 1} A_d(z)\frac{x^d}{d!} = A(z,x) - A_0(z) = A(z,x) - 1,$$

namely $\log A(z,x)$.

On the other hand, grouping permutations with the same underlying partition $\mathrm{s}(\sigma)$, we obtain

$$A(z,x) = \sum_{d\geq 0}\Big(\sum_{\sigma\in\mathfrak{S}_d} z^{\mathrm{exc}(\sigma)}\Big)\frac{x^d}{d!} = \sum_{d\geq 0}\Big(\sum_{\mathrm{X}\vdash[d]}\Big(\sum_{\substack{\sigma\in\mathfrak{S}_d\\ \mathrm{s}(\sigma)=\mathrm{X}}} z^{\mathrm{exc}(\sigma)}\Big)\Big)\frac{x^d}{d!}.$$

Observe that for each partition X of $[d]$,

$$\sum_{\substack{\sigma\in\mathfrak{S}_d\\ \mathrm{s}(\sigma)=\mathrm{X}}} z^{\mathrm{exc}(\sigma)} = \prod_{S\in\mathrm{X}}\Big(\sum_{\sigma_S\in\mathfrak{C}(S)} z^{\mathrm{exc}(\sigma_S)}\Big). \tag{6.3}$$

Thus, the Exponential Formula [Sta99, Corollary 5.1.6] implies

$$A(z,x) = \exp\left(\sum_{d\geq 1}\Big(\sum_{\sigma\in\mathfrak{C}(d)} z^{\mathrm{exc}(\sigma)}\Big)\frac{x^d}{d!}\right).$$

Taking logarithms on both sides yields the result. □

The key idea of the proof is: if $F(x)$ is the generating function counting certain family of objects, each of which is a disjoint union of "irreducible" ones, then $\log F(x)$ is counts these "irreducible" objects. That is the case for permutations, where the "irreducible" permutations are the cyclic ones. We will explore an extension of these ideas for objects of type B, concretely for $\mathfrak{B}_d$, in the following section.

Proof of Theorem 6.1. The permutahedron is a simple polytope. Hence, Theorem 2.13, (6.1) and the multiplicativity of the h polynomial yield

$$\sum_r \xi_{\mathrm{X}}(\Xi_r(\pi))z^r = \sum_r \dim(\Xi_r(\pi_{\mathrm{X}}))z^r = h(\pi_{\mathrm{X}}, z) = A_{|S_1|}(z)\cdot\ldots\cdot A_{|S_k|}(z),$$

where $\mathrm{X} = \{S_1,\ldots,S_k\}$. The relation between ξ and η in (3.4) yields

$$\sum_r \eta_{\mathrm{X}}(\Xi_r(\pi))z^r = \sum_{\mathrm{Y}\,:\,\mathrm{Y}\geq\mathrm{X}} \mu(\mathrm{X},\mathrm{Y})\Big(\sum_r \xi_{\mathrm{Y}}(\Xi_r(\pi))z^r\Big).$$

Using (4.1), we can rewrite the expression above as

$$\sum_r \eta_{\mathrm{X}}(\Xi_r(\pi))z^r = \prod_{S\in\mathrm{X}}\Big(\sum_{\mathrm{Y}=\{T_1,\ldots,T_\ell\}\vdash S} \mu(\perp,\mathrm{Y})A_{|T_1|}(z)\cdot\ldots\cdot A_{|T_\ell|}(z)\Big).$$

Applying Lemma 6.2 and relation (6.3) to this expression, we get

$$\sum_r \eta_{\mathrm{X}}(\Xi_r(\pi))z^r = \prod_{S\in\mathrm{X}}\Big(\sum_{\sigma\in\mathfrak{C}(S)} z^{\mathrm{exc}(\sigma)}\Big) = \sum_{\substack{\sigma\in\mathfrak{S}_d\\ \mathrm{s}(\sigma)=\mathrm{X}}} z^{\mathrm{exc}(\sigma)}.$$

Finally, taking the coefficient of z^r on both sides yields the result. □

Adding over all flats with the same dimension in Theorem 6.1, we conclude the following.

Corollary 6.3. *Let $w \in \Sigma[\mathcal{A}]$ be a characteristic element of non-critical parameter t. Then, the multiplicity of the eigenvalue t^k on $\Xi_r(\pi_d)$ is*

$$\big|\{\sigma \in \mathfrak{S}_d \,:\, |\operatorname{s}(\sigma)| = k,\ \operatorname{exc}(\sigma) = r\}\big|.$$

It follows from the proof of Lemma 6.2 that the exponential generating function of the polynomials

$$\sum_{\sigma \in \mathfrak{S}_d} t^{|\operatorname{s}(\sigma)|} z^{\operatorname{exc}(\sigma)}$$

is

$$\exp(t \log A(z,x)) = A(z,x)^t.$$

This generating function was already discovered by Brenti [Bre00, Proposition 7.3].

6.1.1. *Double-eigenbasis for the Adams element.* Perhaps the most natural characteristic element for the braid arrangement is the *Adams element*

$$\alpha_t = \sum_F \binom{t}{\dim(F)} \mathtt{H}_F.$$

It is invariant with respect to the action of $\mathfrak{S}_d$, and it is closely related with the *convolution powers of the identity map* of a Hopf monoid. The corresponding Eulerian idempotents are [AM17, Theorem 12.75]

$$\mathtt{E}_{\mathrm{X}} = \frac{1}{\dim(\mathrm{X})!} \sum_{\operatorname{s}(F)=\mathrm{X}} \sum_{G \geq F} \frac{(-1)^{\dim(G/F)}}{\deg(G/F)} \mathtt{H}_G,$$

where $\dim(G/F) = \dim(G) - \dim(F)$ and,

$$\deg(G/F) = \prod_{S \in F} \big|G|_S\big|.$$

Theorem 6.1 suggest the existence of a natural basis for $\Xi_r(\pi) \cdot \mathtt{E}_{\mathrm{X}}$ indexed by permutations σ with r excedances and $\operatorname{s}(\sigma) = \mathrm{X}$. In this section we will construct a candidate for such basis.

The standard simplex $\Delta_{[d]} \subseteq \mathbb{R}^d$ is the convex hull of the standard basis $\{e_1, \dots, e_d\}$ of $\mathbb{R}^d$. For a nonempty subset $S \subseteq [d]$, we let Δ_S denote the following face of $\Delta_{[d]}$:

$$\Delta_S = \operatorname{Conv}\{e_i \,:\, i \in S\}.$$

Ardila, Benedetti and Doker showed [ABD10, Proposition 2.4] that every generalized permutahedron $\mathfrak{p}$ can be written uniquely as a *signed Minkowski sum* of simplices

$$\mathfrak{p} = \sum_{J \subseteq [d]} y_J \Delta_J,$$

meaning that

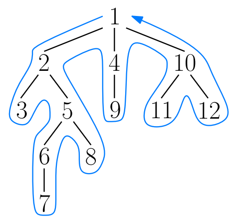
$$\mathfrak{p} + \sum_{y_J < 0} |y_J| \Delta_J = \sum_{y_J > 0} y_J \Delta_J.$$

Thus,

$$\log[\mathfrak{p}] = \sum_{J \subseteq [d]} y_J \log[\Delta_J].$$

Recall that $\log[\Delta_J] = 0$ if $\big|J\big| = 1$, and that $[\mathfrak{p}] = [\mathfrak{q}]$ if and only if $\mathfrak{p}$ is a translate of $\mathfrak{q}$. We conclude that $\{\log[\Delta_J] \,:\, J \subseteq I,\ \big|J\big| \geq 2\}$ is a linear basis for $\Xi_1(\pi)$. This agrees with $\dim(\Xi_1(\pi)) = h_1(\pi) = 2^d - d - 1$.

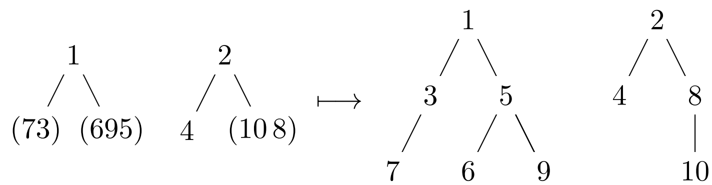
We will use a bijection between *increasing rooted forests* on $[d]$ and permutations in $\mathfrak{S}_d$. An increasing rooted forest is a disjoint union of planar rooted trees where each child is larger than its parent and the children are in strict order from the left to the right. Given a rooted forest t, the corresponding permutation $\sigma(t)$ is read as follows. Each connected component of t corresponds to a cycle of $\sigma(t)$. To form a cycle, traverse the corresponding tree counterclockwise and record a node the *last* time you see it[1].



$\longmapsto \qquad (3\ 7\ 6\ 8\ 5\ 2\ 9\ 4\ 11\ 12\ 10\ 1)$

The inverse can be described inductively by writing each cycle with its minimum element in the last position, and using right to left minima. We omit the details, but provide an example $\sigma \mapsto t(\sigma)$ to illustrate the idea.

(6.4) $\quad (7\ 3\ 6\ 9\ 5\ 1)(4\ 10\ 8\ 2) \longmapsto$



This bijection is such that the connected components of the forest $t(\sigma)$ are the blocks of $\mathrm{s}(\sigma)$. Moreover, the number of leaves of $t(\sigma)$ in $S \in \mathrm{s}(\sigma)$ is $\mathrm{exc}(\sigma|_S)$ (a tree consisting only of its root has zero leaves). Consequently, the total number of leaves $t(\sigma)$ is $\mathrm{exc}(\sigma)$.

Let $\sigma \in \mathfrak{S}_d$ be a permutation with r excedances and $\mathrm{s}(\sigma) = \mathrm{X}$. For $1 \leq i \leq r$, let J_i be the elements on the path from the i^{th} leaf of $t(\sigma)$ to the root of the corresponding tree. Define the element

$$x_\sigma = \Big(\prod_{i=1}^{r} \log[\Delta_{J_i}]\Big) \cdot \mathtt{E}_{\mathrm{X}}.$$

For instance, if σ is the permutation in (6.4), then

$$x_\sigma = \Big(\log[\Delta_{\{7,3,1\}}]\log[\Delta_{\{6,5,1\}}]\log[\Delta_{\{9,5,1\}}]\log[\Delta_{\{4,2\}}]\log[\Delta_{\{10,8,2\}}]\Big)\mathtt{E}_{\{1,3,5,6,7,9\},\{2,4,8,10\}}.$$

Conjecture 6.4. *For fixed* $\mathrm{X} \vdash [d]$ *and* $r \leq d - \big|\mathrm{X}\big|$*, the collection*

$$\{x_\sigma \ : \ \mathrm{s}(\sigma) = \mathrm{X},\ \mathrm{exc}(\sigma) = r\}$$

is a basis of $\Xi_r(\pi) \cdot \mathtt{E}_{\mathrm{X}}$.

It follows from the definition that $x_\sigma \in \Xi_r(\pi) \cdot \mathtt{E}_{\mathrm{X}}$. The content of the conjecture is that these elements are linearly independent. Explicit computations show that this is the case for $d = 2, 3, 4$. Propositions 6.5 and 6.6 below prove the extremal cases $r = 1$ and $r = d - |\mathrm{X}|$ of this conjecture, respectively. A forest with 1 leaf must consist of exactly one *path* and isolated roots. A forest with k trees and $d - r$ leaves must have height 1, meaning that all non-roots are leaves.

Proposition 6.5. *For a subset* $J \subseteq [d]$ *of cardinality at least* 2*, let* $\mathrm{X}_J \vdash [d]$ *be the partition whose only non-singleton block is* J*. Then,*

$$\log[\Delta_J] \cdot \mathtt{E}_{\mathrm{X}_J}$$

[1] A similar bijection is described by Peter Luschny in this OEIS entry.

is a nonzero element. Furthermore, $\{\log[\Delta_J]\cdot \mathtt{E}_{\mathrm{X}_J} : J\subseteq [d], |J|\geq 2\}$ *is a basis of double-eigenvectors for* $\Xi_1(\pi)$.

Proof. First, observe that any cyclic permutation on a set with more than one elements has at least one excedance, and only one cyclic permutation attains this minimum. Hence, a permutation $\sigma\in\mathfrak{S}_d$ has at least as many excedances as non-singleton blocks in $\mathrm{s}(\sigma)$. Moreover, for a fixed X, only one σ with $\mathrm{s}(\sigma)=\mathrm{X}$ attains this minimum. It then follows from Theorem 6.1 that

$$\dim(\Xi_1(\pi)\cdot \mathtt{E}_{\mathrm{X}}) = \begin{cases} 1 & \text{if } \mathrm{X}\vdash [d] \text{ has exactly one non-singleton block,}\\ 0 & \text{otherwise.}\end{cases}$$

Thus, the second statement follows from the first.

We have proved that $\mathcal{B}_1 = \{\log[\Delta_J] : J\subseteq I, \big|J\big|\geq 2\}$ is a linear basis for $\Xi_1(\pi)$. Then, it is enough to write $\log[\Delta_J]\cdot\mathtt{E}_{\mathrm{X}_J}$ in the basis $\mathcal{B}_1$ and check that at least one coefficient is nonzero.

Observe that $\mathrm{X}_J = N(\Delta_J,\Delta_J)$. Therefore, $\Delta_J\cdot\mathtt{H}_G$ is a proper face of Δ_J for any $F<G$ with $\mathrm{s}(F)=\mathrm{X}_J$. Explicitly, if $G=(S_1,S_2,\dots,S_k)$, then $\Delta_J\cdot\mathtt{H}_G=\Delta_{J\cap S_i}$ where i is the first index for which $J\cap S_i$ is nonempty. Hence, the coefficient of $\log[\Delta_J]\in\mathcal{B}_1$ in $\log[\Delta_J]\cdot\mathtt{E}_{\mathrm{X}_J}$ is

$$\frac{1}{\dim(\mathrm{X}_J)!}\sum_{\mathrm{s}(F)=\mathrm{X}_J} 1 = 1.$$

The equality follows since for any flat X of the braid arrangement, $\mathcal{A}^{\mathrm{X}}$ has $\dim(\mathrm{X})!$ chambers. ☐

Note that the element $\log[\Delta_J]\cdot\mathtt{E}_{\mathrm{X}_J}$ in the proposition is precisely the element x_σ for the unique permutation σ with $\mathrm{s}(\sigma)=\mathrm{X}_J$ and $\mathrm{exc}(\sigma)=1$. Similarly, the element x_{X} in the following proposition is precisely x_σ for the unique permutation with $\mathrm{s}(\sigma)=\mathrm{X}$ and $\mathrm{exc}(\sigma)=d-|\mathrm{X}|$.

Proposition 6.6. *For any* $\mathrm{X}=\{S_1,\dots,S_k\}\vdash[d]$*, the space* $\Xi_{d-k}(\pi)\cdot\mathtt{E}_{\mathrm{X}}$ *is* 1*-dimensional. Moreover,*

$$x_{\mathrm{X}} = \prod_{i=1}^{k}\Big(\prod_{j\neq\min(S_i)}\log[\Delta_{\min(S_i),j}]\Big) \tag{6.5}$$

is a nonzero element in $\Xi_{d-k}(\pi)\cdot\mathtt{E}_{\mathrm{X}}$.

Proof. Just like before, observe that any cyclic permutation on a set with s elements has at most $s-1$ excedances, and only one cyclic permutation attains this maximum. Hence, for any $\mathrm{X}\vdash[d]$ there is exactly one permutation with $\mathrm{s}(\sigma)=\mathrm{X}$ and $d-|\mathrm{X}|$ excedances. Therefore,

$$\dim(\Xi_{d-|\mathrm{X}|}(\pi)\cdot\mathtt{E}_{\mathrm{X}})=1$$

for any flat X.

By Lemma 2.8, the element x_{X} is nonzero. We are only left to prove that $x_{\mathrm{X}}\cdot\mathtt{E}_{\mathrm{X}}=x_{\mathrm{X}}$. Let $G\in\sigma[\mathcal{A}]$ with $\mathrm{s}(G)>\mathrm{X}$. Then, for some block $S_i\in\mathrm{X}$ and some $a\in S_i$, a and $\min(S_i)$ are not in the same block of $\mathrm{s}(G)$. Hence $\Delta_{\min(S_i),a}\cdot\mathtt{H}_G$ is a point and $\log[\Delta_{\min(S_i),a}]\cdot\mathtt{H}_G=\log[o]=0$. Therefore,

$$x_{\mathrm{X}}\cdot\mathtt{E}_{\mathrm{X}} = x_{\mathrm{X}}\cdot\Big(\frac{1}{\dim(\mathrm{X})!}\sum_{\mathrm{s}(F)=\mathrm{X}}\mathtt{H}_F\Big) = \frac{1}{\dim(\mathrm{X})!}\sum_{\mathrm{s}(F)=\mathrm{X}} x_{\mathrm{X}}\cdot\mathtt{H}_F = x_{\mathrm{X}},$$

as we wanted to show. ☐

6.2. **Type B.** Let $\mathcal{A} = \mathcal{A}_d^{\pm}$ be signed braid arrangement in $\mathbb{R}^d$. The type B permutahedron $\pi^B = \pi_d^B \subseteq \mathbb{R}^d$ is the convex hull of all the signed permutations of the point $(1, 2, \dots, d)$. It is full-dimensional and a zonotope of $\mathcal{A}$. For a flat $\mathrm{X} = \{S_0, S_1, \overline{S_1}, \dots, S_k, \overline{S_k}\}$ of $\mathcal{A}$,

$$\pi_{\mathrm{X}}^B \cong \pi_{|S_0|/2}^B \times \pi_{|S_1|} \times \cdots \times \pi_{|S_k|} \tag{6.6}$$

is a product of lower-dimensional permutahedra of type A and B, where exactly one factor is of type B.

We now consider the module $\Pi(\pi^B)$. The main result of this section is the following.

Theorem 6.7. *For any flat* $\mathrm{X} \in \mathcal{L}[\mathcal{A}_d^{\pm}]$ *and* $r = 0, 1, \dots, d$,

$$\eta_{\mathrm{X}}(\Xi_r(\pi_d^B)) = \left|\{\sigma \in \mathfrak{B}_d \,:\, \mathrm{s}(\sigma) = \mathrm{X},\ \mathrm{exc}_B(\sigma) = r\}\right|.$$

The following is the analogous of Lemma 6.2 for the hyperoctahedral group. Again, it plays an essential role in the proof of Theorem 6.7.

Lemma 6.8.

$$\sum_{\{S_0,\dots,S_k,\overline{S_k}\}\vdash^B [d]} \mu(\perp, \mathrm{X}) B_{|S_0|/2} A_{|S_1|} \dots A_{|S_k|} = \sum_{\substack{\sigma\in\mathfrak{B}_d \\ \mathrm{s}(\sigma)=\perp}} z^{\mathrm{exc}_B(\sigma)}. \tag{6.7}$$

In the same spirit as the proof of Lemma 6.2, we will compare the type B exponential generating function of both sides of (6.7). An important tool in this proof is the following analog of the compositional formula for type B generating functions.

Proposition 6.9 (Type B Compositional Formula)**.** *Let*

$$f(x) = 1 + \sum_{d\geq 1} f_d \frac{x^d}{(2d)!!} \qquad g(x) = 1 + \sum_{d\geq 1} g_d \frac{x^d}{(2d)!!} \qquad a(x) = \sum_{d\geq 1} a_d \frac{x^d}{d!}.$$

If

$$h(x) = 1 + \sum_{d\geq 1} h_d \frac{x^d}{(2d)!!} \qquad \textit{where} \qquad h_d = \sum_{\{S_0,S_1,\overline{S_1},\dots,S_k,\overline{S_k}\}\vdash^B [d]} f_{|S_0|/2} g_k a_{|S_1|} \dots a_{|S_k|},$$

then

$$h(x) = f(x) g(a(x)).$$

Proof. Using the usual Compositional Formula, the coefficient of $\dfrac{x^d}{(2d)!!}$ in $f(x)g(a(x))$ is

$$\begin{aligned}
&2^d\, d! \sum_{r=0}^{d} \frac{f_r}{2^r\, r!} \left(\frac{1}{(d-r)!} \sum_{\{K_1,\dots,K_k\}\vdash[d-r]} \frac{g_k}{2^k}\, a_{|K_1|} \dots a_{|K_k|} \right) \\
&= \sum_{r=0}^{d} \binom{d}{r} \left(\sum_{\{K_1,\dots,K_k\}\vdash[d-r]} 2^{d-r-k}\, f_r\, g_k\, a_{|K_1|} \dots a_{|K_k|} \right) \\
&= \sum_{\{S_0,S_1,\overline{S_1},\dots,S_k,\overline{S_k}\}\vdash^B [d]} f_{|S_0|/2}\, g_k\, a_{|S_1|} \dots a_{|S_k|},
\end{aligned}$$

this is precisely the coefficient of $\dfrac{x^d}{(2d)!!}$ in $h(x)$. To verify the last equality, note that choosing a type B partition $\{S_0, S_1, \overline{S_1}, \dots, S_k, \overline{S_k}\} \vdash^B [d]$ with $|S_0| = 2r$ is equivalent to:

(a) choosing a subset $K_0 \in \binom{[d]}{r}$ and setting $S_0 = K_0 \sqcup \overline{K_0}$,
(b) choosing a partition $\{K_1, \dots, K_k\}$ of $[d] \setminus K_0$, and

(c) for each $j \in K_i \setminus \{\max K_i\}$, choosing whether $j \in S_i$ or $j \in \overline{S_i}$.

Recall that according to our Convention 4.1, $\max K_i = \max(S_i \cup \overline{S_i}) \in S_i$, so there is no freedom for that element. $\square$

Taking $g_d = 1$ in the Type B Compositional Formula we deduce the following.

Corollary 6.10 (Type B Exponential Formula)**.** *Let $f(x)$ and $a(x)$ be as before. If*

$$h(x) = 1 + \sum_{d\geq 1} h_d \frac{x^d}{(2d)!!} \qquad \text{where} \qquad h_d = \sum_{\mathrm{X} \vdash^B [d]} f_{|S_0|/2} a_{|S_1|} \dots a_{|S_k|},$$

then

$$h(x) = f(x) e^{a(x)/2}.$$

In the proof of Theorem 6.1, we used that for (type A) permutations $\sigma \in \mathfrak{S}_d$, $\mathrm{exc}(\sigma)$ equals the sum of $\mathrm{exc}(\sigma|_S)$ as S runs through the blocks of $\mathrm{s}(\sigma)$. For signed permutations, one easily checks this also holds for the statistics exc and neg. However, it is not obvious at all that the same is true for exc_B, since its definition uses the floor function.

Consider the order $\prec$ of the elements of any subset $S \subseteq \pm[d]$ with $S \cap \overline{S} = \emptyset$ defined as follows:

$$i \prec j \iff \begin{cases} 0 < i < j, \text{ or} \\ i < 0 < j, \text{ or} \\ j < i < 0. \end{cases} \tag{6.8}$$

Proposition 6.11. *Let $\sigma \in \mathfrak{B}_d$ and $\mathrm{s}(\sigma) = \{S_0, S_1, \overline{S_1}, \dots, S_k, \overline{S_k}\}$. Then,*

$$\mathrm{exc}_B(\sigma) = \mathrm{exc}_B(\sigma|_{S_0}) + \mathrm{exc}_\prec(\sigma|_{S_1}) + \dots + \mathrm{exc}_\prec(\sigma|_{S_k}),$$

where $\mathrm{exc}_\prec(\sigma|_{S_i})$ is the number of usual excedances of $\sigma|_{S_i}$ with respect to the order $\prec$.

Proof. Let $i \geq 1$ and write $\sigma|_{S_i} = (j_1 j_2 \dots j_\ell)$ in cycle notation, with $j_1 = \max(S_i \sqcup \overline{S_i}) > 0$. Since $\sigma|_{\overline{S_i}} = (\overline{j_1}\,\overline{j_2} \dots \overline{j_\ell})$, negations of $\sigma|_{S_i \sqcup \overline{S_i}}$ are in correspondence with changes of sign in

$$j_1 \mapsto j_2 \mapsto \dots \mapsto j_\ell \mapsto j_1.$$

It follows that

$$\mathrm{neg}(\sigma|_{S_i \sqcup \overline{S_i}}) = 2 \cdot \left| \{ j \in S_i \,:\, j < 0 < \sigma(j) \} \right|$$

is an even number.

Observe that according to the three cases of definition (6.8), a $\prec$-excedance of $\sigma|_{S_i}$ corresponds to either

$$\begin{cases} \text{an excedance of } \sigma|_{S_i \sqcup \overline{S_i}} \text{ occurring in } S_i, \text{ or} \\ \text{a negation of } \sigma|_{S_i \sqcup \overline{S_i}} \text{ occurring in } \overline{S_i}, \text{ or} \\ \text{an excedance of } \sigma|_{S_i \sqcup \overline{S_i}} \text{ occurring in } \overline{S_i}, \end{cases}$$

respectively. Since exactly half of the negations of $\sigma|_{S_i \sqcup \overline{S_i}}$ occur in $\overline{S_i}$, we deduce that

$$\mathrm{exc}_\prec(\sigma|_{S_i}) = \mathrm{exc}(\sigma|_{S_i \sqcup \overline{S_i}}) + \tfrac{\mathrm{neg}(\sigma|_{S_i \sqcup \overline{S_i}})}{2}.$$

Thus, in view of (4.4),

$$\begin{aligned} \mathrm{exc}_B(\sigma) &= \mathrm{exc}(\sigma) + \lfloor \tfrac{\mathrm{neg}(\sigma)+1}{2} \rfloor \\ &= \mathrm{exc}(\sigma|_{S_0}) + \sum_i \mathrm{exc}(\sigma|_{S_i \sqcup \overline{S_i}}) + \lfloor \tfrac{\mathrm{neg}(\sigma|_{S_0}) + \sum_i \mathrm{neg}(\sigma|_{S_i \sqcup \overline{S_i}}) + 1}{2} \rfloor \\ &= \mathrm{exc}(\sigma|_{S_0}) + \sum_i \mathrm{exc}(\sigma|_{S_i \sqcup \overline{S_i}}) + \lfloor \tfrac{\mathrm{neg}(\sigma|_{S_0})+1}{2} \rfloor + \sum_i \tfrac{\mathrm{neg}(\sigma|_{S_i \sqcup \overline{S_i}})}{2} \\ &= \mathrm{exc}_B(\sigma|_{S_0}) + \mathrm{exc}_\prec(\sigma|_{S_1}) + \dots + \mathrm{exc}_\prec(\sigma|_{S_k}), \end{aligned}$$

as we wanted to show. $\square$

Proof of Lemma 6.8. Recall that $\mu(\perp, \mathrm{X}) = (-1)^k(2k-1)!!$, where $|\mathrm{X}| = 2k+1$. Observe that

$$1 + \sum_{d\geq 1} (-1)^d (2d-1)!! \frac{x^d}{(2d)!!} = \sum_{d\geq 0} \binom{-1/2}{d} x^d = (1+x)^{-1/2}.$$

Using the Type B Compositional formula, we conclude that the type B exponential generating function of the LHS of (6.7) is

$$B(z,x)(1 + (A(z,x)-1))^{-1/2} = \frac{B(z,x)}{\sqrt{A(z,x)}},$$

where $A(z,x)$ and $B(z,x)$ are the generating functions in (4.2) and (4.5), respectively. On the other hand, Proposition 6.11 shows that for each partition $\mathrm{X} = \{S_0, S_1, \overline{S_1}, \ldots, S_k, \overline{S_k}\} \vdash^B [d]$,

$$\sum_{\substack{\sigma\in\mathfrak{B}_d \\ \mathrm{s}(\sigma)=\mathrm{X}}} z^{\mathrm{exc}_B(\sigma)} = \Big(\sum_{\substack{\sigma_0\in\mathfrak{B}(S_0) \\ \mathrm{s}(\sigma_0)=\perp}} z^{\mathrm{exc}_B(\sigma_0)} \Big) \prod_{i=1}^{k} \Big(\sum_{\sigma\in\mathfrak{C}(|S_i|)} z^{\mathrm{exc}(\sigma)} \Big). \tag{6.9}$$

In the proof of Lemma 6.2, we showed that the (usual) exponential generating function of the terms in the product is $\log A(z,x)$. An application of the type B Exponential Formula and (6.9) yields

$$B(z,x) = f(x) e^{\frac{\log A(z,x)}{2}} = f(x)\sqrt{A(z,x)},$$

where $f(x)$ is the type B exponential generating function of the RHS of (6.7). Dividing both sides by $\sqrt{A(z,x)}$ yields the result. $\square$

We are now ready to complete the proof of the main theorem in this section. The steps of the proof mirror those of the type A counterpart.

Proof of Theorem 6.7. Using that π^B is a simple polytope, and the decomposition of its faces in (6.6), we deduce that for any flat $\mathrm{X} = \{S_0, S_1, \overline{S_1}, \ldots, S_k, \overline{S_k}\}$

$$\sum_r \xi_{\mathrm{X}}(\Xi_r(\pi^B)) z^r = h(\pi^B_{\mathrm{X}}, z) = B_{|S_0|/2}(z) A_{|S_1|}(z) \cdot \ldots \cdot A_{|S_k|}(z).$$

Using (4.3) and the expression, we have

$$\sum_r \eta_{\mathrm{X}}(\Xi_r(\pi^B)) z^r = \sum_{\mathrm{Y}\geq\mathrm{X}} \mu(\mathrm{X},\mathrm{Y}) h(\pi^B_{\mathrm{Y}}, z) =$$

$$\Big(\sum_{\substack{\mathrm{Y}\vdash^B S_0 \\ \mathrm{Y}=\{T_0,\ldots,T_\ell,\overline{T_\ell}\}}} \mu(\perp,\mathrm{Y}) B_{|T_0|/2} A_{|T_1|} \ldots A_{|T_\ell|} \Big) \prod_{i=1}^{k} \Big(\sum_{\substack{\mathrm{Y}_i\vdash S_i \\ \mathrm{Y}_i=\{T^i_1,\ldots,T^i_\ell\}}} \mu(\perp,\mathrm{Y}_i) A_{|T^i_1|}(z) \cdot \ldots \cdot A_{|T^i_\ell|}(z) \Big).$$

Using Lemmas 6.2 and 6.8, this becomes

$$\sum_r \eta_{\mathrm{X}}(\Xi_r(\pi^B)) z^r = \Big(\sum_{\substack{\sigma\in\mathfrak{B}(S_0) \\ \mathrm{s}(\sigma)=\perp}} z^{\mathrm{exc}_B(\sigma)} \Big) \prod_{i=1}^{k} \Big(\sum_{\sigma\in\mathfrak{C}(|S_i|)} z^{\mathrm{exc}(\sigma)} \Big) = \sum_{\substack{\sigma\in\mathfrak{B}_d \\ \mathrm{s}(\sigma)=\mathrm{X}}} z^{\mathrm{exc}_B(\sigma)}.$$

The last equality is (6.9). Taking the coefficient of z^k on both sides yields the result. $\square$

Adding over all flats with the same dimension in Theorem 6.7, we conclude the following.

Corollary 6.12. *Let $w \in \Sigma[\mathcal{A}]$ be a characteristic element of non-critical parameter t. Then, the multiplicity of the eigenvalue t^k on $\Xi_r(\pi_d)$ is*

$$\left|\{\sigma \in \mathfrak{S}_d \,:\, |\,\mathrm{s}(\sigma)| = 2k+1,\ \mathrm{exc}_B(\sigma) = r\}\right|.$$

Recall that for the dimension of the flat corresponding to $\mathrm{X} \vdash^B [d]$ with $|\mathrm{X}| = 2k+1$ is k.

Proposition 6.13. *The type B generating function of the polynomials*

$$\sum_{\sigma \in \mathfrak{S}_d} t^{\dim(\mathrm{s}(\sigma))} z^{\mathrm{exc}_B(\sigma)}$$

is

$$B(z,x)A(z,x)^{\frac{t-1}{2}}.$$

Proof. We use the results in the proof of Lemma 6.8. Note that

$$\sum_{\substack{\sigma \in \mathfrak{B}_d \\ \mathrm{s}(\sigma)=\mathrm{X}}} t^{\dim(\mathrm{X})} z^{\mathrm{exc}_B(\sigma)} = \Big(\sum_{\substack{\sigma_0 \in \mathfrak{B}(S_0) \\ \mathrm{s}(\sigma_0)=\perp}} z^{\mathrm{exc}_B(\sigma_0)} \Big) \prod_{i=1}^{k} \Big(t \sum_{\sigma \in \mathfrak{C}(|S_i|)} z^{\mathrm{exc}(\sigma)} \Big).$$

Hence, the type B compositional yields that the type B generating function of the polynomials above is

$$\frac{B(z,x)}{\sqrt{A(z,x)}} \exp\big(\frac{t \log A(z,x)}{2}\big) = B(z,x)A(z,x)^{\frac{t-1}{2}},$$

as we wanted to show. □

Eulerian numbers are defined for any Coxeter group W in terms of W-descents. For the Coxeter groups of type A and B, descents and (B-)excedances are equally distributed, so we can interpret the W-Eulerian polynomials as the generating functions for (B-)excedances. However, the joint distributions of $\big(|\,\mathrm{s}(\cdot)|, \mathrm{des}(\cdot)\big)$ and $\big(|\,\mathrm{s}(\cdot)|, \mathrm{exc}(\cdot)\big)$ do no longer agree. Therefore, Theorems 6.1 and 6.7 cannot be expressed in terms of descents. Extending the results of this section to other Coxeter groups W requires to find the correct notion of W-excedance for other types.

7. Hopf monoid structure

Combinatorial species were originally introduced by Joyal [Joy81] as a tool for studying generating power series from a combinatorial perspective. A comprehensive introduction to the theory of species can be found in the work by Bergeron, Labelle, and Leroux [BLL98]. The category of species possesses more than one monoidal structure. Of central interest are the *Cauchy* and *Hadamard* product. Aguiar and Mahajan [AM10] have explored these structures extensively, and have exploited this rich algebraic structure to obtain outstanding combinatorial results. The first of these structures leads to the definition of Hopf monoids in species, a very active topic of research in the last years.

Aguiar and Ardila introduced the Hopf monoid of generalized permutahedra GP in [AA17]. It contains many other interesting combinatorial Hopf monoids as submonoids. In this section we will shows that the valuation (2.1) and translation invariance (2.2) properties, define a Hopf monoid quotient of GP.

7.1. **Hopf monoids in a nutshell.** Let $\mathsf{set}^\times$ denote the category of finite sets with bijections as morphisms, and Vec the category of vector spaces and linear maps. The category of species Sp is the functor category $[\mathsf{set}^\times, \mathsf{Vec}]$. It is a symmetric monoidal category under the *Cauchy product*. The Cauchy product of two species p and q is

$$(\mathsf{p} \cdot \mathsf{q})[I] = \bigoplus_{I = S \sqcup T} \mathsf{p}[S] \otimes \mathsf{q}[T].$$

We say a species h is a *Hopf monoid* if it is a bimonoid with an *antipode* in this monoidal category.

Let us make these definitions explicit. A *species* p consists of the following data:

i. For each finite set I, a vector space $\mathsf{p}[I]$.
ii. For each bijection $\sigma : I \to J$, a linear isomorphism $\mathsf{p}[\sigma] : \mathsf{p}[I] \to \mathsf{p}[J]$. These linear maps satisfy

$$\mathsf{p}[\sigma \circ \tau] = \mathsf{p}[\sigma] \circ \mathsf{p}[\tau] \quad \text{and} \quad \mathsf{p}[\mathrm{Id}] = \mathrm{Id}\,.$$

A morphism of species $f : \mathsf{p} \to \mathsf{q}$ is a collection of linear maps

$$f_I : \mathsf{p}[I] \to \mathsf{q}[I],$$

one for each finite set I, that commute with bijections. That is, $f_J \circ \mathsf{p}[\sigma] = \mathsf{q}[\sigma] \circ f_I$ for any bijection $\sigma : I \to J$.

A *Hopf monoid* is a species h together a collection of product, coproduct and antipode maps

$$\begin{aligned} \mu_{S,T} : \mathsf{h}[S] \otimes \mathsf{h}[T] &\to \mathsf{h}[I] \\ x \otimes y &\mapsto x \cdot y \end{aligned} \qquad \begin{aligned} \Delta_{S,T} : \mathsf{h}[I] &\to \mathsf{h}[S] \otimes \mathsf{h}[T] \\ z &\mapsto \textstyle\sum z|_S \otimes z/_S \end{aligned} \qquad \begin{aligned} \mathrm{s}_I : \mathsf{h}[I] &\to \mathsf{h}[I] \\ z &\mapsto \mathrm{s}_I(z) \end{aligned}$$

for all finite sets I and decompositions $I = S \sqcup T$. This morphisms satisfy certain naturality, (co)unitality, (co)associativity and compatibility axioms. See [AM13, Section 2] and [AA17] for more details.

7.2. **Generalized permutahedra and the McMullen (co)ideal.** The Hopf monoid of generalized permutahedra GP was introduced by Aguiar and Ardila in [AA17]. As a species, $\mathsf{GP}[I]$ is the vector space with basis

$$\mathrm{GP}[I] = \{\mathfrak{p} \subseteq \mathbb{R}^I \,:\, \mathfrak{p} \text{ is a generalized permutahedra}\}.$$

The product $\mu_{S,T}$ is defined by

$$\mu_{S,T}(\mathfrak{p} \otimes \mathfrak{q}) = \mathfrak{p} \times \mathfrak{q},$$

for all $\mathfrak{p} \in \mathrm{GP}[S]$ and $\mathfrak{q} \in \mathrm{GP}[T]$. In particular, GP is a commutative monoid. Let F be the face of the braid arrangement in $\mathbb{R}^I$ corresponding to the composition (S, T), and let $v \in \mathrm{relint}(F)$. Then, for any $\mathfrak{p} \in \mathrm{GP}[I]$, the face $(\mathfrak{p})_v$ decomposes as a product of generalized permutahedra $\mathfrak{p}|_S \times \mathfrak{p}/_S$, with $\mathfrak{p}|_S \in \mathrm{GP}[S]$ and $\mathfrak{p}/_S \in \mathrm{GP}[T]$. The coproduct is defined by

$$\Delta_{S,T}(\mathfrak{p}) = \mathfrak{p}|_S \otimes \mathfrak{p}/_S.$$

Aguiar and Ardila provide the following grouping-free and cancellation-free formula for its antipode. For a generalized permutahedron $\mathfrak{p} \in \mathrm{GP}[I]$,

$$\mathrm{s}_I(\mathfrak{p}) = (-1)^{|I|} \sum_{\mathfrak{q} \leq \mathfrak{p}} (-1)^{\dim(\mathfrak{q})} \mathfrak{q}.$$

We now introduce the subspecies Mc of GP. The space $\mathsf{Mc}[I] \subseteq \mathsf{GP}[I]$ is the subspace spanned by elements

$$\mathfrak{p} \cup \mathfrak{q} + \mathfrak{p} \cap \mathfrak{q} - \mathfrak{p} - \mathfrak{q} \qquad \text{for} \quad \mathfrak{p}, \mathfrak{q}, \mathfrak{p} \cup \mathfrak{q} \in \mathrm{GP}[I], \tag{7.1}$$

and

$$\mathfrak{p}_{+t} - \mathfrak{p} \qquad \text{for} \quad \mathfrak{p} \in \mathrm{GP}[I] \text{ and } t \in \mathbb{R}^I, \tag{7.2}$$

where $\mathfrak{p}_{+t}$ denotes the Minkowski sum $\mathfrak{p}+\{t\}$. The sums and differences in (7.1) and (7.2) correspond to the vector space structure of $\mathsf{GP}[I]$, not to Minkowski sum. Note that if $\mathfrak{p}$ and $\mathfrak{q}$ are generalized permutahedra such that $\mathfrak{p} \cup \mathfrak{q}$ is a polytope, then $\mathfrak{p} \cup \mathfrak{q}$ is necessarily a generalized permutahedra. This follows since the edge directions of $\mathfrak{p} \cup \mathfrak{q}$ are contained in the edge directions of $\mathfrak{p}$ and $\mathfrak{q}$.

Theorem 7.1. *The subspecies* Mc *is an ideal and a coideal of* GP*. That is,*

$$\mu_{S,T}\big(\mathsf{Mc}[S] \otimes \mathsf{GP}[T]\big) \subseteq \mathsf{Mc}[I] \qquad \text{and} \qquad \Delta_{S,T}\big(\mathsf{Mc}[I]\big) \subseteq \mathsf{Mc}[S] \otimes \mathsf{GP}[T] + \mathsf{GP}[S] \otimes \mathsf{Mc}[T],$$

for any $I = S \sqcup T$*. Therefore, the quotient species* Π *defined by*

$$\Pi[I] = \mathsf{GP}[I]/\mathsf{Mc}[I] \cong \Pi(\pi_I)$$

inherits the Hopf monoid of GP*.*

Proof. For generators of Mc of the form (7.2), the result follows from the following two observations. If $\mathfrak{p} \in \mathrm{GP}[S]$, $\mathfrak{r} \in \mathrm{GP}[T]$ and $t \in \mathbb{R}^S$, then

$$\mathfrak{p}_{+t} \times \mathfrak{r} = (\mathfrak{p} \times \mathfrak{r})_{+(t,0)}.$$

If $\mathfrak{p} \in \mathrm{GP}[I]$ and $t \in \mathbb{R}^T$, then

$$\Delta_{S,T}(\mathfrak{p}_{+t}) = (\mathfrak{p}|_S)_{+t_S} \otimes (\mathfrak{p}|_S)_{+t_T}$$

where t_S and t_T denote the projections of t to $\mathbb{R}^S$ and $\mathbb{R}^T$, respectively.

We will now focus on elements of the form (7.1). Fix an arbitrary finite set I and a nontrivial decomposition $I = S \sqcup T$. Let $v \in \mathbb{R}^I$ be any vector in the interior of the corresponding face of the braid arrangement.

Suppose $\mathfrak{p}, \mathfrak{q}, \mathfrak{p} \cup \mathfrak{q} \in \mathsf{GP}[S]$ and $\mathfrak{r} \in \mathsf{GP}[T]$. Then,

$$(\mathfrak{p} \cup \mathfrak{q}) \times \mathfrak{r} = (\mathfrak{p} \times \mathfrak{r}) \cup (\mathfrak{q} \times \mathfrak{r}), \qquad\qquad (\mathfrak{p} \cap \mathfrak{q}) \times \mathfrak{r} = (\mathfrak{p} \times \mathfrak{r}) \cap (\mathfrak{q} \times \mathfrak{r}),$$

and $(\mathfrak{p} \cup \mathfrak{q}) \times \mathfrak{r} = (\mathfrak{p} \times \mathfrak{r}) \cup (\mathfrak{q} \times \mathfrak{r})$ is a polytope if and only if $\mathfrak{p} \cup \mathfrak{q}$ is. It follows that

$$\mu_{S,T}\big((\mathfrak{p} \cup \mathfrak{q} + \mathfrak{p} \cap \mathfrak{q} - \mathfrak{p} - \mathfrak{q}) \otimes \mathfrak{r}\big) = (\mathfrak{p} \times \mathfrak{r}) \cup (\mathfrak{q} \times \mathfrak{r}) + (\mathfrak{p} \times \mathfrak{r}) \cap (\mathfrak{q} \times \mathfrak{r}) - \mathfrak{p} \times \mathfrak{r} - \mathfrak{q} \times \mathfrak{r} \in \mathsf{Mc}[I].$$

Since GP is commutative, this proves that Mc is an ideal.

Now, let $\mathfrak{p}, \mathfrak{q}, \mathfrak{p} \cup \mathfrak{q} \in \mathsf{GP}[I]$. There are two possibilities:

i. The face $(\mathfrak{p} \cup \mathfrak{q})_v$ of $\mathfrak{p} \cup \mathfrak{q}$ is completely contained in $\mathfrak{p}$ or in $\mathfrak{q}$. Without loss of generality, suppose the former. Then $(\mathfrak{p} \cup \mathfrak{q})_v = \mathfrak{p}_v$ and, necessarily, $(\mathfrak{p} \cap \mathfrak{q})_v = \mathfrak{q}_v$. Hence,

$$\Delta_{S,T}\big(\mathfrak{p} \cup \mathfrak{q} + \mathfrak{p} \cap \mathfrak{q} - \mathfrak{p} - \mathfrak{q}\big) = \Delta_{S,T}(\mathfrak{p}) + \Delta_{S,T}(\mathfrak{q}) - \Delta_{S,T}(\mathfrak{p}) - \Delta_{S,T}(\mathfrak{q}) = 0.$$

ii. The face $(\mathfrak{p} \cup \mathfrak{q})_v$ is not contained in $\mathfrak{p}$ nor in $\mathfrak{q}$. Hence, $(\mathfrak{p} \cup \mathfrak{q})_v = \mathfrak{p}_v \cup \mathfrak{q}_v$ and $(\mathfrak{p} \cap \mathfrak{q})_v = \mathfrak{p}_v \cap \mathfrak{q}_v$. Expanding the first equality we have

$$(\mathfrak{p} \cup \mathfrak{q})|_S \times (\mathfrak{p} \cup \mathfrak{q})/_S = (\mathfrak{p}|_S \times \mathfrak{p}/_S) \cup (\mathfrak{q}|_S \times \mathfrak{q}/_S).$$

The union of two Cartesian products $A \times B$ and $C \times D$ is again a Cartesian product if and only if one contains the other or either $A = C$ or $B = D$. By assumption, the is no containment between $\mathfrak{p}_v$ and $\mathfrak{q}_v$. We can therefore assume without loss of generality that

$$\mathfrak{p}|_S = \mathfrak{q}|_S. \tag{7.3}$$

Projecting to $\mathbb{R}^S$ and $\mathbb{R}^T$, we further see that

$$(\mathfrak{p} \cup \mathfrak{q})|_S = \mathfrak{p}|_S \cup \mathfrak{q}|_S = \mathfrak{p}|_S \quad \text{and} \quad (\mathfrak{p} \cup \mathfrak{q})/_S = \mathfrak{p}/_S \cup \mathfrak{q}/_S. \tag{7.4}$$

In particular, $\mathfrak{p}/_S \cup \mathfrak{q}/_S$ is a generalized permutahedron. On the other hand, expanding $(\mathfrak{p} \cap \mathfrak{q})_v = \mathfrak{p}_v \cap \mathfrak{q}_v$, we have

$$(\mathfrak{p} \cap \mathfrak{q})|_S \times (\mathfrak{p} \cap \mathfrak{q})/_S = (\mathfrak{p}|_S \times \mathfrak{p}/_S) \cap (\mathfrak{p}|_S \times \mathfrak{q}/_S) = \mathfrak{p}|_S \times (\mathfrak{p}/_S \cap \mathfrak{q}/_S).$$

Comparing factors, we deduce

$$(\mathfrak{p} \cap \mathfrak{q})|_S = \mathfrak{p}|_S \quad \text{and} \quad (\mathfrak{p} \cap \mathfrak{q})/_S = \mathfrak{p}/_S \cap \mathfrak{q}/_S. \tag{7.5}$$

Putting together (7.3), (7.4) and (7.5), we conclude

$$\begin{aligned}\Delta_{S,T}\big(\mathfrak{p} \cup \mathfrak{q} + \mathfrak{p} \cap \mathfrak{q} - \mathfrak{p} - \mathfrak{q}\big) &= \Delta_{S,T}(\mathfrak{p} \cup \mathfrak{q}) + \Delta_{S,T}(\mathfrak{q} \cap \mathfrak{q}) - \Delta_{S,T}(\mathfrak{p}) - \Delta_{S,T}(\mathfrak{q}) \\ &= \mathfrak{p}|_S \otimes (\mathfrak{p}/_S \cup \mathfrak{q}/_S) + \mathfrak{p}|_S \otimes (\mathfrak{p}/_S \cap \mathfrak{q}/_S) - \mathfrak{p}|_S \otimes \mathfrak{p}/_S - \mathfrak{p}|_S \otimes \mathfrak{q}/_S \\ &= \mathfrak{p}|_S \otimes \big(\mathfrak{p}/_S \cup \mathfrak{q}/_S + \mathfrak{p}/_S \cap \mathfrak{q}/_S - \mathfrak{p}/_S - \mathfrak{q}/_S\big) \in \mathsf{GP}[S] \otimes \mathsf{Mc}[T].\end{aligned}$$

In either case, $\Delta_{S,T}\big(\mathfrak{p} \cup \mathfrak{q} + \mathfrak{p} \cap \mathfrak{q} - \mathfrak{p} - \mathfrak{q}\big) \in \mathsf{Mc}[S] \otimes \mathsf{GP}[T] + \mathsf{GP}[S] \otimes \mathsf{Mc}[T]$, so Mc is a coideal of GP. □

The antipode formula of GP descends to the quotient Π, but it is no longer grouping-free in general. The Euler map (2.7) allows us to write the antipode formula of GP in a very compact form:

$$\mathrm{s}_I([\mathfrak{p}]) = (-1)^{|I|}[\mathfrak{p}]^*.$$

7.3. **Higher monoidal structures.** We have just proved that Π is a Hopf monoid in the symmetric monoidal category $(\mathsf{Sp}, \cdot)$. The algebra structure of each space $\Pi[I]$ defined by McMullen can also be defined for GP. In both cases, this endows the species with the structure of a monoid in the symmetric monoidal category $(\mathsf{Sp}, \times)$ of species with the *Hadamard product*. The Hadamard product of two species p and q is defined by

$$(\mathsf{p} \times \mathsf{q})[I] = \mathsf{p}[I] \otimes \mathsf{q}[I].$$

Hence, a monoid in $(\mathsf{Sp}, \times)$ consists of a species p with an algebra structure on each space $\mathsf{p}[I]$. For generalized permutahedra, these structures are compatible in a very special way.

Theorem 7.2. *The species of generalized permutahedra* GP *is a* $(2,1)$*-monoid in the 3-monoidal category* $(\mathsf{Sp}, \cdot, \times, \cdot)$.

See [AM10, Chapter 7] for the definition of higher monoidal categories and of monoids in such categories. The notation $(2,1)$ indicates that GP is a monoid with respect to the first two monoidal structures (Cartesian product and Minkowski sum, respectively) and a comonoid with respect to the last (coproduct maps $\Delta_{S,T}$).

Proof. We only verify the remaining compatibility axioms: the compatibility between Cartesian products and Minkowski sum, and the compatibility between Minkowski sums and coproducts.

For $\mathfrak{p}_1, \mathfrak{p}_2 \in \mathrm{GP}[S]$ and $\mathfrak{q}_1, \mathfrak{q}_2 \in \mathrm{GP}[T]$, we have

$$(\mathfrak{p}_1 + \mathfrak{p}_2) \times (\mathfrak{q}_1 + \mathfrak{q}_2) = (\mathfrak{p}_1 \times \mathfrak{q}_1) + (\mathfrak{p}_2 \times \mathfrak{q}_2).$$

On the other hand, for $\mathfrak{p}, \mathfrak{q} \in \mathrm{GP}[I]$, we have

$$(\mathfrak{p} + \mathfrak{q})|_S \otimes (\mathfrak{p} + \mathfrak{q})/_S = (\mathfrak{p}|_S + \mathfrak{q}|_S) \otimes (\mathfrak{p}/_S + \mathfrak{q}/_S).$$

This follows by projecting the identity

$$(\mathfrak{p} + \mathfrak{q})_v = \mathfrak{p}_v + \mathfrak{q}_v$$

to $\mathbb{R}^S$ and $\mathbb{R}^T$, respectively, where v is any vector in the interior of the face $F \in \Sigma[\mathcal{A}]$ corresponding to (S, T). □

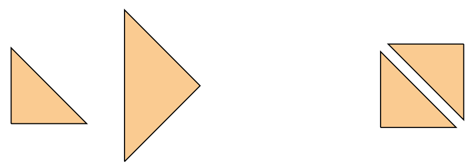
8. Final remarks and questions

1. The results in Section 6.1.1 strongly rely on the existence of a good set of generators for generalized permutahedra modulo translation: the simplices Δ_S with $|S| \geq 2$. The log-classes $\log[\Delta_S]$ of these simplices form a linear basis for the space $\Xi_1(\pi)$. In particular, these classes form a minimal set of generators of the algebra $\Pi(\pi)$. The question about the existence of a nice family of generators in other Coxeter types was already formulated by Ardila, Castillo, Eur, and Postnikov in [ACEP20], in particular for type B.

Already in dimension 2 we have

$$\eta_{\mathrm{H}}(\Xi_1(\pi_2^B)) = 1 \quad \text{for all hyperplanes H of } \mathcal{A}_2^{\pm}, \qquad \text{and} \qquad \eta_{\perp}(\Xi_1(\pi_2^B)) = 2.$$

Indeed, the log-classes of the intervals perpendicular to the hyperplanes and of any two type B triangles with distinct edge directions generate $\Xi_1(\pi_2^B)$. For instance, the type B triangles on the left satisfy this condition, but the triangles on the right do not.



This already shows that, unlike the type A case, such a collection of generators cannot arise from the set of faces of a single (type B) polytope.

2. McMullen [McM89] also studied valuation relations for the collection of polyhedral cones in V. The *full cone group* of V is generated by the classes $[C]$, one for each polyhedral cone $C \subseteq V$. They satisfy the following relation:

$$[C_1 \cup C_2] = [C_1] + [C_2] \tag{8.1}$$

whenever $C_1 \cup C_2$ is a cone and $C_1 \cap C_2$ is a proper face of C_1 and of C_2. Note that this is not to say that the class $[C_1 \cap C_2]$ is zero.

A cone of an arrangement $\mathcal{A}$ is any convex cone obtained as the union of faces of $\mathcal{A}$. The space of formal linear combinations of cones $\Omega[\mathcal{A}]$ is a right $\mathbf{\Sigma}[\mathcal{A}]$ module under the following operation. If C is a cone of $\mathcal{A}$ and $F \in \Sigma[\mathcal{A}]$, then

$$C \cdot \mathtt{H}_F = \begin{cases} T_{\tilde{F}}C & \text{if } F \subseteq C, \\ 0 & \text{otherwise,} \end{cases}$$

where $\tilde{F} \leq C$ is the minimum face of C containing F and $T_{\tilde{F}}C$ denotes the tangent cone of C at $\tilde{F}$. The relation (8.1) is compatible with this action, and defines a quotient module $\overline{\Omega}[\mathcal{A}]$.

Restricting to the case of all the braid arrangements, Ω defines a Hopf monoid in species. The product is defined by means of the Cartesian product. Let C be a cone of the braid arrangement in $\mathbb{R}^I$ and F the face corresponding to the composition $(S,T) \vDash I$. If $F \subseteq C$, the tangent cone $T_{\tilde{F}}C$ decomposes as a product $C|_S \times C/_S$ of cones in $\mathbb{R}^S$ and $\mathbb{R}^T$. The coproduct of Ω is defined as follows:

$$\Delta_{S,T}(C) = \begin{cases} C|_S \otimes C/_S & \text{if } F \subseteq C \\ 0 & \text{otherwise.} \end{cases}$$

With these operations, Ω is isomorphic to the Hopf monoid of preposets $\mathbf{Q}$ considered in [AA17]. Relation (8.1) defines a Hopf monoid quotient $\overline{\Omega}$. Under a suitable change of basis, $\overline{\Omega}$ is isomorphic to the dual Hopf monoid of faces $\mathbf{\Sigma}^*$ defined in [AM10, Chapter 12].

3. There is a Hopf monoid morphism $\mathsf{GP} \to \Omega$, whose components $\mathsf{GP}[I] \to \Omega[I]$ are defined as follows:

$$\mathfrak{p} \longmapsto \sum_{\mathfrak{q} \leq \mathfrak{p}} N(\mathfrak{q}, \mathfrak{p}) = \sum_{C \in \Sigma_{\mathfrak{p}}} C. \tag{8.2}$$

Moreover, this is a morphism of $(2,1)$-monoids in the 3-monoidal category $(\mathsf{Sp}, \cdot, \times, \cdot)$, where the monoidal structure of Ω under the Hadamard product is given by

$$C_1 \cdot C_2 = \begin{cases} C_1 \cap C_2 & \text{if } \operatorname{relint}(C_1) \cap \operatorname{relint}(C_2) \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

That this map defines a morphism of monoids under the Hadamard product is equivalent to the following fact: the normal fan of $\mathfrak{p} + \mathfrak{q}$ is the common refinement of $\Sigma_{\mathfrak{p}}$ and Σ_q.

The map (8.2) does **not** induce a well defined morphism $\Pi \to \overline{\Omega}$. In [McM89, Theorem 5], McMullen shows that

$$\mathfrak{p} \longmapsto \sum_{\mathfrak{q} \leq \mathfrak{p}} \operatorname{vol}(\mathfrak{q}) N(\mathfrak{q}, \mathfrak{p})$$

induces an injective map $\Pi[I] \to \overline{\Omega}[I]$, where $\operatorname{vol}(\mathfrak{q})$ is the *normalized* volume of $\mathfrak{q}$ in the affine space spanned by $\mathfrak{p}$. Moreover, one can verify that the induced morphism $\Pi \to \overline{\Omega}$ is a morphism of Hopf monoids. Is it possible to endow $\overline{\Omega}$ with the structure of a $(2,1)$-monoid so that the morphism above is a morphism of $(2,1)$-monoids?

Such a structure on $\overline{\Omega}[I]$ would contain a subalgebra isomorphic to the Möbius algebra $B^*(M)$ introduced by Huh and Wang in [HW17, Definition 5], where M is the matroid associated with the braid arrangement $\mathcal{A}_I$ in $\mathbb{R}^I$.

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