# ENUMERATION OF GELFAND-CETLIN TYPE REDUCED WORDS 

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#### Abstract

The combinatorics of reduced words and commutation classes plays an important role in geometric representation theory. A string polytope is a lattice polytope associated to each reduced word of the longest element $w_{0}$ in the symmetric group which encodes the character of a certain irreducible representation of a Lie group of type $A$. In this paper, we provide a recursive formula for the number of reduced words of $w_{0}$ such that the corresponding string polytopes are combinatorially equivalent to a Gelfand-Cetlin polytope. The recursive formula involves the number of standard Young tableaux of shifted shape. We also show that each commutation class is completely determined by a list of quantities called indices.


## 1. Introduction

A string polytope, introduced by Littelmann Lit98, is a convex polytope $\Delta_{\mathbf{i}}(\lambda)$ determined by two data: a reduced word $\mathbf{i}$ of the longest element in the Weyl group of a reductive algebraic group and a dominant weight $\lambda$. Its lattice points parametrize the dual canonical basis elements of the irreducible representation with highest weight $\lambda$ so that it can be regarded as a non-abelian generalization of a Newton polytope in toric geometry. The importance of string polytopes has been raised for the study of mirror symmetry of flag varieties (see, for example, BCFKvS00]). We refer the reader to [Lus90, [Kas90, Lit98, GP00, and [BZ01] for various descriptions of string polytopes.

One of the most famous examples of string polytopes is the Gelfand-Cetlin polytope. Similarly to string polytopes, Gelfand-Cetlin polytopes have been used to describe the irreducible representation of $\mathrm{SL}_{n+1}(\mathbb{C})$ with highest weight $\lambda$. We recall the definition of Gelfand-Cetlin polytopes from GC50 and GS83. Let $\lambda=\left(\lambda_{1}, \ldots, \lambda_{n}\right)$ be a sequence of nonnegative integers. For each $\lambda$, define the Gelfand-Cetlin polytope $\mathrm{GC}(\lambda)$ to be the closed convex polytope in $\mathbb{R}^{\bar{n}}$ consisting of the points $\left(x_{k, j}\right)_{1 \leq j \leq k \leq n}$ satisfying the inequalities

$$
x_{k+1, j} \geq x_{k, j} \geq x_{k+1, j+1}, \quad 1 \leq j \leq k \leq n
$$

where $\bar{n}=n(n+1) / 2, x_{n+1, j}=\lambda_{j}+\cdots+\lambda_{n}$ for $1 \leq j \leq n$, and $x_{n+1, n+1}=0$. Note that $\mathrm{GC}(\lambda)$ has the maximum dimension if each $\lambda_{i}$ is positive, i.e., $\lambda$ is regular. In this case, we say that $\mathrm{GC}(\lambda)$ is a full dimensional Gelfand-Cetlin polytope of

[^0]rank $n$. In Lit98, Corollary 5 in Section 5] it is shown that the Gelfand-Cetlin polytope $\mathrm{GC}(\lambda)$ is an example of a string polytope of $\mathrm{SL}_{n+1}(\mathbb{C})$. More precisely, we have
$$
\mathrm{GC}(\lambda) \simeq \Delta_{(1,2,1,3,2,1, \ldots, n, n-1, \ldots, 1)}(\lambda)
$$
where $\simeq$ means the unimodular equivalenc $\}^{1}$. In particular, these two polytopes are combinatorially equivalent. We refer the reader to ACK18 and references therein for more information on the combinatorics of Gelfand-Cetlin polytopes.

The string polytope $\Delta_{\mathbf{i}}(\lambda)$ has the maximum dimension if and only if the weight $\lambda$ is regular. Once the weight is assumed to be regular, the combinatorial type of the string polytope is independent of the choice of the weight. In this paper, we consider string polytopes $\Delta_{\mathbf{i}}(\lambda)$ of type $A$ for a fixed regular dominant weight $\lambda$ so that each $\Delta_{\mathbf{i}}(\lambda)$ is determined by the reduced word $\mathbf{i}$.

The motivation of this paper is to enumerate the string polytopes $\Delta_{\mathbf{i}}(\lambda)$ of type $A$ which are combinatorially equivalent to Gelfand-Cetlin polytopes. To this end, we study the reduced words $\mathbf{i}$ which give rise to such string polytopes $\Delta_{\mathbf{i}}(\lambda)$. To state our results, we introduce some terminologies.

Let $\mathfrak{S}_{n+1}$ be the symmetric group (i.e., the Weyl group of $\mathrm{SL}_{n+1}(\mathbb{C})$ ) of degree $n+1$ and denote by $s_{i}:=(i, i+1)$ the simple transposition which swaps $i$ and $i+1$ and fixes all other elements of $[n+1]=\{1, \ldots, n+1\}$. The set $\left\{s_{1}, \ldots, s_{n}\right\}$ of simple transpositions generates $\mathfrak{S}_{n+1}$, hence every element $w \in \mathfrak{S}_{n+1}$ can be written in the following form:

$$
w=s_{i_{1}} \cdots s_{i_{r}}, \quad i_{1}, \ldots, i_{r} \in[n] .
$$

In this case, the sequence $\mathbf{i}:=\left(i_{1}, \ldots, i_{r}\right)$ is called a word of $w$. The length $\ell(w)$ of $w$ is defined to be the smallest integer $r$ for which $\left(i_{1}, \ldots, i_{r}\right)$ is a word of $w$. A word $\left(i_{1}, \ldots, i_{r}\right)$ of $w$ is reduced if $r=\ell(w)$. We denote by $\mathcal{R}(w)$ the set of reduced words of $w$. There is a unique element $w_{0}^{(n+1)}$, called the longest element, in $\mathfrak{S}_{n+1}$ such that $\ell(w) \leq \ell\left(w_{0}^{(n+1)}\right)$ for all $w \in \mathfrak{S}_{n+1}$.

For $i, j \in[n]$ satisfying $|i-j|>1$, we have $s_{i} s_{j}=s_{j} s_{i}$. This induces an operation on the set $\mathcal{R}(w)$ defined by $(\ldots, i, j, \ldots) \mapsto(\ldots, j, i, \ldots)$, which is called a commutation (or a 2-move). Define an equivalence relation $\sim$ on $\mathcal{R}(w)$ by

$$
\mathbf{i} \sim \mathbf{i}^{\prime} \quad \Leftrightarrow \quad \mathbf{i} \text { is obtained from } \mathbf{i}^{\prime} \text { by a sequence of commutations. }
$$

An element in $[\mathcal{R}(w)]:=\mathcal{R}(w) / \sim$ is called a commutation class for $w$. For a recent account of the study of commutation classes, we refer the reader to B99, STWW17, FMPT18, GMS20 and references therein.

One important fact about commutation classes for our purpose is that two string polytopes $\Delta_{\mathbf{i}}(\lambda)$ and $\Delta_{\mathbf{i}^{\prime}}(\lambda)$ are combinatorially equivalent if (but not necessarily only if) $\mathbf{i}$ and $\mathbf{i}^{\prime}$ are in the same commutation class (see CKLP19a, Lemma 3.1]). Accordingly, studying the elements in $\left[\mathcal{R}\left(w_{0}^{(n+1)}\right)\right]$ is closely related to the classification problem of the combinatorial types of string polytopes.

We say that $\mathbf{i} \in \mathcal{R}\left(w_{0}^{(n+1)}\right)$ is a Gelfand-Cetlin type reduced word if the corresponding string polytope $\Delta_{\mathbf{i}}(\lambda)$ is combinatorially equivalent to a full dimensional Gelfand-Cetlin polytope of rank $n$. Let $\mathrm{gc}(n)$ be the number of Gelfand-Cetlin type reduced words in $\mathcal{R}\left(w_{0}^{(n+1)}\right)$. By the definition of $\mathrm{gc}(n)$, it also counts the number

[^1]of string polytopes $\Delta_{\mathbf{i}}(\lambda)$ with $\mathbf{i} \in \mathcal{R}\left(w_{0}^{(n+1)}\right)$ that are combinatorially equivalent to a full dimensional Gelfand-Cetlin polytope of rank $n$.

The first main result in this paper is the following recurrence relation for $\mathrm{gc}(n)$.
Theorem 1.1 (Theorem 4.13). The number $\operatorname{gc}(n)$ of Gelfand-Cetlin type reduced words in $\mathcal{R}\left(w_{0}^{(n+1)}\right)$ satisfies

$$
\operatorname{gc}(n)=\sum_{k=1}^{n} g^{(n, n-1, \ldots, n-k+1)} \operatorname{gc}(n-k)
$$

where for $\mu=\left(\mu_{1}, \ldots, \mu_{t}\right)$,

$$
g^{\mu}=\frac{|\mu|!}{\mu_{1}!\mu_{2}!\cdots \mu_{t}!} \prod_{i<j} \frac{\mu_{i}-\mu_{j}}{\mu_{i}+\mu_{j}},
$$

which is the number of standard Young tableaux of shifted shape $\mu$.
As a consequence of the proof of the above theorem, we obtain that the number of commutation classes consisting of Gelfand-Cetlin type reduced words in $\mathcal{R}\left(w_{0}^{(n+1)}\right)$ is $2^{n-1}$ (see Corollary 4.10). This result was also proved in a recent paper GMS20. using a different method.

We note that the number of string polytopes $\Delta_{\mathbf{i}}(\lambda)\left(\right.$ for $\left.\mathbf{i} \in \mathcal{R}\left(w_{0}^{(n+1)}\right)\right)$ which are unimodularly equivalent to the Gelfand-Cetlin polytope $\operatorname{GC}(\lambda)$ is the same as the number $\mathrm{gc}(n)$. Accordingly, the above theorem also enumerates the number of string polytopes $\Delta_{\mathbf{i}}(\lambda)$ which are unimodularly equivalent to the Gelfand-Cetlin polytope $\mathrm{GC}(\lambda)$ (see Corollary 4.14).

A crucial object in the proof of Theorem 1.1 is a quantity called $\delta$-index. For a sequence $\delta \in\{\mathrm{A}, \mathrm{D}\}^{n-1}$ of two letters A and D , the $\delta$-index $\operatorname{ind}_{\delta}(\mathbf{i})$ of a reduced word $\mathbf{i} \in \mathcal{R}\left(w_{0}^{(n+1)}\right)$ is an element in $\mathbb{Z}^{n-1}$ which measures how far a given word is from the standard reduced word

$$
(1,2,1,3,2,1, \ldots, n, n-1, \ldots, 1)
$$

of $w_{0}^{(n+1)}$. See Section 3 for the precise definition.
Recently, the first and the third authors together with Kim and Park CKLP19a, Theorem A] classified all Gelfand-Cetlin type reduced words in $\mathcal{R}\left(w_{0}^{(n+1)}\right)$ in terms of $\delta$-indices. More precisely, they showed that $\mathbf{i} \in \mathcal{R}\left(w_{0}^{(n+1)}\right)$ is a Gelfand-Cetlin reduced word if and only if there is a sequence $\delta \in\{\mathrm{A}, \mathrm{D}\}^{n-1}$ such that $\operatorname{ind}_{\delta}(\mathbf{i})=$ $(0, \ldots, 0) \in \mathbb{Z}^{n-1}$.

It turns out that for $\mathbf{i}, \mathbf{j} \in \mathcal{R}\left(w_{0}^{(n+1)}\right)$, if $\operatorname{ind}_{\delta}(\mathbf{i})=\operatorname{ind}_{\delta}(\mathbf{j})=(0, \ldots, 0) \in \mathbb{Z}^{n-1}$ for some $\delta \in\{\mathrm{A}, \mathrm{D}\}^{n-1}$, then $\mathbf{i} \sim \mathbf{j}$. However, the condition $\operatorname{ind}_{\delta}(\mathbf{i})=\operatorname{ind}_{\delta}(\mathbf{j})$ for some $\delta \in\{\mathrm{A}, \mathrm{D}\}^{n-1}$ does not always imply $\mathbf{i} \sim \mathbf{j}$ (see Example 4.9).

The second main result in this paper is the following theorem, which shows that the $\delta$-indices $\operatorname{ind}_{\delta}(\mathbf{i})$ for all $\delta \in\{\mathrm{A}, \mathrm{D}\}^{n-1}$ completely determine the commutation class of $\mathbf{i}$.

Theorem 1.2 (Theorem 5.1. Let $\mathbf{i}, \mathbf{j} \in \mathcal{R}\left(w_{0}^{(n+1)}\right)$. If $\operatorname{ind}_{\delta}(\mathbf{i})=\operatorname{ind}_{\delta}(\mathbf{j})$ for all $\delta \in\{\mathrm{A}, \mathrm{D}\}^{n-1}$, then $\mathbf{i} \sim \mathbf{j}$.

In order to prove our main results, we consider word posets, which are similar to wiring diagrams. A word poset is a poset $P$ together with a function $f_{P}: P \rightarrow$ $\mathbb{Z}_{>0}$. Each commutation class in $\left[\mathcal{R}\left(w_{0}^{(n+1)}\right)\right]$ corresponds to a word poset and the
cardinality of the commutation class is equal to the number of linear extensions of the corresponding word poset.

This paper is organized as follows. In Section 2 we give basic definitions. In Section 3, we recall the operations on $\mathcal{R}\left(w_{0}^{(n+1)}\right)$ called contractions and extensions. Moreover, we provide the definition of indices and prove several properties of them. In Section 4, we study Gelfand-Cetlin type reduced words and provide a proof of Theorem 1.1. In Section 5, we provide a proof of Theorem 1.2 .

## 2. BASIC DEFINITIONS

In this section, we give basic definitions and some of their properties which will be used throughout this paper.
2.1. Commutation classes. Let $w$ be an element in $\mathfrak{S}_{n+1}$ and denote by $\mathcal{R}(w)$ the set of reduced words of $w$, i.e.,

$$
\mathcal{R}(w)=\left\{\left(i_{1}, \ldots, i_{\ell}\right) \in[n]^{\ell} \mid s_{i_{1}} s_{i_{2}} \cdots s_{i_{\ell}}=w\right\}
$$

where $\ell=\ell(w)$ is the length of $w$. It is well known that

$$
\ell(w)=\#\{1 \leq i<j \leq n \mid w(i)>w(j)\} .
$$

We denote by $w_{0}^{(n+1)}$ the longest element $n+1 n \ldots 1$ of $\mathfrak{S}_{n+1}$ (using the one-line notation). The length $\bar{n}$ of the longest element $w_{0}^{(n+1)}$ is given by

$$
\begin{equation*}
\bar{n}=\ell\left(w_{0}^{(n+1)}\right)=\frac{n(n+1)}{2} \quad \text { for } n \in \mathbb{Z}_{>0}:=\{1,2, \ldots\} \tag{2.1}
\end{equation*}
$$

Recall that for a given $\mathbf{i} \in \mathcal{R}(w)$, one can produce new reduced words using braid moves. There are two types of braid moves as follows:

- A 2-move replaces two consecutive elements $i, j$ in $\mathbf{i}$ by $j, i$ for some integers $i$ and $j$ with $|i-j|>1$.
- A 3-move replaces three consecutive elements $i, j, i$ in $\mathbf{i}$ by $j, i, j$ for some integers $i$ and $j$ with $|i-j|=1$.
Note that braid moves do not change the product of the simple transpositions for i since $s_{i} s_{j}=s_{j} s_{i}$ for $|i-j|>1$ and $s_{i} s_{i+1} s_{i}=s_{i+1} s_{i} s_{i+1}$. According to Tits' Theorem Tit69, any two reduced words in $\mathcal{R}(w)$ are connected by a sequence of braid moves. Braid moves and Tits' theorem can be generalized to other Coxeter systems (see BB05, §3.3]).

Define an equivalence relation ' $\sim$ ' on $\mathcal{R}(w)$ by

$$
\mathbf{i} \sim \mathbf{i}^{\prime} \quad \Leftrightarrow \quad \mathbf{i} \text { is obtained from } \mathbf{i}^{\prime} \text { by a sequence of 2-moves. }
$$

We denote by $[\mathcal{R}(w)]:=\mathcal{R}(w) / \sim$ the set of equivalence classes and call an element $[\mathbf{i}] \in[\mathcal{R}(w)]$ a commutation class.

Remark 2.1. There is no known exact formula for the number $\mathrm{c}(n+1)$ of commutation classes of $w_{0}^{(n+1)}$. Some upper and lower bounds for $\mathrm{c}(n)$ were provided by Knuth [Knu92, Section 9]. Felsner and Valtr [FV11, Theorem 2 and Proposition 1] found the following upper and lower bounds for $\mathrm{c}(n+1)$ improving Knuth's results: for a sufficiently large $n$,

$$
2^{0.1887 n^{2}} \leq \mathrm{c}(n) \leq 2^{0.6571 n^{2}}
$$

The first few terms of $c(n)$ are $1,1,2,8,62,908,24698,1232944$, see A006245 in OEIS.


Figure 1. The configurations $G_{1}, G_{2}$, and $G_{3}$ (from left to right) for $n=4$.


Figure 2. The wiring diagrams $G(\mathbf{i})$ (left) and $G(\mathbf{j})$ (right) for $\mathbf{i}=(1,2,1,3,2,1)$ and $\mathbf{j}=(1,3,2,1,3,2)$ in $\mathcal{R}\left(w_{0}^{(4)}\right)$. The 3rd wire in each wiring diagram is colored red. The crossing in row $j$ is labeled by $t_{j}$. The crossing $t_{4}$ in $G(\mathbf{i})$ (respectively, $G(\mathbf{j})$ ) is in column 3 (respectively column 1 ).
2.2. Wiring diagrams. There are several combinatorial models for the commutation classes of the longest element of $\mathfrak{S}_{n+1}$, see DES16 and references therein. We recall a well known combinatorial model, called a wiring diagram (cf. GP93]).
Definition 2.2. Let $\mathbf{i}=\left(i_{1}, i_{2}, \ldots, i_{\ell}\right)$ be a word of $w \in \mathfrak{S}_{n+1}$.
(1) The wiring diagram $G(\mathbf{i})$ of $\mathbf{i}$ is a collection of line segments defined as follows.

- For $1 \leq i \leq n$ and $1 \leq j \leq n+1$, define $G_{i}$ to be the collection of line segments $\left(A_{j}, B_{s_{i}(j)}\right)$ connecting $A_{j}$ and $B_{s_{i}(j)}$, where $A_{j}=(j, 1)$ and $B_{j}=(j, 0)$ are points in $\mathbb{R}^{2}$. The intersection of the segments $\left(A_{i}, B_{i+1}\right)$ and $\left(A_{i+1}, B_{i}\right)$ is called a crossing in column $i$. See Figure 1 .
- Define $G(\mathbf{i})$ to be the configuration obtained by arranging $G_{i_{1}}, G_{i_{2}}, \ldots, G_{i_{\ell}}$ vertically in this order. More precisely,

$$
G(\mathbf{i})=\rho^{\ell-1}\left(G_{i_{1}}\right) \cup \rho^{\ell-2}\left(G_{i_{2}}\right) \cup \cdots \cup \rho^{0}\left(G_{i_{\ell}}\right)
$$

where $\rho$ is the translation by $(0,1)$. See Figure 2 ,
(2) The $j$ th row of $G(\mathbf{i})$ is $\rho^{\ell-j}\left(G_{i_{j}}\right)$. The points $(j, \ell)$ and $(j, 0)$ are called the $j$ th starting point and the $j$ th ending point of $G(\mathbf{i})$, respectively. A wire of $G(\mathbf{i})$ is a path from a starting point to an ending point of $G(\mathbf{i})$ obtained by taking the union of $\ell$ segments one from each row. If a wire starts at the $j$ th starting point, it is called the $j$ th wire of $G(\mathbf{i})$.
(3) We denote by $\mathcal{W D}(w)$ the set of wiring diagrams $G(\mathbf{i})$ for all reduced words $\mathbf{i}$ of $w$.

By the definition of $G(\mathbf{i})$, it is clear that the $j$ th wire is from the $j$ th starting point to the $w(j)$ th ending point. One can reconstruct $\mathbf{i}$ from $G(\mathbf{i})$ because the unique crossing in row $j$ of $G(\mathbf{i})$ is in column $i_{j}$. This gives a bijection between $\mathcal{R}(w)$ and $\mathcal{W D}(w)$. We define the equivalence relation ' $\sim$ ' on $\mathcal{W D}(w)$ by $G(\mathbf{i}) \sim G(\mathbf{j})$ if and only if $\mathbf{i} \sim \mathbf{j}$. Equivalently, we have $G(\mathbf{i}) \sim G(\mathbf{j})$ if and only if $G(\mathbf{i})$ is obtained
from $G(\mathbf{j})$ by a sequence of operations exchanging two adjacent rows in which the crossings are not in adjacent columns.

Let $[\mathcal{W D} \mathcal{D}(w)]=\mathcal{W} \mathcal{D}(w) / \sim$. Then we have an obvious bijection between $[\mathcal{W D}(w)]$ and $[\mathcal{R}(w)]$ induced by the correspondence explained above.
2.3. Word posets. To study commutation classes, we associate a poset and a function on the poset to each reduced word $\mathbf{i} \in \mathcal{R}(w)$.

Definition 2.3. A word poset is a poset $P$ together with a function $f_{P}: P \rightarrow \mathbb{Z}_{>0}$. Two word posets $P$ and $Q$ are isomorphic, denoted by $P \sim Q$, if there is a poset isomorphism $\phi: P \rightarrow Q$ such that $f_{P}(x)=f_{Q}(\phi(x))$ for all $x \in P$.

Definition 2.4. Let $\mathbf{i}=\left(i_{1}, \ldots, i_{\ell}\right) \in \mathcal{R}(w)$. The word poset of $\mathbf{i}$ is a poset $P_{\mathbf{i}}$ on $[\ell]$ together with a function $f_{P_{\mathrm{i}}}: P_{\mathbf{i}} \rightarrow \mathbb{Z}_{>0}$ defined as follows:

- $P_{\mathbf{i}}$ is the transitive closure of the binary relation containing $(r, r)$ for $r \in[\ell]$ and $(j, k)$ for $j, k \in[\ell]$ with $\left|i_{j}-i_{k}\right|=1$ and $j<k$, and
- $f_{P_{\mathbf{i}}}(j)=i_{j}$ for all $j \in P_{\mathbf{i}}$.

Denote by $\mathcal{P}(w)$ the set of word posets $P$ such that $P \sim P_{\mathbf{i}}$ for some $\mathbf{i} \in \mathcal{R}(w)$. We also define $[\mathcal{P}(w)]=\mathcal{P}(w) / \sim$.

We will see later in this subsection that word posets are closely related to wiring diagrams. Throughout this paper the following convention will be used when we draw the Hasse diagram of a word poset.
Convention. Let $P$ be a word poset. For $j \in P$, if $f_{P}(j)=i$, we say that $j$ is in column $i$. When we draw the Hasse diagram of $P$ the element $j$ will be placed in column $i$. For a subset $A$ of $P$, we say that $Q$ is the word poset obtained from $P$ by shifting $A$ to the left (respectively, right) by one column if $P$ and $Q$ are the same as posets and $f_{Q}(x)=f_{P}(x)$ if $x \notin A$, and $f_{Q}(x)=f_{P}(x)-1$ (respectively, $\left.f_{Q}(x)=f_{P}(x)+1\right)$ if $x \in A$.

By definition, every covering relation in a word poset $P \in \mathcal{P}(w)$ occurs between two adjacent columns. In other words, if $x \lessdot_{P} y$, then $\left|f_{P}(x)-f_{P}(y)\right|=1$.

Example 2.5. We illustrate some examples of the word posets $P_{\mathbf{i}}$ associated with some reduced words $\mathbf{i}=\left(i_{1}, \ldots, i_{\ell}\right)$. We will arrange the elements of $P_{\mathbf{i}}$ so that $j$ is in column $f_{P_{\mathrm{i}}}(j)=i_{j}$.
(1) Let $\mathbf{i}=(1,2,1,3,2,1)$. Then the Hasse diagram of the word poset $P_{\mathbf{i}}$ is shown below.


Here the elements $1,3,6$ are in column 1 , the elements 2,5 are in column 2, and the element 4 is in column 3 . Note that if $\mathbf{i}^{\prime}=(1,2,3,1,2,1)$, then $\mathbf{i}^{\prime} \sim \mathbf{i}$ and the word poset $P_{\mathbf{i}^{\prime}}$ shown below is isomorphic to $P_{\mathbf{i}}$.

(2) Let $\mathbf{j}=(1,3,2,1,3,2)$. Then the Hasse diagram of the word poset $P_{\mathbf{j}}$ is given as follows.


A linear extension of a poset $P$ is a permutation $p_{1} p_{2} \ldots p_{n}$ of the elements in $P$ such that $j<k$ whenever $p_{j}<_{P} p_{k}$.

Proposition 2.6. Sam11, Theorem 1.1] Let $\mathbf{i}, \mathbf{j} \in \mathcal{R}(w)$.
(1) There is a bijection between the linear extensions of the poset $P_{\mathbf{i}}$ and the elements in the commutation class $[\mathbf{i}]$ given as follows. A linear extension $p_{1} p_{2} \ldots p_{n}$ of $P_{\mathbf{i}}$ corresponds to the $\operatorname{word}\left(i_{f_{P_{\mathbf{i}}}\left(p_{1}\right)}, i_{f_{P_{\mathbf{i}}}\left(p_{2}\right)}, \ldots, i_{f_{P_{\mathbf{i}}}\left(p_{\ell}\right)}\right)$ in $[\mathbf{i}]$.
(2) We have $\mathbf{i} \sim \mathbf{j}$ if and only if $P_{\mathbf{i}} \sim P_{\mathbf{j}}$.

We note that Proposition 2.6(1) has a similar result on partial commutation monoids, see [Knu98, §5.1.2, Exercise 11], Sta12, Exercise 3.123], or [DBKR19, Proposition 4.11].

By Proposition 2.6(2), we can identify the commutation class [i] with $\left[P_{\mathbf{i}}\right]$.
Proposition 2.7. The map $[\mathbf{i}] \mapsto\left[P_{\mathbf{i}}\right]$ is a bijection from $[\mathcal{R}(w)]$ to $[\mathcal{P}(w)]$.
There is also a direct and natural correspondence between $[\mathcal{W} \mathcal{D}(w)]$ and $[\mathcal{P}(w)]$, which we now explain. Let $[D] \in[\mathcal{W} \mathcal{D}(w)]$. We define the corresponding word poset class $[P] \in[\mathcal{P}(w)]$ as follows.

- The elements of the underlying set $P$ are the crossings in $D$.
- For two distinct elements $a, b \in P$, we have $a<_{P} b$ if there is a downward path from the crossing $a$ to the crossing $b$ in $D$. Here, a downward path means a path following wires (it is allowed to switch between the two wires at a crossing) in the direction that the $y$-coordinate decreases.
- For $a \in P$, define $f_{P}(a)=c$ if the crossing $a$ is in column $c$ in $D$.

For example, the wiring diagrams $G(\mathbf{i})$ and $G(\mathbf{j})$ in Figure 2 correspond to the word posets $P_{\mathbf{i}}$ and $P_{\mathbf{j}}$ in Example 2.5

Proposition 2.8. Let $w \in \mathfrak{S}_{n+1}$. Then the map $[D] \mapsto[P]$ described above is $a$ bijection from $[\mathcal{W D}(w)]$ to $[\mathcal{P}(w)]$.

Proof. We first show that the map $[D] \mapsto[P]$ is well defined. Suppose that $D^{\prime}$ is obtained from $D$ by exchanging two adjacent rows in which the crossings are not in adjacent columns. It is easy to see that a downward path from a crossing $a$ to a
crossing $b$ in $D$ remains a downward path from $a$ to $b$ in $D^{\prime}$. This shows that the images of $[D]$ and $\left[D^{\prime}\right]$ under this map are identical. Thus the map is well defined.

Now we show that the map is a bijection. Since $[\mathcal{W} \mathcal{D}(w)]=\{[G(\mathbf{i})]: \mathbf{i} \in \mathcal{R}(w)\}$ and $[\mathcal{P}(w)]=\left\{\left[P_{\mathbf{i}}\right]: \mathbf{i} \in \mathcal{R}(w)\right\}$ both in bijection with $[\mathcal{R}(w)]$, it suffices to show that $[G(\mathbf{i})] \mapsto\left[P_{\mathbf{i}}\right]$. This is straightforward to check by the construction of the map. We omit the details.

## 3. Contractions, extensions, And indices

In this section, we define $\mathrm{A}^{-}, \mathrm{D}-$, and $\delta$-indices of a word poset, and two operations on word posets, called contractions and extensions. We also provide some results on these objects which will be used in later sections.

From now on, we concentrate on the reduced words of the longest element $w_{0}^{(n+1)}$ in the symmetric group $\mathfrak{S}_{n+1}$ and the word posets in $\mathcal{P}\left(w_{0}^{(n+1)}\right)$. In order to define indices, we prepare the following two lemmas.

Lemma 3.1 (cf. GP00, §2.2]). Let $\mathbf{i} \in \mathcal{R}\left(w_{0}^{(n+1)}\right)$. The wiring diagram $G(\mathbf{i})$ has the following properties.
(1) The ith wire travels from the ith starting point to the $(n+2-i)$ th ending point for $1 \leq i \leq n+1$.
(2) Two different wires meet exactly once.
(3) Every wire has exactly $n$ crossings.
(4) Let $c_{1}, \ldots, c_{n}$ be the crossings that lie on the 1 st wire (respectively, $(n+1)$ st wire) of $G(\mathbf{i})$ in this order, i.e., the row of $c_{i+1}$ is lower than the row of $c_{i}$. Then each $c_{i}$ is in column $i$ (respectively, $n+1-i$ ).

Lemma 3.2. Let $\mathbf{i}=\left(i_{1}, \ldots, i_{\bar{n}}\right) \in \mathcal{R}\left(w_{0}^{(n+1)}\right)$. Then there are unique integers $d_{1}, \ldots, d_{n}$ such that

$$
\begin{equation*}
1 \leq d_{1}<\cdots<d_{n} \leq \bar{n}, \quad\left(i_{d_{1}}, \ldots, i_{d_{n}}\right)=(n, n-1, \ldots, 2,1) \tag{3.1}
\end{equation*}
$$

and there are unique integers $a_{1}, \ldots, a_{n}$ such that

$$
\begin{equation*}
1 \leq a_{1}<\cdots<a_{n} \leq \bar{n}, \quad\left(i_{a_{1}}, \ldots, i_{a_{n}}\right)=(1,2, \ldots, n-1, n) \tag{3.2}
\end{equation*}
$$

Moreover, $\left|\left\{a_{1}, \ldots, a_{n}\right\} \cap\left\{d_{1}, \ldots, d_{n}\right\}\right|=1$.
Proof. We will use the wiring diagram $G(\mathbf{i})$ of i. Let $d_{1}<\cdots<d_{n}$ be the row indices of the crossings in the $(n+1)$ st wire of $G(\mathbf{i})$. Then, by Lemma 3.1, $d_{1}, \ldots, d_{n}$ satisfy (3.1). Similarly, let $a_{1}<\cdots<a_{n}$ be the row indices of the crossings in the 1 st wire of $G(\mathbf{i})$. Then, by Lemma 3.1, $a_{1}, \ldots, a_{n}$ satisfy (3.2). Since the 1 st and the $(n+1)$ st wires meet exactly once, we get $\left|\left\{a_{1}, \ldots, a_{n}\right\} \cap\left\{d_{1}, \ldots, d_{n}\right\}\right|=1$. It remains to show that these integers are unique.

To show the uniqueness of $d_{1}, \ldots, d_{n}$ satisfying (3.1), suppose that $d_{1}^{\prime}, \ldots, d_{n}^{\prime}$ are integers such that $d_{j} \neq d_{j}^{\prime}$ for some $j$ and

$$
1 \leq d_{1}^{\prime}<\cdots<d_{n}^{\prime} \leq \bar{n}, \quad\left(i_{d_{1}^{\prime}}, \ldots, i_{d_{n}^{\prime}}\right)=(n, n-1, \ldots, 2,1)
$$

Let $k$ be the smallest integer such that $d_{j}=d_{j}^{\prime}$ for $j<k$ and $d_{k} \neq d_{k}^{\prime}$. Since the $(n+1)$ st wire passes through the crossing in row $d_{k-1}$ and column $n+2-k$, the crossing in row $d_{k}$ and column $n+1-k$, and the crossing in row $d_{k+1}$ and column $n-k$, there is no crossing in row $j$ and column $n+1-k$ for all $d_{k-1}<j<$ $d_{k+1}$ with $j \neq d_{k}$. Since $d_{k-1}=d_{k-1}^{\prime}<d_{k}^{\prime}$, this shows $d_{k+1}<d_{k}^{\prime}$. See Figure 3 .


Figure 3. An illustration of a part of the $(n+1)$ st wire (colored in red) in a wiring diagram and the crossings in rows $d_{k-1}, d_{k}, d_{k+1}$, and $d_{k}^{\prime}$.

By the same argument, we can deduce that $d_{j+1}<d_{j}^{\prime}$ for $j=k, k+1, \ldots, n-1$. In particular we have $d_{n}<d_{n-1}^{\prime}<d_{n}^{\prime}$. However, by the definition of $d_{n}$, the crossing in row $d_{n}$ and column 1 is the lowest crossing in this column, which is a contradiction to $d_{n}<d_{n}^{\prime}$. This shows that there are no such integers $d_{1}^{\prime}, \ldots, d_{n}^{\prime}$, so the uniqueness of $d_{1}, \ldots, d_{n}$ is proved.

Similarly, we can show the uniqueness of $a_{1}, \ldots, a_{n}$, and the proof is completed.

The previous lemma can be restated in terms of word posets as follows.
Proposition 3.3. Let $P \in \mathcal{P}\left(w_{0}^{(n+1)}\right)$. Then $P$ contains a unique chain $\mathrm{D}(P)$ such that

$$
\mathrm{D}(P)=\left\{d_{1}<_{P} d_{2}<_{P} \cdots<_{P} d_{n}\right\}
$$

and $f_{P}\left(d_{i}\right)=n+1-i$ for all $1 \leq i \leq n$. Similarly, $P$ contains a unique chain $\mathrm{A}(P)$ such that

$$
\mathrm{A}(P)=\left\{a_{1}<_{P} a_{2}<_{P} \cdots<_{P} a_{n}\right\}
$$

and $f_{P}\left(a_{i}\right)=i$ for all $1 \leq i \leq n$. Moreover, we have $|\mathrm{D}(P) \cap \mathrm{A}(P)|=1$.
Proof. Let $P \in \mathcal{P}\left(w_{0}^{(n+1)}\right)$. By Proposition 2.7. there exists $\mathbf{i} \in \mathcal{R}\left(w_{0}^{(n+1)}\right)$ satisfying $P_{\mathbf{i}} \sim P$. Accordingly, it is enough to prove the statements for the word poset $P_{\mathbf{i}}$ for an arbitrary $\mathbf{i}=\left(i_{1}, \ldots, i_{\bar{n}}\right) \in \mathcal{R}\left(w_{0}^{(n+1)}\right)$. Then, using the map in Proposition 2.7, the statements for $P_{\mathbf{i}}$ that we need to prove can be reformulated as the statements for $\mathbf{i}$ in Lemma 3.2. Therefore the proof follows from this lemma.

We call $\mathrm{D}(P)$ the descending chain of $P$ and $\mathrm{A}(P)$ the ascending chain of $P$. We now define an important notion in this paper called indices.

Definition 3.4. Let $P \in \mathcal{P}\left(w_{0}^{(n+1)}\right)$. The D -index of $P$, denoted by $\operatorname{ind}_{\mathrm{D}}(P)$, is the number of elements in $P$ above the descending chain $\mathrm{D}(P)$ in the Hasse diagram
of $P$. Similarly, the A-index of $P$, denoted by $\operatorname{ind}_{\mathrm{A}}(P)$, is the number of elements in $P$ above the ascending chain in the Hasse diagram of $P$. More precisely, if $\mathrm{D}(P)=\left\{d_{1}<_{P} d_{2}<_{P} \cdots<_{P} d_{n}\right\}$ and $\mathrm{A}(P)=\left\{a_{1}<_{P} a_{2}<_{P} \cdots<_{P} a_{n}\right\}$, then

$$
\begin{aligned}
\operatorname{ind}_{\mathrm{D}}(P) & =\sum_{i=1}^{n} \#\left\{k \in[\bar{n}]: k>_{P} d_{i} \text { and } f_{P}(k)=f_{P}\left(d_{i}\right)\right\}, \\
\operatorname{ind}_{\mathrm{A}}(P) & =\sum_{i=1}^{n} \#\left\{k \in[\bar{n}]: k>_{P} a_{i} \text { and } f_{P}(k)=f_{P}\left(a_{i}\right)\right\} .
\end{aligned}
$$

For $\mathbf{i} \in \mathcal{R}\left(w_{0}^{(n+1)}\right)$, we also define

$$
\operatorname{ind}_{\mathrm{D}}(\mathbf{i})=\operatorname{ind}_{\mathrm{D}}\left(P_{\mathbf{i}}\right), \quad \operatorname{ind}_{\mathrm{A}}(\mathbf{i})=\operatorname{ind}_{\mathrm{A}}\left(P_{\mathbf{i}}\right)
$$

Note that in the above definition the summand $\#\left\{k \in[\bar{n}]: k>_{P} d_{i}\right.$ and $f_{P}(k)=$ $\left.f_{p}\left(d_{i}\right)\right\}$ is the number of elements of $P$ above the element $d_{i}$ of $\mathrm{D}(P)$ in column $f_{P}\left(d_{i}\right)=n+1-i$ in the Hasse diagram of $P$.

In CKLP19a, Definition 3.4] the indices $\operatorname{ind}_{\mathrm{D}}(\mathbf{i})$ and $\operatorname{ind}_{\mathrm{A}}(\mathbf{i})$ of $\mathbf{i} \in \mathcal{R}\left(w_{0}^{(n+1)}\right)$ are defined without using word posets. It is not hard to see that the two definitions are equivalent.
Example 3.5. Continuing Example 2.5, let $\mathbf{i}=(1,2,1,3,2,1)$ and $\mathbf{j}=(1,3,2,1,3,2)$. Then we have that

$$
\begin{array}{ll}
\mathrm{A}\left(P_{\mathbf{i}}\right)=1<_{P_{\mathbf{i}}} 2<_{P_{\mathbf{i}}} 4, & \mathrm{D}\left(P_{\mathbf{i}}\right)=4<_{P_{\mathbf{i}}} 5<_{P_{\mathbf{i}}} 6, \\
\mathrm{~A}\left(P_{\mathbf{j}}\right)=1<_{P_{\mathbf{j}}} 3<_{P_{\mathbf{j}}} 5, & \mathrm{D}\left(P_{\mathbf{j}}\right)=2<_{P_{\mathbf{j}}} 3<_{P_{\mathbf{j}}} 4
\end{array}
$$

The chains $\mathrm{D}(P)$ and $\mathrm{A}(P)$ for $P=P_{\mathbf{i}}$ (left) and $P=P_{\mathbf{j}}$ (right) are shown as follows.


Counting the number of elements above $\mathrm{A}(P)$ and $\mathrm{D}(P)$, we obtain

$$
\operatorname{ind}_{\mathrm{D}}(\mathbf{i})=0, \quad \operatorname{ind}_{\mathrm{A}}(\mathbf{i})=3, \quad \operatorname{ind}_{\mathrm{D}}(\mathbf{j})=2, \quad \operatorname{ind}_{\mathrm{A}}(\mathbf{j})=2
$$

The following lemma shows that the elements of $P \in \mathcal{P}\left(w_{0}^{(n+1)}\right)$ in each column form a chain.
Lemma 3.6. Let $P \in \mathcal{P}\left(w_{0}^{(n+1)}\right)$. Then $\left\{x \in P: f_{P}(x)=i\right\}$ is a chain in $P$ for each $i \in[n]$.
Proof. Since $P \in \mathcal{P}\left(w_{0}^{(n+1)}\right)$, there is a reduced word $\mathbf{i}=\left(i_{1}, i_{2}, \ldots, i_{\bar{n}}\right) \in \mathcal{R}\left(w_{0}^{(n+1)}\right)$ such that $P_{\mathbf{i}} \sim P$. Therefore it suffices to show that $\left\{x \in P_{\mathbf{i}}: f_{P_{\mathbf{i}}}(x)=i\right\}$ is a chain. Consider $x, y \in P_{\mathbf{i}}$ with $f_{P_{\mathbf{i}}}(x)=f_{P_{\mathbf{i}}}(y)=i$ and $x<\mathbb{Z} y$. This means $i_{x}=i_{y}=i$. Since $\mathbf{i}$ is reduced there must be an integer $z$ such that $x<z<y$ and $i_{z} \in\{i-1, i+1\}$. Then $x<_{P_{\mathbf{i}}} z<_{P_{\mathbf{i}}} y$ and therefore $x<_{P_{\mathbf{i}}} y$. Since any two elements $x, y \in P_{\mathbf{i}}$ with $f_{P_{\mathbf{i}}}(x)=f_{P_{\mathbf{i}}}(y)$ are comparable, $\left\{x \in P_{\mathbf{i}}: f_{P_{\mathbf{i}}}(x)=i\right\}$ is a chain and the lemma is proved.

Now we define two operations, called contractions, on word posets.
Definition 3.7. Let $P \in \mathcal{P}\left(w_{0}^{(n+1)}\right)$ and let

$$
\begin{aligned}
& \mathrm{D}(P)=\left\{\begin{array}{ll}
d_{1}<_{P} & d_{2}<_{P} \\
\cdots<_{P} & d_{n}
\end{array}\right\} \\
& \mathrm{A}(P)=\left\{\begin{array}{lll}
a_{1}<_{P} & a_{2}<_{P} & \cdots<_{P}
\end{array} a_{n}\right\} .
\end{aligned}
$$

(1) The D-contraction of $P$ is the word poset $C_{\mathrm{D}}(P)$ obtained from $P$ by removing the descending chain $\mathrm{D}(P)$ with the function $f_{C_{\mathrm{D}}(P)}: C_{\mathrm{D}}(P) \rightarrow \mathbb{Z}_{>0}$ given by

$$
f_{C_{\mathrm{D}}(P)}(k)= \begin{cases}f_{P}(k) & \text { if } k \in I_{\mathrm{D}}(P) \\ f_{P}(k)-1 & \text { if } k \in C_{\mathrm{D}}(P) \backslash I_{\mathrm{D}}(P)\end{cases}
$$

where $I_{\mathrm{D}}(P)=\left\{k \in P: k<_{P} d_{i}\right.$ and $f_{P}(k)=f_{P}\left(d_{i}\right)$ for some $\left.i \in[n]\right\}$. Here, the poset structure of $C_{\mathrm{D}}(P)$ is induced from that of $P$, i.e., for $x, y \in C_{\mathrm{D}}(P)$ we have $x<_{C_{\mathrm{D}}(P)} y$ if and only if $x<_{P} y$.
(2) The A-contraction of $P$ is the word poset $C_{\mathrm{A}}(P)$ obtained from $P$ by removing the ascending chain $\mathrm{A}(P)$ with the function $f_{C_{\mathrm{A}}(P)}: C_{\mathrm{A}}(P) \rightarrow \mathbb{Z}_{>0}$ given by

$$
f_{C_{\mathrm{A}}(P)}(k)= \begin{cases}f_{P}(k)-1 & \text { if } k \in I_{\mathrm{A}}(P) \\ f_{P}(k) & \text { if } k \in C_{\mathrm{D}}(P) \backslash I_{\mathrm{A}}(P),\end{cases}
$$

where $I_{\mathrm{A}}(P)=\left\{k \in P: k<_{P} a_{i}\right.$ and $f_{P}(k)=f_{P}\left(a_{i}\right)$ for some $\left.i \in[n]\right\}$. Here, the poset structure of $C_{\mathrm{A}}(P)$ is induced from that of $P$.
Observe that $I_{\mathrm{D}}(P)$ (respectively, $I_{\mathrm{A}}(P)$ ) is the ideal consisting of the elements below the descending chain $\mathrm{D}(P)$ (respectively, ascending chain $\mathrm{A}(P)$ ) in the Hasse diagram of $P$. We call $I_{\mathrm{D}}(P)$ (respectively, $I_{\mathrm{A}}(P)$ ) the D-contraction ideal (respectively, A-contraction ideal) of $P$. Here, an ideal of a poset $P$ means a subset $I$ of $P$ with the property that $x \in I$ and $y<_{P} x$ imply $y \in I$.

One may consider $C_{\mathrm{D}}(P)$ as the word poset whose Hasse diagram is obtained from that of $P$ by removing the descending chain $\mathrm{D}(P)$ and shifting the part $(P \backslash$ $\mathrm{D}(P)) \backslash I_{\mathrm{D}}(P)$ above $\mathrm{D}(P)$ to the left by one column. Similarly one may consider $C_{\mathrm{A}}(P)$ as the word poset whose Hasse diagram is obtained from that of $P$ by removing the ascending chain $\mathrm{A}(P)$ and shifting the part $I_{\mathrm{A}}(P)$ below $\mathrm{A}(P)$ to the left by one column. See Figure 4
Example 3.8. Let $\mathbf{i}=(4,3,4,2,3,4,1,2,5,4,3,2,1,4,5) \in \mathcal{R}\left(w_{0}^{(6)}\right)$. The Hasse diagrams of $P_{\mathbf{i}}, C_{\mathrm{A}}\left(P_{\mathbf{i}}\right)$, and $C_{\mathrm{D}}\left(P_{\mathbf{i}}\right)$ are shown in Figure 4. Note that $\operatorname{ind}_{\mathrm{A}}\left(P_{\mathbf{i}}\right)=2$ and $\operatorname{ind}_{\mathrm{D}}\left(P_{\mathbf{i}}\right)=2$.

Now we define the reverse operations of the contractions.
Definition 3.9. Let $P \in \mathcal{P}\left(w_{0}^{(n)}\right)$ and let $I$ be an ideal of $P$.
(1) The D-extension of $P$ with respect to $I$ is the word poset $E_{\mathrm{D}}(P, I)$ which is a poset on $P \sqcup\left\{d_{1}, d_{2}, \ldots, d_{n}\right\}$ with a function $f_{E_{\mathrm{D}}(P, I)}: E_{\mathrm{D}}(P, I) \rightarrow \mathbb{Z}_{>0}$ defined as follows.

- The function $f_{E_{\mathrm{D}}(P, I)}: E_{\mathrm{D}}(P, I) \rightarrow \mathbb{Z}_{>0}$ is given by

$$
f_{E_{\mathrm{D}}(P, I)}(x)= \begin{cases}f_{P}(x) & \text { if } x \in I \\ n+1-i & \text { if } x=d_{i} \\ f_{P}(x)+1 & \text { if } x \in P \backslash I\end{cases}
$$



Figure 4. The Hasse diagrams of $P_{\mathbf{i}}, C_{\mathrm{D}}\left(P_{\mathbf{i}}\right)$, and $C_{\mathrm{A}}\left(P_{\mathbf{i}}\right)$ for $\mathbf{i}=$ $(4,3,4,2,3,4,1,2,5,4,3,2,1,4,5)$. In each diagram, the element $k$ is in column $f_{P_{\mathbf{i}}}(k), f_{C_{\mathrm{D}}\left(P_{\mathbf{i}}\right)}(k)$, or $f_{C_{\mathrm{A}}\left(P_{\mathbf{i}}\right)}(k)$.

- The covering relations of the poset $E_{\mathrm{D}}(P, I)$ on $P \sqcup\left\{d_{1}, d_{2}, \ldots, d_{n}\right\}$ are given by $x \lessdot_{E_{\mathrm{D}}(P, I)} y$ if and only if one of the following conditions holds:
(a) $x, y \in I$ and $x \lessdot_{P} y$,
(b) $x, y \in P \backslash I$ and $x \lessdot_{P} y$,
(c) $x=d_{i}$ and $y=d_{i+1}$ for some $i \in[n-1]$,
(d) $x$ is a maximal element of $I$ in $P, y \in\left\{d_{1}, \ldots, d_{n}\right\}$, and

$$
\left|f_{E_{\mathrm{D}}(P, I)}(x)-f_{E_{\mathrm{D}}(P, I)}(y)\right|=1, \text { or }
$$

(e) $x \in\left\{d_{1}, \ldots, d_{n}\right\}, y$ is a minimal element of $P \backslash I$ in $P$, and

$$
\left|f_{E_{\mathrm{D}}(P, I)}(x)-f_{E_{\mathrm{D}}(P, I)}(y)\right|=1
$$

(2) The A-extension of $P$ with respect to $I$ is the word poset $E_{\mathrm{A}}(P, I)$ which is a poset on $P \sqcup\left\{a_{1}, a_{2}, \ldots, a_{n}\right\}$ with a function $f_{E_{\mathrm{A}}(P, I)}: E_{\mathrm{A}}(P, I) \rightarrow \mathbb{Z}_{>0}$ defined as follows.

- The function $f_{E_{\mathrm{A}}(P, I)}: E_{\mathrm{A}}(P, I) \rightarrow \mathbb{Z}_{>0}$ is given by

$$
f_{E_{\mathrm{A}}(P, I)}(x)= \begin{cases}f_{P}(x)+1 & \text { if } x \in I \\ i & \text { if } x=a_{i} \\ f_{P}(x) & \text { if } x \in P \backslash I\end{cases}
$$

- The covering relations of the poset $E_{\mathrm{A}}(P, I)$ on $P \sqcup\left\{a_{1}, a_{2}, \ldots, a_{n}\right\}$ are given by $x \lessdot_{E_{\mathrm{A}}(P, I)} y$ if and only if one of the following conditions holds:
(a) $x, y \in I$ and $x \lessdot_{P} y$,
(b) $x, y \in P \backslash I$ and $x \lessdot_{P} y$,
(c) $x=a_{i}$ and $y=a_{i+1}$ for some $i \in[n-1]$,
(d) $x$ is a maximal element of $I$ in $P, y \in\left\{a_{1}, \ldots, a_{n}\right\}$, and

$$
\left|f_{E_{\mathrm{A}}(P, I)}(x)-f_{E_{\mathrm{A}}(P, I)}(y)\right|=1, \text { or }
$$

(e) $x \in\left\{a_{1}, \ldots, a_{n}\right\}, y$ is a minimal element of $P \backslash I$ in $P$, and

$$
\left|f_{E_{\mathrm{A}}(P, I)}(x)-f_{E_{\mathrm{A}}(P, I)}(y)\right|=1
$$

The extensions are the reverse operations of the contractions in the following sense.

Proposition 3.10. Let $P \in \mathcal{P}\left(w_{0}^{(n+1)}\right)$. Then

$$
\begin{aligned}
& P \sim E_{\mathrm{D}}\left(C_{\mathrm{D}}(P), I_{\mathrm{D}}(P)\right), \\
& P \sim E_{\mathrm{A}}\left(C_{\mathrm{A}}(P), I_{\mathrm{A}}(P)\right)
\end{aligned}
$$

Proof. This is straightforward to check using the definitions of contractions and extensions. We omit the details.

In CKLP19a, Definition 3.6], the D-contraction $C_{\mathrm{D}}(\mathbf{i})$ and the A-contraction $C_{\mathrm{A}}(\mathbf{i})$ of a reduced word $\mathbf{i} \in \mathcal{R}\left(w_{0}^{(n+1)}\right)$ are defined using wiring diagrams. For $\mathbf{i} \in$ $\mathcal{R}\left(w_{0}^{(n+1)}\right)$, let $G(\mathbf{i})$ be the corresponding wiring diagram. Removing the $(n+1)$ st wire from $G(\mathbf{i})$ and shifting the part below this wire to the left by one, we get a new wiring diagram such that the $j$ th wire travels from the $j$ th starting point to the $(n+1-j)$ th ending point. Since the number of crossings decreases by $n$, the new wiring diagram represents a reduced word in $\mathcal{R}\left(w_{0}^{(n)}\right)$. Similarly, removing the 1st wire from $G(\mathbf{i})$ also produces the wiring diagram of a reduced word in $\mathcal{R}\left(w_{0}^{(n)}\right)$. One can check that $C_{\mathrm{D}}\left(P_{\mathbf{i}}\right) \sim P_{C_{\mathrm{D}}(\mathbf{i})}$ and $C_{\mathrm{A}}\left(P_{\mathbf{i}}\right) \sim P_{C_{\mathrm{A}}(\mathbf{i})}$. This leads us to the following result.

Proposition 3.11. Let $P \in \mathcal{P}\left(w_{0}^{(n+1)}\right)$. Then $C_{\mathrm{D}}(P)$ and $C_{\mathrm{A}}(P)$ are word posets in $\mathcal{P}\left(w_{0}^{(n)}\right)$.

Finally, we define the $\delta$-index of a word poset.
Definition 3.12. For a sequence $\delta=\left(\delta_{1}, \ldots, \delta_{n-1}\right) \in\{\mathrm{A}, \mathrm{D}\}^{n-1}$, the $\delta$-index of $P \in \mathcal{P}\left(w_{0}^{(n+1)}\right)$ is the integer vector $\operatorname{ind}_{\delta}(P)=\left(I_{1}, \ldots, I_{n-1}\right)$ defined by

$$
I_{k}:=\operatorname{ind}_{\delta_{k}}\left(C_{\delta_{k+1}} \circ C_{\delta_{k+2}} \circ \cdots \circ C_{\delta_{n-1}}(P)\right), \quad 1 \leq k \leq n-1
$$

where $I_{n-1}=\operatorname{ind}_{\delta_{n-1}}(P)$. The $\delta$-index of $\mathbf{i} \in \mathcal{R}\left(w_{0}^{(n+1)}\right)$ is defined by $\operatorname{ind}_{\delta}(\mathbf{i})=$ $\operatorname{ind}_{\delta}\left(P_{\mathbf{i}}\right)$.

Example 3.13. Let $\mathbf{i}=(4,3,4,2,3,4,1,2,5,4,3,2,1,4,5) \in \mathcal{R}\left(w_{0}^{(6)}\right)$. Then, for a sequence $\delta=(\mathrm{A}, \mathrm{A}, \mathrm{A}, \mathrm{A}) \in\{\mathrm{A}, \mathrm{D}\}^{4}$, we obtain $\operatorname{ind}_{\delta}(\mathbf{i})=(1,2,3,2)$ as shown in Figure 5.

## 4. Gelfand-Cetlin type Reduced words

In this section, we introduce Gelfand-Cetlin type reduced words. We give a recursive formula for the number of such reduced words using standard Young tableaux of shifted shapes in Theorem 4.13 .

We first define Gelfand-Cetlin type word posets and Gelfand-Cetlin type reduced words.
Definition 4.1. A word poset $P \in \mathcal{P}\left(w_{0}^{(n+1)}\right)$ is of Gelfand-Cetlin type if there exists $\delta \in\{\mathrm{A}, \mathrm{D}\}^{n-1}$ such that $\operatorname{ind}_{\delta}(P)=(0,0, \ldots, 0) \in \mathbb{Z}^{n-1}$. Denote by $\mathcal{P}_{\mathrm{GC}}(n)$ the set of Gelfand-Cetlin type word posets in $\mathcal{P}\left(w_{0}^{(n+1)}\right)$ and let $\left[\mathcal{P}_{\mathrm{GC}}(n)\right]=\mathcal{P}_{\mathrm{GC}}(n) / \sim$.


Figure 5. The Hasse diagrams of $P_{\mathbf{i}}, C_{\mathrm{A}}\left(P_{\mathbf{i}}\right), C_{\mathrm{A}}\left(C_{\mathrm{A}}\left(P_{\mathbf{i}}\right)\right)$, and $C_{\mathrm{A}}\left(C_{\mathrm{A}}\left(C_{\mathrm{A}}\left(P_{\mathbf{i}}\right)\right)\right)$ for $\mathbf{i}=(4,3,4,2,3,4,1,2,5,4,3,2,1,4,5)$.

Definition 4.2. A reduced word $\mathbf{i} \in \mathcal{R}\left(w_{0}^{(n+1)}\right)$ is of Gelfand-Cetlin type if $P_{\mathbf{i}}$ is of Gelfand-Cetlin type. A commutation class [i] is of Gelfand-Cetlin type if it contains a Gelfand-Cetlin type reduced word. Denote by $\mathcal{R}_{\mathrm{GC}}(n)$ the set of Gelfand-Cetlin type reduced words in $\mathcal{R}\left(w_{0}^{(n+1)}\right)$ and let $\left[\mathcal{R}_{\mathrm{GC}}(n)\right]=\left\{[\mathbf{i}]: \mathbf{i} \in \mathcal{R}_{\mathrm{GC}}(n)\right\}$.
Remark 4.3. One can deduce from CKLP19a, Theroem A] that for $\mathbf{i} \in \mathcal{R}\left(w_{0}^{(n+1)}\right)$, the string polytope $\Delta_{\mathbf{i}}(\lambda)$ is combinatorially equivalent to a full dimensional GelfandCetlin polytope of rank $n$ if and only if $\mathbf{i}$ is of Gelfand-Cetlin type, see the proof of Corollary 4.14. This is why we say that such word posets and reduced words are of Gelfand-Cetlin type.

The following proposition easily follows from Proposition 2.7 and the definitions of $\mathcal{P}_{\mathrm{GC}}(n)$ and $\mathcal{R}_{\mathrm{GC}}(n)$.

Proposition 4.4. The map $[\mathbf{i}] \mapsto\left[P_{\mathbf{i}}\right]$ is a bijection from $\left[\mathcal{R}_{\mathrm{GC}}(n)\right]$ to $\left[\mathcal{P}_{\mathrm{GC}}(n)\right]$. Accordingly,

$$
\left|\left[\mathcal{R}_{\mathrm{GC}}(n)\right]\right|=\left|\left[\mathcal{P}_{\mathrm{GC}}(n)\right]\right| .
$$

We note that if $\operatorname{ind}_{\mathrm{A}}(P)=0$, then $C_{\mathrm{A}}(P)=I_{\mathrm{A}}(P)$. Similarly, we have $C_{\mathrm{D}}(P)=$ $I_{\mathrm{D}}(P)$ when $\operatorname{ind}_{\mathrm{D}}(P)=0$. The succeeding lemma follows immediately from Proposition 3.10 .
Lemma 4.5. Let $P \in \mathcal{P}\left(w_{0}^{(n+1)}\right)$. If $\operatorname{ind}_{\mathrm{A}}(P)=0$, then

$$
P \sim E_{\mathrm{A}}\left(C_{\mathrm{A}}(P), C_{\mathrm{A}}(P)\right)
$$

Similarly, if $\operatorname{ind}_{\mathrm{D}}(P)=0$, then

$$
P \sim E_{\mathrm{D}}\left(C_{\mathrm{D}}(P), C_{\mathrm{D}}(P)\right)
$$

Lemma 4.5 implies that if $\operatorname{ind}_{\mathrm{A}}(P)=0$ (respectively, $\operatorname{ind}_{\mathrm{D}}(P)=0$ ), then $P$ is completely determined by $C_{\mathrm{A}}(P)$ (respectively, $C_{\mathrm{D}}(P)$ ) up to isomorphism.

The following definition will be used frequently throughout this section.
Definition 4.6. For a word poset $P$, an element $x \in P$ is called a top element if $x$ is the largest element in its column. In other words, $x \in P$ is a top element if $y \leq_{P} x$ for all $y \in P$ with $f_{P}(y)=f_{P}(x)$. For $i \in[n]$, denote by $m_{i}(P)$ the top element in column $i$.

The following lemma shows that if $P$ is a Gelfand-Cetlin type word poset, the top elements of $P$ must form a chain.
Lemma 4.7. Let $P \in \mathcal{P}\left(w_{0}^{(n+1)}\right)$. Then

$$
\begin{array}{ll}
m_{1}(P)<_{P} m_{2}(P)<_{P} \cdots<_{P} m_{n}(P) & \text { if } \operatorname{ind}_{\mathrm{A}}(P)=0 \\
m_{1}(P)>_{P} m_{2}(P)>_{P} \cdots>_{P} m_{n}(P) & \text { if } \operatorname{ind}_{\mathrm{D}}(P)=0 .
\end{array}
$$

Moreover, $m_{n}(P)\left(\right.$ respectively, $\left.m_{1}(P)\right)$ is the maximum element of $P$ if $\operatorname{ind}_{\mathrm{A}}(P)=$ 0 (respectively, $\left.\operatorname{ind}_{\mathrm{D}}(P)=0\right)$.

Proof. We will only consider the case $\operatorname{ind}_{\mathrm{A}}(P)=0$ because the other case $\operatorname{ind}_{\mathrm{D}}(P)=$ 0 can be proved similarly.

Since $\operatorname{ind}_{\mathrm{A}}(P)=0$, by Lemma 4.5, we have $P \sim E_{\mathrm{A}}\left(C_{\mathrm{A}}(P), C_{\mathrm{A}}(P)\right)$. By definition, $Q:=E_{\mathrm{A}}\left(C_{\mathrm{A}}(P), C_{\mathrm{A}}(P)\right)$ is the word poset obtained from $C_{\mathrm{A}}(P)$ by adding $n$ elements $a_{1}, \ldots, a_{n}$ with additional covering relations $a_{1} \lessdot_{Q} \cdots \lessdot_{Q} a_{n}$ and $x \lessdot_{Q} a_{i}$ for each maximal element $x$ in $C_{\mathrm{A}}(P)$ and $i \in[n]$ such that $\left|f_{Q}(x)-f_{Q}\left(a_{i}\right)\right|=1$, where $f_{Q}(x)=f_{C_{\mathrm{A}}(P)}(x)$ for $x \in C_{\mathrm{A}}(P)$ and $f_{Q}\left(a_{i}\right)=i$ for $i \in[n]$. By the construction, we have $a_{i}=m_{i}(Q)$ for $i \in[n]$, and therefore $m_{1}(Q)<_{Q} \cdots<_{Q} m_{n}(Q)$. Since $P \sim Q$, this shows the first statement.

For the second statement, let $x$ be an arbitrary element in $Q$. Suppose $f_{Q}(x)=i$. Since $Q \sim P \in \mathcal{P}\left(w_{0}^{(n+1)}\right)$, by Lemma 3.6, $\left\{y \in Q: f_{Q}(y)=i\right\}$ is a chain in $P$ for each $i \in[n]$. By definition $m_{i}(Q)$ is the maximum element in this chain. Then $x \leq_{Q} m_{i}(Q) \leq_{Q} m_{n}(Q)$. Thus $x \leq_{Q} m_{n}(Q)$, and therefore $m_{n}(Q)$ is the maximum element in $Q$. Since $P \sim Q$, this shows the second statement.

By Lemma 4.7, if $P \in \mathcal{P}_{\mathrm{GC}}(n)$ and $n \geq 2$, then we have $\operatorname{ind}_{\mathrm{A}}(P)=0$ or $\operatorname{ind}_{\mathrm{D}}(P)=0$, but not both. This means that there is a unique $\delta=\left(\delta_{1}, \ldots, \delta_{n-1}\right) \in$ $\{\mathrm{A}, \mathrm{D}\}^{n-1}$ such that $\operatorname{ind}_{\delta}(P)=(0, \ldots, 0)$. Therefore the map

$$
\phi:\left[\mathcal{P}_{\mathrm{GC}}(n)\right] \rightarrow\{\mathrm{A}, \mathrm{D}\}^{n-1}
$$

sending $[P]$ to such $\delta$ is well-defined. This map is in fact a bijection.
Proposition 4.8. For $n \geq 2$, the map $\phi:\left[\mathcal{P}_{\mathrm{GC}}(n)\right] \rightarrow\{\mathrm{A}, \mathrm{D}\}^{n-1}$ is a bijection.
Proof. We first show that $\phi$ is injective. Suppose that $P \in \mathcal{P}_{\mathrm{GC}}(n)$ satisfies $\phi([P])=$ $\delta=\left(\delta_{1}, \ldots, \delta_{n-1}\right) \in\{\mathrm{A}, \mathrm{D}\}^{n-1}$. By definition, we have $\operatorname{ind}_{\delta}(P)=(0, \ldots, 0)$. We will show that $[P]$ is determined by $\delta$.

Define the word posets $P_{k} \in \mathcal{P}_{\mathrm{GC}}(k)$ for $k \in[n]$ recursively as follows. First, we set $P_{n}=P$. For $k \in[n-1]$, define

$$
P_{k}=C_{\delta_{k}}\left(P_{k+1}\right)
$$

Since $P_{1} \in \mathcal{P}\left(w_{0}^{(2)}\right), P_{1}$ is a word poset with one element, say $x$, and $f_{P_{1}}(x)=1$. By Lemma 4.5, for $k \in[n-1]$, we have

$$
P_{k+1} \sim E_{\delta_{k}}\left(C_{\delta_{k}}\left(P_{k+1}\right), C_{\delta_{k}}\left(P_{k+1}\right)\right)=E_{\delta_{k}}\left(P_{k}, P_{k}\right)
$$

Thus $P=P_{n}$ is determined uniquely by $\delta$ up to isomorphism. This shows that $\phi$ is injective.

To show that $\phi$ is surjective take an arbitrary sequence $\delta=\left(\delta_{1}, \ldots, \delta_{n-1}\right) \in$ $\{\mathrm{A}, \mathrm{D}\}^{n-1}$. Define the word posets $P_{k} \in \mathcal{P}_{\mathrm{GC}}(k)$ for $k \in[n]$ by $P_{1} \in \mathcal{P}\left(w_{0}^{(2)}\right)$ and

$$
P_{k+1}=E_{\delta_{k}}\left(P_{k}, P_{k}\right) \quad \text { for } k \in[n-1] .
$$



Figure 6. The word posets $P_{(3,2,1,2,3,4,3,2,3,1)}$ and $P_{(1,3,2,1,4,3,4,2,3,1)}$.
Here we may choose any $P_{1} \in \mathcal{P}\left(w_{0}^{(2)}\right)$ because if $P_{1}, P_{1}^{\prime} \in \mathcal{P}\left(w_{0}^{(2)}\right)$ then $P_{1} \sim P_{1}^{\prime}$. It is easy to check that $\phi\left(P_{n}\right)=\delta$. Thus $\phi$ is surjective, which completes the proof.

The proof of the above proposition shows that if $P, Q \in \mathcal{P}\left(w_{0}^{(n+1)}\right)$ satisfy $\operatorname{ind}_{\delta}(P)=\operatorname{ind}_{\delta}(Q)=(0, \ldots, 0)$ for some $\delta \in\{\mathrm{A}, \mathrm{D}\}^{n-1}$, then $P \sim Q$ since both $P$ and $Q$ are determined by $\delta$. In general, the condition $\operatorname{ind}_{\delta}(P)=\operatorname{ind}_{\delta}(Q)$ for some $\delta \in\{\mathrm{A}, \mathrm{D}\}^{n-1}$ does not imply $P \sim Q$ as the following example shows.
Example 4.9. Consider the two words

$$
\mathbf{i}=(3,2,1,2,3,4,3,2,3,1), \quad \mathbf{j}=(1,3,2,1,4,3,4,2,3,1) \in \mathcal{R}\left(w_{0}^{(5)}\right)
$$

For any $\left(\delta_{1}, \delta_{2}\right) \in\{\mathrm{A}, \mathrm{D}\}^{3}$, we have $\operatorname{ind}_{\left(\delta_{1}, \delta_{2}, \mathrm{~A}\right)}\left(P_{\mathbf{i}}\right)=\operatorname{ind}_{\left(\delta_{1}, \delta_{2}, \mathrm{~A}\right)}\left(P_{\mathbf{j}}\right)$, but $P_{\mathbf{i}} \nsim P_{\mathbf{j}}$. See Figure 6 for these word posets. Accordingly, a single $\delta$-index of $P$ does not always determine the word poset class $[P]$.

Although a single $\delta$-index of $P$ is not enough to determine the word poset $[P]$, in the next section, we will show that the $\delta$-indices ind ${ }_{\delta}(P)$ for all $\delta \in\{\mathrm{A}, \mathrm{D}\}^{n-1}$ determine $[P]$ (see Theorem 5.11). Note that in Example 4.9 we have $\operatorname{ind}_{\mathrm{D}}\left(P_{\mathrm{i}}\right)=$ $1 \neq 2=\operatorname{ind}_{\mathrm{D}}\left(P_{\mathbf{j}}\right)$, so $P_{\mathbf{i}}$ and $P_{\mathbf{j}}$ do not have the same $\delta$-indices for all $\delta$.

Proposition 4.8 immediately gives the cardinality of the Gelfand-Cetlin type commutation classes.

Corollary 4.10. For $n \geq 2$, we have

$$
\left|\left[\mathcal{R}_{\mathrm{GC}}(n)\right]\right|=\left|\left[\mathcal{P}_{\mathrm{GC}}(n)\right]\right|=2^{n-1} .
$$

We note that Corollary 4.10 was proved in the recent paper GMS20, Proposition 20] using a different method.

Using the construction in the proof of Proposition 4.8 and standard Young tableaux, we can find a recurrence relation for the number of Gelfand-Cetlin type reduced words. To this end, we need the following lemma, which allows us to draw the Hasse diagram of a Gelfand-Cetlin type word poset $P$ corresponding to $\delta \in\{\mathrm{A}, \mathrm{D}\}^{n-1}$ using only $\delta$.
Lemma 4.11. Let $P \in \mathcal{P}\left(w_{0}^{(n+1)}\right)$ and let $\delta=\left(\delta_{1}, \ldots, \delta_{n-1}\right)$ be the element in $\{\mathrm{A}, \mathrm{D}\}^{n-1}$ satisfying $\operatorname{ind}_{\delta}(P)=(0, \ldots, 0)$. Let $P_{n}=P$, and for $k \in[n-1]$, define

$$
P_{k}=C_{\delta_{k}}\left(P_{k+1}\right) .
$$



Figure 7. The Hasse diagrams of $P_{k}$ and $P_{k+1}$.

Then $P_{1}$ is isomorphic to the unique word poset (up to isomorphism) in $\mathcal{P}\left(w_{0}^{(2)}\right)$ and, for $k \in[n-1]$, the word poset $P_{k+1}$ is constructed as follows (see Figure 7).

- If $\delta_{k-1}=\mathrm{A}$ and $\delta_{k}=\mathrm{A}$, then the Hasse diagram of $P_{k+1}$ is obtained from that of $P_{k}$ by adding the chain $m_{1}\left(P_{k+1}\right) \lessdot_{P_{k+1}} \cdots \lessdot_{P_{k+1}} m_{k+1}\left(P_{k+1}\right)$ with additional covering relations $m_{i}\left(P_{k+1}\right) \gtrdot_{P_{k+1}} m_{i}\left(P_{k}\right)$ for $i \in[k]$.
- If $\delta_{k-1}=\mathrm{A}$ and $\delta_{k}=\mathrm{D}$, then the Hasse diagram of $P_{k+1}$ is obtained from that of $P_{k}$ by adding the chain $m_{1}\left(P_{k+1}\right) \gtrdot_{P_{k+1}} \cdots \gtrdot_{P_{k+1}} m_{k+1}\left(P_{k+1}\right)$ with an additional covering relation $m_{k}\left(P_{k}\right) \lessdot P_{k+1} m_{k+1}\left(P_{k+1}\right)$.
- If $\delta_{k-1}=\mathrm{D}$ and $\delta_{k}=\mathrm{A}$, then the Hasse diagram of $P_{k+1}$ is obtained from that of $P_{k}$ by adding the chain $m_{1}\left(P_{k+1}\right) \lessdot_{P_{k+1}} \cdots \lessdot_{P_{k+1}} m_{k+1}\left(P_{k+1}\right)$ with an additional covering relation $m_{1}\left(P_{k+1}\right) \gtrdot_{P_{k+1}} m_{1}\left(P_{k}\right)$.
- If $\delta_{k-1}=\mathrm{D}$ and $\delta_{k}=\mathrm{D}$, then the Hasse diagram of $P_{k+1}$ is obtained from that of $P_{k}$ by adding the chain $m_{1}\left(P_{k+1}\right) \gtrdot_{P_{k+1}} \cdots \gtrdot_{P_{k+1}} m_{k+1}\left(P_{k+1}\right)$ with additional covering relations $m_{i}\left(P_{k}\right) \lessdot \bigodot_{P_{k+1}} m_{i+1}\left(P_{k+1}\right)$ for $i \in[k]$.

Proof. Consider the case that $\delta_{k-1}=\mathrm{A}$ and $\delta_{k}=\mathrm{A}$. Since $P_{k}=C_{\delta_{k}}\left(P_{k+1}\right)=$ $C_{\mathrm{A}}\left(P_{k+1}\right)$ and $\operatorname{ind}_{\mathrm{A}}\left(P_{k+1}\right)=0$, by Lemma 4.5, we have

$$
P_{k+1} \sim E_{\mathrm{A}}\left(P_{k}, P_{k}\right)
$$

Then it is straightforward to check that we obtain the desired description for $P_{k+1}$ by the definition of $A$-extension in Definition 3.9 . The other three cases can be checked similarly.

Now we define standard Young tableaux of shifted shape.
Definition 4.12. A partition of $n$ is a weakly decreasing sequence $\mu=\left(\mu_{1}, \ldots, \mu_{t}\right)$ of positive integers summing to $n$. A partition $\mu=\left(\mu_{1}, \ldots, \mu_{t}\right)$ is strict if $\mu_{1}>$ $\cdots>\mu_{t}$. For a strict partition $\mu=\left(\mu_{1}, \ldots, \mu_{t}\right)$, the shifted diagram of $\mu$, denoted $\mu^{*}$, is the set

$$
\mu^{*}:=\left\{(i, j) \mid 1 \leq i \leq t, i \leq j \leq \mu_{i}+i-1\right\}
$$

We will identify $\mu^{*}$ as an array of squares where there is a square in row $i$ and column $j$ for each $(i, j) \in \mu^{*}$. For a strict partition $\mu=\left(\mu_{1}, \ldots, \mu_{t}\right)$ of $n$, a standard Young tableau of shifted shape $\mu$ is a bijection $T: \mu^{*} \rightarrow[n]$ such that $T(i, j) \leq T\left(i^{\prime}, j^{\prime}\right)$ if $i \leq i^{\prime}$ and $j \leq j^{\prime}$. We will represent a standard Young tableau $T$ of shifted shape $\mu$ by filling $T(i, j)$ in the square in row $i$ and column $j$ of $\mu^{*}$. Denote by $g^{\mu}$ the number of standard Young tableaux of shifted shape $\mu$.

For example, the shifted diagram of shape $\mu=(3,2,1)$ is drawn as follows.

$$
\mu^{*}=\square \square
$$

There are two standard Young tableaux of shifted shape $\mu=(3,2,1)$ :

$$
\begin{array}{|l|l|l|l|l|l|}
\hline 1 & 2 & 3 \\
\hline & 4 & 5 \\
\hline & & 6 \\
\hline
\end{array} \quad \begin{array}{ll}
\hline 1 & 2
\end{array} 4 \begin{aligned}
& 4 \\
& \hline
\end{aligned}
$$

Thrall Thr52] showed that the number $g^{\mu}$ of standard Young tableaux of shifted shape $\mu=\left(\mu_{1}, \ldots, \mu_{t}\right)$ is given as follows:

$$
\begin{equation*}
g^{\mu}=\frac{|\mu|!}{\mu_{1}!\mu_{2}!\cdots \mu_{t}!} \prod_{i<j} \frac{\mu_{i}-\mu_{j}}{\mu_{i}+\mu_{j}} \tag{4.1}
\end{equation*}
$$

For instance, if $\mu=(3,2,1)$, then we have that

$$
g^{(3,2,1)}=\frac{6!}{3!2!1!} \cdot \frac{1 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3}=2
$$

There is another formula for $g^{\mu}$ called the (shifted) hook length formula, see Mac95, p.267, eq.(2)].

For a strict partition $\mu=\left(\mu_{1}, \ldots, \mu_{t}\right)$, we define $Q_{\mu}$ to be the poset on $\mu^{*}$ with relations $(i, j) \leq_{Q_{\mu}}\left(i^{\prime}, j^{\prime}\right)$ if $i \leq i^{\prime}$ and $j \leq j^{\prime}$. For example, if $\mu=(4,3,2)$, then the poset $Q_{\mu}$ is given as follows.


There is a natural bijection between the standard Young tableaux of shifted shape $\mu$ and the linear extensions of $Q_{\mu}$. Therefore the number of linear extensions of $Q_{\mu}$ is equal to $g^{\mu}$.

Now we are ready to state our first main theorem.
Theorem 4.13. Let $\operatorname{gc}(n)$ be the number of Gelfand-Cetlin type reduced words in $\mathcal{R}\left(w_{0}^{(n+1)}\right)$, i.e., $\operatorname{gc}(n)=\left|\mathcal{R}_{\mathrm{GC}}(n)\right|$. Then $\operatorname{gc}(0)=\mathrm{gc}(1)=1$ and for $n \geq 2$, we have

$$
\operatorname{gc}(n)=\sum_{i=1}^{n} g^{(n, n-1, \ldots, n-i+1)} \operatorname{gc}(n-i)
$$

where $g^{\mu}$ is the number of standard Young tableaux of shifted shape $\mu$ (see (4.1)).
Proof. Clearly, we have $\operatorname{gc}(0)=\operatorname{gc}(1)=1$. Suppose $n \geq 2$. Observe that

$$
\begin{equation*}
\operatorname{gc}(n)=\left|\mathcal{R}_{\mathrm{GC}}(n)\right|=\sum_{[\mathbf{i}] \in\left[\mathcal{R}_{\mathrm{GC}}(n)\right]}|[\mathbf{i}]| . \tag{4.2}
\end{equation*}
$$

By Proposition 4.4, the map $[\mathbf{i}] \mapsto\left[P_{\mathbf{i}}\right]$ is a bijection from $\left[\mathcal{R}_{\mathrm{GC}}(n)\right]$ to $\left[\mathcal{P}_{\mathrm{GC}}(n)\right]$. Moreover, by Proposition 2.6, $|[\mathbf{i}]|$ is equal to the number of linear extensions of $P_{\mathbf{i}}$. This shows that we can rewrite 4.2 as

$$
\begin{equation*}
\operatorname{gc}(n)=\sum_{[P] \in\left[\mathcal{P}_{\mathrm{GC}}(n)\right]} e(P), \tag{4.3}
\end{equation*}
$$

where $e(P)$ is the number of linear extensions of $P$.
Define $\mathrm{a}(n)$ and $\mathrm{d}(n)$ by

$$
\mathrm{a}(n)=\sum_{\substack{[P] \in\left[\mathcal{P}_{\mathrm{GC}}(n)\right], \operatorname{ind}_{\mathrm{A}}(P)=0}} e(P), \quad \mathrm{d}(n)=\sum_{\substack{[P] \in\left[\mathcal{P}_{\mathrm{GC}}(n)\right], \\ \text { ind }(P)=0}} e(P) .
$$

We claim that

$$
\begin{align*}
& \mathrm{a}(n)=\sum_{i=1}^{n} g^{(n, n-1, \ldots, n-i+1)} \mathrm{d}(n-i)  \tag{4.4}\\
& \mathrm{d}(n)=\sum_{i=1}^{n} g^{(n, n-1, \ldots, n-i+1)} \mathrm{a}(n-i) \tag{4.5}
\end{align*}
$$

Since $\mathrm{gc}(n)=\mathrm{a}(n)+\mathrm{d}(n)$, the identity in this theorem is obtained by adding 4.4 and 4.5). Thus it suffices to show these two identities. We will only show 4.4 because 4.5 can be shown similarly.

To show (4.4), consider $[P] \in\left[\mathcal{P}_{\mathrm{GC}}(n)\right]$ with $\operatorname{ind}_{\mathrm{A}}(P)=0$. By Proposition 4.8, there is a unique $\delta \in\{\mathrm{A}, \mathrm{D}\}^{n-1}$ such that $\operatorname{ind}_{\delta}(P)=(0, \ldots, 0)$. Let $P_{n}=P$, and for $k \in[n-1]$, define

$$
P_{k}=C_{\delta_{k}}\left(P_{k+1}\right)
$$

Then, by Lemma 4.11, $P_{k+1}$ is obtained from $P_{k}$ by adding a descending or ascending chain of length $k+1$ depending on $\delta_{k-1}$ and $\delta_{k}$. Since $\operatorname{ind}_{\mathrm{A}}(P)=0$, there is a unique integer $i \in[n-1]$ such that $\delta_{n-1}=\delta_{n-2}=\cdots=\delta_{n-i}=\mathrm{A}$ and $\delta_{n-i-1}=\mathrm{D}$, where the second condition is ignored if $i=n-1$. By Lemma 4.11, one can easily see that $P=P_{n}$ is obtained from $P_{n-i}$ by adding the poset $Q_{(n, n-1, \ldots, n-i+1)}$ above it as shown in Figure 8. Since every element of $P_{n-i}$ is smaller than every element of $Q_{(n, n-1, \ldots, n-i+1)}$, we have

$$
e(P)=e\left(P_{n-i}\right) e\left(Q_{(n, n-1, \ldots, n-i+1)}\right)=e\left(P_{n-i}\right) g^{(n, n-1, \ldots, n-i+1)}
$$



Figure 8. The Hasse diagram of a word poset $P \in \mathcal{P}_{\mathrm{GC}}(n)$ such that $\operatorname{ind}_{\delta}(P)=(0, \ldots, 0)$ for $\delta=(\mathrm{A}, \mathrm{A}, \mathrm{D}, \mathrm{D}, \mathrm{A}, \mathrm{A}, \mathrm{A})$. In this case, $\delta_{n-1}=\delta_{n-2}=\cdots=\delta_{n-i}=\mathrm{A}$ and $\delta_{n-i}=\mathrm{D}$, where $n=8$ and $i=3$. For $r=2,3, \ldots, n$, the set $P_{r} \backslash P_{r-1}$ forms is an ascending chain $\mathrm{A}_{r}$ or a descending chain $\mathrm{D}_{r}$. The word poset $P=P_{n}$ is decomposed into two parts $P_{n-i}$ and $P_{n} \backslash P_{n-i} \sim Q_{(n, n-1, \ldots, n-i+1)}$.

Note that $\left[P_{n-i}\right] \in\left[\mathcal{P}_{\mathrm{GC}}(n-i)\right]$ and $\operatorname{ind}_{D}\left(P_{n-i}\right)=0$. Conversely, for any such $P_{n-i}$, one can construct $P$ in this way. This shows that

$$
\mathrm{a}(n)=\sum_{i=1}^{n} g^{(n, n-1, \ldots, n-i+1)} \sum_{\substack{\left[P_{n-i}\right] \in\left[\mathcal{P}_{\mathrm{GC}}(n-i)\right] \\ \operatorname{ind}_{\mathrm{D}}(P)=0}} e\left(P_{n-i}\right)
$$

which is the same as 4.4. Similarly, we obtain the formula 4.5 and the proof is completed.

By 4.1), we have

$$
\begin{aligned}
& g^{(2,1)}=1, \quad g^{(2)}=1 \\
& g^{(3,2,1)}=2, \quad g^{(3,2)}=2, \quad g^{(3)}=1 \\
& g^{(4,3,2,1)}=12, \quad g^{(4,3,2)}=12, \quad g^{(4,3)}=5, \quad g^{(4)}=1
\end{aligned}
$$

Applying Theorem 4.13 we can compute $\mathrm{gc}(n)$ for $n=2,3,4$ as follows.

$$
\begin{aligned}
& \operatorname{gc}(2)=g^{(2,1)} \mathrm{gc}(0)+g^{(2)} \mathrm{gc}(1)=1+1=2, \\
& \operatorname{gc}(3)=g^{(3,2,1)} \operatorname{gc}(0)+g^{(3,2)} \operatorname{gc}(1)+g^{(3)} \operatorname{gc}(2)=2+2+2=6, \\
& \operatorname{gc}(4)=g^{(4,3,2,1)} \operatorname{gc}(0)+g^{(4,3,2)} \operatorname{gc}(1)+g^{(4,3)} \operatorname{gc}(2)+g^{(4)} \operatorname{gc}(3)=12+12+10+6=40 .
\end{aligned}
$$

We present the first few terms of $\mathrm{gc}(n)$ in Table 1 .

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\operatorname{gc}(n)$ | 1 | 1 | 2 | 6 | 40 | 916 | 102176 | 68464624 | 317175051664 |
| TABLE 1. The first few terms of $\operatorname{gc}(n)$ |  |  |  |  |  |  |  |  |  |

We close this section by presenting the following corollary of Theorem 4.13.
Corollary 4.14. Let $\lambda$ be a regular dominant weight of $\mathrm{SL}_{n+1}(\mathbb{C})$. The number of reduced words $\mathbf{i} \in \mathcal{R}\left(w_{0}^{(n+1)}\right)$ such that the string polytope $\Delta_{\mathbf{i}}(\lambda)$ is unimodularly equivalent to the Gelfand-Cetlin polytope $\mathrm{GC}(\lambda)$ is the same as $\mathrm{gc}(n)$.

Proof. We first recall the known result from CKLP19a, Theorem A] that for $\mathbf{i} \in \mathcal{R}\left(w_{0}^{(n+1)}\right)$, the string polytope $\Delta_{\mathbf{i}}(\lambda)$ is unimodularly equivalent to the GelfandCetlin polytope $\mathrm{GC}(\lambda)$ if and only if the string polytope $\Delta_{\mathbf{i}}(\lambda)$ has exactly $n(n+1)$ facets. Here, facets are codimension one faces. We note that the number of facets of any full dimensional Gelfand-Cetlin polytope of rank $n$ is $n(n+1)$ (cf. ACK18). Accordingly, if the string polytope $\Delta_{\mathbf{i}}(\lambda)$ is combinatorially equivalent to a full dimensional Gelfand-Cetlin polytope of rank $n$, then it is also unimodularly equivalent to $\mathrm{GC}(\lambda)$ because it has $n(n+1)$ facets. This proves the corollary.

## 5. Word posets are determined by $\delta$-Indices

In this section, we prove that the $\delta$-indices of $P$ for all $\delta \in\{\mathrm{A}, \mathrm{D}\}^{n-1}$ completely determine $P \in \mathcal{P}\left(w_{0}^{(n+1)}\right)$ up to isomorphism. Equivalently, the $\delta$-indices of $\mathbf{i} \in$ $\mathcal{R}\left(w_{0}^{(n+1)}\right)$ for all $\delta \in\{\mathrm{A}, \mathrm{D}\}^{n-1}$ determine the commutation class $[\mathbf{i}] \in\left[\mathcal{R}\left(w_{0}^{(n+1)}\right)\right]$.

Theorem 5.1. Let $P, Q \in \mathcal{P}\left(w_{0}^{(n+1)}\right)$. If $\operatorname{ind}_{\delta}(P)=\operatorname{ind}_{\delta}(Q)$ for all $\delta \in\{\mathrm{A}, \mathrm{D}\}^{n-1}$, then $P \sim Q$.

In order to prove Theorem 5.1, we need the following two lemmas.
Lemma 5.2. Let $P \in \mathcal{P}\left(w_{0}^{(n+1)}\right)$. Then

$$
\begin{aligned}
& \mathrm{A}(P) \cap C_{\mathrm{D}}(P)=\mathrm{A}\left(C_{\mathrm{D}}(P)\right) \\
& \mathrm{D}(P) \cap C_{\mathrm{A}}(P)=\mathrm{D}\left(C_{\mathrm{A}}(P)\right)
\end{aligned}
$$

In other words, the ascending chain of $P$ restricted to $C_{\mathrm{D}}(P)$ is the ascending chain of $C_{\mathrm{D}}(P)$, and similarly, the descending chain of $P$ restricted $C_{\mathrm{A}}(P)$ is the descending chain of $C_{\mathrm{A}}(P)$.

Proof. By definition we can write

$$
\left.\begin{array}{l}
\mathrm{D}(P)=\left\{\begin{array}{llll}
d_{1}<_{P} & d_{2}<_{P} & \cdots<_{P} & d_{n}
\end{array}\right\}, \\
\mathrm{A}(P)=\left\{\begin{array}{lll}
a_{1}<_{P} & a_{2}<_{P} & \cdots<_{P}
\end{array} a_{n}\right.
\end{array}\right\},
$$

where $f_{P}\left(a_{i}\right)=i$ and $f_{P}\left(d_{i}\right)=n+1-i$ for $1 \leq i \leq n$. By Proposition 3.3 , $\mathrm{D}(P) \cap \mathrm{A}(P)$ has a unique element, say $a_{k}$. By definition, $C_{\mathrm{D}}(P)=P \backslash \mathrm{D}$ and

$$
\begin{equation*}
\left\{a_{1}<_{C_{\mathrm{D}}(P)} \cdots<_{C_{\mathrm{D}}(P)} a_{k-1}<_{C_{\mathrm{D}}(P)} a_{k+1}<_{C_{\mathrm{D}}(P)} \cdots<_{C_{\mathrm{D}}(P)} a_{n}\right\} \tag{5.1}
\end{equation*}
$$

Moreover, since $a_{1}, \ldots, a_{k-1}$ are below the descending chain $\mathrm{D}(P)$ and $a_{k+1}, \ldots, a_{n}$ are above $\mathrm{D}(P)$ in the Hasse diagram of $P$, we have $f_{C_{\mathrm{D}}(P)}\left(a_{i}\right)=i$ for $1 \leq i \leq k-1$ and $f_{C_{\mathrm{D}}(P)}\left(a_{i}\right)=i-1$ for $k+1 \leq i \leq n$, see Figure 9 . Therefore (5.1) is the ascending chain of $C_{\mathrm{D}}(P)$, which shows the first identity. The second identity can be proved similarly.

The following lemma shows that an ideal of a word poset $P \in \mathcal{P}\left(w_{0}^{(n+1)}\right)$ is determined by the number of elements in each column.


Figure 9. The ascending chain $\mathrm{A}(P)$ induces the ascending chain $A\left(C_{\mathrm{D}}(P)\right)$.

Lemma 5.3. Let $P \in \mathcal{P}\left(w_{0}^{(n+1)}\right)$. Suppose that $I$ and $J$ are ideals of $P$ such that

$$
\left|\left\{x \in I: f_{P}(x)=i\right\}\right|=\left|\left\{x \in J: f_{P}(x)=i\right\}\right|
$$

for all $1 \leq i \leq n$. Then we have $I=J$.
Proof. Consider the ideals $I$ and $J$ in the statement of this lemma. Observe that $I$ is the disjoint union of $\left\{x \in I: f_{P}(x)=i\right\}$ for $1 \leq i \leq n$. Thus it suffices to show that

$$
\begin{equation*}
\left\{x \in I: f_{P}(x)=i\right\}=\left\{x \in J: f_{P}(x)=i\right\} \tag{5.2}
\end{equation*}
$$

Fix $1 \leq i \leq n$, and let

$$
r=\left|\left\{x \in I: f_{P}(x)=i\right\}\right|=\left|\left\{x \in J: f_{P}(x)=i\right\}\right| .
$$

By Lemma 3.6, for each $i \in[n],\left\{x \in P: f_{P}(x)=i\right\}$ is a chain. Accordingly, we can write

$$
C:=\left\{x \in P: f_{P}(x)=i\right\}=\left\{c_{1}<_{P} c_{2}<_{P} \cdots<_{P} c_{t}\right\} .
$$

Since $I$ is an ideal, $I \cap C$ is also an ideal of $C$. Because $C$ is a chain this means

$$
I \cap C=\left\{c_{1}<_{P} c_{2}<_{P} \cdots<_{P} c_{r}\right\} .
$$

By the same argument we also have

$$
J \cap C=\left\{c_{1}<_{P} c_{2}<_{P} \cdots<_{P} c_{r}\right\} .
$$

Therefore $I \cap C=J \cap C$, which is (5.2). This completes the proof.
We are now ready to prove Theorem 5.1 .
Proof of Theorem 5.1. Let $P, Q \in \mathcal{P}\left(w_{0}^{(n+1)}\right)$ such that $\operatorname{ind}_{\delta}(P)=\operatorname{ind}_{\delta}(Q)$ for all $\delta \in\{\mathrm{A}, \mathrm{D}\}^{n-1}$. We will prove $P \sim Q$ by induction on $n$.

Since $P, Q \in \mathcal{P}\left(w_{0}^{(n+1)}\right)$, there are reduced words $\mathbf{i}, \mathbf{j} \in \mathcal{R}\left(w_{0}^{(n+1)}\right)$ with $P \sim P_{\mathbf{i}}$ and $Q \sim P_{\mathbf{j}}$. If $n=1$, then $\mathcal{R}\left(w_{0}^{(n+1)}\right)$ has only one element (1). Thus $\mathbf{i}=\mathbf{j}$, and $P \sim P_{\mathbf{i}}=P_{\mathbf{j}} \sim Q$. If $n=2$, there are two reduced words $(1,2,1)$ and $(2,1,2)$ in


Figure 10. An illustration of $I_{\mathrm{D}}(R) \backslash I_{\mathrm{A}}(R), I_{\mathrm{D}}(R) \cap I_{\mathrm{A}}(R), I_{\mathrm{A}}(R) \backslash$ $I_{\mathrm{D}}(R), a_{i}(R), b_{k}(R)$, and $c_{j}(R)$.
$\mathcal{R}\left(w_{0}^{(n+1)}\right)$. Since $\operatorname{ind}_{\mathrm{A}}(1,2,1)=1$ and $\operatorname{ind}_{\mathrm{A}}(2,1,2)=0$, if $\operatorname{ind}_{\delta}\left(P_{\mathbf{i}}\right)=\operatorname{ind}_{\delta}\left(P_{\mathbf{j}}\right)$ for all $\delta \in\{\mathrm{A}, \mathrm{D}\}^{n-1}$, we must have $\mathbf{i}=\mathbf{j}$. Therefore we also have $P \sim P_{\mathbf{i}}=P_{\mathbf{j}} \sim Q$.

Now let $n>2$ and suppose that the statement holds for $n-1$. Since $\operatorname{ind}_{\delta}(P)=$ $\operatorname{ind}_{\delta}(Q)$ for all $\delta \in\{\mathrm{A}, \mathrm{D}\}^{n-1}$, by the definition of $\delta$-indices, we have

$$
\begin{array}{ll}
\operatorname{ind}_{\delta}\left(C_{\mathrm{D}}(P)\right)=\operatorname{ind}_{\delta}\left(C_{\mathrm{D}}(Q)\right) & \text { for all } \delta \in\{\mathrm{A}, \mathrm{D}\}^{n-2} \\
\operatorname{ind}_{\delta}\left(C_{\mathrm{A}}(P)\right)=\operatorname{ind}_{\delta}\left(C_{\mathrm{A}}(Q)\right) & \text { for all } \delta \in\{\mathrm{A}, \mathrm{D}\}^{n-2}
\end{array}
$$

Thus, by the induction hypothesis, we have $C_{\mathrm{D}}(P) \sim C_{\mathrm{D}}(Q)$ and $C_{\mathrm{A}}(P) \sim C_{\mathrm{A}}(Q)$.
By Proposition 3.10

$$
\begin{aligned}
& P \sim E_{\mathrm{D}}\left(C_{\mathrm{D}}(P), I_{\mathrm{D}}(P)\right), \\
& Q \sim E_{\mathrm{D}}\left(C_{\mathrm{D}}(Q), I_{\mathrm{D}}(Q)\right)
\end{aligned}
$$

Since $C_{\mathrm{D}}(P) \sim C_{\mathrm{D}}(Q)$, in order to show $P \sim Q$, it suffices to show that the word poset isomorphism $C_{\mathrm{D}}(P) \sim C_{\mathrm{D}}(Q)$ induces $I_{\mathrm{D}}(P) \sim I_{\mathrm{D}}(Q)$. By Lemma 5.3 in order to show $I_{\mathrm{D}}(P) \sim I_{\mathrm{D}}(Q)$, it suffices to show the following claim: for all $i \in[n]$,

$$
\left|\left\{x \in I_{\mathrm{D}}(P): f_{C_{\mathrm{D}}(P)}(x)=i\right\}\right|=\left|\left\{x \in I_{\mathrm{D}}(Q): f_{C_{\mathrm{D}}(Q)}(x)=i\right\}\right|
$$

Since $f_{R}(x)=f_{C_{\mathrm{D}}(R)}(x)$ for all $x \in I_{\mathrm{D}}(R)$, where $R$ is $P$ or $Q$, the claim can be rewritten as

$$
\begin{equation*}
\left|\left\{x \in I_{\mathrm{D}}(P): f_{P}(x)=i\right\}\right|=\left|\left\{x \in I_{\mathrm{D}}(Q): f_{Q}(x)=i\right\}\right| . \tag{5.3}
\end{equation*}
$$

Let $R$ be either $P$ or $Q$. By Proposition $3.3, \mathrm{D}(R) \cap \mathrm{A}(R)$ has a unique element, say $z$. Suppose $f_{R}(z)=s$. For $i \in[n]$, define

$$
\begin{aligned}
a_{i}(R) & =\left|\left\{x \in I_{\mathrm{D}}(R) \backslash I_{\mathrm{A}}(R): f_{R}(x)=i\right\}\right|, \\
b_{i}(R) & =\left|\left\{x \in I_{\mathrm{D}}(R) \cap I_{\mathrm{A}}(R): f_{R}(x)=i\right\}\right|, \\
c_{i}(R) & =\left|\left\{x \in I_{\mathrm{A}}(R) \backslash I_{\mathrm{D}}(R): f_{R}(x)=i\right\}\right| .
\end{aligned}
$$

See Figure 10.
By definition,

$$
\left|\left\{x \in I_{\mathrm{D}}(R): f_{R}(x)=i\right\}\right|= \begin{cases}a_{i}(R)+b_{i}(R) & \text { if } i<s  \tag{5.4}\\ b_{i}(R) & \text { if } i \geq s\end{cases}
$$

By Lemma 5.2, $\mathrm{D}(R) \backslash\{z\}$ is the descending chain of $C_{\mathrm{A}}(R)$, which is obtained from $R$ by removing $\mathrm{A}(R)$ and shifting the elements below $\mathrm{A}(R)$ to the left by one column. This shows that

$$
\left|\left\{x \in I_{\mathrm{D}}\left(C_{\mathrm{A}}(R)\right): f_{C_{\mathrm{A}}(R)}(x)=i\right\}\right|= \begin{cases}a_{i}(R)-1+b_{i+1}(R) & \text { if } i<s  \tag{5.5}\\ b_{i+1}(R) & \text { if } i \geq s\end{cases}
$$

Similarly, we have

$$
\left|\left\{x \in I_{\mathrm{A}}\left(C_{\mathrm{D}}(R)\right): f_{C_{\mathrm{D}}(R)}(x)=i\right\}\right|= \begin{cases}b_{i}(R) & \text { if } i<s  \tag{5.6}\\ c_{i+1}(R)-1+b_{i+1}(R) & \text { if } i \geq s\end{cases}
$$

Since $C_{\mathrm{A}}(P) \sim C_{\mathrm{A}}(Q)$ (respectively, $C_{\mathrm{D}}(P) \sim C_{\mathrm{D}}(Q)$ ), the left hand side of 5.5 (respectively, 5.6) is the same for both cases $R=P$ and $R=Q$. Comparing the right hand sides of (5.5) and (5.6 for the cases $R=P$ and $R=Q$, we obtain $b_{i}(P)=b_{i}(Q)$ for all $1 \leq i \leq n, a_{i}(P)=a_{i}(Q)$ for all $1 \leq i \leq s-1$, and $c_{i}(P)=c_{i}(Q)$ for all $s+1 \leq i \leq n$. By 5.4, this implies the claim 5.3) and the proof is completed.

We note that Bédard B99 studied the combinatorics of commutation classes for Weyl groups of any Lie types by introducing a level function on a certain subset of positive roots of the corresponding root system. Indeed, for each reduced word a level function is defined, and this function distinguishes commutation classes, i.e., $\mathbf{i} \sim \mathbf{j}$ if and only if the corresponding level functions are the same.

Question 5.4. Recall that the indices have been defined for the reduced words of the longest element in $\mathfrak{S}_{n+1}$, which is the Weyl group of Lie type $A$. The level functions introduced by Bédard [B99] and string polytopes are defined for any Lie type. In this regard, we may ask whether one can generalize the definitions of indices to other Lie types to provide more fruitful understanding of the combinatorics of string polytopes.

Remark 5.5. We have seen that the indices of reduced words are used to classify the string polytopes combinatorially equivalent to a Gelfand-Cetlin polytope. Recently, the combinatorics of string polytopes associated with reduced words of small indices has been studied in CKLP19b. A reduced word $\mathbf{i} \in \mathcal{R}\left(w_{0}^{(n+1)}\right)$ has small indices if $\operatorname{ind}_{\delta}(\mathbf{i})=(0, \ldots, 0, k)$ for some $\delta \in\{\mathrm{A}, \mathrm{D}\}^{n-1}$ and $k \leq \kappa\left(\delta_{n-1}, \delta_{n}\right)$. Here, $\kappa\left(\delta_{n-1}, \delta_{n}\right)=2$ if $\delta_{n}=\delta_{n-1}$; and $\kappa\left(\delta_{n-1}, \delta_{n}\right)=n-1$ otherwise. In CKLP19b, Cho et al. found the number of codimension one faces and the description of the vertices for the string polytopes associated with reduced words having small indices. These examples show that the notion of indices may have a potential role to study the combinatorics of string polytopes.

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[^1]:    ${ }^{1}$ Given integral polytopes $P \subset \mathbb{R}^{d}$ and $Q \subset \mathbb{R}^{d}$, we say that $P$ and $Q$ are unimodularly equivalent if there exist a matrix $U \in \mathrm{M}_{d \times d}(\mathbb{Z})$ satisfying $\operatorname{det} U= \pm 1$ and an integral vector $\mathbf{v} \in \mathbb{Z}^{d}$ such that $Q=f_{U}(P)+\mathbf{v}$. Here, $f_{U}$ is the linear transformation defined by $U$.

