# ON FIBONACCI PARTITIONS 

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To Carl Pomerance, a legend of number theory


#### Abstract

We prove an exact formula for OEIS A000119, which counts partitions into distinct Fibonacci numbers. We also establish an exact formula for its mean value, and determine the asymptotic behaviour.


## 1. Introduction

For $n \in \mathbb{Z}_{\geqslant 0}$, let $R(n)$ be the number of solutions to

$$
x_{1}+\cdots+x_{s}=n,
$$

where $s \in \mathbb{Z}_{\geqslant 0}$, and $x_{1}<x_{2}<\cdots<x_{s}$ are Fibonacci numbers. Note that $R(0)=1$. We call $R$ the Fibonacci partition function, as it counts partitions into distinct Fibonacci numbers. It has existed since the very first volume of the Fibonacci Quarterly in 1963, see [10], and its values comprise the sequence OEIS A000119. In 1968, Leonard Carlitz [5, Theorem 2] showed that

$$
\begin{equation*}
R\left(F_{m}\right)=\lfloor m / 2\rfloor \quad(m=2,3, \ldots) \tag{1.1}
\end{equation*}
$$

where $F_{1}=F_{2}=1, F_{3}=2$, and if $m \in \mathbb{N}$ then $F_{m}$ denotes the $m^{\text {th }}$ Fibonacci number. Many authors have investigated Fibonacci partitions, and the topic has received attention over several decades [2, 3, 5, 10, 12, 16, 18, 20]. In general $R(n)$ behaves erratically.


Figure 1. $R(n)$ against $n$ for $n=0,1, \ldots, 6765$
Our first result is an exact formula for $R(n)$. Recall Zeckendorf's theorem [13, 15, 22], which asserts that each positive integer has a unique representation as a sum of nonconsecutive Fibonacci numbers, called the Zeckendorf expansion.
Theorem 1.1. Let

$$
H=F_{m_{0}}+F_{m_{1}}+\cdots+F_{m_{k}}
$$

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be the Zeckendorf expansion of $H \in \mathbb{N}$, where

$$
m_{i-1}-m_{i} \geqslant 2 \quad(1 \leqslant i \leqslant k), \quad m_{k} \geqslant 2 .
$$

Write

$$
x_{\ell}=F_{m_{\ell}}+\cdots+F_{m_{k}} \quad(0 \leqslant \ell \leqslant k+1)
$$

and

$$
t_{i}=\left\lfloor\frac{m_{i-1}-m_{i}+2}{2}\right\rfloor, \quad \varepsilon_{i}=2 t_{i}-1-m_{i-1}+m_{i} \quad(1 \leqslant i \leqslant k) .
$$

Finally, let

$$
a_{0}=1, \quad a_{1}=t_{1}, \quad a_{\ell+1}=t_{\ell+1} a_{\ell}-\varepsilon_{\ell} a_{\ell-1} \quad(1 \leqslant \ell \leqslant k-1)
$$

Then

$$
R(H)= \begin{cases}a_{k}\left\lfloor m_{k} / 2\right\rfloor-\varepsilon_{k} a_{k-1}, & \text { if } k \geqslant 1  \tag{1.2}\\ \left\lfloor m_{0} / 2\right\rfloor, & \text { if } k=0 .\end{cases}
$$

Throughout, we adopt the standard convention that empty sums are 0 , so $x_{k+1}=0$ above.
Carlitz [5] had a recursive formula, but on attempting to produce a non-recursive formula found that "the general case is very complicated". Robbins [16] had a simpler recursive formula, leading to an algorithm used to produce some initial values of $R(H)$, but also did not write down a non-recursive formula. Weinstein [20] obtained a non-recursive expression, albeit a complicated one. The nicest formula that we could find in the literature is that of Berstel [3, Proposition 3.1] which, being a product of $2 \times 2$ matrices, is quite similar to Theorem 1.1. Our formula $\sqrt{1.2}$ is extremely efficient in practice. For example, it can compute $R\left(10^{100}\right)$ in less than one second on a standard laptop computer. Mathematica [21] code for this is provided in Appendix A. Theorem 1.1 follows readily from Robbins's recursion, so it is not our main result by any means.

We also study the mean value

$$
M(H):=H^{-1} \sum_{n=0}^{H} R(n) \quad(H \in \mathbb{N})
$$

or equivalently the summatory function

$$
A(H):=\sum_{n=0}^{H} R(n) \quad(H \in \mathbb{Z}) .
$$

For $t \in \mathbb{N}$, let

$$
\begin{equation*}
f(t)=1+\frac{2\left(4^{t-1}-1\right)}{3} \tag{1.3}
\end{equation*}
$$

We establish the following exact formula for $A(H)$.
Theorem 1.2. Let $H \in \mathbb{N}$, and let the values of the $x_{\ell}, t_{i}, \varepsilon_{i}$ and $a_{\ell}$ be as in Theorem 1.1. Then for $\ell=1,2, \ldots, k$ we have

$$
\begin{equation*}
A(H)=a_{\ell} A\left(x_{\ell}\right)-\varepsilon_{\ell} a_{\ell-1} A\left(x_{\ell+1}\right)+\sum_{i \leqslant \ell} a_{i-1} f\left(t_{i}\right) 2^{m_{i-1}-2 t_{i}} \tag{1.4}
\end{equation*}
$$

In particular

$$
A(H)= \begin{cases}a_{k}\left\lfloor\frac{2^{m_{k}}}{6}+\frac{m_{k}+1}{2}\right\rfloor-\varepsilon_{k} a_{k-1}+\sum_{i \leqslant k} a_{i-1} f\left(t_{i}\right) 2^{m_{i-1}-2 t_{i}}, & \text { if } k \geqslant 1  \tag{1.5}\\ \left\lfloor\frac{2^{m_{0}}}{6}+\frac{m_{0}+1}{2}\right\rfloor, & \text { if } k=0\end{cases}
$$

This enables us to understand the asymptotic behaviour of $A(H)$ and $M(H)$. Put

$$
\varphi=\frac{1+\sqrt{5}}{2}, \quad \lambda=\frac{\log 2}{\log \varphi} \approx 1.44
$$

and define

$$
c_{1}=\liminf _{H \rightarrow \infty} \frac{A(H)}{H^{\lambda}}, \quad c_{2}=\limsup _{H \rightarrow \infty} \frac{A(H)}{H^{\lambda}} .
$$

We now present our main result.
Theorem 1.3 (Main Theorem). We have

$$
c_{1}=0.52534 \ldots, \quad c_{2}=0.54338 \ldots,
$$

and more precisely

$$
\begin{equation*}
0.525347<c_{1}<0.525349, \quad 0.5433878<c_{2}<0.5433893 \tag{1.6}
\end{equation*}
$$

It follows that $A(H) \asymp H^{\lambda}$. Subject to hardware constraints, our method computes $c_{1}$ and $c_{2}$ to arbitrary precision.


Figure 2. $A(H) / H^{\lambda}$ against $H$ for $H=0,1, \ldots, 75025$, with horizontal lines at 0.525348 and 0.543388 .

The Fibonacci partition function behaves very differently to the usual partition function $p(n)$, for which there is a nice asymptotic formula

$$
p(n) \sim \frac{1}{4 n \sqrt{3}} \exp (\pi \sqrt{2 n / 3})
$$

going back to Hardy and Ramanujan [9]; see also [1, §5]. Our work shows that even the mean value $M(H)$ of the Fibonacci partition function does not have a 'nice' asymptotic formula, however we are able to describe the asymptotic behaviour fairly well.

The logarithmic average of $R(n) n^{1-\lambda}$, namely

$$
B(H):=(\log H)^{-1} \sum_{n \leqslant H} \frac{R(n)}{n^{\lambda}} \quad(H \geqslant 2),
$$

might be better behaved. Breaking into ranges $I_{m}=\left(F_{m}, F_{m+1}\right]$, wherein

$$
\frac{A\left(F_{m+1}\right)-A\left(F_{m}\right)}{F_{m+1}^{\lambda}} \leqslant \sum_{n \in I_{m}} \frac{R(n)}{n^{\lambda}} \leqslant \frac{A\left(F_{m+1}\right)-A\left(F_{m}\right)}{F_{m}^{\lambda}}
$$

it follows from Theorem 1.3 that

$$
B(H) \asymp 1
$$

Though $B(H)$ is not decreasing, it does exhibit a clear downward trend.

Conjecture 1.4. There exists $B>0$ such that

$$
B(H) \rightarrow B \quad(H \rightarrow \infty)
$$

We also invite the enthusiastic reader to consider:
(1) Higher moments of the Fibonacci partition function
(2) Lucas partitions [6, 11]
(3) Partitions into distinct terms of a sequence $\left(\left\lfloor\tau^{m}\right\rfloor\right)_{m=1}^{\infty}$, where $\tau \in(1,2)$ is fixed
(4) Partitions into distinct terms of a Piatetski-Shapiro sequence $\left(\left\lfloor m^{\tau}\right\rfloor\right)_{m=1}^{\infty}$, where $\tau>1$ is fixed, cf. for polynomials [7, 8]
(5) Partitions into distinct Piatetski-Shapiro primes, cf. [14, 19].

Methods. We deduce Theorem 1.1 by iterating Robbins's recursion [16, Theorem 4]. For Theorem 1.2 , we begin with the observation that $A(H)$ counts sets of distinct Fibonacci numbers whose sum is at most $H$. This enables us to prove a combinatorial recursion analogous to that of Robbins. By systematic applications of our recursion, we prove an exact formula for $A(H)$ in terms of the Zeckendorf expansion of $H$. Finally, for $m \in \mathbb{N}$ large, we subdivide $\left[F_{m}, F_{m+1}\right) \cap \mathbb{Z}$ into many discrete subintervals, according to the initial Zeckendorf digits. By estimating $A(H)$ at the endpoints of these subintervals, we are able to compute $c_{1}$ and $c_{2}$ to arbitrary precision, subject to hardware constraints. We used the software Mathematica [21] to perform the calculations, leading to Theorem 1.3.

Notation. As usual, empty sums are 0 . We adopt the following standard asymptotic notations: if $f, g: \mathbb{N} \rightarrow \mathbb{R}_{>0}$, we write

$$
\begin{gathered}
f(m) \sim g(m) \quad \text { if } \quad \lim _{m \rightarrow \infty} \frac{f(m)}{g(m)}=1, \\
f(m)=o(g(m)) \quad \text { if } \quad \lim _{m \rightarrow \infty} \frac{f(m)}{g(m)}=0
\end{gathered}
$$

and

$$
f(m) \asymp g(m) \quad \text { if } \quad 0<\liminf _{m \rightarrow \infty} \frac{f(m)}{g(m)} \leqslant \limsup _{m \rightarrow \infty} \frac{f(m)}{g(m)}<\infty
$$

In words, the first notion is that $f$ is asymptotic to $g$, the second notion is that $f$ has a smaller asymptotic order of magnitude than $g$, and the third notion is that $f$ and $g$ have the same asymptotic order of magnitude.

Organisation. We prove Theorems 1.1, 1.2 and 1.3 in Sections 2, 3 and 4, respectively. The appendices contain the code that we used for the computations.

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## 2. An exact formula for Fibonacci partitions

In this section, we prove Theorem 1.1. With the notation of Theorem 1.1, Robbins [16, Theorem 4] established the following recursion.

Lemma 2.1 (Robbins). If $H \geqslant 2$ and $k \geqslant 1$ then

$$
R(H)=t_{1} R\left(x_{1}\right)-\varepsilon_{1} R\left(x_{2}\right) .
$$

We induct on $\ell$ to show that if $\ell=1,2, \ldots, k$ then

$$
\begin{equation*}
R(H)=a_{\ell} R\left(x_{\ell}\right)-\varepsilon_{\ell} a_{\ell-1} R\left(x_{\ell+1}\right) \tag{2.1}
\end{equation*}
$$

The base case $\ell=1$ is Lemma 2.1. Now suppose $1 \leqslant \ell \leqslant k-1$, and that (2.1) holds. Then

$$
\begin{aligned}
R(H) & =a_{\ell} R\left(x_{\ell}\right)-\varepsilon_{\ell} a_{\ell-1} R\left(x_{\ell+1}\right) \\
& =a_{\ell}\left(t_{\ell+1} R\left(x_{\ell+1}\right)-\varepsilon_{\ell+1} R\left(x_{\ell+2}\right)\right)-\varepsilon_{\ell} a_{\ell-1} R\left(x_{\ell+1}\right) \\
& =a_{\ell+1} R\left(x_{\ell+1}\right)-\varepsilon_{\ell+1} a_{\ell} R\left(x_{\ell+2}\right),
\end{aligned}
$$

which is (2.1) with $\ell+1$ in place of $\ell$. Thus, we have established (2.1) by induction.
For $k \geqslant 1$, applying (2.1) with $\ell=k$, and then applying (1.1) with $m=m_{k}$, gives

$$
R(H)=a_{k} R\left(m_{k}\right)-\varepsilon_{k} a_{k-1}=a_{k}\left\lfloor m_{k} / 2\right\rfloor-\varepsilon_{k} a_{k-1}
$$

The $k=0$ case of (1.2) is (1.1), which was already established by Carlitz [5, Theorem 2]. This completes the proof of Theorem 1.1.

## 3. The summatory function

In this section, we prove Theorem 1.2 .
3.1. A combinatorial recursion. Recall that the Fibonacci sequence enjoys the recursive relation $F_{m+1}=F_{m}+F_{m-1}$. Our starting point is the following recursion for the summatory function $A(H)$.

Lemma 3.1. If $m \in \mathbb{Z}_{\geqslant 3}$ and $F_{m} \leqslant H<F_{m+1}$ then

$$
A(H)=A\left(H-F_{m}\right)+A\left(H-F_{m-1}\right)-A\left(H-2 F_{m-1}\right)+2^{m-3}
$$

Proof. Observe that $A(H)$ counts tuples $\left(x_{1}, \ldots, x_{s}\right)$ of Fibonacci numbers such that

$$
s \geqslant 0, \quad x_{1}<\cdots<x_{s}, \quad x_{1}+\cdots+x_{s} \leqslant H
$$

Note that $x_{1}, \ldots, x_{s} \in\left\{F_{2}, \ldots, F_{m}\right\}$, since $F_{1}=F_{2}$. There are $A\left(H-F_{m}\right)$ such tuples for which $x_{s}=F_{m}$, since $H-F_{m}<F_{m}$.

If $x_{s}=F_{m-1}$, then we have

$$
x_{1}+\cdots+x_{s-1} \leqslant H-F_{m-1}<F_{m}<2 F_{m-1} .
$$

There would be $A\left(H-F_{m-1}\right)$ solutions to this if $x_{s-1}$ were allowed to equal $F_{m-1}$, but since $x_{s-1}<x_{s}$ this is forbidden, and we need to subtract $A\left(H-2 F_{m-1}\right)$. Thus, there are

$$
A\left(H-F_{m-1}\right)-A\left(H-2 F_{m-1}\right)
$$

valid tuples for which $x_{s}=F_{m-1}$.
Finally, if $x_{1}<x_{2}<\ldots<x_{s} \leqslant F_{m-2}$ are Fibonacci numbers, then we always have

$$
x_{1}+\cdots+x_{s} \leqslant F_{2}+\cdots+F_{m-2}<F_{m} \leqslant H,
$$

owing to the well-known identity

$$
F_{1}+F_{2}+\cdots+F_{n-2}=F_{n}-1 \quad(n \in \mathbb{N})
$$

the proof of which is a straightforward exercise in mathematical induction. As there are $2^{m-3}$ subsets of $\left\{F_{2}, \ldots, F_{m-2}\right\}$, there are $2^{m-3}$ valid tuples for which $x_{s} \leqslant F_{m-2}$.

Summing the contributions from the three cases completes the proof of the lemma.

Next, we provide a simple argument to show that

$$
\begin{equation*}
A(H) \asymp H^{\lambda} \tag{3.1}
\end{equation*}
$$

recalling our notational convention that this describes the asymptotic order of magnitude as $H \rightarrow+\infty$. Let $m \geqslant 4$ be an integer. If $m$ is odd then, by Lemma 3.1, we have

$$
\begin{aligned}
A\left(F_{m}\right) & =2^{m-3}+1+A\left(F_{m-2}\right)=\ldots \\
& =\left(2^{m-3}+1\right)+\left(2^{m-5}+1\right)+\cdots+\left(2^{2}+1\right)+A\left(F_{3}\right) \\
& =\left(1+2^{2}+\cdots+2^{m-3}\right)+(m+1) / 2 \\
& =\frac{2^{m-1}-1}{3}+\frac{m+1}{2}=\left\lfloor\frac{2^{m}}{6}+\frac{m+1}{2}\right\rfloor .
\end{aligned}
$$

Similarly, when $m \geqslant 4$ is even we reach the same eventual conclusion, and we can check directly that it also holds when $m=2,3$. Thus, we have

$$
\begin{equation*}
A\left(F_{m}\right)=\left\lfloor\frac{2^{m}}{6}+\frac{m+1}{2}\right\rfloor \quad(m \geqslant 2) \tag{3.2}
\end{equation*}
$$

and

$$
\begin{equation*}
A\left(F_{m}\right) \sim \frac{2^{m}}{6} \tag{3.3}
\end{equation*}
$$

Therefore

$$
A\left(F_{m}\right) \sim c F_{m}^{\lambda}, \quad c=\frac{1}{6} \sqrt{5}{ }^{\lambda}
$$

Note also that if $F_{m} \leqslant H<F_{m+1}$ then

$$
\varphi^{m}(1+o(1))=F_{m} \sqrt{5} \leqslant H \sqrt{5}<F_{m+1} \sqrt{5}=\varphi^{m+1}(1+o(1)),
$$

and consequently

$$
A(H)<A\left(F_{m+1}\right)=\frac{2^{m+1}}{6}(1+o(1)) \leqslant \frac{1}{3}(H \sqrt{5})^{\lambda}(1+o(1))
$$

and

$$
A(H) \geqslant A\left(F_{m}\right)=\frac{2^{m}}{6}(1+o(1)) \geqslant \frac{1}{12}(H \sqrt{5})^{\lambda}(1+o(1)) .
$$

These calculations furnish (3.1), in the stronger form

$$
c / 2 \leqslant c_{1} \leqslant c_{2} \leqslant 2 c .
$$

Example 3.2. By Lemma 3.1, as $m \rightarrow \infty$ we have

$$
\begin{aligned}
A\left(2 F_{m-1}\right) & =A\left(2 F_{m-1}-F_{m}\right)+A\left(F_{m-1}\right)-A(0)+2^{m-3} \\
& =A\left(F_{m-3}\right)+A\left(F_{m-1}\right)+2^{m-3}-1 \sim \frac{11}{48} 2^{m} \\
& \sim \frac{11}{24}\left(F_{m-1} \sqrt{5}\right)^{\lambda}=\frac{11(\sqrt{5} / 2)^{\lambda}}{24}\left(2 F_{m-1}\right)^{\lambda}, \quad \frac{11(\sqrt{5} / 2)^{\lambda}}{24} \approx 0.538
\end{aligned}
$$

and

$$
\begin{aligned}
A\left(F_{m}+F_{m-2}\right) & =A\left(F_{m-2}\right)+A\left(2 F_{m-2}\right)-A\left(F_{m-4}\right)+2^{m-3} \\
& \sim \frac{2^{m-2}}{6}+\frac{11}{48} 2^{m-1}-\frac{2^{m-4}}{6}+2^{m-3} \sim \frac{13}{48} 2^{m} \\
& \sim \frac{13}{48}\left(\sqrt{5} \frac{F_{m}+F_{m-2}}{1+\varphi^{-2}}\right)^{\lambda}=\frac{13 \sqrt{5}^{\lambda}}{48\left(1+\varphi^{-2}\right)^{\lambda}}\left(F_{m}+F_{m-2}\right)^{\lambda} \\
& =\frac{13 \varphi^{\lambda}}{48}\left(F_{m}+F_{m-2}\right)^{\lambda}=\frac{13}{24}\left(F_{m}+F_{m-2}\right)^{\lambda}, \quad \frac{13}{24} \approx 0.542
\end{aligned}
$$

3.2. An exact formula for the summatory function. Recall (1.3). Applying Lemma 3.1 several times provides the following more elaborate recursion.

Lemma 3.3. If $t \geqslant 2, m \geqslant 2 t$, and $F_{m-2 t+1} \leqslant x<F_{m-2 t+3}$, then

$$
A\left(F_{m}+x\right)=t A(x)-A\left(x-F_{m-2 t+2}\right)+f(t) 2^{m-2 t}
$$

Proof. For the base case $t=2$ of our induction, for $m \geqslant 4$ and $F_{m-3} \leqslant x<F_{m-1}$ we have

$$
\begin{aligned}
A\left(F_{m}+x\right) & =A(x)+A\left(F_{m-2}+x\right)-A\left(x-F_{m-3}\right)+2^{m-3} \\
& =2 A(x)-A\left(x-F_{m-2}\right)+2^{m-4}+2^{m-3}=2 A(x)-A\left(x-F_{m-2}\right)+\frac{3}{16} 2^{m} \\
& =2 A(x)-A\left(x-F_{m-2}\right)+f(2) 2^{m-4}
\end{aligned}
$$

Now let $t \geqslant 3$, and suppose the result holds with $t-1$ in place of $t$. Then for $m \geqslant 2 t$ and $x \in\left[F_{m-2 t+1}, F_{m-2 t+3}\right)$ we have

$$
\begin{aligned}
A\left(F_{m}+x\right) & =A(x)+A\left(F_{m-2}+x\right)+2^{m-3} \\
& =t A(x)-A\left(x-F_{m-2-2(t-1)+2}\right)+\left(1+\frac{2\left(4^{t-2}-1\right)}{3}\right) 2^{m-2-2(t-1)}+2^{m-3} \\
& =t A(x)-A\left(x-F_{m-2 t+2}\right)+\left(1+\frac{2^{2 t-3}-2}{3}+2^{2 t-3}\right) 2^{m-2 t} \\
& =t A(x)-A\left(x-F_{m-2 t+2}\right)+\left(1+\frac{2\left(4^{t-1}-1\right)}{3}\right) 2^{m-2 t} \\
& =t A(x)-A\left(x-F_{m-2 t+2}\right)+f(t) 2^{m-2 t}
\end{aligned}
$$

The following immediate consequence is analogous to Lemma 2.1.
Corollary 3.4. Let $t \geqslant 2, m \geqslant 2 t$, and $F_{m-2 t+1} \leqslant x<F_{m-2 t+3}$. Set

$$
(\varepsilon, y)= \begin{cases}\left(1, x-F_{m-2 t+2}\right) & \text { if } F_{m-2 t+2} \leqslant x<F_{m-2 t+3} \\ \left(0, x-F_{m-2 t+1}\right) & \text { if } F_{m-2 t+1} \leqslant x<F_{m-2 t+2}\end{cases}
$$

Then

$$
A\left(F_{m}+x\right)=t A(x)-\varepsilon A(y)+f(t) 2^{m-2 t}
$$

We now establish (1.4) for $1 \leqslant \ell \leqslant k$. For the base case $\ell=1$ of our induction, we know from Corollary 3.4 that

$$
A(H)=a_{1} A\left(x_{1}\right)-\varepsilon_{1} a_{0} A\left(x_{2}\right)+a_{0} f\left(t_{1}\right) 2^{m_{0}-2 t_{1}} .
$$

Now suppose that for some $\ell \in\{1,2, \ldots, k-1\}$ we have (1.4). Then

$$
\begin{aligned}
A(H)= & a_{\ell}\left(t_{\ell+1} A\left(x_{\ell+1}\right)-\varepsilon_{\ell+1} A\left(x_{\ell+2}\right)+f\left(t_{\ell+1}\right) 2^{m_{\ell}-2 t_{\ell+1}}\right)-\varepsilon_{\ell} a_{\ell-1} A\left(x_{\ell+1}\right) \\
& +\sum_{i \leqslant \ell} a_{i-1} f\left(t_{i}\right) 2^{m_{i-1}-2 t_{1}} \\
= & a_{\ell+1} A\left(x_{\ell+1}\right)-\varepsilon_{\ell+1} a_{\ell} A\left(x_{\ell+2}\right)+\sum_{i \leqslant \ell+1} a_{i-1} f\left(t_{i}\right) 2^{m_{i-1}-2 t_{i}} .
\end{aligned}
$$

We have proved (1.4) by induction on $\ell$.
For $k \geqslant 1$, inserting (3.2) into the $\ell=k$ case of (1.4) yields (1.5). Meanwhile, the $k=0$ case of (1.5) is precisely (3.2). This completes the proof of Theorem 1.2.
Example 3.5. Let $m$ be large, and consider

$$
H=F_{m}+F_{m-7}+F_{m-12}+F_{m-19} .
$$

In this case

$$
t_{1}=4, \quad t_{2}=3, \quad t_{3}=4, \quad \varepsilon_{0}=\varepsilon_{1}=\varepsilon_{2}=0
$$

Therefore

$$
\begin{aligned}
A(H) & =48\left\lfloor\frac{2^{m-19}}{6}+\frac{m-18}{2}\right\rfloor+f(4) 2^{m-8}+4 f(3) 2^{m-13}+12 f(4) 2^{m-20} \\
& \sim\left(16+43 \times 2^{12}+44 \times 2^{7}+12 \times 43\right) 2^{m-20}=\frac{45573}{262144} 2^{m}
\end{aligned}
$$

Thus, as $m \rightarrow \infty$, we have

$$
\frac{A(H)}{H^{\lambda}} \rightarrow \frac{45573}{262144}\left(\frac{\sqrt{5}}{1+\varphi^{-7}+\varphi^{-12}+\varphi^{-19}}\right)^{\lambda} \approx 0.525352
$$

## 4. Subdivision

In this section, we prove Theorem 1.3. Let $m$ be a large positive integer. We subdivide the discrete interval $\left[F_{m}, F_{m+1}\right) \cap \mathbb{Z}$ into subintervals

$$
\left[p_{j}, p_{j+1}\right) \cap \mathbb{Z} \quad(0 \leqslant j \leqslant 317810)
$$

according to the initial Zeckendorf digits. The left endpoints $p_{0}, \ldots, p_{317810}$ are given by

$$
F_{m}+\sum_{i \leqslant \ell} F_{m-a_{i}},
$$

where

$$
\ell \geqslant 0, \quad a_{1}, a_{2}-a_{1}, \ldots, a_{\ell}-a_{\ell-1} \geqslant 2, \quad a_{\ell} \leqslant 27
$$

The right endpoints have the same form, except $p_{317811}=F_{m+1}$. Using Theorem 1.2, we can show that

$$
A\left(p_{j}\right) \sim v_{j} 2^{m}, \quad p_{j} \sim w_{j} \varphi^{m} \quad(0 \leqslant j \leqslant 317811)
$$

as $m \rightarrow \infty$, for some computable values of $v_{j}$ and $w_{j}$. Then

$$
(1+o(1)) L_{j} \leqslant \frac{A(H)}{H^{\lambda}} \leqslant(1+o(1)) U_{j} \quad\left(p_{j} \leqslant H<p_{j+1}\right)
$$

where

$$
L_{j}=\frac{v_{j}}{w_{j+1}^{\lambda}}, \quad U_{j}=\frac{v_{j+1}}{w_{j}^{\lambda}} \quad(0 \leqslant j \leqslant 317810) .
$$

We carried out these computations using the software Mathematica [21]; the code is provided in Appendix B. Then

$$
c_{1} \geqslant \min _{j} L_{j}, \quad c_{2} \leqslant \max _{j} U_{j} .
$$

The software also told us which subintervals attaining the least $L_{j}$ and the greatest $U_{j}$, namely

$$
j=19401, \quad\left(a_{1}, \ldots, a_{\ell}\right)=(7,12,18,25)
$$

and

$$
j=184839, \quad\left(a_{1}, \ldots, a_{\ell}\right)=(3,5,8,10,12,16,18,21,23,26)
$$

respectively. Since

$$
\frac{A\left(p_{j}\right)}{p_{j}^{\lambda}} \sim \frac{v_{j}}{w_{j}^{\lambda}},
$$

we thereby also obtained an upper bound for $c_{1}$ and a lower bound for $c_{2}$. These calculations delivered (1.6), completing the proof of Theorem 1.3 .

## Appendix A. Code for $R(H)$

Lines 2-7 are Rosetta code [17], available for general use under the GNU Free Documentation License, version 1.2. The value of $H$ in the first line can be changed.

```
H = 1234;
zeckendorf[0] = 0;
zeckendorf[n_Integer] :=
    10^(# - 1) + zeckendorf[n - Fibonacci[# + 1]] &@
        LengthWhile[
            Fibonacci /@
            Range[2, Ceiling@Log[GoldenRatio, n Sqrt@5]], # <= n &];
Z = IntegerDigits[zeckendorf [H]];
l = Total[Z];
X = ConstantArray[0, 1];
t = 1;
If [l == 1, Floor[(Length[Z] + 1)/2],
    For[i = 1, i < Length[Z] + 1, i++,
    If[Z[[i]] == 1, X[[t]] = Length[Z] - i + 2; t++;,]
    ];
T = ConstantArray[0, l - 1];
Ep = ConstantArray[0, l - 1];
For[i = 1, i < l, i++,
    T[[i]] = Floor[(X[[i]] - X[[i + 1]] + 2)/2];
    Ep[[i]] = 2 T[[i]] - 1 - X[[i]] + X[[i + 1]];
    ];
    a = ConstantArray[1, l];
    a[[2]] = T[[1]];
For[i = 3, i < l + 1, i++,
    a[[i]] = T[[i - 1]] a[[i - 1]] - Ep[[i - 2]] a[[i - 2]]
    ];
    a[[1]]*Floor[X[[1]]/2] - a[[l - 1]]*Ep[[1 - 1]]
]
```

Appendix B. Code for $A(H)$

```
P = (1 + Sqrt[5])/2;
L = Log[2]/Log[P];
l = 27;
X = ConstantArray[0, {Fibonacci[l + 1], Floor[l/2]}];
t = 1;
X[[2, 1]] = l;
```

```
For[i = 3, i < Fibonacci[l + 1] + 1, i++,
    If[X[[i - 1, t]] == l || X[[i - 1, t]] == l - 1,
    If[(t > 1 && X[[i - 1, t]] - X[[i - 1, t - 1]] == 2),
        t--;
        For[j = t, j > 0, j--,
            If [j == 1, t = j;
            For[k = 1, k < t, k++, X[[i, k]] = X[[i - 1, k]]];
            X[[i, t]] = X[[i - 1, t]] - 1; j = 0,
            If[X[[i - 1, j]] - X[[i - 1, j - 1]] != 2, t = j;
                For[k = 1, k < t, k++, X[[i, k]] = X[[i - 1, k]]];
                X[[i, t]] = X[[i - 1, t]] - 1; j = 0]
            ]
        ]
        X[[i]] = X[[i - 1]];
        X[[i, t]]--;
        ],
    t++;
    X[[i]] = X[[i - 1]];
    X[[i, t]] = l;
    ]
]
T = ConstantArray[0, {Fibonacci[l + 1], Floor[l/2]}];
For[i = 2, i < Fibonacci[l + 1] + 1, i++,
    T[[i, 1]] = Floor[(X[[i, 1]] + 2)/2]
    ]
For[i = 2, i < Fibonacci[l + 1] + 1, i++,
    For[j = 2, j < Floor[l/2] + 1, j++,
    If[X[[i, j]] == 0, ,
        T[[i, j]] = Floor[(X[[i, j]] - X[[i, j - 1]] + 2)/2]
        ]
    ]
]
Ep = ConstantArray[0, {Fibonacci[l + 1], Floor[l/2]}];
For[i = 2, i < Fibonacci[l + 1] + 1, i++,
    Ep[[i, 1]] = 2 T[[i, 1]] - 1 - X[[i, 1]]
    ]
For[i = 2, i < Fibonacci[l + 1] + 1, i++,
    For[j = 2, j < Floor[l/2] + 1, j++,
        If[X[[i, j]] == 0, ,
        Ep[[i, j]] = 2 T[[i, j]] - 1 - X[[i, j]] + X[[i, j - 1]]
        ]
    ]
]
a = ConstantArray[0, {Fibonacci[l + 1], Floor[l/2] + 1}];
For[i = 1, i < Fibonacci[l + 1] + 1, i++, a[[i, 1]] = 1];
For[i = 2, i < Fibonacci[l + 1] + 1, i++, a[[i, 2]] = T[[i, 1]]];
For[i = 2, i < Fibonacci[l + 1] + 1, i++,
    For[j = 3, j < Floor[l/2] + 2, j++,
    If[X[[i, j - 1]] == 0, ,
        a[[i, j]] =
            T[[i, j - 1]] a[[i, j - 1]] - Ep[[i, j - 2]] a[[i, j - 2]]
        ]
        ]
]
f = Function[t, 1 + (2/3) (4^(t - 1) - 1)];
k = ConstantArray[0, {Fibonacci[1 + 1] + 1}];
k[[1]] = (1/6);
k[[Fibonacci[l + 1] + 1]] = (1/3);
For[i = 2, i < Fibonacci[l + 1] + 1, i++,
    For[j = Floor[l/2], j > 0, j--,
    If[X[[i, j]] == 0, ,
```

```
    k[[i]] = (a[[i, j + 1]]/(6*2^(X[[i, j]]))) +
        Sum[(a[[i, l]] f[T[[i, l]]]/2^(X[[i, l - 1]] + 2 T[[i, l]])), {l,
            2, j}] + (a[[i, 1]] f[T[[i, 1]]]/2^(2 T[[i, 1]]));
        j = 0
    ]
    ]
]
p = ConstantArray[0, {Fibonacci[1 + 1] + 1}];
p[[1]] = 1; p[[Fibonacci[l + 1] + 1]] = P;
For[i = 2, i < Fibonacci[l + 1] + 1, i++,
    For[j = Floor[l/2], j > 0, j--,
    If[X[[i, j]] == 0, ,
        p[[i]] = 1 + Sum[P^(-X[[i, k]]), {k, 1, j}];
        j = 0
        ]
    ]
]
LU = ConstantArray[0, {Fibonacci[l + 1], 2}];
For[i = 1, i < Fibonacci[l + 1] + 1, i++,
    LU[[i, 1]] = k[[i]]*(Sqrt[5]/p[[i + 1]])^L;
    LU[[i, 2]] = k[[i + 1]]*(Sqrt[5]/p[[i]])^L
]
NumberForm[N[Min[LU]], 8]
NumberForm[N[Max[LU]], 8]
Position[LU, Min[LU]]
Position[LU, Max[LU]]
X[[19401]]
X[[184839]]
NumberForm[N[k[[19401]]*(Sqrt[5]/p[[19402]])^L], 8]
NumberForm[N[k[[19401]]*(Sqrt[5]/p[[19401]])^L], 8]
NumberForm[N[k[[184840]]*(Sqrt[5]/p[[184839]])^L], 8]
NumberForm[N[k[[184839]]*(Sqrt[5]/p[[184839]])^L], 8]
```


## References

[1] G. E. Andrews, The theory of partitions, Cambridge University Press, Cambridge, 1998.
[2] F. Ardila, On the coefficients of a Fibonacci power series, Fibonacci Quart. 42 (2004), 202-204.
[3] J. Berstel, An exercise on Fibonacci representations, Theoret. Informatics Appl. 35 (2001), 491-498.
[4] J. L. Brown, Jr., Zeckendorf's theorem and some applications, Fibonacci Quart. 2 (1964), 162-168.
[5] L. Carlitz, Fibonacci representations, Fib. Quart. 6 (1968), 193-220.
[6] L. Carlitz, R. Scoville and V. E. Hoggatt, Jr., Lucas representations, Fibonacci Quart. 10 (1972), 29-42, 70, 112.
[7] A. Dunn and N. Robles, Polynomial partition asymptotics, J. Appl. Math. Anal. Appl. 459 (2018), 359-384.
[8] A. Gafni, Power partitions, J. Number Theory 163 (2016), 19-42.
[9] G. H. Hardy and S. Ramanujan, Asymptotic formule in combinatory analysis, Proc. London Math. Soc. (2) $\mathbf{1 7}$ (1918), 75-115.
[10] V. E. Hoggatt, Jr. and S. L. Basin, Representations by complete sequences - Part I (Fibonacci), Fibonacci Quart. 1 (1963), 1-14.
[11] D. A. Klarner, Representations of $N$ as a sum of distinct elements from special sequences, Fib. Quart. 4 (1966), 289-306, 322.
[12] D. A. Klarner, Partitions of $n$ into distinct Fibonacci numbers, Fibonacci Quart. 6 (1968), 235-243.
[13] D. E. Knuth, Fibonacci multiplication, Appl. Math. Lett. 1 (1988), 57-60.
[14] A. Kumchev, On the Piatetski-Shapiro-Vinogradov theorem, J. Théor. Nombres Bordeaux 9 (1997), 11-23.
[15] C. G. Lekkerkerker, Voorstelling van natuurlijke getollen door een som van getallen van Fibonacci, Simon Stevin 29 (1952), 190-195.
[16] N. Robbins, Fibonacci partitions, Fibonacci Quart. 34 (1996), 306-313.
[17] https://rosettacode.org/wiki/Zeckendorf_number_representation\#Mathematica, accessed August 13, 2020.
[18] P. K. Stockmeyer, A smooth tight upper bound for the Fibonacci representation function $R(n)$, Fibonacci Quart. 46/47 (2008/2009), 103-106.
[19] R C. Vaughan, On the number of partitions into primes, Ramanujan J. 15 (2008), 109-121.
[20] F. V. Weinstein, Notes on Fibonacci partitions, Exp. Math. 25 (2016), 482-499.
[21] Wolfram Research, Inc., Mathematica, Version 12.0, Champaign, IL, 2019.
[22] E. Zeckendorf, Représentation des nombres naturels par une somme de nombres de Fibonacci ou de nombres de Lucas, Bull. Soc. Roy. Sci. Liège 41 (1972), 179-182.

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