# Energy difference between the lowest doublet and quartet states of the boron atom 

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The energies of the lowest ${ }^{2} P_{u},{ }^{4} P_{g}$ and $2 D_{g}$ states of the boron atom are calculated with $\mu$ hartree accuracy, in the basis of symmetrized, explicitly correlated Gaussian lobe functions. Finite nuclear mass and scalar relativistic corrections are taken into account. This study contributes to the problem of the energy differences between doublet and quartet states of boron, which have not been measured to date. It is found that the ${ }^{2} P_{u} \rightarrow{ }^{4} P_{g}$ excitation energy, recommended in the Atomic Spectra Database, appears underestimated by more than $300 \mathrm{~cm}^{-1}$.

## I. INTRODUCTION

Highly accurate calculations, carried out within the well-grounded theory of quantum mechanics, are currently possible for few-electron atoms and molecules. The results are usually compared with spectroscopic data. This collation verifies the theory and computational methods, but may also stimulate improvements of the experiment. History of the studies on the rovibrational spectrum of the hydrogen molecule is a good example of such positive feedback [1,2]. The calculations may also provide reliable results where experimental data are missing. For the boron atom, intersystem radiative transitions were not observed, therefore the energy differences between the spin doublet and quartet states, listed in the Atomic Spectra Database (ASD) [3] are based on numerical extrapolation of the transition energies known for heavier, isoelectronic ions [4].

According to this extrapolation, the lowest ${ }^{4} P_{g}$ term has the energy higher by $28644.3 \mathrm{~cm}^{-1}$, than the ground state term $\left({ }^{2} P_{u}\right)$. The $J$ quantum number is omitted, because the fine structure is not considered in the present work. The energy of a non-splitted term is not observable, and is
computed from experimental data, as weighted average over associated, $J$-dependent term energies. Calculations of this energy difference were also carried out in the past, but the results do not agree with that "experimental" value. The short review is limited to most recent articles, because the results of earlier calculations [5, 6] were simply too innaccurate for a comparison with spectroscopic data. Froese Fischer and coworkers [7] used the multiconfiguration Hartree-Fock (MCHF) method, with finite nuclear mass and scalar relativistic corrections taken into account, and obtained the excitation energy amounting to $28959(5) \mathrm{cm}^{-1}$. It is to be noted that their computational method was validated for the carbon cation, with theoretical result different from experimental one by only $7 \mathrm{~cm}^{-1}$. Chen [8] predicted $28719.46 \mathrm{~cm}^{-1}$, using Configuration Interaction wave function, and also including relativistic and finite nuclear mass corrections. Nakatsuji and coworkers [9] employed the free-complement chemical-formula-theory (FC-CFT) method. The value of $28826 \mathrm{~cm}^{-1}$ is obtained, with their nonrelativistic, fixed-nucleus energies, and assuming that the respective corrections would contribute c.a. $50 \mathrm{~cm}^{-1}$, similarly as in the calculations by Chen and Froese Fischer. The largest discrepancy between theoretical and experimental excitation energy exceeds $300 \mathrm{~cm}^{-1}$. Computational results are however rather scattered and a decisive calculation requires a wave function that provides sufficiently accurate absolute electronic energies. Apart of the ${ }^{2} P_{u}$ and ${ }^{4} P_{g}$ states, the lowest ${ }^{2} D_{g}$ state is also the subject of the present study, because the transition energies to the latter, from the ground state, are known and may serve for estimation of uncertainty of final results. Experiment-based energy difference between ${ }^{2} P_{u}$ and ${ }^{2} D_{g}$ terms amounts to $47846.74 \mathrm{~cm}^{-1}$ [3].

In theoretical studies of the boron atom, not necessarily aimed at the ${ }^{2} P_{u} \rightarrow{ }^{4} P_{g}$ excitation, most efforts to date were devoted to the ground state [10-12]. Preliminary Hylleraas-CI calculations were reported by Ruiz [13]. Highly accurate, nonrelativistic energies were obtained with the explicitly correlated $\mathrm{r}_{12}$-MR-CI method [14], and in the diffusion Monte Carlo (DMC) simulations [15]. There is a masterpiece of CI calculations, by Almora-Diaz and Bunge [16], with the orbital basis containing functions corresponding to the $l$ quantum number reaching 20 ( $z$-type orbitals), yielding the energy only $31 \mu$ hartree above the variational limit. Well-hit extrapolation to complete basis set missed this limit by $6 \mu$ hartree. Best results were obtained with explicitly correlated Gaussian functions (ECG) [17, 18]. The estimated error of nonrelativistic energy of this state was smaller than $1 \mu$ hartree. Similar accuracy was achieved for the ${ }^{2} S_{g}$ states, and the transition energies between the ground state and $S$-symmetry states were reproduced within a fraction of $\mathrm{cm}^{-1}$, with finite nuclear mass, relativistic (including fine and hyperfine structure for the ground
state term) and leading radiative corrections taken into account.
The wavefunctions and energies of comparable accuracy are missing for the ${ }^{4} P_{g}$ and ${ }^{2} D_{g}$ states, and the results are scarce in the literature [5-7, 9]. The present paper is aimed at filling in this hole, and contributing to final resolution of the discrepancies concerning the energy differences between the spin doublet and quartet terms of the boron atom.

Nonrelativistic wavefunctions, expressed as linear combinations of symmetry-adapted, explicitly correlated Gaussian functions, and variational energies with scalar relativistic corrections are obtained for the lowest ${ }^{2} P_{u},{ }^{4} P_{g}$, and ${ }^{2} D_{g}$ states. Atomic units are used unless stated otherwise. Conversion factor to the energy unit used commonly in spectroscopy amounts to 1 hartree $=219474.63 \mathrm{~cm}^{-1}$

## II. METHOD

The stationary Schrödinger equation for $n$-electron atom is solved with the nonrelativistic Hamiltonian

$$
\begin{equation*}
\hat{H}=-\frac{\nabla_{n u c}^{2}}{2 m_{n u c}}+\sum_{i=1}^{n}\left(-\frac{\nabla_{i}^{2}}{2}-\frac{Z}{r_{i}}\right)+\sum_{i>j=1}^{n} \frac{1}{r_{i j}} \tag{1}
\end{equation*}
$$

where $i$ and $j$ count the electrons. Details of the method have been introduced in earlier papers devoted to the lithium and carbon atoms [19, 20], and various states of many-electron harmonium [21-24]. The wavefunction

$$
\begin{equation*}
\Psi\left(\mathbf{r}_{1}, s_{1}, \ldots, \mathbf{r}_{n}, s_{n}\right)=\sum_{I=1}^{K} C_{I} \hat{A} \Theta_{I}\left(s_{1}, \ldots, s_{n}\right) \hat{P} \chi_{I}\left(\mathbf{r}_{1}, \ldots, \mathbf{r}_{n}\right) \tag{2}
\end{equation*}
$$

is expressed as linear combination of explicitly correlated Gaussian primitives (lobes)

$$
\begin{equation*}
\chi_{I}\left(\mathbf{r}_{1}, \ldots, \mathbf{r}_{n}\right)=\exp \left[-\sum_{i=1}^{n} a_{I, i}\left(\mathbf{r}_{i}-\mathbf{R}_{I, i}\right)^{2}-\sum_{i>j=1}^{n} b_{I, i j} r_{i j}^{2}\right], \tag{3}
\end{equation*}
$$

symmetrized by the spatial symmetry projector $\hat{P}$, proper for chosen one-dimensional, irreducible representation of selected finite point group. This wavefunction is not an eigenfunction of the square of angular momentum operator ( $\hat{L^{2}}$ ), for non-zero $\mathbf{R}_{I, i}$ vectors. The deviation from exact $L(L+1)$ eigenvalue is effectively diminished by the procedure of variational energy minimization, in which the parameters (linear $C_{I}$ and nonlinear $a_{I, i}, b_{I, i j}$, and $\mathbf{R}_{I, i}$ ) are established. Action of $\hat{P}$ upon $\chi_{I}$ annihilates from the wavefunction, a finite subset of unwanted components, whose symmetry properties are specific to some other representations of the $K_{h}$ point group, and ensures
convergence towards desired state. $\Theta_{I}\left(s_{1}, \ldots, s_{n}\right)$ is the spin function, common for all basis functions for given state, which is sufficient, because the spatial functions are nonorthogonal. Namely,

$$
\begin{equation*}
\Theta_{I}\left(s_{1}, \ldots, s_{5}\right)=[\alpha(1) \beta(2)-\beta(1) \alpha(2)][\alpha(3) \beta(4)-\beta(3) \alpha(4)] \alpha(5) \tag{4}
\end{equation*}
$$

is used for both doublets, and

$$
\begin{equation*}
\Theta_{I}\left(s_{1}, \ldots, s_{5}\right)=[\alpha(1) \beta(2)-\beta(1) \alpha(2)] \alpha(3) \alpha(4) \alpha(5) \tag{5}
\end{equation*}
$$

for the quartet. $\hat{A}$ is the antisymmetrizer, which ensures proper permutational symmetry of the wavefunction.

The relativistic energy of a resting system may be written as the power series of the fine structure constant $\alpha=\frac{1}{4 \pi \varepsilon_{0}} \frac{e^{2}}{\hbar c}$. Omitting the rest mass contribution,

$$
\begin{equation*}
E_{r e l}=E_{n r}+E^{(2)}+E^{(3)}+\cdots \tag{6}
\end{equation*}
$$

where $E_{n r}$ is the nonrelativistic energy, $E^{(2)}$ contains the Breit-Pauli relativistic corrections and higher order terms are known as the radiative (QED) corrections. All these corrections may be calculated in perturbative manner, as expectation values of respective operators, with known nonrelativistic wavefunction. The Breit-Pauli Hamiltonian may be split to the relativistic shift $\hat{H}_{R S}$ operator, with expectation value $E_{R S}$, and the fine and hyperfine structure operators, which contain spin-orbit and spin-spin coupling terms. Only the former is considered in this work. It is convenient to write it down as the sum of following terms:

$$
\begin{equation*}
\hat{H}_{R S}=\hat{H}_{1}+\hat{H}_{1 n}+\hat{H}_{2}+\hat{H}_{3}+\hat{H}_{4}+\hat{H}_{4 n} . \tag{7}
\end{equation*}
$$

These operators describe respectively the electronic mass-velocity correction

$$
\begin{equation*}
\hat{H}_{1}=-\frac{1}{8 c^{2}} \sum_{i=1}^{n} \nabla_{i}^{4} \tag{8}
\end{equation*}
$$

the electron-nucleus Darwin term

$$
\begin{equation*}
\hat{H}_{2}=\frac{Z \pi}{2 c^{2}} \sum_{i=1}^{n} \delta\left(\mathbf{r}_{i}\right) \tag{9}
\end{equation*}
$$

the sum of the electron-electron Darwin term and spin-spin Fermi contact interaction (both have the same mathematical form, after integration over spin variables [25])

$$
\begin{equation*}
\hat{H}_{3}=\frac{\pi}{c^{2}} \sum_{i>j=1}^{n} \delta\left(\mathbf{r}_{i j}\right) \tag{10}
\end{equation*}
$$

and the electron orbit-orbit term

$$
\begin{equation*}
\hat{H}_{4}=\frac{1}{2 c^{2}} \sum_{i>j=1}^{n}\left(\frac{\nabla_{i} \cdot \nabla_{j}}{r_{i j}}+\frac{\mathbf{r}_{i j} \cdot\left[\left(\mathbf{r}_{i j} \cdot \nabla_{i}\right) \nabla_{j}\right]}{r_{i j}^{3}}\right) \tag{11}
\end{equation*}
$$

which describes the interaction of magnetic dipoles arising from orbital motion of the electrons. There are two terms in equation 7, that have non-zero value only for finite nuclear mass, namely the nuclear mass-velocity correction

$$
\begin{equation*}
\hat{H}_{1 n}=-\frac{1}{8 m_{n u c}^{3} c^{2}} \nabla_{n u c}^{4} \tag{12}
\end{equation*}
$$

and the nucleus-electron contribution to orbit-orbit magnetic interaction energy

$$
\begin{equation*}
\hat{H}_{4 n}=-\frac{Z}{2 m_{n u c} c^{2}} \sum_{i=1}^{n}\left(\frac{\nabla_{i} \cdot \nabla_{n u c}}{r_{i j}}+\frac{\mathbf{r}_{i} \cdot\left[\left(\mathbf{r}_{i} \cdot \nabla_{i}\right) \nabla_{n u c}\right]}{r_{i}^{3}}\right) . \tag{13}
\end{equation*}
$$

Distinction of the cases of fixed and non-fixed nucleus requires only the modification of the nuclear mass in all Hamiltonians, from infinity to the one proper for given isotope of boron. The wavefunction given by Eqs. 2 and 3 is expressed in relative coordinates - $\mathbf{r}_{i}$ denotes the position of $\mathrm{i}^{\text {th }}$ electron relatively to the nucleus. Therefore explicit transformation of the operators, both nonrelativistic and relativistic, from laboratory to center-of-mass coordinate frame, is not necessary. Only relative coordinates appear in these operators explicitly. Each differrentiation over a coordinate in Cartesian laboratory frame, may be written as properly weighted sum of differentiations over respective relative and center-of-mass coordinates. Differentiation of a function, which is dependent on relative coordinates only, over a center-of-mass coordinate, gives zero, so the final result is the same with non-transformed operators as with explicit elimination of the center of mass motion [26].

## III. NUMERICAL RESULTS

In the first step, nonrelativistic wavefunctions are constructed. The ground state wavefunction of the boron atom has $P_{u}$ symmetry. Assuming the magnetic quantum number equal to 0 , this symmetry is effectively represented by the $A_{u}$ representation of the $C_{i}$ point group, with the projector

$$
\begin{equation*}
\hat{P}=\hat{E}-\hat{i} \tag{14}
\end{equation*}
$$

and all $\mathbf{R}_{I, i}$ vectors placed at the $z$-axis of the coordinate frame. The $C_{4 v}$ point group is employed for both excited states, with $\mathbf{R}_{I, i}$ vectors confined to the $x y$ plane. The projector proper for the $A_{2}$

TABLE I: Nonrelativistic energies, deviations of $<L^{2}>$ from $L(L-1)$, and extrapolated energies, for fixed nucleus. For extrapolated ( $E_{\text {extr }}$ ) results, standard deviations of the least significant digits are given in parentheses

| K | $E_{n r}$ | $\left\langle L^{2}\right\rangle-L(L+1)$ | K | $E_{n r}$ | $\left\langle L^{2}\right\rangle-L(L+1)$ |
| ---: | :--- | :--- | ---: | :---: | :---: |
| ${ }^{2} P_{u}(\mathrm{~L}=1)$ |  |  |  |  |  |
| 277 | -24.653001970 | $7.81 \cdot 10^{-6}$ | 2745 | -24.653862346 | $1.73 \cdot 10^{-7}$ |
| 406 | -24.653462344 | $5.76 \cdot 10^{-6}$ | 4022 | -24.653865404 | $8.82 \cdot 10^{-8}$ |
| 595 | -24.653681184 | $3.48 \cdot 10^{-6}$ | 5679 | -24.653867017 | $4.97 \cdot 10^{-8}$ |
| 872 | -24.653785377 | $1.83 \cdot 10^{-6}$ | 7456 | -24.653867660 | $3.54 \cdot 10^{-8}$ |
| 1278 | -24.653833991 | $7.33 \cdot 10^{-7}$ | 10304 | -24.653868064 | $2.12 \cdot 10^{-8}$ |
| 1873 | -24.653854171 | $3.30 \cdot 10^{-7}$ | $E_{\text {extr }}$ | $-24.65386890(14)$ | 0 |
| ${ }^{4} P_{g}(\mathrm{~L}=1)$ |  |  |  |  |  |
| 277 | -24.521826756 | $1.10 \cdot 10^{-5}$ | 1873 | -24.522039020 | $2.26 \cdot 10^{-7}$ |
| 406 | -24.521944458 | $7.17 \cdot 10^{-6}$ | 2733 | -24.522040459 | $1.11 \cdot 10^{-7}$ |
| 595 | -24.521999781 | $3.07 \cdot 10^{-6}$ | 3580 | -24.522041147 | $5.47 \cdot 10^{-8}$ |
| 872 | -24.522023448 | $1.47 \cdot 10^{-6}$ | 4672 | -24.522041430 | $3.49 \cdot 10^{-8}$ |
| 1278 | -24.522035395 | $4.74 \cdot 10^{-7}$ | $E_{\text {extr }}$ | $-24.52204180(5)$ | 0 |
| ${ }^{2} D_{g}(\mathrm{~L}=2)$ |  |  |  |  |  |
| 277 | -24.434439490 | $1.86 \cdot 10^{-4}$ | 2745 | -24.435961389 | $5.34 \cdot 10^{-6}$ |
| 406 | -24.435110865 | $1.43 \cdot 10^{-4}$ | 4023 | -24.435972976 | $2.69 \cdot 10^{-6}$ |
| 595 | -24.435568403 | $7.98 \cdot 10^{-5}$ | 5858 | -24.435978480 | $1.30 \cdot 10^{-6}$ |
| 872 | -24.435789658 | $4.04 \cdot 10^{-5}$ | 8231 | -24.435981009 | $5.35 \cdot 10^{-7}$ |
| 1278 | -24.435896508 | $1.90 \cdot 10^{-5}$ |  |  | 0 |
| 1873 | -24.435941219 | $1.01 \cdot 10^{-5}$ | $E_{\text {extr }}$ | $-24.43598347(63)$ | 0 |

representation,

$$
\begin{equation*}
\hat{P}=\hat{E}+\hat{C}_{4}^{1}+\hat{C}_{2}+\hat{C}_{4}^{3}-\hat{\sigma}_{v 1}-\hat{\sigma}_{v 2}-\hat{\sigma}_{d 1}-\hat{\sigma}_{d 2} \tag{15}
\end{equation*}
$$

produces effectively the $P_{g}$ symmetry of the quartet state, and the $B_{1}$ representation, with

$$
\begin{equation*}
\hat{P}=\hat{E}-\hat{C}_{4}^{1}+\hat{C}_{2}-\hat{C}_{4}^{3}+\hat{\sigma}_{v 1}+\hat{\sigma}_{v 2}-\hat{\sigma}_{d 1}-\hat{\sigma}_{d 2} \tag{16}
\end{equation*}
$$

is adequate for the $D_{g}$ state, producing the wavefunction converging to the normalized sum of eigenfunctions of $\hat{L}_{z}$, pertaining to $m_{L}=2$ and $m_{L}=-2$.

The accuracy of nonrelativistic energies is assessed, exploiting the convergence of $\left\langle\hat{L}^{2}\right\rangle$, whose known exact limits amount to $L(L+1)$. Basis sets were extended stepwise, beginning with 1,2 and 3 ECGs and then appending functions optimized two steps back in the process, to the current set. Optimization of all variational parameters of the new basis followed, aimed at energy minimization. Successive basis sizes formed thus initially the Narayana's cows sequence [27]. For

FIG. 1: Energy extrapolation using deviation of $\left\langle L^{2}\right\rangle$ from $L(L+1)$, for the ${ }^{2} P_{u}$ state

large bases, functions appeared that contributed too little to the energy, and these functions were removed from the set. The threshold value was set to $1,0.5$ or 0.2 nanohartree, dependent on the estimated distance to the variational limit. The values of nonrelativistic energies and $\left\langle L^{2}\right\rangle$, calculated for infinite-mass nucleus, with $K$ basis functions, are collected in table It is noticed that the energy depends smoothly on the error of the square of angular momentum, $\left\langle L^{2}\right\rangle-L(L+1)-$ similarly as for the carbon atom [20]. This observation, which has no theoretical background and may be related to the method of construction of consecutive basis sets, gives rise to an assumption that the rotational energy error becomes nearly constant fraction of the total energy error. Either linear (for the ground state, Fig. (1) or quadratic (for both excited states, Figs. 2] and 3) functions are fitted to five best points, giving estimations of complete basis set limits of the electronic energies. Variational energies look converged to a fraction of $\mu$ hartree for ${ }^{2} P_{u}$ and ${ }^{4} P_{g}$ states, while the accuracy for the ${ }^{2} D_{g}$ state is a little worse, with the distance to the estimated limit still amounting to c.a. $2.5 \mu$ hartree. The wavefunction of this state has apparently more complicated character, but calculation with a significantly larger basis set was not feasible.

Comparison with literature data, in table [II, reveals that the variational energy of the ground state, obtained in the present work with 7456 basis functions, is lower than the best previous result [18] by $0.5 \mu$ hartree, and with 10304 basis functions surpasses also the old estimate of the complete basis set limit. The $\left\langle L^{2}\right\rangle$-based extrapolation lowers this limit by $0.85 \mu$ hartree. There

FIG. 2: Energy extrapolation using deviation of $\left\langle L^{2}\right\rangle$ from $L(L+1)$, for the ${ }^{4} P_{g}$ state


FIG. 3: Energy extrapolation using deviation of $\left\langle L^{2}\right\rangle$ from $L(L+1)$, for the ${ }^{2} D_{g}$ state

are no published energies of comparable accuracies, for both excited states. The calculation by Nakatsuji [9] yielded the energy of the ground state, higher by 0.135 mhartree than the present result. On the contrary, the energy of the ${ }^{4} P_{g}$ state was too low, overstepping the variational limit by 0.58 mhartree. The MCHF energies by Froese Fischer [7] look more balanced, being higher by $0.345\left({ }^{2} P_{u}\right.$ state) and $0.219\left({ }^{4} P_{g}\right.$ state) mhartree. Most accurate nonrelativistic energy of the ${ }^{2} D_{g}$

TABLE II: Comparison of nonrelativistic energies with published results

| method | ${ }^{2} P_{u}$ | ${ }^{4} P_{g}$ | ${ }^{2} D_{g}$ |
| :---: | :---: | :---: | :---: |
| MCHF $\left(l_{\text {max }}=7\right)[5]$ | -24.651009 |  | -24.431353 |
| VMC [6] | -24.64502(6) | -24.51581(6) | -24.42486(5) |
| CI $\left(l_{\text {max }}=6\right.$, selected) [8] | -24.652032 | -24.521401 | -24.433575 |
| MCHF $\left(l_{\text {max }}=5\right) ~[7]$ | -24.653523595 | -24.521822334 |  |
| FC-CFT [9] | -24.653734(103) | $-24.522622(50)$ |  |
| $\mathrm{r}_{12}$-MR-CI [14] | -24.653787 |  |  |
| DMC [15] | -24.65379(3) |  |  |
| CI $\left(l_{\text {max }}=20\right)$ [16] | -24.65383733 |  |  |
| CI, extrapolated [16] | -24.653862(2) |  |  |
| ECG, K=5100 [17] | -24.65386608 |  |  |
| ECG, K=8192 [18] | -24.653867537 |  |  |
| ECG, extrapolated [18] | -24.65386805(45) |  |  |
| ECG lobes (present work) | -24.653868064 | -24.522041430 | -24.435981009 |
| $E_{\text {extr }}$ (present work) | -24.65386890(14) | -24.52204180(5) | -24.43598347 (63) |

state, published to date [8], is by more than 2 mhartree higher than the present one.
Concerning the components of relativistic corrections (table III), the convergence of the massvelocity and electron-nucleus Darwin terms is still unsatisfactory for all states, with differences of few $\mu$ hartree, between two most accurate wavefunctions. This inaccuracy is due to $\nabla^{4}$ and $\delta(\mathbf{r})$ operators, whose expectation values converge very slowly in the basis of Gaussian functions, which do not represent properly the wavefunctions at coalescence points (cusps). Fortunately, the errors of $\left\langle\hat{H}_{1}\right\rangle$ and $\left\langle\hat{H}_{2}\right\rangle$ have opposite signs and cancel to a significant extent. The number of stable significant digits of $\left\langle\hat{H}_{3}\right\rangle$ is even smaller than that of $\left\langle\hat{H}_{2}\right\rangle$, but the absolute value is smaller by two orders of magnitude. On the other hand, the orbit-orbit magnetic interaction energies look accurate within one nanohartree. Total relativistic corrections (last column of table III), calculated with two largest basis sets, differ by less than $0.1 \mu$ hartree for all states, although there is no way to extrapolate these results and estimate the error margin more rigorously. For the ground state, the results by Puchalski [18] are available, obtained with the method that involves regularization of the $\nabla^{4}$ and $\delta(\mathbf{r})$ operators, which leads to much better convergence, and yields the scalar relativistic correction amounting to -7.515977 mhartree. This means that the error of best present calculation amounts to $0.141 \mu$ hartree.

In order to compare the computed excitation energies with experimental data, nuclear mass proper for particular isotope has to be taken into account. The most abundant isotopes of boron

TABLE III: Scalar relativistic corrections (in mhartree), for fixed nucleus

| K | $\left\langle\hat{H}_{1}\right\rangle$ | $\left\langle\hat{H}_{2}\right\rangle$ | $\left\langle\hat{H}_{3}\right\rangle$ | $\left\langle\hat{H}_{4}\right\rangle$ | $E_{R S}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| ${ }^{2} P_{u}$ |  |  |  |  |  |
| 277 | -36.599999 | 29.743913 | -0.601678 | -0.057897 | -7.515662 |
| 406 | -36.728411 | 29.867256 | -0.597815 | $-0.057872$ | $-7.516842$ |
|  | -36.806576 | 29.938863 | -0.595500 | $-0.057843$ | $-7.521055$ |
|  | -36.834510 | 29.969803 | $-0.594004$ | $-0.057833$ | $-7.516544$ |
| 1278 | -36.863671 | 29.998464 | $-0.593480$ | $-0.057827$ | $-7.516515$ |
| 1873 | -36.873790 | 30.008434 | -0.592950 | $-0.057823$ | -7.516129 |
| 2745 | -36.893141 | 30.027586 | $-0.592638$ | $-0.057822$ | $-7.516015$ |
|  | $-36.897900$ | 30.032285 | -0.592543 | $-0.057821$ | -7.515980 |
| 679 | -36.904331 | 30.038645 | $-0.592389$ | $-0.057820$ | $-7.515896$ |
| 7456 | $-36.910731$ | 30.044909 | $-0.592249$ | $-0.057820$ | 1 |
| 103 | -36.913837 | 30 | -0 | -0. |  |
| ${ }^{4} P_{g}$ |  |  |  |  |  |
| 27 | -36.062593 | 29.388974 | -0.578999 | 2 |  |
| 406 | -36.062306 | 29.387961 | -0.576977 | $-0.027940$ | -7.279263 |
|  | -36.120148 | 29.442882 | $-0.576059$ | $-0.027944$ | -7.281270 |
|  | -36.135572 | 29.460497 | $-0.575635$ | $-0.027944$ | -7.278654 |
| 1278 | -36.155086 | 29.479579 | -0.575179 | $-0.027944$ | -7.278630 |
| 1873 | -36.166969 | 29.491458 | -0.574979 | -0.027944 | $-7.278433$ |
| 2733 | -36.172481 | 29.496815 | -0.574859 | -0.027943 | $-7.278469$ |
|  | -36.179044 | 29. | -0.574758 | -0.027943 | -7.278371 |
| 4672 | -36.181062 | 29.505364 | $-0.574684$ | -0.027943 | 5 |
| ${ }^{2} D_{g}$ |  |  |  |  |  |
|  | 7 | 29.276227 | -0.585496 | - | 3 |
| 40 | $-36.038120$ | 29.340083 | $-0.583894$ | $-0.043104$ | -7.325035 |
| 59 | -36.141397 | 29.439580 | $-0.582434$ | $-0.043201$ | $-7.327453$ |
| 872 | -36.182799 | 29.478829 | $-0.581222$ | $-0.043269$ | $-7.328461$ |
| 1278 | -36.222970 | 29.517760 | $-0.580282$ | $-0.043298$ | $-7.328790$ |
| 1873 | -36.263293 | 29.557494 | $-0.579795$ | $-0.043309$ | -7.328902 |
| 2745 | -36.281367 | 29.575076 | -0.579327 | $-0.043314$ | $-7.328933$ |
| 4023 | -36.290237 | 29.583798 | $-0.579005$ | $-0.043317$ | $-7.328761$ |
| 5858 | -36.308459 | 29.601703 | $-0.578769$ | $-0.043319$ | $-7.328843$ |
| 8231 | -36.313784 | 29.606950 | $-0.578644$ | -0.043319 | -7.328797 |

are ${ }^{11} B$ and ${ }^{10} B$, whose nuclear masses amount to 20063.7375 a.u. and 18247.4689 a.u., respectively. The same basis sets are used in the calculations, as for fixed nucleus - only the linear parameters are allowed to vary. Table IV lists the nonrelativistic energies and all components of scalar relativistic corrections, for the largest basis, for each state. Extrapolations to complete ba-

TABLE IV: Variationally bound, and extrapolated nonrelativistic energies (in hartree), and scalar relativistic corrections (in mhartree) for ${ }^{11} B$ and ${ }^{10} B$ isotopes of boron

|  | ${ }^{2} P_{u}\left({ }^{11} B\right)$ | ${ }^{2} P_{u}\left({ }^{10} B\right)$ | ${ }^{4} P_{g}\left({ }^{11} B\right)$ | ${ }^{4} P_{g}\left({ }^{10} B\right)$ | ${ }^{2} D_{g}\left({ }^{11} B\right)$ | ${ }^{2} D_{g}\left({ }^{10} B\right)$ |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
| $E_{n r}$ | -24.652625854 | -24.652502219 | -24.520826909 | -24.520706030 | -24.434765075 | -24.434644055 |
| $E_{\text {extr }}$ | -24.65262669 | -24.65250305 | -24.52082728 | -24.52070640 | -24.43476754 | -24.43464652 |
| $\left\langle\hat{H}_{\rangle}\right\rangle$ | -36.906350 | -36.905605 | -36.173742 | -36.173014 | -36.306284 | -36.300219 |
| $\left\langle\hat{H}_{1 n}\right\rangle$ | $-6.5 \cdot 10^{-12}$ | $-8.6 \cdot 10^{-12}$ | $-6.3 \cdot 10^{-12}$ | $-8.4 \cdot 10^{-12}$ | $-6.3 \cdot 10^{-12}$ | $-8.4 \cdot 10^{-12}$ |
| $\left\langle\hat{H}_{2}\right\rangle$ | 30.043429 | 30.042975 | 29.500886 | 29.500440 | 29.602345 | 29.596644 |
| $\left\langle\hat{H}_{3}\right\rangle$ | -0.592094 | -0.592086 | -0.574607 | -0.574600 | -0.578563 | -0.578680 |
| $\left\langle\hat{H}_{4}\right\rangle$ | -0.057750 | -0.057743 | -0.027879 | -0.027873 | -0.043248 | -0.043241 |
| $\left\langle\hat{H}_{4 n}\right\rangle$ | -0.003040 | -0.003342 | -0.002961 | -0.003255 | -0.002977 | -0.003273 |
| $E_{R S}$ | -7.515805 | -7.515802 | -7.278303 | -7.278301 | -7.328727 | -7.328720 |

sis sets are carried out with the same corrections as for fixed nucleus. Concerning the terms not appearing for fixed nucleus, $\left\langle\hat{H}_{1 n}\right\rangle$ is damped effectively by third power of the nuclear mass in the denominator, and amounts to few femtohartree only, which is negligible at the accuracy level achieved in present calculations. On the other hand, $\left\langle\hat{H}_{4 n}\right\rangle$ amount to few $\mu$ hartree. Other components' values however change in such extent that total scalar relativistic corrections differ from those obtained for fixed nucleus by few nanohartree only.

The wavenumbers proper for excitations from the ground state to the lowest ${ }^{4} P_{g}$ and ${ }^{2} D_{g}$ states, calculated for ${ }^{11} B$, and not accountig for the fine structure, amount to $28978.75 \mathrm{~cm}^{-1}$ and $47855.62 \mathrm{~cm}^{-1}$, respectively. The latter differs from the experiment-based one by $9 \mathrm{~cm}^{-1}$, which is comparable with the energy difference between the ${ }^{2} P_{1 / 2}$ and ${ }^{2} P_{3 / 2}$ states (fine structure, $15 \mathrm{~cm}^{-1}$ ) [3]. Similar accuracy is expected for the excitation energy to the ${ }^{4} P_{g}$ state.

The isotopic shifts may be easily calculated from present results. The differences of term energies, between ${ }^{11} B$ and ${ }^{10} B$, computed with the same basis, remain very stable as the basis size is increased - similarly as for the carbon atom [20]. They are given in table V] with larger number of significant digits than total energy, for two largest basis sets. Isotopic shift of $-0.57316 \mathrm{~cm}^{-1}$ is obtained for the ${ }^{2} P_{u} \rightarrow{ }^{2} D_{g}$ excitiation, while the measured value, averaged over two spectral lines, is equal to $-0.569(3) \mathrm{cm}^{-1}$ [28]. $-0.60502 \mathrm{~cm}^{-1}$ is predicted for the ${ }^{2} P_{u} \rightarrow{ }^{4} P_{g}$ transitions.

TABLE V: Isotopic shifts for term energies (components in hartree, total in $\mathrm{cm}^{-1}$ )

| K |  |  |  |
| :--- | :---: | :---: | :---: |
| $E_{n r}\left({ }^{10} B\right)-E_{n r}\left({ }^{11} B \mid E_{R S}\left({ }^{10} B\right)-E_{R S}\left({ }^{11} B\right)\right.$ |  |  |  |
| ${ }^{2} P_{u}$ | $E_{\text {rel }}\left({ }^{10} B\right)-E_{\text {rel }}\left({ }^{11} B\right)$ |  |  |
| 7456 | 0.0001236349 | $3.0 \cdot 10^{-9}$ | 27.13538 |
| 10304 | 0.0001236348 | $3.1 \cdot 10^{-9}$ | 27.13538 |
| ${ }^{4} P_{g}$ |  |  |  |
| 3580 | 0.0001208790 | $2.2 \cdot 10^{-9}$ | 26.53036 |
| 4672 | 0.0001208790 | $2.2 \cdot 10^{-9}$ | 26.53036 |
| ${ }^{2} D_{g}$ |  |  |  |
| 5858 | 0.0001210195 | $6.8 \cdot 10^{-9}$ | 26.56220 |
| 8231 | 0.0001210195 | $6.9 \cdot 10^{-9}$ | 26.56222 |

## IV. CONCLUSIONS

The present work provides most accurate to date, nonrelativistic energies of the lowest ${ }^{2} P_{u},{ }^{4} P_{g}$ abd ${ }^{2} D_{g}$ states of the boron atom. With scalar relativistic corrections and finite nuclear mass taken into account, term energies are obtained, whose main source of remaining error is the missing fine structure. The measured fine splitting amounts to c.a. $15 \mathrm{~cm}^{-1}$ for the ${ }^{2} P_{u}$ term, c.a. $11 \mathrm{~cm}^{-1}$ for the ${ }^{4} P_{g}$ term, and less than $1 \mathrm{~cm}^{-1}$ for the ${ }^{2} D_{g}$ term [3]. The computed ${ }^{2} P_{u} \rightarrow{ }^{2} D_{g}$ excitation energy confirms the experiment-based result within c.a. $11 \mathrm{~cm}^{-1}$, and comparable accuracy is expected for the ${ }^{2} P_{u} \rightarrow{ }^{4} P_{g}$ excitation. This reveals gross inaccuracy of the latter excitation energy, based on experimental data for heavier, isoelectronic ions. This inaccuracy exceeds $300 \mathrm{~cm}^{-1}$, therefore an update of the content of Atomic Spectra Database [3] would be recommended, concerning the energies of the quartet states of boron atom. It is worth noting that the predictions of the MCHF study [7] were accurate within $20 \mathrm{~cm}^{-1}$. Further calculations that would include splitting of energy levels due to magnetic spin-orbit and spin-spin couplings are desired.

On technical side of the work, it is proven again that the symmetrized, explicitly correlated Gaussian lobe functions form an efficient basis for atomic states, in spite of not being eigenfunctions of the $\hat{L}^{2}$ operator. Lower variational energies are obtained at shorter expansions, than with basis functions having exact symmetry properties.

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