ON THE ENUMERATION OF HOPPY'S WALKS

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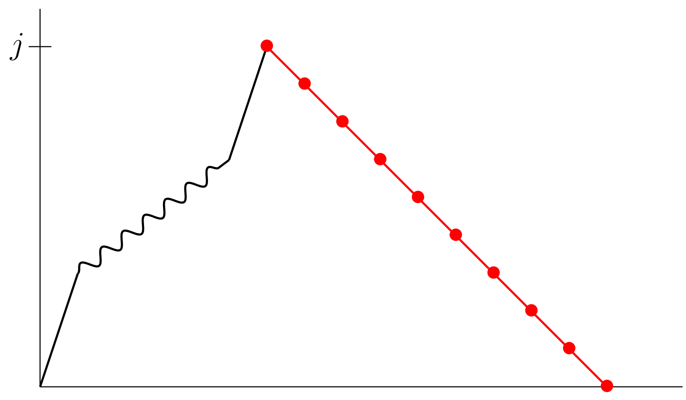
1. Hoppy walks

Deng and Mansour [1] introduce a rabbit named Hoppy and let him move according to certain rules. At that stage, we don't need to know the rules. Eventually, the enumeration problem is one about k-Dyck paths. The up-steps are $(1, k)$ and the down-steps are $(1, -1)$.



The question is about the length of the sequence of down-steps printed in red. Or, phrased differently, how many k-Dyck paths end on level j, after m up-steps, the last step being an up-step. The recent paper [6] contains similar computations, although without the restriction that the last step must be an up-step.

Counting the number of up-steps is enough, since in total, there are $m + km = (k + 1)m$ steps. The original description of Deng and Mansour is a reflection of this picture, with up-steps of size 1 and down-steps of sice $-k$, but we prefer it as given here, since we are going to use the adding-a-new-slice method, see [2, 5]. A slice is here a run of down-steps, followed by an up-step. The first up-step is treated separately, and then $m - 1$ new slices are added. We keep track of the level after each slice, using a variable u. The variable z is used to count the number of up-steps.

Deng and Mansour work out a formula which comprises $O(m)$ terms. Our method leads only to a sum of $O(j)$ terms.

The following substitution is essential for adding a new slice:

$$u^j \longrightarrow z \sum_{0\le h\le j} u^{h+k} = \frac{zu^k}{1-u}(1-u^{j+1}).$$

Now let $F_m(z,u)$ we the generating function according to m runs of down-steps. The substitution leads to

$$F_{m+1}(z,u) = \frac{zu^k}{1-u}F_m(z,1) - \frac{zu^{k+1}}{1-u}F_m(z,u), \quad F_0(z,u) = zu^k.$$

Let $F = \sum_{m\ge0} F_m$, then

$$F(z,u) = zu^k + \frac{zu^k}{1-u}F(z,1) - \frac{zu^{k+1}}{1-u}F(z,u),$$

or

$$F(z,u)\frac{1-u+zu^{k+1}}{1-u} = zu^k + \frac{zu^k}{1-u}F(z,1).$$

The equation $1-u+zu^{k+1}=0$ is famous when enumerating $(k+1)$-ary trees. Its relevant combinatorial solution (also the only one being analytic at the origin) is

$$\overline{u} = \sum_{\ell\ge0} \frac{1}{1+\ell(k+1)}\binom{1+\ell(k+1)}{\ell} z^\ell.$$

Since $u-\overline{u}$ is a factor of the LHS, is must also be a factor of the RHS, and we can compute (by dividing out the factor $(u-\overline{u})$) that

$$\frac{zu^k(1-u+F(z,1))}{u-\overline{u}} = -zu^k.$$

Thus

$$F(z,u) = zu^k\frac{\overline{u}-u}{1-u+zu^{k+1}}.$$

The first factor has even a combinatorial interpretation, as a description of the first step of the path. It is also clear from this that the level reached is $\ge k$ after each slice. We don't care about the factor zu^k anymore, as it produces only a simple shift. The main interest is now how to get to the coefficients of

$$\frac{\overline{u}-u}{1-u+zu^{k+1}}$$

in an efficient way. There is also the formula

$$1-u+zu^{k+1} = (\overline{u}-u)\Big(1 - z\frac{u^{k+1}-\overline{u}^{k+1}}{u-\overline{u}}\Big),$$

but it does not seem to be useful here.

First we deal with the denominators

$$S_j := [u^j]\frac{1}{1-u+zu^{k+1}} = \sum_{0\le m\le j/k}(-1)^m\binom{j-km}{m}z^m.$$

One way to see this formula is to prove by induction that the sums S_j satisfy the recursion

$$S_j - S_{j-1} + zS_{j-k-1} = 0$$

and initial conditions $S_0 = \cdots = S_k = 1$. In [6] such expressions also appear as determinants. Summarizing,

$$\frac{1}{1-u+zu^{k+1}} = \sum_{m\ge 0}(-1)^m z^m \sum_{j\ge km}\binom{j-km}{m}u^j.$$

Now we read off coefficients:

$$[u^j]\frac{\overline{u}}{1-u+zu^{k+1}} \\ = \sum_{0\le m\le j/k}(-1)^m\binom{j-km}{m}z^m\sum_{\ell\ge 0}\frac{1}{1+\ell(k+1)}\binom{1+\ell(k+1)}{\ell}z^\ell$$

and further

$$[z^n][u^j]\frac{\overline{u}}{1-u+zu^{k+1}} \\ = \sum_{0\le m\le j/k}(-1)^m\binom{j-km}{m}\frac{1}{1+(n-m)(k+1)}\binom{1+(n-m)(k+1)}{n-m}.$$

The final answer to the Deng-Mansour enumeration (without the shift) is

$$\sum_{0\le m\le j/k}(-1)^m\binom{j-km}{m}\frac{1}{1+(n-m)(k+1)}\binom{1+(n-m)(k+1)}{n-m} \\ -(-1)^n\binom{j-1-kn}{n}.$$

If one wants to take care of the factor zu^k as well, one needs to do the replacements $n\to n+1$ and $j\to j+k$ in the formula just derived. That enumerates then the k-Dyck paths ending at level j after n up-steps, where the last step is an up-step.

2. An application

The encyclopedia of integer sequences [4] has the sequences A334680, A334682, A334719, (with a reference to [3]) which is the total number of down-steps of the last down-run, for $k = 2, 3, 4$. So, if the path ends on level j, the contribution to the total is j.

All we have to do here is to differentiate

$$F(z,u) = zu^k \frac{\overline{u} - u}{1 - u + zu^{k+1}}.$$

w.r.t. u, and then replace u by 1. The result is

$$\frac{\overline{u}}{z} - \overline{u} - \frac{1}{z},$$

and the coefficient of z^m therein is

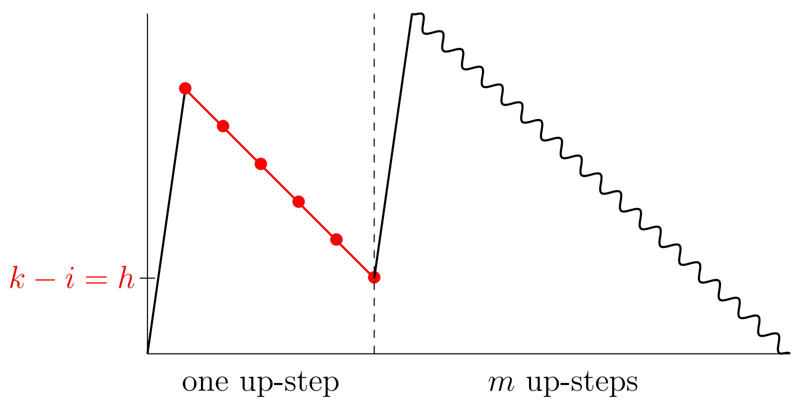
$$\frac{1}{1+(m+1)(k+1)}\binom{1+(m+1)(k+1)}{m+1} - \frac{1}{1+m(k+1)}\binom{1+m(k+1)}{m}.$$

I don't know how this was derived in [3], but it is more fun to figure out things for oneself!

We hope to report about more applications soon.

3. Hoppy's early adventures

Now we investigate what Hoppy does after his first up-step; he might follow with $0, 1, \dots, k$ down-steps. Eventually, we want to sum all these steps (red in the picture).



A new slice is now an up-step, followed by a sequence of down-steps. The substitution of interest is:

$$u^i \to z \sum_{0 \le h \le i+k} u^h = \frac{z}{1-u} - \frac{zu^{i+k+1}}{1-u}.$$

Furthermore

$$F_{h+1}(z,u) = \frac{z}{1-u}F_h(z,1) - \frac{zu^{k+1}}{1-u}F_h(z,u),$$

and $F_0 = u^h$, the starting level.

We have

$$H(z,u) = \sum_{h\ge 0} F_h(z,u) = u^h + \frac{z}{1-u}H(z,1) - \frac{zu^{k+1}}{1-u}H(z,u)$$

or

$$H(z,u)(1-u+zu^{k+1}) = u^h(1-u) + zH(z,1)$$

Plugging in $\overline{u}$ into the RHS gives 0:

$$zH(z,1) = -\overline{u}^h(1-\overline{u}),$$

and

$$H(z,u) = \frac{u^h(1-u) - \overline{u}^h(1-\overline{u})}{1-u+zu^{k+1}}.$$

But we only need $H(z,0)$, since we return to the x-axis at the end:

$$H(z,0) = [h=0] + \overline{u}^{h+1} - \overline{u}^h.$$

The total contribution of red steps is then

$$k + \sum_{h=0}^{k}(k-h)(\overline{u}^{h+1} - \overline{u}^h) = \sum_{h=1}^{k}\overline{u}^h;$$

the coefficient of z^m in this is the total contribution. Since $\overline{u} = 1 + z\overline{u}^{k+1}$, there is the further simplification

$$-1 + \frac{1}{z} + \frac{1}{1-\overline{u}} = \sum_{m\ge 1}\frac{k}{m+1}\binom{(k+1)m}{m}z^m.$$

The proof of this is as follows. Let $m \ge 1$, then

$$\begin{aligned}
[z^m]\Big(-1 + \frac{1}{z} + \frac{1}{1-\overline{u}}\Big) &= -[z^m]\frac{1}{z\overline{u}^{k+1}} \\
&= -[z^{m+1}]\sum_{\ell\ge 0}\frac{-(k+1)}{(k+1)\ell-(k+1)}\binom{(k+1)\ell-(k+1)}{\ell}z^\ell \\
&= [z^{m+1}]\sum_{\ell\ge 0}\frac{(k+1)}{(k+1)(\ell-1)}\binom{(k+1)(\ell-1)}{\ell}z^\ell \\
&= \frac{(k+1)}{(k+1)m}\binom{(k+1)m}{m+1} = \frac{k}{m+1}\binom{(k+1)m}{m}.
\end{aligned}$$

We did not expect such a simple answer $\frac{k}{m+1}\binom{(k+1)m}{m}$ to this question about Hoppy's early adventures!

This analysis of Hoppy's early adventures covers sequences A007226, A007228, A124724 of [4], which references to [3].

References

[1] Eva Deng and Toufik Mansour, Three Hoppy path problems and ternary paths, Discrete Applied Mathematics 156, 2008, 770–779.
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[6] Prodinger, Helmut, On k-Dyck paths with a negative boundary.

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