ORTHOGONAL POLYNOMIALS OF TYPE R_I

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Dedicated to Dick Askey, our great mentor.

Abstract. A combinatorial theory for type R_I orthogonal polynomials is given. The ingredients include weighted generalized Motzkin paths, moments, continued fractions, determinants, and histories. Several explicit examples in the Askey scheme are given.

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1. Introduction

Ismail and Masson [11] defined orthogonal polynomials of type R_I by generalizing the three-term recurrence relation that classical orthogonal polynomials satisfy. In this paper we develop the combinatorial theory of type R_I orthogonal polynomials. This theory parallels the Flajolet–Viennot development [7, 18], for classical orthogonal polynomials, using weighted paths. We also give type R_I versions of several classical polynomials in the Askey scheme.

The orthogonal polynomials $P_n(x)$ of type R_I are defined recursively. Let $\{P_n(x)\}_{n\ge 0}$ be defined by $P_{-1}(x)=0$, $P_0(x)=1$ and for $n\ge 0$,

$$P_{n+1}(x) = (x-b_n)P_n(x) - (a_n x+\lambda_n)P_{n-1}(x). \tag{1.1}$$

This defines $P_n(x)$ as a monic polynomial in x of degree n and also a polynomial in the recurrence coefficients, $\{b_k\}$, $\{a_k\}$, and $\{\lambda_k\}$. Let

$$d_m(x) := \prod_{i=1}^{m}(a_i x+\lambda_i), \quad \textit{and} \quad Q_m(x) := \frac{P_m(x)}{d_m(x)}.$$

The orthogonality relation for $P_n(x)$ is defined by a linear functional $\mathcal{L}$ on a certain vector space of rational functions:

$$\mathcal{L}(P_n(x)Q_m(x)) = 0, \quad \text{if} \quad 0\le n<m.$$

The motivation for this paper is to explain this orthogonality combinatorially. Our combinatorial models for $\mathcal{L}$ and $P_n(x)$ have weighted objects, where the weights depend on the coefficients b_n, a_n, and λ_n. Note that $a_n=0$ is the classical orthogonal polynomial case. In this case the objects of weight 0 may be deleted, and Viennot's theory is recovered.

Throughout this paper we assume that $a_n\ne 0$ and $P_n(-\lambda_n/a_n)\ne 0$ for all $n\ge 0$ unless otherwise stated.

In the classical theory, the orthogonality takes place via a linear functional $\mathcal{L}$ on the vector space of polynomials. For type R_I orthogonality, we will need to extend the definition of $\mathcal{L}$ from the space of polynomials to a larger vector space

$$V = \operatorname{span}\{x^n Q_m(x) : n,m\ge 0\}$$

of rational functions.

In Section 2 we find several bases of the vector space V. We show that there is a unique linear functional $\mathcal{L}$ on V with respect to which the type R_I orthogonal polynomials $P_n(x)$ are orthogonal. This is a slight improvement of a result of Ismail and Masson [11].

In Section 3 we give combinatorial interpretations for $\mu_n=\mathcal{L}(x^n)$, $\mu_{n,m}=\mathcal{L}(x^nQ_m(x))$, $\mu_{n,m,\ell}=\mathcal{L}(x^nP_m(x)Q_\ell(x))$, and $\rho_{n,m,\ell}=\mathcal{L}(x^nP_m(x)P_\ell(x))$ in terms of lattice paths called Motzkin-Schröder paths. We find the infinite continued fraction for the moment generating function. We also give a recursive formula for $\nu_{n,m}=\mathcal{L}\left(x^n/d_m(x)\right)$.

In Section 4 we study Laurent biorthogonal polynomials $P_n(x)$, which are type R_I orthogonal polynomials with $\lambda_n=0$. Kamioka [12] combinatorially studied these polynomials. In this case there is another linear functional $\mathcal{F}$ that gives a different type of orthogonality for $P_n(x)$. We find a simple connection between our linear functional $\mathcal{L}$ and the other linear functional $\mathcal{F}$ using inverted polynomials. We then review Kamioka's results and generalize them.

In Section 5 we express the generating function for Motzkin-Schröder paths with bounded height in terms of inverted polynomials where the indices of the sequences a_n, b_n, and λ_n are shifted. There are finite continued fractions for these rational functions which are explicitly given by the three-term recurrence coefficients.

In Section 6 we find determinant formulas for $P_n(x)$ and $Q_n(x)$ using $\nu_{i,j}$. We also consider some Hankel determinants using μ_n.

In Sections 7, 8, and 9 we give examples of type R_I orthogonal polynomials, including Askey–Wilson and q-Racah polynomials.

In Section 10 we study combinatorial aspects of some type R_I polynomials in the previous sections.

2. The orthogonality of type R_I polynomials

In this section we find several bases of the vector space

$$V := \mathrm{span}\{x^n Q_m(x) : n, m \geq 0\}. \tag{2.1}$$

We then give a detailed proof of the following result of Ismail and Masson [11].

Theorem 2.1. [11, Theorem 2.1] *There is a unique linear functional $\mathcal{L}$ on V satisfying $\mathcal{L}(1) = 1$ and*

$$\mathcal{L}(x^n Q_m(x)) = 0 \qquad \text{if } 0 \leq n < m.$$

We note that Theorem 2.1 is stated slightly differently in [11, Theorem 2.1]. Instead of stating the uniqueness of $\mathcal{L}$, they write that the values of $\mathcal{L}$ on the elements in the set $\{x^n : n \geq 0\} \cup \{1/d_m(x) : m \geq 1\}$ are uniquely determined. We will show that this set is a basis of V, hence the two statements are equivalent. In the proof of [11, Theorem 2.1] they implicitly use the fact that $\{x^{m-1} Q_m(x) : m \geq 2\} \cup \{Q_m(x) : m \geq 0\}$ is a basis of V without a proof. In this section we give a detailed proof of Theorem 2.1 by showing this fact.

Observe that dividing both sides of (1.1) by $d_n(x)$ gives

$$(a_{n+1}x + \lambda_{n+1})Q_{n+1}(x) = (x - b_n)Q_n(x) - Q_{n-1}(x). \tag{2.2}$$

Let

$$V' := \mathrm{span}\{x^n / d_m(x) : n, m \geq 0\}. \tag{2.3}$$

We first find a basis of V' and then show that $V = V'$.

Lemma 2.2. *The vector space V' has a basis*

$$\{x^n : n \geq 0\} \cup \{1/d_m(x) : m \geq 1\}. \tag{2.4}$$

Proof. Let B be the given set. Since the elements in B are linearly independent, it suffices to show that B spans V'. To do this, it suffices to show that $p(x)/d_j(x) \in \mathrm{span}(B)$ for any polynomial $p(x)$ and any integer $j \geq 1$.

By dividing $p(x)$ by $d_j(x)$, we can write

$$\frac{p(x)}{d_j(x)} = q(x) + \frac{r(x)}{d_j(x)}, \tag{2.5}$$

where $q(x)$ and $r(x)$ are polynomials and $\deg r(x) < j$. If $r(x)$ is constant, we have $p(x)/d_j(x) \in \mathrm{span}(B)$. Otherwise, we can write

$$\frac{r(x)}{d_j(x)} = \frac{(a_j x + \lambda_j) r_1(x) + c}{d_j(x)} = \frac{r_1(x)}{d_{j-1}(x)} + \frac{c}{d_j(x)}, \tag{2.6}$$

where $r_1(x)$ is a polynomial with $\deg r_1(x) < j - 1$ and c is a constant. By iterating (2.6) we can express $r(x)/d_j(x)$ as a linear combination of the elements in B. Then by (2.5) we have $p(x)/d_j(x) \in \mathrm{span}(B)$ as desired. □

Lemma 2.3. *We have $V = V'$.*

Proof. By the definitions (2.1) and (2.3) of V and V', it is clear that $V \subseteq V'$ and $x^n \in V$ for $n \geq 0$. By Lemma 2.2, the set in (2.4) is a basis of V'. Thus it suffices to show that $1/d_m(x) \in V$ for all $m \geq 0$. We prove this by induction on m, where the base case $m = 0$ is clear.

For $m \geq 1$, let $U_{m-1}(x)$ be the quotient of $P_m(x)$ when divided by $a_m x + \lambda_m$:

$$P_m(x) = (a_m x + \lambda_m)U_{m-1}(x) + P_m(-\lambda_m/a_m).$$

Then

$$\frac{P_m(x)}{d_m(x)} = \frac{U_{m-1}(x)}{d_{m-1}(x)} + \frac{P_m(-\lambda_m/a_m)}{d_m(x)},$$

and

$$\frac{1}{d_m(x)} = \frac{1}{P_m(-\lambda_m/a_m)} \left(\frac{P_m(x)}{d_m(x)} - \frac{U_{m-1}(x)}{d_{m-1}(x)} \right). \tag{2.7}$$

Since $U_{m-1}(x)$ is a polynomial of degree $m-1$, by iterating (2.6), we can write

$$\frac{U_{m-1}(x)}{d_{m-1}(x)} = \sum_{i=0}^{m-1} \frac{c_i}{d_i(x)},$$

for some constants $c_0, c_1, \ldots, c_{m-1}$. Then by the induction hypothesis, $U_{m-1}(x)/d_{m-1}(x) \in V$. Since $P_m(x)/d_m(x) = Q_m(x) \in V$, (2.7) shows that $1/d_m(x) \in V$, completing the proof. □

Observe that every element of V is of the form $p(x)/d_m(x)$ for some polynomial $p(x)$ and an integer $m \geq 0$. One can find many bases of V as follows.

Proposition 2.4. *Let $\{t_n : n \geq 1\}$ be a sequence of nonnegative integers and let $\{f_n(x) : n \geq 1\}$ and $\{g_n(x) : n \geq 0\}$ be sequences of polynomials satisfying the following conditions:*

- $\deg(f_n(x)) = n + t_n$ *for all* $n \geq 1$, *and*
- $\deg(g_n(x)) \leq n$ *and* $g_n(-\lambda_n/a_n) \neq 0$ *for all* $n \geq 0$.

Then V has a basis given by

$$B = \{f_n(x)/d_{t_n}(x) : n \geq 1\} \cup \{g_n(x)/d_n(x) : n \geq 0\}.$$

Proof. We first show that the elements in B are linearly independent. Suppose that

$$\sum_{i=1}^{n} C_i \frac{f_i(x)}{d_{t_i}(x)} + \sum_{j=0}^{m} D_j \frac{g_j(x)}{d_j(x)} = 0, \tag{2.8}$$

where C_i and D_j are constants. We must show that the coefficients C_i and D_j are all zero. Dividing both sides of (2.8) by x^n and taking the limit $x \to \infty$, we obtain

$$C_n \frac{\text{lead}(f_n(x))}{\text{lead}(d_{t_n}(x))} = 0,$$

where lead$(p(x))$ is the leading coefficient of $p(x)$. This shows $C_n = 0$. Repeating this process with x^n replaced by x^i, for $i = n-1, n-2, \ldots, 1$, we obtain that $C_i = 0$ for all $1 \leq i \leq n$. Then (2.8) becomes

$$\sum_{j=0}^{m} D_j \frac{g_j(x)}{d_j(x)} = 0. \tag{2.9}$$

Let k be the number of integers $1 \leq j \leq m$ such that $-\lambda_j/a_j = -\lambda_m/a_m$. Multiplying both sides of (2.9) by $(a_m x + \lambda_m)^k$ and substituting $x = -\lambda_m/a_m$ we obtain $D_m = 0$. In this way one can show that $D_j = 0$ for all $1 \leq j \leq m$. Then we also have $D_0 = 0$. Therefore the elements in B are linearly independent.

Now we show that B spans V. By Lemmas 2.2 and 2.3, it suffices to show that $x^n, 1/d_m(x) \in \operatorname{span}(B)$ for all $n \geq 1$ and $m \geq 0$.

We claim that, for any $m \geq 0$ and any polynomial $p(x)$,

$$\frac{p(x)}{d_m(x)} \in \operatorname{span}(B) \qquad \text{if } \deg p(x) \leq m. \tag{2.10}$$

We proceed by induction on m, where the base case $m = 0$ is true because $p(x)/d_0(x)$ is a constant and $g_0(x)/d_0(x) \in B$ is a nonzero constant. Let $m \geq 1$ and suppose (2.10) holds for all integers less than m. Then by the same argument as in the proof of 2.3 we can write

$$\frac{p(x)}{d_m(x)} = \sum_{i=0}^{m} \frac{c_i}{d_i(x)}$$

for some constants c_i. By the induction hypothesis, we have $c_i/d_i(x) \in \operatorname{span}(B)$, and therefore to show $p(x)/d_m(x) \in \operatorname{span}(B)$, it is enough to show that $1/d_m(x) \in \operatorname{span}(B)$. Dividing $g_m(x)$ by $(a_m x + \lambda_m)$, we have

$$\frac{g_m(x)}{d_m(x)} = \frac{q(x)}{d_{m-1}(x)} + \frac{g_m(-a_m/\lambda_m)}{d_m(x)}, \tag{2.11}$$

where $q(x)$ is a polynomial with $\deg q(x) = \deg g_m(x) - 1 \leq m - 1$. By the induction hypothesis, we have $q(x)/d_{m-1}(x) \in \operatorname{span}(B)$. Then (2.11) shows that $1/d_m(x) \in \operatorname{span}(B)$ because $g_m(-a_m/\lambda_m) \neq 0$. Thus (2.10) is also true for m and the claim is proved.

By (2.10), we have $1/d_m(x) \in \operatorname{span}(B)$ for $m \geq 0$. Therefore it remains to show that $x^n \in \operatorname{span}(B)$ for $n \geq 1$. Dividing $f_n(x)$ by $d_{t_n}(x)$, we have

$$\frac{f_n(x)}{d_{t_n}(x)} = q_n(x) + \frac{r_n(x)}{d_{t_n}(x)}, \tag{2.12}$$

where $q_n(x)$ and $r_n(x)$ are polynomials with $\deg q_n(x) = n$ and $\deg r_n(x) < t_n$. By (2.10), we have $r_n(x)/d_{t_n}(x) \in \operatorname{span}(B)$, and therefore (2.12) shows $q_n(x) \in \operatorname{span}(B)$. This implies that

$$\operatorname{span}\{x^n : n \geq 0\} = \operatorname{span}(\{1\} \cup \{q_n(x) : n \geq 1\}) \subseteq \operatorname{span}(B).$$

Hence we have $x^n \in \operatorname{span}(B)$ for all $n \geq 0$, which completes the proof. □

As a corollary of Proposition 2.4 we present three notable bases of V. We use the following three choices in Proposition 2.4:

$$\begin{aligned}
f_n(x) &= x^n d_n(x), & t_n &= n, & g_n(x) &= 1,\\
f_n(x) &= P_n(x) d_n(x), & t_n &= n, & g_n(x) &= P_n(x),\\
f_n(x) &= x^n P_{n+1}(x), & t_n &= n+1, & g_n(x) &= P_n(x).
\end{aligned}$$

Corollary 2.5. *The following are bases of V:*

$$\{x^n : n \geq 1\} \cup \{1/d_m(x) : m \geq 0\},$$

$$\{P_n(x) : n \geq 1\} \cup \{Q_m(x) : m \geq 0\},$$

$$\{x^n Q_{n+1}(x) : n \geq 1\} \cup \{Q_m(x) : m \geq 0\}.$$

Now we give a detailed proof of Theorem 2.1. Our proof is basically the same as the proof in [11] with some details provided.

Proof of Theorem 2.1. By Corollary 2.5,

$$B := \{x^{m-1}Q_m(x) : m \geq 2\} \cup \{Q_m(x) : m \geq 0\}.$$

is a basis of V. Since the values of $\mathcal{L}$ on $\{1\} \cup \{x^n Q_m(x) : 0 \leq n < m\}$ are given, and this set contains B, if $\mathcal{L}$ exists, it is unique. For the existence, we construct $\mathcal{L}$ by defining its values on the basis B as follows:

$$\mathcal{L}(1) = 1, \qquad \mathcal{L}(Q_m(x)) = \mathcal{L}(x^{m-1}Q_m(x)) = 0, \qquad \text{for } m \geq 1.$$

It suffices to show that $\mathcal{L}$ satisfies

$$\mathcal{L}(x^n Q_m(x)) = 0 \qquad \text{if } 0 \leq n < m. \tag{2.13}$$

We prove (2.13) by induction on (n, m). The base case $n = 0$ is true by definition of $\mathcal{L}$. Let $1 \leq n < m$ and assume that (2.13) is true all pairs $(n', m') \neq (n, m)$ such that $0 \leq n' < m'$, $n' \leq n$ and $m' \leq m$. We now show that it is also true for (n, m). Since it is true for $(m-1, m)$ by definition of $\mathcal{L}$, we may assume $1 \leq n \leq m-2$. By (2.2) we have

$$(a_m x + \lambda_m)Q_m(x) = (x - b_{m-1})Q_{m-1}(x) - Q_{m-2}(x).$$

Multiplying both sides of the above equation by x^{n-1} we have

$$a_m x^n Q_m(x) + \lambda_m x^{n-1} Q_m(x) = x^n Q_{m-1}(x) - b_{m-1}x^{n-1}Q_{m-1}(x) - x^{n-1}Q_{m-2}(x).$$

Since $0 \leq n-1 \leq m-3$, by the induction hypothesis, taking $\mathcal{L}$ on both sides gives

$$a_m \mathcal{L}(x^n Q_m(x)) = 0.$$

Since $a_m \neq 0$, we obtain $\mathcal{L}(x^n Q_m(x)) = 0$. Hence (2.13) is also true for (n, m) and the proof is completed by induction. □

3. Moments of type R_I orthogonal polynomials

Recall from Theorem 2.1 that there is a unique linear functional $\mathcal{L}$ on V satisfying $\mathcal{L}(1) = 1$ and

$$\mathcal{L}(x^n Q_m(x)) = 0 \qquad \text{if } 0 \leq n < m.$$

In this section we give combinatorial interpretations for

$$\begin{aligned} \mu_n &:= \mathcal{L}(x^n), \\ \mu_{n,m} &:= \mathcal{L}(x^n Q_m(x)), \\ \mu_{n,m,\ell} &:= \mathcal{L}(x^n P_m(x) Q_\ell(x)), \\ \rho_{n,m,\ell} &:= \mathcal{L}(x^n P_m(x) P_\ell(x)) \end{aligned}$$

in terms of lattice paths called Motzkin-Schröder paths. We also give a recursive formula for

$$\nu_{n,m} = \mathcal{L}\left(x^n / d_m(x)\right).$$

Note that $Q_n(x)Q_m(x) \notin V$ in general, so we do not consider $\mathcal{L}(Q_n(x)Q_m(x))$.

3.1. Combinatorial interpretations for μ_n and $\mu_{n,m}$.

In this subsection we give combinatorial interpretations for μ_n and $\mu_{n,m}$ using recursive formulas.

Note that $\mu_{n,m}$ has the initial conditions given by

$$\mu_{0,0} = 1, \qquad \mu_{n,m} = 0 \quad \text{for } 0 \leq n < m. \tag{3.1}$$

The following lemma gives a recurrence, which determines $\mu_{n,m}$ since the recurrence decreases either n or $n - m$

Lemma 3.1. *For $n,m \geq 0$ with $(n,m) \neq (0,0)$, we have*

$$\mu_{n,m} = \begin{cases} 0, & \text{if } n < m, \\ a_{m+1}\mu_{n,m+1} + b_m\mu_{n-1,m} + \mu_{n-1,m-1} + \lambda_{m+1}\mu_{n-1,m+1}, & \text{if } n \geq m, \end{cases}$$

where $\mu_{i,j} = 0$ if $j < 0$.

Proof. We have already seen that $\mu_{n,m} = 0$ for $n < m$. Suppose $n \geq m$. Since $(n,m) \neq (0,0)$ we have $n \geq 1$. Then

$$\begin{aligned} a_{m+1}\mu_{n,m+1} &= \mathcal{L}\left(\frac{a_{m+1}x^nP_{m+1}(x)}{d_{m+1}(x)}\right) \\ &= \mathcal{L}\left(\frac{x^{n-1}(a_{m+1}x+\lambda_{m+1})P_{m+1}(x)}{d_{m+1}(x)} - \frac{\lambda_{m+1}x^{n-1}P_{m+1}(x)}{d_{m+1}(x)}\right) \\ &= \mathcal{L}\left(\frac{x^{n-1}((x-b_m)P_m(x)-(a_mx+\lambda_m)P_{m-1}(x))}{d_m(x)}\right) - \lambda_{m+1}\mu_{n-1,m+1} \\ &= \mu_{n,m} - b_m\mu_{n-1,m} - \mu_{n-1,m-1} - \lambda_{m+1}\mu_{n-1,m+1}. \end{aligned}$$

Thus

$$\mu_{n,m} = a_{m+1}\mu_{n,m+1} + b_m\mu_{n-1,m} + \mu_{n-1,m-1} + \lambda_{m+1}\mu_{n-1,m+1},$$

as desired. □

Definition 3.2. A *Motzkin path* is a path on or above the x-axis consisting of up steps $U = (1,1)$, horizontal steps $H = (1,0)$, and down steps $D = (1,-1)$. A *Schröder path* is a path on or above the x-axis consisting of up steps $U = (1,1)$, horizontal steps $H = (1,0)$, and vertical step $V = (0,-1)$.

Remark 3.3. A Schröder path is more commonly defined as a path consisting of up steps $(1,1)$, double horizontal steps $(2,0)$, and down steps $(1,-1)$. One can easily convert from our definition of a Schröder path to this one by changing each horizontal step $(1,0)$ to a double horizontal step $(2,0)$, and each vertical step $(0,-1)$ to a down step $(1,-1)$.

Now we define another lattice path, which contains both Motzkin and Schröder paths.

Definition 3.4. A *Motzkin-Schröder path* is a path on or above the x-axis consisting of up steps $U = (1,1)$, horizontal steps $H = (1,0)$, vertical down step $V = (0,-1)$, and diagonal down steps $D = (1,-1)$. For a Motzkin-Schröder path π, the *weight* $\mathrm{wt}(\pi)$ of π is the product of the weight of each step, where every up step has weight 1, a horizontal step starting at height k has weight b_k, a vertical down step $(0,-1)$ starting at height k has weight a_k, and a diagonal down step $(1,-1)$ starting at height k has weight λ_k. See Figure 1.

Remark 3.5. In [13, 14], Kim found a combinatorial interpretation for moments of biorthogonal polynomials using lattice paths. The steps are up steps, horizontal steps, and d types of down steps. If $d = 2$, these lattice paths are equivalent to our Motzkin-Schröder paths.

Let $\mathrm{MS}((a,b) \to (c,d))$ denote the set of Motzkin-Schröder paths from (a,b) to (c,d) and let

$$\mathrm{MS}_{n,m} := \mathrm{MS}((0,0) \to (n,m)), \qquad \mathrm{MS}_n := \mathrm{MS}_{n,0}\,.$$

We also denote by $\mathrm{Motz}((a,b) \to (c,d))$ (resp. $\mathrm{Sch}((a,b) \to (c,d))$) the set of Motzkin (resp. Schröder) paths from (a,b) to (c,d). The sets $\mathrm{Motz}_{n,m}$, Motz_n, $\mathrm{Sch}_{n,m}$, and Sch_n are defined similarly.

Theorem 3.6. *For $n,m \geq 0$, we have*

$$\mu_{n,m} = \mathcal{L}(x^nQ_m(x)) = \sum_{\pi \in \mathrm{MS}_{n,m}} \mathrm{wt}(\pi).$$

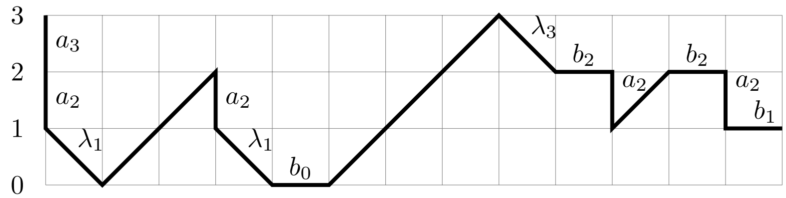

FIGURE 1. A Motzkin-Schröder path π from $(0,3)$ to $(13,1)$ with $\mathrm{wt}(\pi) = a_2^4 a_3 b_0 b_1 b_2^2 \lambda_1^2 \lambda_3$.

Proof. Let $\mu'_{n,m}$ be the right hand side. Then one can easily check that $\mu'_{n,m}$ satisfies the same initial conditions in (3.1) and the recurrence relation in Lemma 3.1 as $\mu_{n,m}$. Therefore $\mu_{n,m} = \mu'_{n,m}$ for all $n, m \geq 0$. □

Corollary 3.7. *We have*

$$\mu_n = \sum_{\pi \in \mathrm{MS}_n} \mathrm{wt}(\pi).$$

Equivalently,

$$\sum_{n\geq 0} \mu_n x^n = \cfrac{1}{1 - b_0 x - \cfrac{a_1 x + \lambda_1 x^2}{1 - b_1 x - \cfrac{a_2 x + \lambda_2 x^2}{\ddots}}}.$$

Proof. The first statement is the special case $m = 0$ of Theorem 3.6. The second statement follows from the first using Flajolet's theory [7]. □

For example, by Corollary 3.7, the moments μ_n for $n = 0, 1, 2$ can be computed as

$$\begin{aligned}
\mu_0 &= 1, \\
\mu_1 &= b_0 + a_1, \\
\mu_2 &= b_0^2 + \lambda_1 + 2a_1 b_0 + a_2 a_1 + b_1 a_1 + a_1^2.
\end{aligned}$$

Remark 3.8. Lattice paths containing Motzkin-Schröder paths are considered in [5]. The number of Motzkin-Schröder paths is listed in [16, A064641]. If $b_n = a_n = 0$ and $\lambda_n = 1$, then $\mu_{2n+1} = 0$ and $\mu_{2n} = C_n$, the nth Catalan number. If $b_n = \lambda_n = 0$ and $a_n = 1$, then $\mu_n = C_n$.

By taking $a_n = b_n = \lambda_n = 1$ for all n in Corollary 3.7 we obtain the following result, which also appears in [5].

Proposition 3.9. *The generating function for the number of Motzkin-Schröder paths is given by*

$$\sum_{n\geq 0} |\,\mathrm{MS}_n\,| x^n = \frac{1 - x - \sqrt{1 - 6x - 3x^2}}{2(x + x^2)} = 1 + 2x + 7x^2 + 29x^3 + 133x^4 + 650x^5 + \cdots .$$

3.2. Combinatorial interpretations for $\mu_{n,m,\ell}$ and $\rho_{n,m,\ell}$.

In this subsection we give a combinatorial interpretation for $P_n(x)$ in terms of tilings (Theorem 3.11). This allows us to write $\mu_{n,m,\ell} = \mathcal{L}(x^n P_m(x) Q_\ell(x))$ as a signed sum of $\mu_{n,m} = \mathcal{L}(x^n P_m(x))$. We will show that these are positive sums by finding a sign-reversing involution which cancel all negative terms. Using a similar argument we will also find a positive formula for $\rho_{n,m,\ell} = \mathcal{L}(x^n P_m(x) P_\ell(x))$.

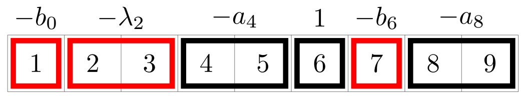

FIGURE 2. A Favard tiling $T \in \mathrm{FT}_9$ with $\mathrm{wt}(T) = -a_4a_8b_0b_6\lambda_2$. This tile contributes $\mathrm{wt}(T)x^{\mathsf{bm}(T)+\mathsf{bd}(T)} = -a_4a_8b_0b_6\lambda_2x^3$ to $P_8(x)$.

Definition 3.10. A *(bicolored) Favard tiling of size* n is a tiling of a $1 \times n$ square board with tiles where each tile is a domino or a monomino and is colored black or red. We label the squares in the $1 \times n$ board by $1, 2, \dots, n$ from left to right. The set of Favard tilings of size n is denoted by FT_n.

For $T \in \mathrm{FT}_n$, we define $\mathsf{bm}(T)$ (resp. $\mathsf{bd}(T)$, $\mathsf{rm}(T)$, and $\mathsf{rd}(T)$) to be the number of black monominos (resp. black dominos, red monominos, and black monominos) in T. We also define

$$\mathrm{wt}(T) = \prod_{\tau \in T} \mathrm{wt}(\tau),$$

where

$$\mathrm{wt}(\tau) = \begin{cases} 1 & \text{if } \tau \text{ is a black monomino,} \\ -b_{i-1} & \text{if } \tau \text{ is a red monomino with (largest) entry } i, \\ -a_{i-1} & \text{if } \tau \text{ is a black domino with largest entry } i, \\ -\lambda_{i-1} & \text{if } \tau \text{ is a red domino with largest entry } i. \end{cases}$$

For example, see Figure 2.

The recurrence (1.1) gives the following combinatorial interpretation for $P_n(x)$.

Theorem 3.11. *For $n \geq 0$, we have*

$$P_n(x) = \sum_{T \in \mathrm{FT}_n} \mathrm{wt}(T)x^{\mathsf{bm}(T)+\mathsf{bd}(T)}.$$

Proof. Let $U_n(x)$ denote the right hand side. Then by definition one can easily check that, for $n \geq 0$, we have

$$U_{n+1}(x) = (x - b_n)U_n(x) - (a_nx + \lambda_n)U_{n-1}(x)$$

where $U_{-1}(x) = 0$ and $U_0(x) = 1$. Therefore, by (1.1), $T_n(x)$ and $P_n(x)$ satisfy the same recurrence relation and the same initial conditions. This show the theorem. □

Now we give a combinatorial interpretation for $\mu_{n,m,\ell}$.

Theorem 3.12. *For $n, m, \ell \geq 0$, we have*

$$\mu_{n,m,\ell} = \mathcal{L}(x^nP_m(x)Q_\ell(x)) = \sum_{\pi \in \mathrm{MS}((0,m)\to(n,\ell))} \mathrm{wt}(\pi).$$

Proof. We will find a sign-reversing involution on a larger set whose fixed point set is given by the Motzkin-Schröder paths in this theorem.

Applying Theorem 3.11 to $P_m(x)$ and using Theorem 3.6, we have

$$\mathcal{L}(x^nP_m(x)Q_\ell(x)) = \sum_{T \in \mathrm{FT}_m} \mathrm{wt}(T)\mathcal{L}(x^{n+\mathsf{bm}(T)+\mathsf{bd}(T)}Q_\ell(x)) = \sum_{(\pi,T)\in X} \mathrm{wt}(\pi)\,\mathrm{wt}(T), \tag{3.2}$$

where X is the set of pairs (π, T) of a Motzkin-Schröder path $\pi \in \mathrm{MS}_{t,\ell}$ and a Favard tiling $T \in \mathrm{FT}_m$ satisfying $t = n + \mathsf{bm}(T) + \mathsf{bd}(T)$. The sign-reversing involution on X will remove or add a horizontal step or a peak ((U, V) or (U, D)) in π, and modify T accordingly.

Consider $(\pi, T) \in X$ and write $\pi = S_1 \dots S_r$ as a sequence of steps. Suppose that i and j are the largest integers such that π starts with i up steps and T starts with j black monominos.

Case 1: $j \ge i+1$. In this case we have $i+1 \le \mathsf{bm}(T) \le n + \mathsf{bm}(T) + \mathsf{bd}(T) = t$. Therefore π must have the $(i+1)$st step. We define π' and T' in the following three cases depending on the step S_{i+1}.

Case 1-a: S_{i+1} is a horizontal step. In this case let

$$\pi' = S_1 \dots \widehat{S}_{i+1} \dots S_r,$$

and define T' to be the Favard tiling obtained from T by replacing the black monomino at position $i+1$ by a red monomino. Here the notation $\widehat{S}_{i+1}$ means that S_{i+1} is removed from the sequence. See Figure 3.

Case 1-b: S_{i+1} is a vertical step. In this case let

$$\pi' = S_1 \dots \widehat{S}_i \widehat{S}_{i+1} \dots S_r,$$

and define T' to be the Favard tiling obtained from T by replacing the two black monominos at positions i and $i+1$ by a black domino. See Figure 4.

Case 1-c: S_{i+1} is a down step. In this case let

$$\pi' = S_1 \dots \widehat{S}_i \widehat{S}_{i+1} \dots S_r,$$

and define T' to be the Favard tiling obtained from T by replacing the two black monominos at positions i and $i+1$ by a red domino. See Figure 5.

Case 2: $j \le i$ and $j < m$. In this case T contains a tile, say A, with entry $j+1$. We define π' and T' in the following three cases depending on the tile A.

Case 2-a: A is a red monomino. In this case let

$$\pi' = S_1 \dots S_j H S_{j+1} \dots S_r,$$

and define T' to be the Favard tiling obtained from T by replacing A by a black monomino. See Figure 3.

Case 2-b: A is a black domino. In this case let

$$\pi' = S_1 \dots S_j U V S_{j+1} \dots S_r,$$

and define T' to be the Favard tiling obtained from T by replacing A by two black monominos. See Figure 4.

Case 2-c: A is a red domino. In this case let

$$\pi' = S_1 \dots S_j U D S_{j+1} \dots S_r,$$

and define T' to be the Favard tiling obtained from T by replacing A by two black monominos. See Figure 5.

Case 3: $j \le i$ and $j = m$. In this case define $\pi' = \pi$ and $T' = T$. See Figure 6.

It is straightforward to verify that $(\pi, T) \mapsto (\pi', T')$ is a sign-reversing involution on X whose fixed points are the pairs (π, T) with $\pi \in \mathrm{MS}((0,0) \to (m+n, \ell))$ and $T \in \mathrm{FT}_m$ such that the first m steps of π are up steps and T consists of m black monominos. Note that if (π, T) is a fixed point, then $\mathrm{wt}(T) = 1$ and $\mathrm{wt}(\pi) = \mathrm{wt}(\pi_{\ge m})$, where $\pi_{\ge m}$ is the subpath of π from (m, m) to $(m+n, \ell)$. This shows that

$$\sum_{(\pi,T)\in X} \mathrm{wt}(\pi)\,\mathrm{wt}(T) = \sum_{\pi \in \mathrm{MS}((m,m)\to(m+n,\ell))} \mathrm{wt}(\pi) = \sum_{\pi \in \mathrm{MS}((0,m)\to(n,\ell))} \mathrm{wt}(\pi). \tag{3.3}$$

Then the theorem follows from (3.2) and (3.3). $\square$

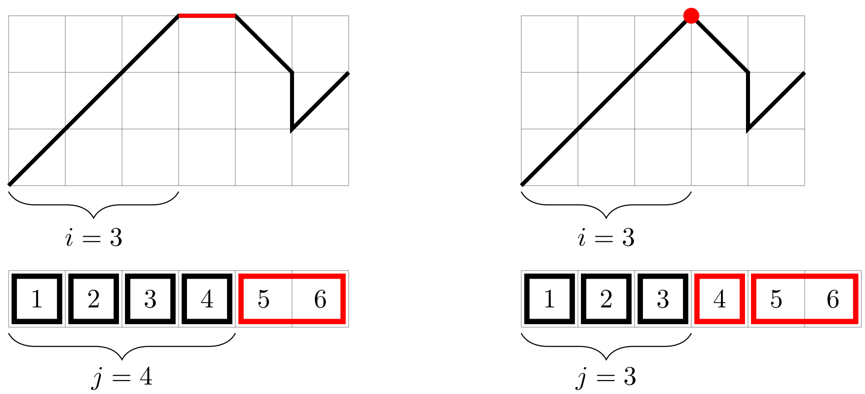

FIGURE 3. A pair $(\pi, T) \in X$ in Case 1-a on the left and the corresponding pair (π', T') in Case 2-a on the right, for $(n, m, \ell) = (2, 6, 2)$. The horizontal step starting at $(3, 3)$ in π is collapsed to a point.

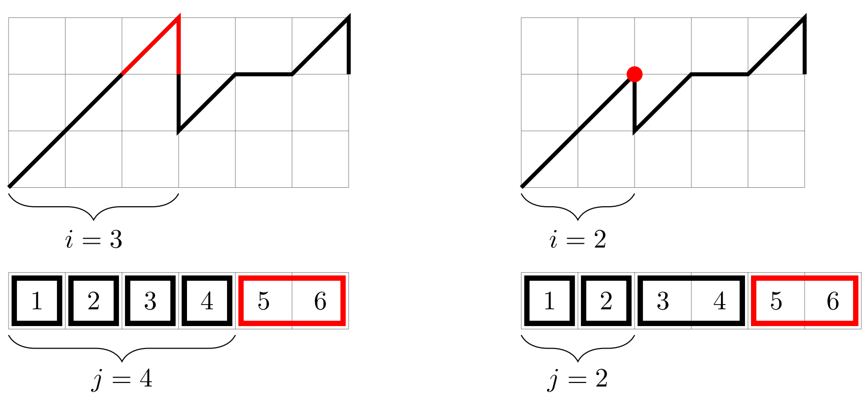

FIGURE 4. A pair $(\pi, T) \in X$ in Case 1-b on the left and the corresponding pair (π', T') in Case 2-b on the right, for $(n, m, \ell) = (2, 6, 2)$. The peak (U, V) starting at $(2, 2)$ in π is collapsed to a point.

Now we list a number of special cases of Theorem 3.12.
First of all, if $a_n = 0$, we obtain Viennot's result.

Corollary 3.13. [18, Proposition 17 on page I-15] *We have*

$$\mathcal{L}(x^n P_m(x) P_\ell(x)) = \lambda_1 \dots \lambda_\ell \sum_{\pi \in \mathrm{Motz}((0,m) \to (n,\ell))} \mathrm{wt}(\pi).$$

In the next section we will show that if $\lambda_n = 0$ and $\ell = 0$ in Theorem 3.12, then we obtain Kamioka's result [12, Lemma 3.1] on Laurent biorthogonal polynomials.
If $m = 0$ or $\ell = 0$ in Theorem 3.12, we obtain the following corollary.

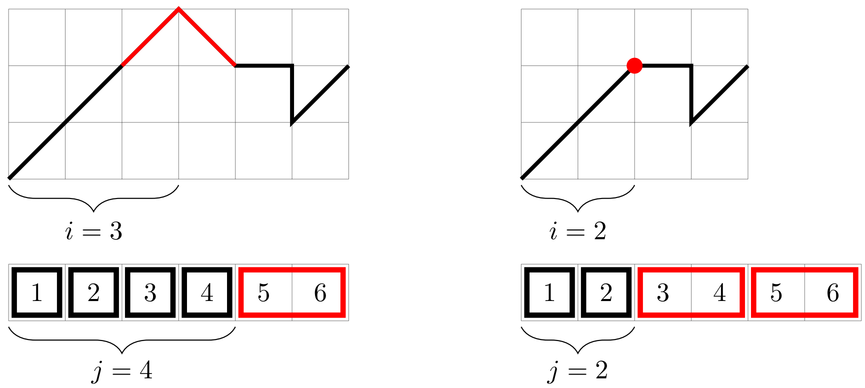

FIGURE 5. A pair $(\pi, T) \in X$ in Case 1-c on the left and the corresponding pair (π', T') in Case 2-c on the right, for $(n, m, \ell) = (2, 6, 2)$. The peak (U, D) starting at $(2, 2)$ in π is collapsed to a point.

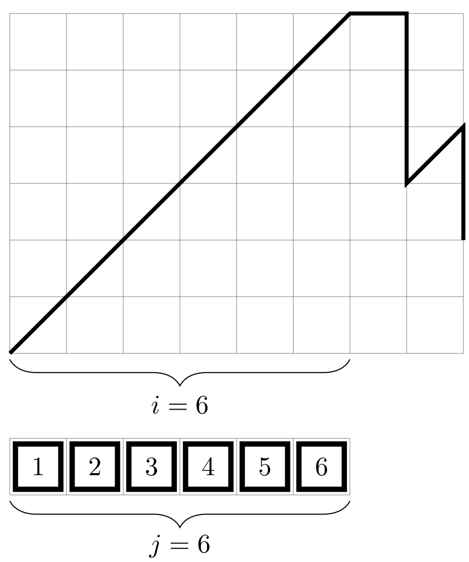

FIGURE 6. A pair $(\pi, T) \in X$ in Case 3 for $(n, m, \ell) = (2, 6, 2)$. In this case $(\pi, T) = (\pi', T')$ is a fixed point.

Corollary 3.14. *For $n, m \geq 0$, we have*

$$\mathcal{L}(x^n P_m(x)) = \sum_{\pi \in \mathrm{MS}((0,m)\to(n,0))} \mathrm{wt}(\pi),$$

$$\mathcal{L}(x^n Q_m(x)) = \sum_{\pi \in \mathrm{MS}((0,0)\to(n,m))} \mathrm{wt}(\pi).$$

If $n = 0$ in Theorem 3.12, we obtain the following corollary.

Corollary 3.15. *We have*

$$\mathcal{L}(P_n(x)Q_m(x)) = \begin{cases} 0 & \text{if } n < m, \\ a_{m+1}a_{m+2}\dots a_n & \text{if } n \geq m. \end{cases}$$

In particular,

$$\mathcal{L}(P_n(x)Q_m(x)) = \delta_{n,m}, \qquad \text{if } 0 \leq n \leq m,$$
$$\mathcal{L}(P_n(x)) = a_1a_2\cdots a_n,$$
$$\mathcal{L}(P_n(x)(Q_m(x) - a_{m+1}Q_{m+1}(x))) = \delta_{n,m}.$$

If $m = 0$ and $n = \ell$, or $n = 0$ and $m = \ell$, we obtain the following corollary, which is equivalent to [11, Corollary 2.2].

Corollary 3.16. *We have*

$$\mathcal{L}(x^nQ_n(x)) = \mathcal{L}(P_n(x)Q_n(x)) = 1.$$

Using Corollary 3.15 we can find the coefficients in the expansion of an arbitrary polynomial as a linear combination of $P_n(x)$.

Proposition 3.17. *Let $p(x)$ be a polynomial in x, and expand*

$$p(x) = \sum_{m=0}^{\infty} c_m P_m(x).$$

Then

$$c_m = \mathcal{L}\left(p(x)(Q_m(x) - a_{m+1}Q_{m+1}(x))\right).$$

Proof. Corollary 3.15 implies

$$\mathcal{L}(p(x)Q_\ell(x)) = \sum_{m\geq\ell} c_m a_m a_{m-1}\cdots a_{\ell+1},$$

and thus

$$\mathcal{L}(p(x)(Q_\ell(x) - a_{\ell+1}Q_{\ell+1}(x))) = c_\ell,$$

as desired. □

The following theorem implies that $\rho_{n,m,\ell} = \mathcal{L}(x^nP_m(x)P_\ell(x))$ is a positive polynomial in a_k, b_k, and λ_k.

Theorem 3.18. *For $n, m, \ell \geq 0$, we have*

$$\rho_{n,m,\ell} = \mathcal{L}(x^nP_m(x)P_\ell(x)) = \sum_{\pi} \operatorname{wt}(\pi),$$

where the sum is over all Motzkin-Schröder paths from $(0,m)$ to $(n+\ell,0)$ such that the last ℓ steps consist only of vertical steps and down steps.

Proof. By Theorem 3.11 and Theorem 3.12, we have

$$\mathcal{L}(x^nP_m(x)P_\ell(x)) = \sum_{T\in\mathrm{FT}_\ell} \operatorname{wt}(T)\mathcal{L}(x^{n+\mathsf{bm}(T)+\mathsf{bd}(T)}P_m(x)) = \sum_{(\pi,T)\in X} \operatorname{wt}(\pi)\operatorname{wt}(T),$$

where X is the set of pairs (π, T) of a Motzkin-Schröder path π from $(0,m)$ to $(n+t,0)$ and a Favard tiling $T \in \mathrm{FT}_\ell$ satisfying $t = n + \mathsf{bm}(T) + \mathsf{bd}(T)$.

By the same argument as in the proof of Theorem 3.12, we can find a sign-reversing involution on X whose fixed points are exactly the Motzkin-Schröder paths described in this theorem. The only difference in the construction of the sign-reversing involution is that we write $\pi =$

$S_r S_{r-1} \dots S_1$ and let i be the largest integer such that the last i steps $S_1, \dots, S_i$ consist of vertical and down steps. We omit the details. □

If $\ell = 0$ in Theorem 3.18, we obtain Corollary 3.14. If $n = 0$ in Theorem 3.18, we obtain the following corollary.

Corollary 3.19. *For $m, n \geq 0$, we have*

$$\mathcal{L}(P_m(x)P_n(x)) = \sum_{\pi} \mathrm{wt}(\pi),$$

where the sum is over all Motzkin-Schröder paths from $(0, m)$ to $(n, 0)$ such that the last n steps consist only of vertical steps and down steps.

Example 3.20. If $(n, m, \ell) = (0, 1, 1)$, we have

$$\mathcal{L}(P_1(x)P_1(x)) = a_1 a_2 + a_1 b_1 + \lambda_1.$$

One path is eliminated: the path VH, because the last step is a horizontal step, which is neither a vertical step nor a down step.

If $(n, m, \ell) = (0, 2, 1)$, we have

$$\mathcal{L}(P_2(x)P_1(x)) = a_1^2 a_2 + a_1 a_2^2 + a_1 a_2 a_3 + a_1 a_2 b_1 + a_1 a_2 b_2 + a_2 \lambda_1 + a_1 \lambda_2.$$

One path is eliminated: the path VVH, because the last step is a horizontal step, which is neither a vertical step nor a down step.

3.3. **The moments $\nu_{n,m}$.**

We will find a recurrence for $\nu_{n,m} = \mathcal{L}\left(x^n / d_m(x)\right)$ and a generating function for them. We do not have a combinatorial interpretation for them.

For $m \geq 1$, define $U_m(x)$ to be the quotient of $P_m(x)$ when divided by $a_m x + \lambda_m$:

$$P_m(x) = (a_m x + \lambda_m) U_m(x) + P_m(-\lambda_m / a_m).$$

Let $f_{m,i}$ be the coefficients of $U_m(x)$:

$$U_m(x) = \sum_{i=0}^{m-1} f_{m,i} x^i.$$

The following Lemma 3.21 with $\nu_{0,0} = 1$ allows us to compute $\nu_{n,m}$ for all $n, m \geq 0$.

Lemma 3.21. *For $n, m \geq 1$ we have*

$$\nu_{n,0} = \mu_{n,0} = \mu_n, \tag{3.4}$$

$$\nu_{0,m} = -\frac{1}{P_m(-\lambda_m / a_m)} \sum_{i=0}^{m-1} f_{m,i} \nu_{i,m-1}, \tag{3.5}$$

$$\nu_{n,m} = \frac{1}{a_m} \nu_{n-1,m-1} - \frac{\lambda_m}{a_m} \nu_{n-1,m}. \tag{3.6}$$

Proof. The first identity is immediate from the definitions of $\nu_{n,m}$ and $\mu_{n,m}$. The second identity follows from

$$0 = \mathcal{L}\left(\frac{P_m(x)}{d_m(x)}\right) = \mathcal{L}\left(\frac{U_m(x)}{d_{m-1}(x)} + \frac{P_m(-\lambda_m/a_m)}{d_m(x)}\right).$$

The third identity follows from

$$a_m \nu_{n,m} = \mathcal{L}\left(\frac{x^{n-1}(a_m x + \lambda_m)}{d_m} - \frac{\lambda_m x^{n-1}}{d_m}\right) = \nu_{n-1,m-1} - \lambda_m \nu_{n-1,m}.$$

□

For $m \geq 0$, let

$$V_m(x) = \sum_{n\geq 0} \nu_{n,m} x^n.$$

Proposition 3.22. *For an integer $m \geq 1$, we have*

$$V_m(x) = \frac{a_m \nu_{0,m}}{a_m + \lambda_m x} + \frac{x V_{m-1}(x)}{a_m + \lambda_m x}.$$

Proof. By (3.6), for $n \geq 1$,

$$a_m \nu_{n,m} + \lambda_m \nu_{n-1,m} = \nu_{n-1,m-1}.$$

By multiplying both sides by x^n and summing over $n \geq 1$, we obtain

$$a_m(V_m(x) - \nu_{0,m}) + \lambda_m x V_m(x) = x V_{m-1}(x),$$

which is equivalent to the desired equation. □

By iterating the equation in Proposition 3.22 and observing the fact $V_0(x) = \sum_{n\geq 0} \mu_n x^n$ we obtain the following corollary.

Corollary 3.23. *For $m \geq 1$, we have*

$$V_m(x) = \frac{x^m \sum_{n\geq 0} \mu_n x^n}{\prod_{j=1}^m (a_j + \lambda_j x)} + \sum_{i=1}^m \frac{\nu_{0,i} x^{m-i}}{\prod_{j=i}^m (1 + \lambda_j x / a_j)}.$$

4. Moments of Laurent biorthogonal polynomials

In this section we study Laurent biorthogonal polynomials $P_n(x)$, which are type R_I orthogonal polynomials with $\lambda_n = 0$. Kamioka [12] combinatorially studied this case. There is another linear functional $\mathcal{F}$ that gives a different type of orthogonality for $P_n(x)$. We will first study the connection between our linear functional $\mathcal{L}$ and Kamioka's linear functional $\mathcal{F}$. We then review Kamioka's results and show that these are special cases of Theorem 3.12.

In this section we consider the case $\lambda_n = 0$ for all $n \geq 0$, so that the polynomials $P_n(x)$ are defined by $P_{-1}(x) = 0$, $P_0(x) = 1$, and for $n \geq 0$,

$$P_{n+1}(x) = (x - b_n) P_n(x) - a_n x P_{n-1}(x). \tag{4.1}$$

Throughout this section we assume that $P_n(0) \neq 0$ and $a_n \neq 0$ for all $n \geq 0$. Since $P_n(0) = (-1)^n b_0 b_1 \dots b_{n-1}$, we must have $b_n \neq 0$ for all $n \geq 0$.

For $n \geq 0$, let

$$Q_n(x) = \frac{P_n(x)}{a_1 \dots a_n x^n},$$

where $Q_0(x) = 1$. Then $V = \operatorname{span}\{x^n Q_m(x) : n, m \geq 0\}$ is the vector space of Laurent polynomials.

Kamioka showed the following Favard-type theorem.

Theorem 4.1. [12, Theorem 2.1] *There is a unique linear functional $\mathcal{F}$ on V such that $\mathcal{F}(1) = 1$ and*

$$\mathcal{F}(x^{-n} P_m(x)) = 0, \qquad 0 \leq n < m.$$

Remark 4.2. We note that the original statement of [12, Theorem 2.1] is that there is a unique linear functional $\mathcal{F}$ on V such that $\mathcal{F}(1) = 1$ and

$$\mathcal{F}(x^{-n} P_m(x)) = h_n \delta_{n,m}, \qquad 0 \leq n \leq m,$$

for some constants $h_n \neq 0$. Since $\mathrm{span}(\{1\} \cup \{x^{-n}P_m(x) : 0 \leq n < m\}) = V$ and $x^{-k}P_k(x)$, for $k \geq 0$, is a linear combination of the elements in the spanning set $\{1\} \cup \{x^{-n}P_m(x) : 0 \leq n < m\}$, where the coefficient of 1 is nonzero, the two statements are equivalent.

Using Theorem 2.1 we obtain a slightly different Favard-type theorem.

Theorem 4.3. *There is a unique linear functional $\mathcal{L}$ on V such that $\mathcal{L}(1) = 1$ and*

$$\mathcal{L}(x^{-n}P_m(x)) = 0, \qquad 0 < n \leq m.$$

Proof. By Theorem 2.1, there is a unique linear functional $\mathcal{L}$ on V satisfying the orthogonality

$$\mathcal{L}\left(x^{n-m}P_m(x)/a_1 \dots a_m\right) = 0, \qquad 0 \leq n < m.$$

Replacing n by $m - n$ in the above equation gives the theorem. □

Note that, since $P_1(x) = x - b_0$, if $n = 0$ and $m = 1$ in Theorem 4.1 we obtain

$$\mathcal{F}(x) = b_0. \tag{4.2}$$

Similarly, if $n = m = 1$ in Theorem 4.9, we obtain

$$\mathcal{L}(x^{-1}) = b_0^{-1}. \tag{4.3}$$

We now show that the linear functionals $\mathcal{F}$ and $\mathcal{L}$ in the above two Favard-type theorems have a simple connection.

Proposition 4.4. *For all $f(x) \in V$, we have*

$$\mathcal{F}(f(x)) = b_0 \cdot \mathcal{L}(x^{-1}f(x)).$$

Proof. We will show the equivalent statement

$$\mathcal{L}(f(x)) = b_0^{-1} \cdot \mathcal{F}(xf(x)). \tag{4.4}$$

Let $\mathcal{L}'(f(x))$ be the right hand side of (4.4). Then by Theorem 4.9 it suffices to show that $\mathcal{L}'(1) = 1$ and

$$\mathcal{L}'(x^{-n}P_m(x)) = 0, \qquad 0 < n \leq m.$$

By definition of $\mathcal{L}'$ and (4.2), we have $\mathcal{L}'(1) = b_0^{-1}\mathcal{F}(x) = 1$. For $0 < n \leq m$, we have

$$\mathcal{L}'(x^{-n}P_m(x)) = b_0^{-1} \cdot \mathcal{F}(x^{-(n-1)}P_m(x)) = 0$$

because $0 \leq n - 1 < m$. This completes the proof. □

We will find another connection between the linear functionals $\mathcal{F}$ and $\mathcal{L}$ using inverted polynomials.

Definition 4.5. The *inverted polynomial* of $P_n(x)$ is defined by $P_n^*(x) := x^n P_n(x^{-1})/P_n(0)$. For any linear functional $\mathcal{M}$ on V define $\mathcal{M}^*$ by

$$\mathcal{M}^*(f(x)) := \mathcal{M}(f(x^{-1})).$$

Using (4.1) and $P_n(0) = (-1)^n b_0 b_1 \dots b_{n-1}$, we have

$$P_{n+1}^*(x) = (x - b_n^*)P_n^*(x) - a_n^* x P_{n-1}^*(x),$$

where

$$b_n^* := \frac{1}{b_n}, \qquad a_n^* := \frac{a_n}{b_{n-1}b_n}.$$

It is easy to check that the map $X \mapsto X^*$ is an involution, i.e., $X^{**} = X$, for each $X \in \{P_n, a_n, b_n, \mathcal{M}\}$.

Let $P = \{P_n(x)\}_{n\ge 0}$ be the sequence of polynomials given by (4.1). By Theorems 4.1 and 4.9 there are unique linear functionals, denoted by $\mathcal{L}_P$ and $\mathcal{F}_P$, satisfying $\mathcal{L}(1) = \mathcal{F}(1) = 1$ and

$$\mathcal{L}_P(x^{-n}P_m(x)) = 0, \qquad 0 < n \le m,$$
$$\mathcal{F}_P(x^{-n}P_m(x)) = 0, \qquad 0 \le n < m.$$

We will sometime write $\mathcal{L}$ in place of $\mathcal{L}_P$.

Proposition 4.6. *Let $P = \{P_n(x)\}_{n\ge 0}$ be the sequence of polynomials given by* (4.1) *and let $P^* = \{P_n^*(x)\}_{n\ge 0}$. Then we have*

$$\mathcal{F}_{P^*} = \mathcal{L}_P^*, \qquad \mathcal{L}_{P^*} = \mathcal{F}_P^*.$$

Proof. For $0 \le n < m$, by the definition of $P_m^*(x)$, we have

$$\mathcal{L}_P^*(x^{-n}P_m^*(x)) = \mathcal{L}_P(x^nP_m^*(x^{-1})) = \mathcal{L}_P(x^{-(m-n)}P_m(x)/P_m(0)) = 0,$$

where the last equality follows from Theorem 4.9 since $0 < m - n \le m$. By Theorem 4.1, this shows the first identity $\mathcal{F}_{P^*} = \mathcal{L}_P^*$.

Applying the first identity to P^*, we have $\mathcal{L}_{P^*}^* = \mathcal{F}_{(P^*)^*}$. Then

$$\mathcal{L}_{P^*} = (\mathcal{L}_{P^*}^*)^* = (\mathcal{F}_{(P^*)^*})^* = \mathcal{F}_P^*,$$

which gives the second identity. □

Kamioka [12, Lemma 3.1] showed that (in our notation) for $n, m \ge 0$,

$$\mathcal{F}(x^{n+1}P_m(x)) = \mathcal{F}(x) \sum_{\pi\in \mathrm{Sch}((0,m)\to(n,0))} \mathrm{wt}(\pi), \tag{4.5}$$

$$\mathcal{F}^*(x^nP_m^*(x)) = \sum_{\pi\in \mathrm{Sch}((0,m)\to(n,0))} \mathrm{wt}^*(\pi), \tag{4.6}$$

where $\mathrm{wt}^*(\pi)$ is the same weight $\mathrm{wt}(\pi)$ with b_n and a_n replaced by b_n^* and a_n^*, respectively.

Remark 4.7. Kamioka's definition of Schröder paths is different from ours; his definition is the one explained in Remark 3.3. In [12, Lemma 3.1] it is written that "both the sums range over all Schröder paths from $(-n, -n)$ to $(2k, 0)$." Here $(-n, -n)$ is a typo for $(-n, n)$.

By Proposition 4.4, the first identity (4.5) is equivalent to

$$\mathcal{L}(x^nP_m(x)) = \sum_{\pi\in \mathrm{Sch}((0,m)\to(n,0))} \mathrm{wt}(\pi). \tag{4.7}$$

By Proposition 4.6, the left hand side of (4.6) can be rewritten as

$$\mathcal{F}^*(x^nP_m^*(x)) = \mathcal{L}_{P^*}(x^nP_m^*(x)),$$

and therefore the second identity (4.6) is also equivalent to (4.7) with P replaced by P^*.

The following theorem, which is the special case $\lambda_n = 0$ of Theorem 3.12, is a generalization of (4.7), hence a generalization of Kamioka's results (4.5) and (4.6).

Theorem 4.8. *For $n, m, \ell \ge 0$, we have*

$$\mathcal{L}(x^nP_m(x)Q_\ell(x)) = \sum_{\pi\in \mathrm{Sch}((0,m)\to(n,\ell))} \mathrm{wt}(\pi).$$

We give a similar formula for $\mathcal{L}(x^nP_m(x)Q_\ell(x))$ when n is negative.

Theorem 4.9. *For $n, m, \ell \geq 0$, we have*

$$\mathcal{L}(x^{-n-1}Q_m(x)P_\ell(x)) = \frac{a_1^* \dots a_\ell^* P_m(0)P_\ell(0)}{a_1 \dots a_m b_0} \sum_{\pi \in \mathrm{Sch}((0,m)\to(n,\ell))} \mathrm{wt}^*(\pi).$$

Proof. Since $Q_\ell(x) = x^{-\ell}P_\ell(x)/a_1 \dots a_\ell$, applying Theorem 4.1 to the inverted polynomials $P_k^*(x)$, we obtain

$$\mathcal{L}_{P^*}\left(x^n P_m^*(x) \frac{x^{-\ell}P_\ell^*(x)}{a_1^* \dots a_\ell^*}\right) = \sum_{\pi \in \mathrm{Sch}((0,m)\to(n,\ell))} \mathrm{wt}(\pi). \tag{4.8}$$

Observe that

$$\mathcal{L}_{P^*}(f(x)) = \mathcal{F}_P^*(f(x)) = \mathcal{F}_P(f(x^{-1})) = b_0\mathcal{L}_P(x^{-1}f(x^{-1})),$$

where the first, second, and third equalities follow from Proposition 4.6, the definition of $\mathcal{F}_P^*$, and Proposition 4.4, respectively. Therefore the left hand side of (4.8) can be rewritten as

$$b_0\mathcal{L}_P\left(x^{-1-n}P_m^*(x^{-1})\frac{x^\ell P_\ell^*(x^{-1})}{a_1^* \dots a_\ell^*}\right) = \frac{a_1 \dots a_m b_0}{a_1^* \dots a_\ell^* P_m(0)P_\ell(0)}\mathcal{L}\left(x^{-1-n}Q_m(x)P_\ell(x)\right), \tag{4.9}$$

where the following identities are used:

$$P_m^*(x^{-1}) = \frac{x^{-m}P_m(x)}{P_m(0)} = \frac{a_1 \dots a_m Q_m(x)}{P_m(0)}, \qquad P_\ell^*(x^{-1}) = \frac{x^{-\ell}P_\ell(x)}{P_\ell(0)}.$$

Using (4.8) and (4.9) we obtain the desired identity. □

If $m = \ell = 0$ in Theorems 4.8 and 4.9 we obtain the following result of Kamioka.

Corollary 4.10. [12, Theorem 3.1] *For $n \geq 0$, we have*

$$\mathcal{L}(x^n) = \sum_{\pi \in \mathrm{Sch}_n} \mathrm{wt}(\pi),$$

$$\mathcal{L}(x^{-n-1}) = b_0^{-1} \sum_{\pi \in \mathrm{Sch}_n} \mathrm{wt}^*(\pi).$$

5. Lattice paths with bounded height

In this section we express the generating function for Motzkin-Schröder paths with bounded height as quotients of inverted polynomials where the indices of the sequences $b = \{b_n\}_{n\geq 0}, a = \{a_n\}_{n\geq 0}$, and $\lambda = \{\lambda_n\}_{n\geq 0}$ are shifted.

Recall that $P_n(x)$ are defined in (1.1). We will denote this polynomial by $P_n(x; b, a, \lambda)$ to indicate that the three-term recurrence coefficients are taken from the sequences $b = \{b_n\}_{n\geq 0}, a = \{a_n\}_{n\geq 0}$, and $\lambda = \{\lambda_n\}_{n\geq 0}$. The inverted polynomial $P_n^*(x) = x^n P_n(1/x)$ will also be written as $P_n^*(x; b, a, \lambda) = x^n P_n(1/x; b, a, \lambda)$. Note that $P_n^*(x) = x^n P_n(1/x)$ satisfy $P_{-1}^*(x) = 0, P_1^*(x) = 1$ and

$$P_{n+1}^*(x) = (1 - b_n x)P_n^*(x) - (a_n x + \lambda_n x^2)P_{n-1}^*(x). \tag{5.1}$$

Definition 5.1. For a sequence $s = \{s_n\}_{n\geq 0}$ define $\delta s = \{s_{n+1}\}_{n\geq 0}$. For $P_n(x) = P_n(x; b, a, \lambda)$, we also define

$$\delta P_n(x) = \delta P_n(x; b, a, \lambda) = P_n(x; \delta b, \delta a, \delta\lambda),$$

$$\delta P_n^*(x) = \delta P_n^*(x; b, a, \lambda) = P_n^*(x; \delta b, \delta a, \delta\lambda).$$

Definition 5.2. We denote by $\mathrm{MS}_{n,r,s}^{\le k}$ the set of Motzkin-Schröder paths from $(0,r)$ to (n,s) such that the y-coordinate of every point is at most k. Define $\mathrm{MS}_{n}^{\le k} := \mathrm{MS}_{n,0,0}^{\le k}$ and

$$\mu_{n,r,s}^{\le k} := \sum_{\pi \in \mathrm{MS}_{n,r,s}^{\le k}} \mathrm{wt}(\pi),$$
$$\mu_{n}^{\le k} := \sum_{\pi \in \mathrm{MS}_{n}^{\le k}} \mathrm{wt}(\pi).$$

The goal of this section is to prove the following theorem, which is a generalization of Viennot's results [18, (27) on page V-19] on orthogonal polynomials (the case $a_n = 0$).

Theorem 5.3. *If $r \le s$, then*

$$\sum_{n\ge 0} \mu_{n,r,s}^{\le k} x^n = \frac{P_r^*(x)\delta^{s+1} P_{k-s}^*(x)}{P_{k+1}^*(x)} \cdot x^{s-r}. \tag{5.2}$$

If $r > s$, then

$$\sum_{n\ge 0} \mu_{n,r,s}^{\le k} x^n = \frac{P_s^*(x)\delta^{r+1} P_{k-r}^*(x)}{P_{k+1}^*(x)} \cdot \prod_{i=s+1}^{r} (a_i + \lambda_i x). \tag{5.3}$$

If $r = s = 0$ in Theorem 5.3 we obtain the following corollary.

Corollary 5.4. *We have*

$$\sum_{n\ge 0} \mu_{n}^{\le k} x^n = \frac{\delta P_k^*(x)}{P_{k+1}^*(x)}.$$

On the other hand, using Flajolet's argument [7], we obtain a continued fraction expression for the generating function for $\mu_n^{\le k}$.

Proposition 5.5. *We have*

$$\sum_{n\ge 0} \mu_{n}^{\le k} x^n = \cfrac{1}{1 - b_0 x - \cfrac{a_1 x + \lambda_1 x^2}{1 - b_1 x - \cfrac{a_2 x + \lambda_2 x^2}{1 - b_2 x - \ddots - \cfrac{a_k x + \lambda_k x^2}{1 - b_k x}}}}.$$

Remark 5.6. Combining Corollary 5.4 and Proposition 5.5 gives

$$\frac{\delta P_k^*(x)}{P_{k+1}^*(x)} = \cfrac{1}{1 - b_0 x - \cfrac{a_1 x + \lambda_1 x^2}{1 - b_1 x - \cfrac{a_2 x + \lambda_2 x^2}{1 - b_2 x - \ddots - \cfrac{a_k x + \lambda_k x^2}{1 - b_k x}}}}, \tag{5.4}$$

which can also be shown using the following fundamental recurrence relations for continued fractions, see [3, Chapter III, §2].

For the remainder of this section we give a proof of Theorem 5.3. To do this we give a combinatorial meaning to

$$P_{k+1}^*(x) \sum_{n\ge 0} \mu_{n,r,s}^{\le k} x^n.$$

First we need a combinatorial interpretation for $P_n^*(x)$. Similarly to Theorem 3.11, the recurrence (5.1) gives the following proposition.

Proposition 5.7. *For $n \ge 0$, we have*

$$P_n^*(x) = \sum_{T\in \mathrm{FT}_n} \mathrm{wt}(T) x^{\mathsf{bm}(T)+\mathsf{rm}(T)+2\mathsf{rd}(T)}.$$

Let

$$\mathrm{MS}_{*,r,s}^{\le k} := \bigcup_{n\ge 0} \mathrm{MS}_{n,r,s}^{\le k}.$$

Then by Proposition 5.7 we have

$$P_{k+1}^*(x) \sum_{n\ge 0} \mu_{n,r,s}^{\le k} x^n = \sum_{T\in \mathrm{FT}_{k+1}} \sum_{\pi\in \mathrm{MS}_{*,r,s}^{\le k}} \mathrm{wt}(T)\,\mathrm{wt}(\pi) x^{|\pi|+\mathsf{rm}(T)+\mathsf{bm}(T)+2\mathsf{bd}(T)}.$$

In what follows we construct a sign-reversing involution on $\mathrm{MS}_{*,r,s}^{\le k} \times \mathrm{FT}_{k+1}$, which cancels many terms in the above equation. The basic idea is similar to that in Section 3: for $(\pi, T) \in \mathrm{MS}_{*,r,s}^{\le k} \times \mathrm{FT}_{k+1}$ we add or remove a horizontal step, a peak ((U,V) or (U,D)), or a valley ((V,U) or (D,U)) in π and modify the corresponding tile(s) in T. We first need several terminologies.

For $\pi \in \mathrm{MS}_{*,r,s}^{\le k}$, we define $|\pi|$ to be n if $\pi \in \mathrm{MS}_{n,r,s}^{\le k}$. A *valley* of π is a pair (D,U) or (V,U) of consecutive steps in π and a *peak* of π is a pair (U,D) or (U,V) of consecutive steps in π.

Let $(\pi, T) \in \mathrm{MS}_{*,r,s}^{\le k} \times \mathrm{FT}_{k+1}$ and write $\pi = S_1 \dots S_m$ as a sequence of steps. A *removable point* of π is a point (j,h) on π satisfying one of the following conditions:

- π has a horizontal step starting at (j,h),
- $h \ge \min(r,s)$ and π has a peak starting at (j,h), or
- $h \le \min(r,s)$ and π has a valley starting at (j,h).

In other words, a removable point of π is the starting point of a horizontal step, a peak above the line $y = \min(r,s)$, or a valley below the line $y = \min(r,s)$. Let $\mathsf{remove}(\pi)$ denote the smallest integer $i \ge 0$ such that the ending point of $S_1 \dots S_i$ is a removable point. (If $i = 0$, the ending point of $S_1 \dots S_i$ means the starting point of π, which is $(0,r)$.) If there is no such integer i, we define $\mathsf{remove}(\pi) = \infty$. See Figures 7 and 8.

An *addable point* of (π, T) is a point (j,h) on π satisfying one of the following conditions:

- T has a red monomino containing $h+1$,
- $h \ge \min(r,s)$ and T has a (red or black) domino containing $h+1, h+2$, or
- $h \le \min(r,s)$ and T has a (red or black) domino containing $h, h+1$.

In other words, an addable point of (π, T) is an intersection of π with the line $y = h$ for some h such that T has a red monomino with $h+1$, a domino with $h+1, h+2$ and $h \ge \min(r,s)$, or a domino with $h, h+1$ and $h \le \min(r,s)$. Let $\mathsf{add}(\pi, T)$ denote the smallest integer $i \ge 0$ such that the ending point of $S_1 \dots S_i$ is an addable point. If there is no such integer i, we define $\mathsf{add}(\pi, T) = \infty$. See Figures 9 and 10.

We are now ready to define a map $\phi : \mathrm{MS}_{*,r,s}^{\le k} \times \mathrm{FT}_{k+1} \to \mathrm{MS}_{*,r,s}^{\le k} \times \mathrm{FT}_{k+1}$. Let $(\pi, T) \in \mathrm{MS}_{*,r,s}^{\le k} \times \mathrm{FT}_{k+1}$ and write $\pi = S_1 \dots S_m$ as a sequence of steps. Then $\phi(\pi, T) = (\pi', T')$ is defined as follows.

Case 1: $\mathsf{remove}(\pi) = \mathsf{add}(\pi, T) = \infty$. In this case, $(\pi', T') = (\pi, T)$.

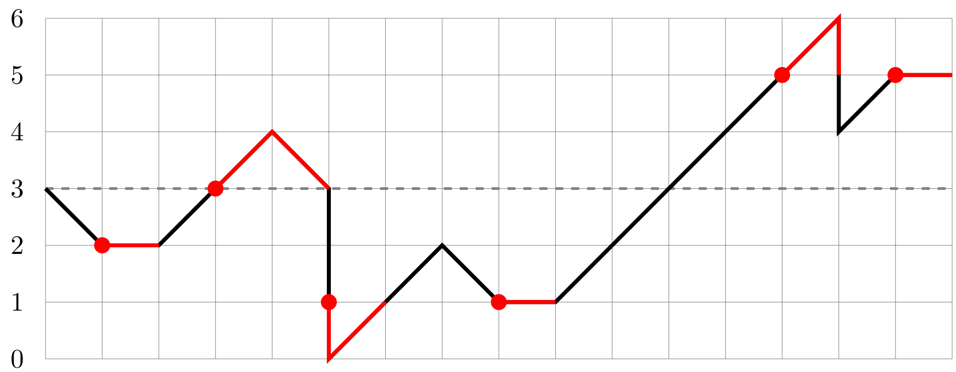

FIGURE 7. A Motzkin-Schröder path π in $\mathrm{MS}_{*,r,s}^{\le k}$ for $r = 3, s = 5$ and $k = 6$. The dashed line is the line $y = \min(r, s)$. The red dots are the removable points of π. The horizontal step, a peak, or a valley starting at each removable point is colored red. In this case $\mathsf{remove}(\pi) = 1$ because the first removable point occurs after the first step.

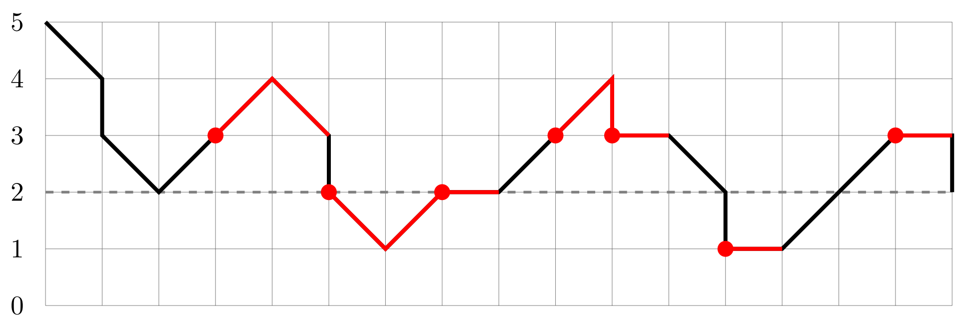

FIGURE 8. A Motzkin-Schröder path π in $\mathrm{MS}_{*,r,s}^{\le k}$ for $r = 5, s = 2$ and $k = 5$. The dashed line is the line $y = \min(r, s)$. The red dots are the removable points of π. The horizontal step, a peak, or a valley starting at each removable point is colored red. In this case $\mathsf{remove}(\pi) = 4$ because the first removable point occurs after the fourth step.

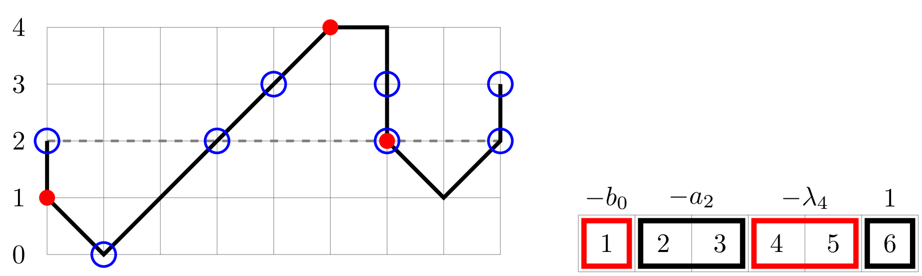

FIGURE 9. An element $(\pi, T) \in \mathrm{MS}_{*,r,s}^{\le k} \times \mathrm{FT}_{k+1}$ for $r = 2, s = 3, k = 5$. The dashed line is the line $y = \min(r, s)$. The removable points of π are the red dots and $\mathsf{remove}(\pi) = 1$. The addable points of (π, T) are circled. Since the first addable point occurs at the beginning of π, we have $\mathsf{add}(\pi, T) = 0$.

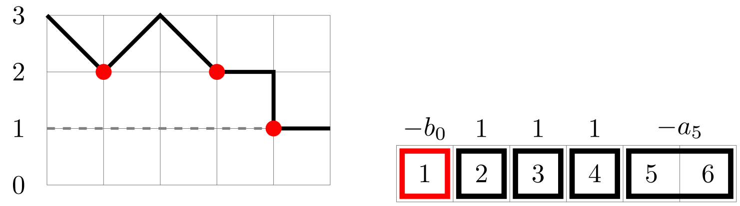

FIGURE 10. An element $(\pi, T) \in \mathrm{MS}^{\le k}_{*,r,s} \times \mathrm{FT}_{k+1}$ for $r = 3, s = 1, k = 5$. The dashed line is the line $y = \min(r, s)$. The removable points of π are the red dots and $\mathsf{remove}(\pi) = 1$. Since there are no addable points, we have $\mathsf{add}(\pi, T) = \infty$.

Case 2: $\mathsf{remove}(\pi) \ge \mathsf{add}(\pi, T)$. Suppose that $i = \mathsf{add}(\pi, T)$ and $S_1 \dots S_i$ ends at (j, h), which is an addable point. Let τ be the tile of T containing $h + 1$. Then

$$\pi' = \begin{cases} S_1 \dots S_i H S_{i+1} \dots S_m, & \text{if } \tau \text{ is a red monomino,} \\ S_1 \dots S_i U D S_{i+1} \dots S_m, & \text{if } \tau \text{ is a red domino and } h \ge \min(r, s), \\ S_1 \dots S_i U V S_{i+1} \dots S_m, & \text{if } \tau \text{ is a black domino and } h \ge \min(r, s), \\ S_1 \dots S_i D U S_{i+1} \dots S_m, & \text{if } \tau \text{ is a red domino and } h \le \min(r, s), \\ S_1 \dots S_i V U S_{i+1} \dots S_m, & \text{if } \tau \text{ is a black domino and } h \le \min(r, s), \end{cases}$$

and T' is the tiling obtained from T by replacing τ by one or two black monomino(s) according to the size of τ.

Case 3: $\mathsf{remove}(\pi) < \mathsf{add}(\pi, T)$. Suppose that $i = \mathsf{remove}(\pi, T)$ and $S_1 \dots S_i$ ends at (j, h), which is a removable point. Then

$$\pi' = \begin{cases} S_1 \dots \widehat{S}_{i+1} \dots S_m, & \text{if } S_{i+1} = H, \\ S_1 \dots \widehat{S}_{i+1} \widehat{S}_{i+2} \dots S_m, & \text{otherwise,} \end{cases}$$

$$T' = \begin{cases} T - B_{h+1} + R_{h+1}, & \text{if } S_{i+1} = H, \\ T - B_{h+1} - B_{h+2} + R_{h+1,h+2}, & \text{if } (S_{i+1}, S_{i+2}) = (U, D) \text{ and } h \ge \min(r, s), \\ T - B_{h+1} - B_{h+2} + B_{h+1,h+2}, & \text{if } (S_{i+1}, S_{i+2}) = (U, V) \text{ and } h \ge \min(r, s), \\ T - B_h - B_{h+1} + R_{h,h+1}, & \text{if } (S_{i+1}, S_{i+2}) = (D, U) \text{ and } h \le \min(r, s), \\ T - B_h - B_{h+1} + B_{h,h+1}, & \text{if } (S_{i+1}, S_{i+2}) = (D, V) \text{ and } h \le \min(r, s). \end{cases}$$

Here, for example, $T - B_h - B_{h+1} + R_{h,h+1}$ means the tiling obtained from T by removing a black monomino with h and a black monomino with $h + 1$ and adding a red domino with $h, h + 1$.

Example 5.8. If (π, T) is the element in Figure 9, then $\phi(\pi, T)$ is the element in Figure 11. If (π, T) is the element in Figure 10, then $\phi(\pi, T)$ is the element in Figure 12.

Lemma 5.9. *The map* $\phi : \mathrm{MS}^{\le k}_{*,r,s} \times \mathrm{FT}_{k+1} \to \mathrm{MS}^{\le k}_{*,r,s} \times \mathrm{FT}_{k+1}$ *is a sign-reversing involution, i.e., if* $\phi(\pi, T) = (\pi', T')$ *with* $(\pi, T) \ne (\pi', T')$*, then* $\mathrm{wt}(\pi', T') = -\mathrm{wt}(\pi, T)$*, where*

$$\mathrm{wt}(\pi, T) := \mathrm{wt}(T)\,\mathrm{wt}(\pi) x^{|\pi| + \mathsf{rm}(T) + \mathsf{bm}(T) + 2\mathsf{bd}(T)}.$$

Moreover, the set of fixed points of ϕ *is given by*

$$\mathrm{Fix}(\phi) = \{(\phi, T) : \mathsf{remove}(\pi) = \mathsf{add}(\pi, T) = \infty\}.$$

Proof. This is a straightforward verification using the definition of ϕ. We omit the details. $\square$

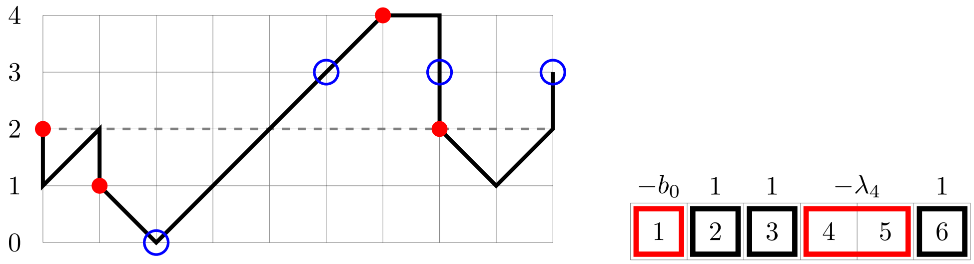

FIGURE 11. An element $(\pi, T) \in \mathrm{MS}^{\le k}_{*,r,s} \times \mathrm{FT}_{k+1}$ for $r = 2, s = 3, k = 5$. The dashed line is the line $y = \min(r, s)$. The removable points of π are the red dots and $\mathsf{remove}(\pi) = 0$. The addable points of (π, T) are circled and $\mathsf{add}(\pi, T) = 4$.

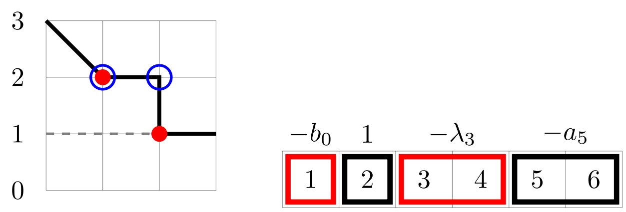

FIGURE 12. An element $(\pi, T) \in \mathrm{MS}^{\le k}_{*,r,s} \times \mathrm{FT}_{k+1}$ for $r = 3, s = 1, k = 5$. The dashed line is the line $y = \min(r, s)$. The removable points of π are the red dots and $\mathsf{remove}(\pi) = 1$. The removable points of (π, T) are circled and $\mathsf{add}(\pi, T) = 1$.

Now we are ready to prove Theorem 5.3.

Proof of Theorem 5.3. The theorem can be reformulated as follows:

$$P^*_{k+1}(x) \sum_{n\ge 0} \mu^{\le k}_{n,r,s} x^n = \begin{cases} P^*_r(x)\delta^{s+1} P^*_{k-s}(x) x^{s-r}, & \text{if } r \le s, \\ P^*_s(x)\delta^{s+1} P^*_{k-r}(x) \prod_{i=s+1}^{r} (a_i + \lambda_i x), & \text{if } r > s. \end{cases} \tag{5.5}$$

By Lemma 5.9 we have

$$\begin{aligned} P^*_{k+1}(x) \sum_{n\ge 0} \mu^{\le k}_{n,r,s} x^n &= \sum_{T\in \mathrm{FT}_{k+1}} \sum_{\pi \in \mathrm{MS}^{\le k}_{*,r,s}} \mathrm{wt}(T)\,\mathrm{wt}(\pi) x^{|\pi| + \mathsf{rm}(T) + \mathsf{bm}(T) + 2\mathsf{bd}(T)} \\ &= \sum_{(\pi,T)\in \mathrm{Fix}(\phi)} \mathrm{wt}(T)\,\mathrm{wt}(\pi) x^{|\pi| + \mathsf{rm}(T) + \mathsf{bm}(T) + 2\mathsf{bd}(T)}, \end{aligned}$$

where $\mathrm{Fix}(\phi)$ is the set of pairs $(\pi, T) \in \mathrm{MS}^{\le k}_{*,r,s} \times \mathrm{FT}_{k+1}$ such that $\mathsf{remove}(\pi) = \mathsf{add}(\pi, T) = \infty$.

We first consider the case $r \le s$. Suppose $(\pi, T) \in \mathrm{Fix}(\phi)$. It is easy to see that there is a unique π satisfying $\mathsf{remove}(\pi) = \infty$, namely $\pi = UU \dots U$ consisting of $s - r$ up steps. This π contributes the factor x^{s-r} in (5.5). Moreover, since $\mathsf{add}(\pi, T) = \infty$, we must have that the tile in T containing h must be a red monomino for every $r+1 \le h \le s+1$. On the other hand, there is no restriction on the tiles containing $i \le r$ and $j \ge s+2$. If we sum over all $(\pi, T) \in \mathrm{Fix}(\phi)$, the part of T consisting of tiles with entries in $\{1, 2, \dots, r\}$ (resp. $\{s+2, s+3, \dots, k+1\}$) contributes the factor $P^*_r(x)$ (resp. $\delta^{s+1} P^*_{k-s}(x)$) in (5.5). This shows (5.5) when $r \le s$.

Now we consider the case $r < s$. This can be shown similarly as in the previous case. The only difference is that π is not unique, but π can be any path with $r - s$ steps, where each step is either U or V. The contribution of such π's is the factor $\prod_{i=s+1}^{r}(a_i+\lambda_i x)$ in (5.5). This completes the proof. □

6. Determinants for type R_I polynomials

Classically the orthogonal polynomials may be given by a quotient of Hankel determinants:

$$P_n(x) = \frac{1}{\det(\mu_{i+j})_{i,j=0}^{n-1}} \det \begin{pmatrix} (\mu_{i+j})_{\substack{0\le i\le n-1\\ 0\le j\le n}} \\ (x^j)_{0\le j\le n} \end{pmatrix}. \tag{6.1}$$

Moreover, we know that ([3, p. 16]) the denominator determinant factors

$$\Delta_n := \det(\mu_{i+j})_{i,j=0}^{n} = \prod_{k=1}^{n} \lambda_k^{n+1-k}. \tag{6.2}$$

In this section we find analogues of (6.1) and (6.2) for type R_I polynomials $P_n(x)$ and $Q_n(x)$.

One may ask why such determinantal formulas arise. In the classical case the orthogonality relations may be considered as linear equations for the coefficients of $P_n(x)$. These may be solved by Cramer's rule to give a quotient of determinants [3, p. 12]. These are exactly the Hankel determinants in (6.1). The type R_I orthogonality relations may also be considered as linear relations for expansion coefficients of $P_n(x)$ and $Q_n(x)$. The same method would gives the determinant in this section, but not the product formulas for the denominator determinants. Thus we do not use Cramer's rule in this section.

6.1. Quotients of determinants for $P_n(x)$ and $Q_n(x)$.

We shall give in Theorem 6.1 three quotients of determinants for $P_n(x)$ and $Q_n(x)$. We also factor the three denominator determinants in Theorem 6.4, these are

$$\Delta'_n := \det(\nu_{i+j,n})_{0\le i,j\le n},$$
$$\Delta''_n := \det(\nu_{i+j,j})_{0\le i,j\le n},$$
$$\Delta'''_n := \det(\nu_{i,j})_{0\le i,j\le n}.$$

Theorem 6.4 shows that Δ'_n and Δ'''_n are always nonzero and Δ''_n is nonzero if $\lambda_k \ne 0$ for all $k \ge 1$.

Theorem 6.1. *We have*

$$P_n(x) = \frac{1}{\Delta'_n} \det \begin{pmatrix} \nu_{0,n} & \nu_{1,n} & \cdots & \nu_{n,n} \\ \nu_{1,n} & \nu_{2,n} & \cdots & \nu_{n+1,n} \\ \vdots & \vdots & \ddots & \vdots \\ \nu_{n-1,n} & \nu_{n,n} & \cdots & \nu_{2n-1,n} \\ 1 & x & \cdots & x^n \end{pmatrix} = \frac{1}{\Delta'_n} \det \begin{pmatrix} (\nu_{i+j,n})_{\substack{0\le i\le n-1\\ 0\le j\le n}} \\ (x^j)_{0\le j\le n} \end{pmatrix}, \tag{6.3}$$

$$Q_n(x) = \frac{1}{\Delta''_n} \det \begin{pmatrix} \nu_{0,0} & \nu_{1,1} & \cdots & \nu_{n,n} \\ \nu_{1,0} & \nu_{2,1} & \cdots & \nu_{n+1,n} \\ \vdots & \vdots & \ddots & \vdots \\ \nu_{n-1,0} & \nu_{n,1} & \cdots & \nu_{2n-1,n} \\ 1 & \frac{x}{d_1(x)} & \cdots & \frac{x^n}{d_n(x)} \end{pmatrix} = \frac{1}{\Delta''_n} \det \begin{pmatrix} (\nu_{i+j,j})_{\substack{0\le i\le n-1\\ 0\le j\le n}} \\ \left(\frac{x^j}{d_j(x)}\right)_{0\le j\le n} \end{pmatrix}, \tag{6.4}$$

$$(6.5)\qquad Q_n(x) = \frac{1}{\Delta_n'''} \det \begin{pmatrix} \nu_{0,0} & \nu_{0,1} & \cdots & \nu_{0,n} \\ \nu_{1,0} & \nu_{1,1} & \cdots & \nu_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ \nu_{n-1,0} & \nu_{n-1,1} & \cdots & \nu_{n-1,n} \\ 1 & \frac{1}{d_1(x)} & \cdots & \frac{1}{d_n(x)} \end{pmatrix} = \frac{1}{\Delta_n'''} \det \begin{pmatrix} (\nu_{i,j})_{\substack{0\le i\le n-1\\ 0\le j\le n}} \\ \left(\frac{1}{d_j(x)}\right)_{0\le j\le n} \end{pmatrix}.$$

Proof. Let $f(x)$ be the determinant in the right hand side of (6.3), which is a polynomial in x of degree n. Multiplying the last row of the determinant by $x^j/d_n(x)$, and applying $\mathcal{L}$, yields a last row equal to the $j+1$ row, so $\mathcal{L}(f(x)x^j/d_n(x)) = 0$ for $0 \le j < n$.

Note that $P_n(x)$ is uniquely determined, up to a multiple, as the polynomial of degree n such that $\mathcal{L}(g(x)P_n(x)/d_n(x)) = 0$ for any polynomial $g(x)$ of degree at most $n-1$. Thus there is a constant c satisfying

$$(6.6)\qquad P_n(x) = cf(x).$$

Multiplying both sides of (6.6) by $x^n/d_n(x)$ and applying $\mathcal{L}$ yield

$$\mathcal{L}(x^n Q_n(x)) = c\Delta_n'.$$

Since $\mathcal{L}(x^n Q_n(x)) = 1$, we have $c = 1/\Delta_n'$, which proves (6.3).

The second and third identities can be proved similarly. □

Since $P_n(x)$ is a monic polynomial, the coefficient of x^n is 1. The only contribution of x^n occurs in the cofactor of x^n in the determinant in (6.3). Thus we have the following corollary.

Corollary 6.2. *We have*

$$\Delta_n' = \det(\nu_{i+j,n})_{0\le i,j\le n} = \det(\nu_{i+j,n})_{0\le i,j\le n-1}.$$

Dividing both sides of (6.3) by $d_n(x)$, we may obtain another formula for $Q_n(x)$.

Remark 6.3. We can take special cases of Theorem 6.1 to recover the classical results and Kamioka's results.

If $a_k = 0$ for all $k \ge 0$, then $d_k(x) = \lambda_1 \dots \lambda_k$, $\nu_{i,j} = \mu_i/\lambda_1 \dots \lambda_j$, and therefore

$$(6.7)\qquad \Delta_n' = \frac{\Delta_n}{(\lambda_1\dots\lambda_n)^{n+1}}, \qquad \Delta_n'' = \frac{\Delta_n}{\lambda_1^n \lambda_2^{n-1} \dots \lambda_n^1}, \qquad \Delta_n''' = 0.$$

In this case $P_n(x)$ become the usual orthogonal polynomials and (6.4) reduces to (6.1).

If $\lambda_k = 0$ for all $k \ge 0$, then $d_k(x) = a_1 \dots a_k x^k$, $\nu_{i,j} = \mathcal{L}(x^{i-j})/a_1 \dots a_j$, and

$$(6.8)\qquad \Delta_n' = \frac{\det(\mathcal{L}(x^{i+j-n}))_{0\le i,j\le n}}{(a_1\dots a_n)^{n+1}}, \qquad \Delta_n'' = 0, \qquad \Delta_n''' = \frac{\det(\mathcal{L}(x^{i-j}))_{0\le i,j\le n}}{a_1^n a_2^{n-1} \dots a_n^1}.$$

In this case the polynomials $P_n(x)$ become biorthogonal Laurent polynomials and (6.5) reduces to

$$(6.9)\qquad P_n(x) = \frac{a_1\dots a_n}{\det(\mathcal{L}(x^{i-j}))_{0\le i,j\le n}} \det \begin{pmatrix} (\mathcal{L}(x^{i-j}))_{\substack{0\le i\le n-1\\ 0\le j\le n}} \\ (x^{n-j})_{0\le j\le n} \end{pmatrix},$$

which is slightly different but equivalent to Kamioka's formula [12, (2.8)].

Here is how the three denominator determinants factor.

Theorem 6.4. *We have*

$$\Delta'_n = \prod_{k=1}^{n} \frac{1}{(-a_k)^k P_k(-\lambda_k/a_k)}, \tag{6.10}$$

$$\Delta''_n = \prod_{k=1}^{n} \frac{\lambda_k^k}{(-a_k)^k P_k(-\lambda_k/a_k)}, \tag{6.11}$$

$$\Delta'''_n = \prod_{k=1}^{n} \frac{1}{P_k(-\lambda_k/a_k)}. \tag{6.12}$$

Remark 6.5. Note that if $a_k = 0$, by (6.7), both (6.10) and (6.11) reduce to the formula (6.2) for the Hankel determinant of orthogonal polynomials.

If $\lambda_k = 0$, the constant term of $P_k(x)$ is $P_k(0) = (-1)^k b_0 b_1 \dots b_{k-1}$. In this case we have

$$\Delta'_n = \prod_{k=1}^{n} \frac{1}{a_k^k b_{k-1}^{n-k+1}}, \qquad \Delta''_n = 0, \qquad \Delta'''_n = (-1)^{\binom{n+1}{2}} \prod_{k=1}^{n} \frac{1}{b_{k-1}^{n-k+1}}.$$

Since $\nu_{i,j} = \mathcal{L}(x^i/d_j(x)) = \mathcal{L}(x^{i-j}/a_1 \dots a_j)$, the above formulas for Δ'_n and Δ'''_n are equivalent to the following Toeplitz determinant formula due to Kamioka [12, 2.14b]:

$$\det(\mathcal{L}(x^{i-j}))_{0\le i,j\le n} = (-1)^{\binom{n+1}{2}} \prod_{k=1}^{n} \left(\frac{a_k}{b_{k-1}}\right)^{n-k+1}. \tag{6.13}$$

The rest of this subsection is devoted to proving Theorem 6.4. We shall use determinantal facts to reduce the determinant to one involving the orthogonality relations, (6.14). We need several lemmas.

Lemma 6.6. *Let $\mathcal{L}$ be any linear functional defined on the vector space spanned by $x^i/r_j(x)$ for $i, j \ge 0$, where $r_j(x)$ is a fixed polynomial for each $j \ge 0$. Suppose that for $k \ge 0$, $p_k(x)$ and $q_k(x)$ are given by*

$$p_k(x) = \sum_{i=0}^{k} p_{k,i} x^i, \qquad q_k(x) = \sum_{i=0}^{k} \frac{q_{k,i} x^i}{r_i(x)},$$

where $p_{k,k} \ne 0$ and $q_{k,k} \ne 0$. Then we have

$$\det\left(\mathcal{L}\left(\frac{x^{i+j}}{r_j(x)}\right)\right)_{i,j=0}^{n} = \frac{1}{\prod_{i=0}^{n} p_{i,i} q_{i,i}} \det\left(\mathcal{L}\left(p_i(x) q_j(x)\right)\right)_{i,j=0}^{n}.$$

Proof. The proof is similar to that of Lemma 6.12. The matrices $P = (p_{i,j})_{i,j=0}^n$ and $Q = (q_{j,i})_{i,j=0}^n$ are upper and lower triangular, respectively. Thus, letting $M = (\mathcal{L}(x^{i+j}/r_j(x)))_{i,j=0}^n$, we have

$$\det(\mathcal{L}(x^{i+j}/r_j(x)))_{i,j=0}^n = \frac{1}{\prod_{i=0}^{n} p_{i,i} q_{i,i}} \det(PMQ)_{i,j=0}^n.$$

Since the (k, s)-entry of PMQ is

$$\sum_{i,j=0}^{n} \mathcal{L}(p_{ki} x^i q_{sj} x^j / r_j(x)) = \mathcal{L}(p_k(x) q_s(x)),$$

we are done. □

Lemma 6.7. *For any polynomial $f(x)$ of degree n, there are unique numbers $A_0, \dots, A_n$ satisfying*

$$f(x) = \sum_{i=0}^{n} A_i x^i \prod_{j=i+1}^{n} (a_j x + \lambda_j).$$

Moreover,

$$A_n = (-a_n/\lambda_n)^n f(-\lambda_n/a_n).$$

Proof. The expansion exists because the ith term in the sum has terms from x^i to x^n. The factor $(a_n x + \lambda_n)$ exists for all terms except $i = n$. So putting $x = -\lambda_n/a_n$ gives only the A_n term. □

Now we prove the second equation (6.11) in Theorem 6.4. The other two equations in Theorem 6.4 will be proved using this equation.

Proof of (6.11). Let

$$P_k(x) = \sum_{i=0}^{k} p_{k,i} x^i, \qquad Q_k(x) = \sum_{i=0}^{k} \frac{q_{k,i} x^i}{d_i(x)}.$$

Then by Lemma 6.6 with $p_k(x) = P_k(x)$, $q_k(x) = Q_k(x)$, and $r_k(x) = d_k(x)$, we have

$$\Delta''_n = \det\left(\mathcal{L}\left(\frac{x^{i+j}}{d_j(x)}\right)\right)_{i,j=0}^{n} = \frac{1}{\prod_{i=0}^{n} p_{i,i} q_{i,i}} \det\left(\mathcal{L}\left(P_i(x) Q_j(x)\right)\right)_{i,j=0}^{n}. \tag{6.14}$$

Since $P_k(x)$ are monic and $\mathcal{L}(P_i(x)Q_j(x)) = \delta_{i,j}$ for $0 \le i \le j$, we have

$$\Delta''_n = \prod_{i=0}^{n} q_{i,i}^{-1}. \tag{6.15}$$

On the other hand, by Lemma 6.7, we can write

$$P_k(x) = \sum_{i=0}^{k} A_{ki} x^i \prod_{j=i+1}^{k} (a_j x + \lambda_j),$$

where $A_{kk} = (-a_k/\lambda_k)^k P_k(-\lambda_k/a_k)$. Then

$$Q_k(x) = \frac{P_k(x)}{d_k(x)} = \sum_{i=0}^{k} \frac{A_{ki} x^i}{d_i(x)}.$$

Therefore $q_{k,k} = A_{kk} = (-a_k/\lambda_k)^k P_k(-\lambda_k/a_k)$ and substituting this in (6.15) gives (6.11). □

We will prove (6.10) by relating the two determinants in (6.3) and (6.4). First note that dividing (6.3) by $d_n(x)$ gives a determinant formula for $Q_n(x)$ with last row of

$$1/d_n(x), x/d_n(x), \dots, x^n/d_n(x), \tag{6.16}$$

while the determinant formula (6.4) for $Q_n(x)$ has last row

$$1, x/d_1(x), x^2/d_2(x), \dots, x^n/d_n(x), \tag{6.17}$$

and the last columns of the two matrices are the same. So the plan is to change the last row from (6.17) to (6.16). To this end we need the following lemma.

Lemma 6.8. *There exists a unique upper triangular matrix $(B_{i,j})_{i,j=0}^n$ such that for all $0 \le k \le n$,*

$$\sum_{j=k}^{n} B_{k,j} x^j / d_j(x) = x^k / d_n(x). \tag{6.18}$$

Moreover, the entries $B_{i,j}$ also satisfy

$$B_{k,k} = (\lambda_{k+1} \dots \lambda_n)^{-1}, \qquad 0 \le k \le n, \tag{6.19}$$

$$\sum_{j=k}^{n} B_{k,j} \nu_{s+j,j} = \nu_{s+k,n}, \qquad s \ge 0. \tag{6.20}$$

Proof. We can rewrite (6.18) as

$$1 = \sum_{j=k}^{n} B_{k,j} x^{j-k} d_n(x)/d_j(x). \tag{6.21}$$

Since $x^{j-k} d_n(x)/d_j(x)$ is a polynomial with lowest term x^{j-k} with nonzero coefficient $\lambda_{j+1} \cdots \lambda_n$, one can uniquely determine $B_{k,j}$ for $j = k, k+1, \dots, n$. This proves the first statement of the lemma. Setting $x = 0$ in (6.21) we obtain $1 = B_{k,k} d_n(0)/d_j(0)$. Therefore (6.19) follows from $d_j(0) = \lambda_j \dots \lambda_n$. Lastly, by (6.21) we have

$$\nu_{s+k,n} = \mathcal{L}\left(\frac{x^{s+k}}{d_n(x)}\right) = \mathcal{L}\left(x^{s+k} \sum_{j=k}^{n} B_{k,j} \frac{x^{j-k}}{d_j(x)}\right) = \sum_{j=k}^{n} B_{k,j} \mathcal{L}\left(\frac{x^{s+j}}{d_j(x)}\right) = \sum_{j=k}^{n} B_{k,j} \nu_{s+j,j},$$

which proves (6.20). □

We are now ready to prove (6.10).

Proof of (6.10). Dividing (6.3) by $d_n(x)$ and comparing with (6.4), we obtain

$$\frac{1}{\Delta_n'} \det \begin{pmatrix} (\nu_{i+j,n})_{\substack{0 \le i \le n-1 \\ 0 \le j \le n}} \\ \left(\frac{x^j}{d_n(x)}\right)_{0 \le j \le n} \end{pmatrix} = \frac{1}{\Delta_n''} \det \begin{pmatrix} (\nu_{i+j,j})_{\substack{0 \le i \le n-1 \\ 0 \le j \le n}} \\ \left(\frac{x^j}{d_j(x)}\right)_{0 \le j \le n} \end{pmatrix}. \tag{6.22}$$

Let L (resp. R) be the matrix in the left (resp. right) hand side of (6.22). Let C_j be the jth column of R and $(B_{i,j})_{i,j=0}^n$ the upper triangular matrix in Lemma 6.8. For $k = 0, 1, \dots, n-1$ in this order, replace the kth column of R by $\sum_{j=k}^n B_{k,j} C_j$. Then by Lemma 6.8, the resulting matrix is L. Therefore

$$\det(L) = \det(R) \prod_{k=0}^{n-1} B_{k,k}. \tag{6.23}$$

By (6.22), (6.23), and (6.19), we have

$$\Delta_n' = \Delta_n'' \prod_{k=1}^{n} \lambda_k^k.$$

Then (6.10) follows from (6.11). □

We note that the matrices for Δ_n'' and for Δ_n''' can be obtained from each other using column operations using Lemma 6.9.

Lemma 6.9. *There exists a unique lower triangular matrix $(C_{i,j})_{i,j=0}^n$ such that for all $0 \le k \le n$,*

$$\sum_{j=0}^{k} C_{k,j} \frac{x^j}{d_j(x)} = \frac{1}{d_k(x)}. \tag{6.24}$$

Moreover, the entries $C_{i,j}$ also satisfy

$$C_{k,k} = (-a_k/\lambda_k)^k, \qquad 0 \le k \le n, \tag{6.25}$$

$$\sum_{j=0}^{k} C_{k,j} \nu_{s+j,j} = \nu_{s,k}, \qquad s \ge 0, \quad 0 \le k \le n. \tag{6.26}$$

Proof. We can rewrite (6.24) as

$$1 = \sum_{j=0}^{k} C_{k,j} x^j \prod_{j=i+1}^{k} (a_j x + \lambda_j). \tag{6.27}$$

By Lemma 6.7, there are unique numbers $C_{k,0}, C_{k,1}, \dots, C_{k,k}$ satisfying (6.27) and (6.25) holds. By (6.24), we have

$$\nu_{s,k} = \mathcal{L}\left(\frac{x^s}{d_k(x)}\right) = \mathcal{L}\left(x^s \sum_{j=0}^{k} C_{k,j} \frac{x^j}{d_j(x)}\right) = \sum_{j=0}^{k} C_{k,j} \mathcal{L}\left(\frac{x^{s+j}}{d_j(x)}\right) = \sum_{j=0}^{k} C_{k,j} \nu_{s+j,j},$$

which proves (6.26). □

Finally we prove the last equation (6.12) in Theorem 6.4.

Proof of (6.12). The proof is similar to that of (6.10). By (6.4) and (6.5), we have

$$\frac{1}{\Delta_n'''} \det \begin{pmatrix} (\nu_{i,j})_{\substack{0\le i\le n-1\\ 0\le j\le n}} \\ \left(\frac{1}{d_j(x)}\right)_{0\le j\le n} \end{pmatrix} = \frac{1}{\Delta_n''} \det \begin{pmatrix} (\nu_{i+j,j})_{\substack{0\le i\le n-1\\ 0\le j\le n}} \\ \left(\frac{x^j}{d_j(x)}\right)_{0\le j\le n} \end{pmatrix}. \tag{6.28}$$

Let L (resp. R) be the matrix in the left (resp. right) hand side of (6.28). Let D_j be the jth column of R and $(C_{i,j})_{i,j=0}^n$ the upper triangular matrix in Lemma 6.9. For $k = 0, 1, \dots, n-1$ in this order, replace the kth column of R by $\sum_{j=0}^{k} C_{k,j} D_j$. Then by Lemma 6.9, the resulting matrix is L. Therefore

$$\det(L) = \det(R) \prod_{k=0}^{n-1} C_{k,k}. \tag{6.29}$$

By (6.28), (6.29), and (6.25), we have

$$\Delta_n''' = \Delta_n'' \prod_{k=1}^{n} (-a_k/\lambda_k)^k.$$

Then (6.12) follows from (6.11). □

6.2. More determinants.

In this subsection we consider more general determinants. Let

$$\Delta'_{n,s} = \det\left(\nu_{s+i+j,s+n}\right)_{i,j=0}^{n},$$
$$\Delta''_{n,s} = \det\left(\nu_{s+i+j,s+j}\right)_{i,j=0}^{n},$$
$$\Delta'''_{n,s} = \det\left(\nu_{i,s+j}\right)_{i,j=0}^{n}.$$

Then $\Delta'_n = \Delta'_{n,0}$, $\Delta''_n = \Delta''_{n,0}$, and $\Delta'''_n = \Delta'''_{n,0}$. Hence Theorem 6.4 can be restated as follows:

$$\Delta'_{n,0} = \frac{(-1)^{\binom{n+1}{2}}}{\prod_{k=1}^{n} a_k^k P_k(-\lambda_k/a_k)}, \tag{6.30}$$

$$\Delta''_{n,0} = \frac{(-1)^{\binom{n+1}{2}}\lambda_1^1\lambda_2^2\dots\lambda_n^n}{\prod_{k=1}^{n} a_k^k P_k(-\lambda_k/a_k)}, \tag{6.31}$$

$$\Delta'''_{n,0} = \frac{1}{\prod_{k=1}^{n} P_k(-\lambda_k/a_k)}. \tag{6.32}$$

Letting $x=0$ in (6.3) and (6.4), we have

$$P_n(0) = (-1)^n \frac{\Delta'_{n-1,1}}{\Delta'_{n,0}}, \tag{6.33}$$

$$Q_n(0) = (-1)^n \frac{\Delta''_{n-1,1}}{\Delta''_{n,0}}. \tag{6.34}$$

Since $\lim_{x\to\infty} Q_n(x) = 1/a_1\dots a_n$, letting $x\to\infty$ in (6.5), we have

$$\frac{1}{a_1\dots a_n} = (-1)^n \frac{\Delta'''_{n-1,1}}{\Delta'''_{n,0}}. \tag{6.35}$$

Theorem 6.10. *We have*

$$\Delta'_{n-1,1} = \frac{(-1)^{\binom{n}{2}} P_n(0)}{\prod_{k=1}^{n} a_k^k P_k(-\lambda_k/a_k)}, \tag{6.36}$$

$$\Delta''_{n-1,1} = \frac{(-1)^{\binom{n}{2}}\lambda_1^0\lambda_2^1\dots\lambda_n^{n-1} P_n(0)}{\prod_{k=1}^{n} a_k^k P_k(-\lambda_k/a_k)}, \tag{6.37}$$

$$\Delta'''_{n-1,1} = \frac{(-1)^n}{\prod_{k=1}^{n} a_k P_k(-\lambda_k/a_k)}. \tag{6.38}$$

Proof. By (6.30) and (6.33), we have the first identity. By (6.31), (6.34), we obtain the second identity. By (6.32) and (6.35), we obtain the last identity. □

Remark 6.11. If $\lambda_k = 0$ for all $k\ge 1$, then $P_n(0) = (-1)^n b_0 b_1 \dots b_{n-1}$ and $d_j(x) = a_1\dots a_j$. In this case (6.32) and (6.38) reduce to

$$\det(\mathcal{L}(x^{i-j}))_{0\le i,j\le n} = (-1)^{\binom{n+1}{2}} \prod_{k=1}^{n} \left(\frac{a_k}{b_{k-1}}\right)^{n+1-k}, \tag{6.39}$$

$$\det(\mathcal{L}(x^{i-j-1}))_{0\le i,j\le n-1} = \frac{(-1)^{\binom{n}{2}}}{a_1\dots a_n} \prod_{k=1}^{n} \left(\frac{a_k}{b_{k-1}}\right)^{n+1-k}, \tag{6.40}$$

which are equivalent to Kamioka's results [12, 2.14a and 2.14b].

6.3. **Hankel determinants for μ_n.**

Recall from (6.2) that Δ_n factors in the classical case ($a_k = 0$). In this subsection we will show in Theorem 6.14 that Δ_n always factors if the three-term recurrence coefficients are all constants.

We need the following well-known lemma.

Lemma 6.12. *Let $p_k(x)$ and $q_k(x)$ be monic polynomials of degree k for $0 \le k \le n$. Let $\mathcal{L}$ be any linear functional defined on polynomials. Then we have*

$$\det(\mathcal{L}(x^{i+j}))_{i,j=0}^n = \det(\mathcal{L}(p_i(x)q_j(x)))_{i,j=0}^n.$$

Proof. Let $p_k(x) = \sum_{i=0}^k p_{k,i}x^i$ and $q_k(x) = \sum_{i=0}^k q_{k,i}x^i$. Then $P = (p_{i,j})_{i,j=0}^n$ is a lower unitriangular matrix and $Q = (q_{j,i})_{j,i=0}^n$ is an upper unitriangular matrix. Thus, letting $M = (\mathcal{L}(x^{i+j}))_{i,j=0}^n$, we have

$$\det(\mathcal{L}(x^{i+j}))_{i,j=0}^n = \det(PMQ)_{i,j=0}^n.$$

Since the (k,s)-entry of PMQ is

$$\sum_{i=0}^n \sum_{j=0}^n \mathcal{L}(p_{ki}x^i q_{sj}x^j) = \mathcal{L}(p_k(x)q_s(x)),$$

we are done. □

We first show that Δ_n factors if a_k and b_k are constants and $\lambda_k = 0$.

Lemma 6.13. *If $a_k = 1$, $b_k = t$, and $\lambda_k = 0$ for all $k \ge 0$, then*

$$\Delta_n = (1+t)^{\binom{n+1}{2}}. \tag{6.41}$$

If $a_k = A$, $b_k = B$, and $\lambda_k = 0$ for all $k \ge 0$, then

$$\Delta_n = (A^2 + AB)^{\binom{n+1}{2}}. \tag{6.42}$$

Proof. Using the Lindström–Gessel–Viennot lemma [15, 10], one obtains that the first identity (6.41) is equivalent to the result of Sulanke and Xin [17, Lemma 2.2] on non-intersecting Schröder paths where each horizontal step has weight t. The second identity (6.42) then follows from the first using the fact that μ_i is a polynomial in A and B of degree i. □

Now we show that Δ_n factors if b_k, a_k, and λ_k are constants.

Theorem 6.14. *If $a_k = A$, $b_k = B$, $\lambda_k = C$, for all $k \ge 0$, then*

$$\Delta_n = (A^2 + AB + C)^{\binom{n+1}{2}}.$$

Proof. Let $y = x + C/A$ and $\widetilde{P}_n(y) = P_n(x) = P_n(y - C/A)$. Since

$$P_{n+1}(x) = (x - B)P_n(x) - (Ax + C)P_{n-1}(x),$$

we have

$$\widetilde{P}_{n+1}(y) = (y - C/A - B)\widetilde{P}_n(y) - Ay\widetilde{P}_{n-1}(y).$$

Moreover, since $d_n(x) = (Ax + C)^n$ and $\widetilde{d}_n(y) = A^n y^n$, we have $\widetilde{d}_n(y) = d_n(x)$ and $\widetilde{Q}_n(y) = Q_n(x)$. Thus the orthogonality

$$\mathcal{L}(x^j Q_n(x)) = 0, \qquad \text{if } 0 \le j < n,$$

implies that

$$\mathcal{L}(y^j \widetilde{Q}_n(y)) = 0, \qquad \text{if } 0 \le j < n.$$

Therefore $\widetilde{P}_n(y)$ are type R_I orthogonal polynomials with the same linear functional $\mathcal{L}$.

Now the original Hankel determinant is

$$\det\left(\mu_{i+j}\right)_{i,j=0}^{n} = \det\left(\mathcal{L}(x^{i+j})\right)_{i,j=0}^{n} = \det\left(\mathcal{L}((y-C/A)^{i+j})\right)_{i,j=0}^{n}.$$

By Lemma (6.12) with $p_k(y) = q_k(y) = (y - C/A)^k$, we have

$$\det\left(\mathcal{L}((y-C/A)^{i+j})\right)_{i,j=0}^{n} = \det\left(\mathcal{L}(y^{i+j})\right)_{i,j=0}^{n}.$$

By (6.42), we have

$$\det\left(\mathcal{L}(y^{i+j})\right)_{i,j=0}^{n} = (A^2 + A(C/A+B))^{\binom{n+1}{2}},$$

which finishes the proof. □

The following proposition provides another proof of Theorem 6.14 via (6.2) because the $\lambda_n = A^2 + AB + C$ are constant.

Proposition 6.15. *Let μ_n be the moments for the type R_I polynomials with $a_k = A$, $b_k = B$, and $\lambda_k = C$. Then μ_n are also the moments for classical orthogonal polynomials defined by*

$$B_0 = A + B, \quad B_n = 2A + B, \quad \Lambda_n = A^2 + AB + C, \quad n \ge 1.$$

Sketch of Proof. Because the type R_I coefficients are constant, the type R_I continued fraction for the moment generating function in Corollary 3.7 satisfies a quadratic equation and may be solved. This also occurs in the classical case except for the B_0 term. Dealing with this term, and explicitly solving, shows that each moment generating function is

$$\frac{1 - Bx - \sqrt{1 - 4Ax - 2Bx + B^2x^2 - 4Cx^2}}{2x(A+Cx)}.$$

Note that Proposition 3.9 is the $A = B = C = 1$ special case. □

Remark 6.16. Note that in Theorem 6.14, the entries in the determinant of $\Delta_n = \det\left(\mu_{i+j}\right)_{i,j=0}^{n}$ are positive polynomials in A, B, and C. It would be interesting to prove this theorem using nonintersecting lattice paths such as the proof of the Aztec diamond theorem due to Eu and Fu [6]. Brualdi and Kirkland [2] evaluated the Hankel determinant of Schröder numbers using J-fractions.

If $C = B^2$, we can cancel a crossing of an up step and a down step with a pair of parallel horizontal steps. Thus we obtain the following corollary.

Corollary 6.17. *If $a_k = A$, $b_k = B$, $\lambda_k = B^2$, for all $k \ge 0$, then*

$$\sum_{\pi_0,\pi_1,\dots,\pi_n} \mathrm{wt}(\pi_1)\cdots\mathrm{wt}(\pi_n) = (A^2 + AB + B^2)^{\binom{n+1}{2}},$$

where the sum is over all nonintersecting Motzkin-Schröder paths $\pi_0, \pi_1, \dots, \pi_n$ such that π_i is from $(-i, 0)$ to $(i, 0)$ and there are no pairs of horizontal steps starting at (x, y) and $(x, y+1)$.

Proof. By Theorem 6.14, we have

$$\det\left(\mu_{i+j}\right)_{i,j=0}^{n} = (A^2 + AB + B^2)^{\binom{n+1}{2}}.$$

The left hand side can be written as

$$\sum_{\sigma,\pi_0,\pi_1,\dots,\pi_n} \mathrm{sign}(\sigma)\,\mathrm{wt}(\pi_0)\,\mathrm{wt}(\pi_1)\cdots\mathrm{wt}(\pi_n), \tag{6.43}$$

where the sum is over all permutations σ of $\{0, 1, \dots, n\}$ and Motzkin-Schröder paths $\pi_0, \pi_1, \dots, \pi_n$ such that π_i is from $(-i, 0)$ to $(\sigma(i), 0)$ for all $i \in \{0, 1, \dots, n\}$.

If there are two distinct paths π_i and π_j sharing a lattice point $(r,s) \in \mathbb{Z}^2$, find the smallest pair (i,j) in lexicographic order and then the smallest pair (r,s) in lexicographic order. By swapping the parts in π_i and π_j after the point (r,s) we obtain a different term $\operatorname{sign}(\sigma') \operatorname{wt}(\pi'_0) \operatorname{wt}(\pi'_1) \cdots \operatorname{wt}(\pi'_n)$ in the sum and

$$\operatorname{sign}(\sigma) \operatorname{wt}(\pi_0) \operatorname{wt}(\pi_1) \cdots \operatorname{wt}(\pi_n) = -\operatorname{sign}(\sigma') \operatorname{wt}(\pi'_0) \operatorname{wt}(\pi'_1) \cdots \operatorname{wt}(\pi'_n). \tag{6.44}$$

Canceling these terms, we may assume that no distinct paths π_i and π_j share a lattice point in the sum (6.43). Then if two distinct paths π_i and π_j intersect, each intersection must occur between an up step in one path and a down step in the other path. By replacing the up step and the down step by a horizontal step and swapping the parts after these steps we similarly obtain a different term $\operatorname{sign}(\sigma') \operatorname{wt}(\pi'_0) \operatorname{wt}(\pi'_1) \cdots \operatorname{wt}(\pi'_n)$ in the sum satisfying (6.44). After canceling such configurations we obtain the desired formula. □

Problem 6.18. Find a bijective proof of Corollary 6.17.

7. Explicit type R_I polynomials

In this section we explain the methods for finding the explicit type R_I polynomials later in Sections 8 and 9.

We need to define a linear functional $\mathcal{L}$ on V. Recall that V is the vector space whose basis is

$$\{x^m : m \geq 0\} \cup \{1/d_n(x) : n \geq 1\}.$$

Note that we could replace the monomial x^m by any set of polynomials, one for each degree.

In Section 8 we start by taking a classical $\mathcal{L}$ on the polynomial part of V. We extend $\mathcal{L}$ to the larger space V by explicitly defining $\mathcal{L}(1/d_n(x))$. Good choices of the denominator polynomials $d_n(x)$, found by "gluing" onto a weight function representing $\mathcal{L}$, allow the extension to be explicitly defined by shifting parameters in the linear functional $\mathcal{L}$. The explicit type R_I polynomials will be obtained by also shifting parameters.

As an example of this phenomenon, take the classical Jacobi polynomials on $[-1,1]$,

$$\mathcal{L}_{a,b}(f) = \frac{\Gamma(a+b+2)}{2^{a+b+1}\Gamma(a+1)\Gamma(b+1)} \int_{-1}^{1} w(x) f(x) dx, \quad w(x) = (1-x)^a (1+x)^b$$

which includes the parameters a and b. For the integral to converge for all polynomials $f(x)$, one needs $a, b > -1$. However one may extend these values to all real numbers by defining $\mathcal{L}_{a,b}$ on a basis

$$\mathcal{L}_{a,b}((1-x)^k) = 2^k \frac{(a+1)_k}{(a+b+2)_k}, \quad k \geq 0.$$

The choice of $d_n(x) = (1+x)^n$ glues to $w(x)$ so that one defines

$$\mathcal{L}_{a,b}(1/d_n(x)) = 2^{-n} \frac{(b-n+1)_n}{(a+b-n+2)_n}, \quad n \geq 1.$$

This defines an extension for $\mathcal{L}_{a,b}$ to V by defining it on a basis, without referring to the integral. We shall see in Section 8 that the type R_I polynomials for this $\mathcal{L}_{a,b}$ and $d_n(x)$ are the corresponding shifted Jacobi polynomials.

Since $\mathcal{L}$ extended in this way always has the same moment sequence, we have equality of their moment generating functions, which are continued fractions. Each of the examples in Section 8 satisfies this theorem.

Theorem 7.1. *Suppose that $P_n(x)$ is a type R_I orthogonal polynomial*

$$P_{n+1}(x) = (x - b_n)P_n(x) - (a_n x + \lambda_n)P_{n-1}(x), \quad P_{-1}(x) = 0, \quad P_0(x) = 1,$$

whose linear functional extends that for an orthogonal polynomial

$$p_{n+1}(x) = (x - B_n)p_n(x) - \Lambda_n p_{n-1}(x), \quad p_{-1}(x) = 0, \quad p_0(x) = 1.$$

Then we have the formal power series equality between continued fractions

$$\sum_{n\geq 0} \mu_n x^n = \cfrac{1}{1 - b_0 x - \cfrac{a_1 x + \lambda_1 x^2}{1 - b_1 x - \cfrac{a_2 x + \lambda_2 x^2}{1 - b_2 x - \cdots}}}$$

$$= \cfrac{1}{1 - B_0 x - \cfrac{\Lambda_1 x^2}{1 - B_1 x - \cfrac{\Lambda_2 x^2}{1 - B_2 x - \cdots}}}.$$

In Section 9 our second method for explicit type R_I polynomials chooses $d_n(x) = (1 + ax)^n$, an nth power, for some special orthogonal polynomials. This inserts $1 + ax$ into the three-term recurrence (9.2), which does change the linear functional $\mathcal{L}$. Yet there is a relation between the type R_I and classical moment generating functions, see Proposition 9.3 and Theorem 9.4.

8. Gluing

We shall show gluing works by considering the Jacobi polynomials $P_n^{(a,b)}(x)$ on $[-1, 1]$, whose weight function is $(1 - x)^a(1 + x)^b$, and $\mathcal{L}_{a,b}$ defined as before. Then we use the same technique on classical orthogonal polynomials in the subsections.

The linear functional on the vector space of polynomials is

$$\mathcal{L}_{a,b}(f(x)) = \frac{\Gamma(a + b + 2)}{2^{a+b+1}\Gamma(a + 1)\Gamma(b + 1)} \int_{-1}^{1} (1 - x)^a(1 + x)^b f(x)dx, \tag{8.1}$$

where

$$\mathcal{L}_{a,b}(1) = 1, \quad \mathcal{L}_{a,b}(P_n^{(a,b)}(x)P_m^{(a,b)}(x)) = 0, \quad m \neq n.$$

As in Section 7 we extend $\mathcal{L}_{a,b}$ to the vector space V by defining $\mathcal{L}_{a,b}$ on the basis elements $1/d_n(x) = 1/(1 + x)^n$. Even though the integral may not exist, if $b < n$, we may define the linear functional by what the integral would give by shifting b to $b - n$

$$\mathcal{L}_{a,b}(1/d_n(x)) = 2^{-n} \frac{(b - n + 1)_n}{(a + b + 2 - n)_n}. \tag{8.2}$$

Proposition 8.1. *We have for any polynomial $p(x)$*

$$\mathcal{L}_{a,b}(p(x)/d_n(x)) = 2^{-n} \frac{(b - n + 1)_n}{(a + b + 2 - n)_n} \mathcal{L}_{a,b-n}(p(x)). \tag{8.3}$$

Proof. Write

$$\frac{p(x)}{d_n(x)} = t(x) + \sum_{i=1}^{n} \frac{c_i}{d_i(x)} \tag{8.4}$$

for some polynomial $t(x)$. Assume that $b - n > -1$ and $a > -1$. Then applying $\mathcal{L}_{a,b}$ to (8.4) may be done integration which is linear, so (8.3) holds. Each term is a rational function of a and b, thus (8.3) is true in general. ∎

Consider the type R_I orthogonality for a fixed $n \geq 1$

$$\mathcal{L}_{a,b}\left(x^k \frac{P_n(x)}{(1+x)^n}\right) = 0, \quad 0 \leq k \leq n-1.$$

This is by Proposition 8.4

$$\mathcal{L}_{a,b-n}\left(x^k P_n(x)\right) = 0, \quad 0 \leq k \leq n-1.$$

If $b-n > -1$, this is the usual orthogonal polynomial orthogonality, so $P_n(x)$ must be a multiple of the Jacobi polynomial $P_n^{(a,b-n)}(x)$. Note also that $P_n^{(a,b-n)}(-1) \neq 0$, which is required.

Once we know an explicit formula for the type R_I polynomials, we can find their three term recurrence by considering the higher term coefficients.

To summarize, all we need to do is find the shift in parameters when $d_n(x)$ is glued onto a known weight $w(x)$. Then apply this shift to the parameters in the usual orthogonal polynomials. This is carried out on the next 7 subsections. The degeneracy condition $P_n(-\lambda_n/a_n) \neq 0$ holds because the polynomials may be explicitly evaluated at these values.

8.1. **Jacobi polynomials on** $[-1, 1]$**.**

Recall that the Jacobi polynomials are

$$P_n^{(a,b)}(x) = \frac{(a+1)_n}{n!}\, {}_2F_1\left(\begin{matrix}-n, a+b+1\\ a+1\end{matrix} \;\middle|\; \frac{1-x}{2}\right).$$

In Section 7 we took $d_n(x) = (1+x)^n$. Here we record the case $d_n(x) = (1-x)^n$, which shifts a to $a-n$. The values of the linear functional are given by a beta function evaluation.

Definition 8.2. Let $\mathcal{L}_{a,b}$ be the linear functional on V such that

$$\mathcal{L}_{a,b}\left(\frac{(1+x)^k}{(1-x)^n}\right) = 2^{k-n}\frac{(b+1)_k}{(a-n+1)_n(a+b+2)_{k-n}}, \quad k, n \geq 0. \tag{8.5}$$

Theorem 8.3. *Up to a constant c_n, the type R_I polynomials for $w(x) = (1-x)^a(1+x)^b$ on $[-1, 1]$ and $d_n(x) = (1-x)^n$ are shifted Jacobi polynomials*

$$p_n(x) = c_n P_n^{(a-n,b)}(x).$$

Proposition 8.4. *We have for the monic type R_I polynomials, $\hat{p}_n(x)$,*

$$\hat{p}_{n+1}(x) = (x-b_n)\hat{p}_n(x) - (1-x)\lambda_n \hat{p}_{n-1}(x).$$

where

$$b_n = \frac{b-a+3n+1}{a+b+n+1}, \quad \lambda_n = \frac{2n(n+b)}{(a+b+n)(a+b+n+1)}.$$

Remark 8.5. The moments are the same as the moments for the usual Jacobi polynomials $P_n^{(a,b)}(x)$ on $[-1, 1]$,

$$\mathcal{L}_{a,b}(x^n) = \sum_{s=0}^{n} \binom{n}{s} (-2)^s \frac{(a+1)_s}{(a+b+2)_s}.$$

The usual three-term recurrence coefficients for the monic Jacobi polynomials are

$$B_n = \frac{(b^2-a^2)}{(2n+a+b)(2n+a+b+2)},$$
$$\Lambda_n = \frac{4n(n+a)(n+b)(n+a+b)}{(2n+a+b-1)(2n+a+b)^2(2n+a+b+1)}$$

and Theorem 7.1 holds.

Remark 8.6. If $a=b=1/2$, then the Catalan numbers $C_k=\frac{1}{k+1}\binom{2k}{k}$ are moments

$$\{4^k\mathcal{L}_{1/2,1/2}(x^{2k})=C_k : k\ge 0\}=\{1,1,2,5,14,42,\dots\}.$$

If $a=b=-1/2$, then the central binomial coefficients are moments

$$\left\{4^k\mathcal{L}_{-1/2,-1/2}(x^{2k})=\binom{2k}{k} : k\ge 0\right\}=\{1,2,6,20,70,\dots\}.$$

8.1.1. *A mixed Jacobi formula.*

One may alternate inserting $(1-x)$ with $(1+x)$ into the denominator by taking

$$d_n(x)=(1-x)^{\lceil n/2\rceil}(1+x)^{\lfloor n/2\rfloor}.$$

Definition 8.7. Let $\mathcal{L}_{a,b}$ be the linear functional on V such that

$$\mathcal{L}_{a,b}\left(\frac{(1-x)^k}{(1-x)^{\lceil n/2\rceil}(1+x)^{\lfloor n/2\rfloor}}\right)=2^{k-n}\frac{(a+1)_{k-\lceil n/2\rceil}}{(b-\lfloor n/2\rfloor+1)_{\lfloor n/2\rfloor}(a+b+2)_{k-n}},\quad k,n\ge 0.$$

Theorem 8.8. *Up to a constant, the type R_I polynomials for $w(x)=(1-x)^a(1+x)^b$ on $[-1,1]$ and $d_n(x)=((1-x)^{\lceil n/2\rceil}(1+x)^{\lfloor n/2\rfloor}$ are shifted Jacobi polynomials*

$$p_n(x)=c_nP_n^{(a-\lceil n/2\rceil,b-\lfloor n/2\rfloor)}(x)=c_n'\,{}_1F_2\left(\begin{matrix}-n,a+b+1\\a-\lceil n/2\rceil+1\end{matrix}\;\middle|\;\frac{1-x}{2}\right).$$

Proposition 8.9. *For the monic type R_I polynomials $\hat{p}_n(x)$, we have*

$$\hat{p}_{n+1}(x)=(x-b_n)\hat{p}_n(x)-(1-(-1)^{n-1}x)\lambda_n\hat{p}_{n-1}(x),$$

where

$$b_n=\frac{1}{a+b+n+1}\begin{cases}\times(b-a+1), & \text{if } n \text{ is even}\\ \times(b-a), & \text{if } n \text{ is odd,}\end{cases}$$

and

$$\lambda_n=\frac{2n}{(a+b+n)(a+b+n+1)}\begin{cases}\times(a+n/2), & \text{if } n \text{ is even}\\ \times(b+(n+1)/2), & \text{if } n \text{ is odd.}\end{cases}$$

The moments are again given by Remarks 8.5 and 8.6.

8.2. Jacobi polynomials on $[0,1]$.

The Jacobi polynomials on $[0,1]$ are

$$P_n^{(a,b)}(1-2x)=\frac{(a+1)_n}{n!}\,{}_2F_1\left(\begin{matrix}-n,n+a+b+1\\a+1\end{matrix}\;\middle|\;x\right)$$

and have the weight function $w(x)=x^a(1-x)^b$ given by the linear functional

$$\mathcal{M}_{a,b}(f(x))=\frac{\Gamma(a+b+2)}{\Gamma(a+1)\Gamma(b+1)}\int_0^1 x^a(1-x)^bf(x)dx.$$

8.2.1. $d_n(x) = (1-x)^n$.

Let's choose $d_n(x) = (1-x)^n$ which naturally glues onto the weight $w(x)$. So we see that the modified weight function $w'(x)$ occurs by replacing b by $b-n$ in $w(x)$.

As before this beta integral may be evaluated.

Definition 8.10. Let $\mathcal{M}_{a,b}$ be the linear functional on V such that

$$\mathcal{M}_{a,b}\left(\frac{x^k}{(1-x)^n}\right) = \frac{(a+1)_k}{(b-n+1)_n(a+b+2)_{k-n}}, \quad k,n \geq 0.$$

Theorem 8.11. *Up to a constant c_n, the type R_I polynomials for $w(x) = x^a(1-x)^b$ on $[0,1]$ and $d_n(x) = (1-x)^n$ are shifted Jacobi polynomials*

$$p_n(x) = c_n P_n^{(a,b-n)}(1-2x) = c_n'\, {}_2F_1\left(\begin{matrix} -n, a+b+1 \\ a+1 \end{matrix} \,\middle|\, x\right).$$

Proposition 8.12. *We have for the monic type R_I polynomials, $\hat{p}_n(x)$,*

$$\hat{p}_{n+1}(x) = (x-b_n)\hat{p}_n(x) - (1-x)\lambda_n\hat{p}_{n-1}(x).$$

where

$$b_n = \frac{a+2n+1}{a+b+n+1}, \quad \lambda_n = \frac{n(n+a)}{(a+b+n)(a+b+n+1)}.$$

Remark 8.13. The moments are the same as the moments for the usual Jacobi polynomials $P_n^{(a,b)}(x)$, on $[0,1]$ instead of $[-1,1]$, evaluated by beta functions,

$$\mathcal{M}_{a,b}(x^k) = \frac{(a+1)_k}{(a+b+2)_k}.$$

Remark 8.14. If $a = b = 1/2$, then the Catalan numbers are moments

$$\{4^k\mathcal{M}_{1/2,1/2}(x^k) = C_{k+1} : k \geq 0\} = \{1, 2, 5, 14, 42, \dots\}.$$

8.2.2. $d_n(x) = x^n$.

Theorem 8.15. *Up to a constant the type R_I polynomials for $w(x) = x^a(1-x)^b$ on $[0,1]$ and $d_n(x) = x^n$ are shifted Jacobi polynomials*

$$p_n(x) = c_n P_n^{(a-n,b)}(1-2x) = c_n'\, {}_2F_1\left(\begin{matrix} -n, a+b+1 \\ a-n+1 \end{matrix} \,\middle|\, x\right).$$

Proposition 8.16. *We have for the monic type R_I polynomials, $\hat{p}_n(x)$,*

$$\hat{p}_{n+1}(x) = (x-b_n)\hat{p}_n(x) - xa_n\hat{p}_{n-1}(x).$$

where

$$b_n = \frac{a-n}{a+b+n+1}, \quad a_n = \frac{n(b+n)}{(a+b+n)(a+b+n+1)}.$$

8.3. **Laguerre polynomials.**

The Laguerre polynomials

$$L_n^a(x) = \frac{(a+1)_n}{n!}\, {}_1F_1\left(\begin{matrix} -n \\ a+1 \end{matrix} \,\middle|\, x\right)$$

have

$$\mathcal{L}_a(f(x)) = \frac{1}{\Gamma(a+1)}\int_0^\infty x^a e^{-x} f(x)dx.$$

Thus the choice of $d_n(x) = x^n$ shifts a.

Definition 8.17. Extend the linear functional $\mathcal{L}_a$ to V by

$$\mathcal{L}_a\left(\frac{1}{d_n(x)}\right) = \frac{1}{(a-n+1)_n}, \quad n \geq 1.$$

Theorem 8.18. *Up to a constant the type R_I polynomials for $w(x) = x^a e^{-x}$ on $[0, \infty)$ and $d_n(x) = x^n$ are shifted Laguerre polynomials*

$$p_n(x) = c_n L_n^{a-n}(x) = c'_n \, {}_1F_1\left(\begin{matrix} -n \\ a-n+1 \end{matrix} \middle| \, x\right).$$

Proposition 8.19. *We have for the monic type R_I polynomials, $\hat{p}_n(x)$,*

$$\hat{p}_{n+1}(x) = (x - b_n)\hat{p}_n(x) - x a_n \hat{p}_{n-1}(x).$$

where

$$b_n = a - n, \quad a_n = n.$$

The monic Laguerre polynomials have

$$B_n = 2n + a + 1, \quad \Lambda_n = n(n+a).$$

Remark 8.20. The moments are

$$L(x^k) = (a+1)_k.$$

8.4. **Meixner polynomials.**

The Meixner polynomials

$$M_n(x; b, c) = {}_2F_1\left(\begin{matrix} -n, -x \\ b \end{matrix} \middle| \, 1 - \frac{1}{c}\right)$$

have for their linear functional

$$\mathcal{L}_{b,c}(f(x)) = (1-c)^b \sum_{x=0}^{\infty} \frac{(b)_x}{x!} c^x f(x)$$

Thus the choice of $d_n(x) = (x+b-1)(x+b-2)\cdots(x+b-n)$ shifts b to $b-n$.

Definition 8.21. Extend the linear functional $\mathcal{L}_{b,c}$ to V by

$$\mathcal{L}_{b,c}\left(\frac{1}{d_n(x)}\right) = \frac{(1-c)^n}{(b-n)_n}, \quad n \geq 1.$$

Theorem 8.22. *Up to a constant, the type R_I polynomials for $\mathcal{L}_{b,c}$ and $d_n(x) = (x+b-n)_n$ are shifted Meixner polynomials*

$$p_n(x) = c_n M_n(x; b-n, c) = c_n \, {}_2F_1\left(\begin{matrix} -n, -x \\ b-n \end{matrix} \middle| \, 1 - \frac{1}{c}\right).$$

Proposition 8.23. *We have for the monic type R_I polynomials, $\hat{p}_n(x)$,*

$$\hat{p}_{n+1}(x) = (x - b_n)\hat{p}_n(x) - (x+b-n)\lambda_n \hat{p}_{n-1}(x),$$

where

$$b_n = \frac{n - (2n+1)c + bc}{1-c}, \quad \lambda_n = \frac{cn}{1-c}.$$

The monic Meixner polynomials have

$$B_n = \frac{n + (n+b)c}{1-c}, \quad \Lambda_n = \frac{n(n+b-1)c}{(1-c)^2}.$$

Remark 8.24. The moments are

$$\mathcal{L}_{b,c}(x^k) = \sum_{j=1}^{k} S(k,j)(b)_j \left(\frac{c}{1-c}\right)^j,$$

where $S(k,j)$ are the Stirling numbers of the second kind.

8.5. Little q-Jacobi polynomials.

The little q-Jacobi polynomials are

$$p_n(x;a,b|q) = {}_2\phi_1\left(\begin{matrix} q^{-n}, abq^{n+1} \\ aq \end{matrix} \,\middle|\, q; qx\right).$$

The linear functional for the orthogonal polynomial orthogonality is

$$\mathcal{L}_{a,b}(f(x)) = \frac{(aq)_\infty}{(abq^2)_\infty} \sum_{x=0}^{\infty} \frac{(bq)_x}{(q)_x} (aq)^x f(q^x). \tag{8.6}$$

Choosing $d_n(x) = (bx;q^{-1})_n$ we see that b shifts to bq^{-n}

$$\frac{(bq)_x}{d_n(q^x)} = \frac{(bq^{1-n})_x}{(bq^{1-n})_n},$$

so the next theorem results using the extension

$$\mathcal{L}_{a,b}\left(\frac{1}{d_n(x)}\right) = \frac{(abq^{2-n})_n}{(bq^{1-n})_n}, \quad n \geq 1.$$

Theorem 8.25. *Up to a constant, the type R_I polynomials for the little q-Jacobi polynomials with $\mathcal{L}_{a,b}$ given by (8.6) and $d_n(x) = (bx;q^{-1})_n$ are shifted little q-Jacobi polynomials*

$$p_n(x) = c_n p_n(x;a,bq^{-n}|q).$$

Proposition 8.26. *We have for the type R_I monic polynomials, $\hat{p}_n(x)$,*

$$\hat{p}_{n+1}(x) = (x-b_n)\hat{p}_n(x) - \lambda_n(1-bxq^{1-n})\hat{p}_{n-1}(x),$$

where

$$b_n = q^n \frac{1+a-aq^n-aq^{n+1}}{1-abq^{n+1}}, \quad \lambda_n = q^{2n-1}\frac{(1-q^n)(1-aq^n)}{(1-abq^n)(1-abq^{n+1})}.$$

Remark 8.27. The moments are

$$\mathcal{L}_{a,b}(x^k) = \frac{(aq;q)_k}{(abq^2;q)_k}.$$

8.6. Big q-Jacobi polynomials.

The big q-Jacobi polynomials are

$$P_n(x;a,b,c;q) = {}_3\phi_2\left(\begin{matrix} q^{-n}, abq^{n+1}, x \\ aq, cq \end{matrix} \,\middle|\, q; q\right).$$

The linear functional for orthogonality is given by a q-integral

$$\mathcal{L}_{a,b,c}(f(x)) = \frac{1}{aq(1-q)} \frac{(aq,bq,cq,abq/c;q)_\infty}{(q,abq^2,c/a,aq/c;q)_\infty} \int_{cq}^{aq} \frac{(x/a,x/c;q)_\infty}{(x,bx/c;q)_\infty} f(x) d_q(x). \tag{8.7}$$

There are two choices for $d_n(x)$ which shift parameters

$$d_n(x) = (bx/cq;q^{-1})_n,\ b \to bq^{-n},$$
$$d_n(x) = (x/a;q)_n,\ a \to aq^{-n}.$$

Extending $\mathcal{L}_{a,b,c}$ may be accomplished via

$$\mathcal{L}_{a,b}\left(\frac{1}{d_n(x)}\right)=\begin{cases}\frac{(abq^{2-n})_n}{(bq^{1-n})_n(abq^{1-n}/c)_n}, & \text{if } d_n(x)=(bx/cq;q^{-1})_n, n\geq 1\\ \frac{(abq^{2-n})_n(aq^{1-n}/c)_n}{(aq^{1-n})_n(abq^{1-n}/c)_n(c/a)_n}, & \text{if } d_n(x)=(x/a;q)_n, n\geq 1.\end{cases}$$

Theorem 8.28. *Up to a constant, the type R_I polynomials for the big q-Jacobi polynomials with $\mathcal{L}_{a,b,c}$ given by* (8.7) *and $d_n(x)=(bx/cq;q^{-1})_n$ are shifted big q-Jacobi polynomials*

$$p_n(x)=c_np_n(x;a,bq^{-n};q).$$

Also choosing $d_n(x)=(x/a;q)_n$ we obtain the shifted big q-Jacobi polynomials

$$p_n(x)=c_np_n(x;aq^{-n},b;q).$$

Proposition 8.29. *We have for the type R_I monic polynomials, $\hat{p}_n(x)$,*

$$\hat{p}_{n+1}(x)=(x-b_n)\hat{p}_n(x)-(1-bxq^{-n}/c)\lambda_n\hat{p}_{n-1}(x),\quad d_n(x)=(bx/cq;q^{-1})_n,$$

where

$$b_n=-q\frac{ab-aq^n-cq^n-acq^n+acq^{2n}+acq^{2n+1}}{1-abq^{n+1}},$$

$$\lambda_n=-acq^{n+1}\frac{(1-q^n)(1-aq^n)(1-cq^n)}{(1-abq^n)(1-abq^{n+1})}.$$

Proposition 8.30. *We have for the type R_I monic polynomials, $\hat{p}_n(x)$,*

$$\hat{p}_{n+1}(x)=(x-b_n)\hat{p}_n(x)-(1-xq^{n-1}/a)\lambda_n\hat{p}_{n-1}(x),\quad d_n(x)=(x/a;q)_n,$$

where

$$b_n=q^{-n}\frac{a+aq-aq^{n+1}-abq^{n+1}-acq^{n+1}+cq^{2n+1}}{1-abq^{n+1}},$$

$$\lambda_n=a^2q^{2-2n}\frac{(1-q^n)(1-bq^n)(1-cq^n)}{(1-abq^n)(1-abq^{n+1})}.$$

8.7. **The Askey–Wilson polynomials.**

Here we consider separately the absolutely continuous case and the purely discrete case.

8.7.1. *The continuous case.*

The Askey–Wilson polynomials are defined by

$$p_n(x;a,b,c,d|q)=\frac{(ab,ac,ad;q)_n}{a^n}\,{}_4\phi_3\left(\begin{matrix}q^{-n},abcdq^{n-1},az,a/z\\ ab,ac,ad\end{matrix}\;\middle|\;q;q\right),$$

$$z=e^{i\theta},\quad x=\cos\theta=(z+1/z)/2.$$

Note that

$$(Az,A/z;q)_n=\prod_{j=0}^{n-1}(1-2Axq^j+A^2q^{2j})$$

is a polynomial in x of degree n. Thus $p_n(x;a,b,c,d|q)$ is a function of x.

The weight function for the Askey–Wilson polynomials is

$$\mathcal{L}_{a,b,c,d}(r(x))=\frac{(q,ab,ac,ad,bc,bd,cd)_\infty}{2\pi(abcd)_\infty}\int_0^\pi r((e^{i\theta}+e^{-i\theta})/2)w(\theta,a,b,c,d)d\theta,\quad \mathcal{L}_{a,b,c,d}(1)=1,$$

where

$$w(\theta,a,b,c,d)=\frac{(e^{2i\theta},e^{-2i\theta})_\infty}{(ae^{i\theta},ae^{-i\theta},be^{i\theta},be^{-i\theta},ce^{i\theta},ce^{-i\theta},de^{i\theta},de^{-i\theta})_\infty}.$$

Let

$$d_n(x) = (bz/q, b/zq; q^{-1})_n = \prod_{j=0}^{n-1}(1 - 2bxq^{-1-j} + b^2q^{-2-2j}).$$

We next define the extension of $\mathcal{L}_{a,b,c,d}$ to V using the Askey–Wilson integral and

$$\frac{w(\theta, a, b, c, d)}{d_n(x)} = w(\theta, a, bq^{-n}, c, d).$$

Definition 8.31. Suppose $\mathcal{L}_{a,b,c,d}$ is a linear functional on V such that

(8.8)

$$\mathcal{L}_{a,b,c,d}\left(\frac{(cz, c/z; q)_j(az, a/z; q)_k}{(bzq^{-n}, bq^{-n}/z; q)_n}\right) = \frac{(cd; q)_j(ac; q)_{k+j}(ad; q)_k}{(abq^{k-n}; q)_{n-k}(bcq^{j-n}; q)_{n-j}(bdq^{-n}; q)_n(abcd; q)_{k+j-n}}.$$

Theorem 8.32. *The type R_I polynomials for $\mathcal{L}_{a,b,c,d}$ and denominator polynomials*

$$d_n(x) = \prod_{j=0}^{n-1}(1 - 2bxq^{-1-j} + b^2q^{-2-2j})$$

are shifted Askey–Wilson polynomials

$$p_n(x; a, bq^{-n}, c, d|q) = \frac{(abq^{-n}, ac, ad; q)_n}{a^n}\, {}_4\phi_3\left(\begin{matrix} q^{-n}, abcd/q, az, a/z \\ ac, ad, abq^{-n}\end{matrix}\;\middle|\; q; q\right).$$

Note that

$$d_n(x)/d_{n-1}(x) = 1 - 2bxq^{-n} + b^2q^{-2n}$$

which is the factor in the type R_I recurrence relation.

Proposition 8.33. *We have for the monic type R_I polynomials, $\hat{p}_n(x)$,*

$$\hat{p}_{n+1}(x) = (x - b_n)\hat{p}_n(x) - \lambda_n(1 - 2bxq^{-n} + b^2q^{-2n})\hat{p}_{n-1}(x),$$

where

$$b_n = \frac{bq^{-n} + q^n/b}{2} - \frac{q^{1+2n}}{2b(1 - abcdq^{n-1})}\Big((1 - abq^{-1-n})(1 - bcq^{-1-n})(1 - bdq^{-1-n})$$
$$- (1 - q^{-1-n})(1 - abcd/q)(1 - b^2q^{-1-2n})\Big),$$
$$\lambda_n = \frac{(1 - q^n)(1 - acq^{n-1})(1 - adq^{n-1})(1 - cdq^{n-1})}{4(1 - abcdq^{n-2})(1 - abcdq^{n-1})}.$$

Sketch of Proof. The value $z_n = bq^{-n}$ puts $x_n = 1/2(bq^{-n} + q^n/b)$, and $d_n(x_n)/d_{n-1}(x_n) = 0$. For this value of x_n, $p_n(x_n)$ is evaluable as a product by the 1-balanced ${}_3\phi_2$ evaluation. This choice also allows $p_{n+1}(x_n)$ to be a sum of 2 terms, using the Sears transformation for a 1-balanced ${}_4\phi_3$. This determines the value of b_n, and λ_n can be found by finding the coefficients of x^n. □

8.7.2. *The q-Racah case.*

For completeness we record the analogous results for the q-Racah polynomials.

Definition 8.34. For $0 \le n \le N$, let $p_n(X; b, c, d, N; q)$ be the polynomial of degree n in $X = \mu(x) = q^{-x} + cdq^{x+1}$

$$p_n(\mu(x); b, c, d, N; q) = {}_4\phi_3\left(\begin{matrix} q^{-n}, bq^{n-N}, q^{-x}, cdq^{x+1} \\ q^{-N}, bdq, cq\end{matrix}\;\middle|\; q; q\right).$$

Since

$$(q^{-x};q)_j(cdq^{x+1};q)_j=\prod_{s=0}^{j-1}(1-q^s\mu(x)+cdq^{1+2s})$$

$p_n(\mu(x);b,c,d,N;q)$ is a polynomial in $\mu(x)=X$ of degree n.

Let

$$d_n(X)=\prod_{j=0}^{n-1}(1-Xq^j/bd+q^{2j+1}c/b^2d),$$
$$d_n(\mu(x))=(q^{-x}/bd,cq^{x+1}/b;q)_n.$$

Definition 8.35. Let $\mathcal{M}_{b,c,d,N}$ be the linear functional defined on V by

$$\mathcal{M}_{b,c,d,N}(r(X))=\sum_{x=0}^{N}vwp(x,b,c,d,N)r(\mu(x)),$$

where

$$vwp(x,b,c,d,N)=\frac{(dq)_N(cq/b)_N}{(cdq^2)_N(1/b)_N}\frac{(cdq)_x}{(q)_x}\frac{1-cdq^{1+2x}}{1-cdq}\frac{(q^{-N})_x}{(cdq^{N+2})_x}\frac{(cq)_x}{(dq)_x}\frac{(bdq)_x}{(cq/b)_x}(q^N/b)^x.$$

See [9, (II-21)]. Here the constants have been chosen, using the very well poised ${}_6\phi_5$ summation theorem, so that $\mathcal{M}_{b,c,d,N}(1)=1$.

The classical orthogonal polynomials for $\mathcal{M}_{b,c,d,N}$ are q-Racah polynomials

$$\mathcal{M}_{b,c,d,N}\left(p_n(X;b,c,d,N;q)p_m(X;b,c,d,N;q)\right)=0 \qquad \text{if } n\neq m.$$

Note that gluing does occur

$$\frac{vwp(x,b,c,d,N)}{d_n(\mu(x))}=c(n,b,c,d,N)\times vwp(x,bq^{-n},c,d,N),$$

for some constant $c(n,b,c,d,N)$ independent of x.

Theorem 8.36. *The polynomials $p_n(X;bq^{-n},c,d,N;q)$, $0\le n\le N$, are the type R_I polynomials for the q-Racah linear functional $\mathcal{M}_{b,c,d,N}$ with*

$$d_n(X)=\prod_{j=0}^{n-1}(1-Xq^j/bd+q^{2j+1}c/b^2d),$$
$$d_n(\mu(x))=(q^{-x}/bd,cq^{x+1}/b;q)_n.$$

Proposition 8.37. *The monic type R_I q-Racah polynomials satisfy for $0\le n\le N-1$*

$$\hat{p}_{n+1}(X)=(X-b_n)\hat{p}_n(X)-\lambda_n(1-Xq^{n-1}/bd+q^{2n-1}c/b^2d)\hat{p}_{n-1}(X)$$

where

$$\begin{aligned}b_n&=-(-b+bd(-1+q^{N-n}+q^{N-n+1}-q^{N+1})+q^n\\&\qquad+c(q^n-q^{2n}-q^{2n+1}+q^{N+n+1})-bcdq^{N+1}+cdq^{N+n+1})/(bq^n-q^N),\\ \lambda_n&=dq^{1-2n}\frac{(1-q^n)(1-cq^n)(1-q^{N-n+1})(1-dq^{N-n+1})}{(1-q^{N-n}/b)(1-q^{N-n+1}/b)}.\end{aligned}$$

9. $d_n(x)$ AS nTH POWERS

In this section we consider the case $b_n = 0$ for orthogonal polynomials, so that the polynomials are either even or odd. Hermite polynomials are one example. We choose $d_n(x) = (1+ax)^n$, then modify the three term recurrence by inserting the factor $1 + ax$, independent of n, for a type R_I polynomial, see (9.2). The new linear functional $\mathcal{L}_a$ has new values on the polynomials, so the moments do change, unlike the previous examples.

General results for the type R_I polynomials and their moments, in terms of the original orthogonal polynomials, are given in Proposition 9.1, Proposition 9.2, and Theorem 9.4. We do not know a representing measure for the new linear functional $\mathcal{L}_a$ in terms of an original measure, even in the Hermite case.

9.1. General results.

Let $p_n(x)$ be orthogonal polynomials defined by

$$p_{n+1}(x) = xp_n(x) - \lambda_n p_{n-1}(x) \tag{9.1}$$

with non-zero moments $\mu_{2n} = \mathcal{L}(x^{2n})$.

Consider a type R_I version of p_n

$$r_{n+1}(x) = xr_n(x) - \lambda_n(1 + ax)r_{n-1}(x). \tag{9.2}$$

These polynomials are rescaled versions of the original orthogonal polynomials. Proposition 9.1 follows easily by rescaling.

Proposition 9.1. *We have*

$$r_n(x) = p_n\left(\frac{x}{\sqrt{1+ax}}\right)(\sqrt{1+ax})^n.$$

Next we see how the moments are related. Let $\mathcal{L}_a$ be the linear functional for these polynomials. Let μ_n be the moments for the orthogonal polynomials in (9.1), and let $\theta_n = \mathcal{L}_a(x^n)$ be the moments for the type R_I polynomials (9.2).

Proposition 9.2. *The moments $\theta_n = \mathcal{L}_a(x^n)$ for the above type R_I polynomials are given by*

$$\theta_{2n} = \sum_{k \text{ even}} \binom{n+k/2}{k} a^k \mu_{2n+k},$$
$$\theta_{2n+1} = \sum_{k \text{ odd}} \binom{n+(k+1)/2}{k} a^k \mu_{2n+1+k}.$$

Proof. We use the combinatorial interpretation of θ_n as weighted Motzkin-Schröder paths and μ_{2n} as weighted Dyck paths, where a Dyck path is a Motzkin path with no horizontal steps.

Take the paths for θ_{2n}, which start at $(0,0)$ end at $(2n,0)$, and stay at or above the x-axis. There are no horizontal steps, as $b_n = 0$ in (9.2). There are down steps starting at y-coordinate n with weight λ_n, and vertical steps starting at y-coordinate n with weight $a\lambda_n$. If there are k vertical steps, where k must be even, these contribute a weight of a^k. We can change these k steps to down steps to obtain a weighted Dyck path from $(0,0)$ to $(2n+k,0)$. This is a term in the combinatorial expansion for μ_{2n+k}. But each such Dyck path for μ_{2n+k} occurs $\binom{n+k/2}{k}$ times, by choosing which of the $n+k/2$ down steps are switched to vertical steps.

The proof for θ_{2n+1} is basically the same. ∎

These moments may also be connected via Chebyshev polynomials.

Proposition 9.3. *The moments θ_n satisfy*

$$\theta_n = \mathcal{L}_a(x^n) = \mathcal{L}(x^n w_n(x,a)),$$

where

$$w_{2n}(x,a) = \sum_{k\ even} \binom{n+k/2}{k} a^k x^k = {}_2F_1\left(\begin{matrix}-n, n+1\\ 1/2\end{matrix} \,\middle|\, -a^2x^2/4\right) = U_{2n}(aix/2),$$

$$w_{2n+1}(x,a) = \sum_{k\ odd} \binom{n+(k+1)/2}{k} a^k x^k = (n+1)ax\, {}_2F_1\left(\begin{matrix}-n, n+2\\ 3/2\end{matrix} \,\middle|\, -a^2x^2/4\right).$$

The moment generating functions, which are given by the continued fractions in Theorem 7.1 are related.

Theorem 9.4. *As formal power series in t, we have for the usual orthogonal polynomials*

$$\sum_{n=0}^{\infty} \mu_{2n} t^{2n} = \mathcal{L}\left(\frac{1}{1-xt}\right) = \mathcal{L}\left(\frac{1}{1-x^2t^2}\right),$$

and for the type R_I polynomials

$$\sum_{n=0}^{\infty} \theta_n t^n = \mathcal{L}_a\left(\frac{1}{1-xt}\right) = \mathcal{L}\left(\frac{1}{1-x^2t(a+t)}\right).$$

9.2. **Explicit examples.**

9.2.1. *Chebyshev polynomials.*

Proposition 9.5. *If $b_n = b$, $a_n = a$ and $\lambda_n = \lambda$ are constant, then the type R_I monic polynomials are*

$$p_n(x) = U_n\left(\frac{x-b}{2\sqrt{ax+\lambda}}\right)(\sqrt{ax+\lambda})^n.$$

Note that these are the polynomials we considered in Theorem 6.14.

9.2.2. *Hermite polynomials.*

Proposition 9.6. *If $b_n = 0$, $a_n = an$ and $\lambda_n = n$, then the type R_I monic polynomials are $p_n(x)$ satisfy*

$$\sum_{n=0}^{\infty} \frac{p_n(x)}{n!} t^n = e^{xt-(1+ax)t^2/2}.$$

Also

$$p_n(x) = \mathrm{He}_n\left(\frac{x}{\sqrt{1+ax}}\right)(\sqrt{1+ax})^n,$$

where $\mathrm{He}_n(x)$ are the monic Hermite polynomials normalized by

$$\mathrm{He}_{n+1}(x) = x\mathrm{He}_n(x) - n\mathrm{He}_{n-1}(x), \quad \mathrm{He}_{-1}(x) = 0, \quad \mathrm{He}_0(x) = 1.$$

10. Combinatorics

In this section we study combinatorial aspects of some type R_I orthogonal polynomials considered in Section 8 and 9.

10.1. **Hermite polynomials.**

The classical Hermite $\mathrm{He}_n(x)$ polynomials are the generating functions for matchings (or involutions) on n points. Their moments count perfect matchings on n points. In this section we give the corresponding results for the type R_I Hermite polynomials in Section 9.2.

Let $H_n(x,a)$ be the type R_I Hermite polynomials given by Proposition 9.6. The following result follows from either Theorem 3.11 or the exponential generating function for $H_n(x,a)$.

Proposition 10.1. *$H_n(x,a)$ is the generating function for involutions of length n, where 1-cycles are weighted by x, and 2-cycles are two colored, with weights -1 and $-ax$.*

The combinatorics of the moments can be given using Proposition 9.2.

Proposition 10.2. *The moments $\theta_n = \mathcal{L}(x^n)$ for the type R_I Hermite polynomials are the generating functions for 2-coloring the edges of the following perfect matchings, red and blue.*

(1) *If n is even, the perfect matching is on $2n+2K$ points, and $2K$ of these $n+K$ edges are colored red, each of weight a, for some $0 \leq K \leq n$.*

(2) *If n is odd, the perfect matching is on $2n+2K+2$ points, and $2K+1$ of these $n+K+1$ edges are colored red, each of weight a, for some $0 \leq K \leq n$.*

Here is a simple linearization result:

$$H_n(x,a)H_m(x,a) = \sum_{s=0}^{min(m,n)} \binom{n}{s}\binom{m}{s} s!(1+ax)^s H_{n+m-2s}(x,a).$$

10.2. **Laguerre polynomials.**

The classical Laguerre polynomials have moments $\mu_n = (a+1)_n = (A)_n$ if $A = a+1$. When $A = 1$ this is $\mu_n = n!$, so the number of weighted Motzkin paths is the number of permutations of length n. Viennot [18] used *Laguerre histories* to give a bijection which explained this fact, and implied many weighted versions.

In Section 8.3 the type R_I Laguerre polynomials are given. The moments remain $\mu_n = (a+1)_n = (A)_n$, but we now have weighted Motzkin-Schröder paths with different weights,

$$b_n = a-n, \quad a_n = n, \quad \lambda_n = 0.$$

Note that since $\lambda_n = 0$, Motzkin-Schröder paths become Schröder paths. In terms of weighted lattice paths we have the following proposition.

Proposition 10.3. *For $b_n = a-n$, $a_n = n$, and $\lambda_n = 0$, we have*

$$\sum_{\pi \in \mathrm{Sch}_n} \mathrm{wt}(\pi) = (a+1)_n.$$

In this subsection we give a combinatorial proof of Proposition 10.3 using a *type R_I Laguerre history.*

Let Sch'_n denote the set of Schröder paths $\pi \in \mathrm{Sch}_n$ that contain no peaks (U,V). For $\pi \in \mathrm{Sch}'_n$, define $\mathrm{wt}'(\pi)$ to be the weight of π with respect to $b_n = a+1$, $a_n = n$, and $\lambda_n = 0$.

Lemma 10.4. *We have*

$$\sum_{\pi \in \mathrm{Sch}_n} \mathrm{wt}(\pi) = \sum_{\pi \in \mathrm{Sch}'_n} \mathrm{wt}'(\pi).$$

Proof. For $\pi, \sigma \in \mathrm{Sch}_n$ define a relation $\pi \sim \sigma$ if π is obtained from σ by a sequence of replacing a peak (U, V) by a horizontal step H or vice versa. Since a peak (U, V) and a horizontal step H have the same starting and ending points, it is easy to see that $\sim$ is an equivalence relation on Sch_n and Sch'_n is a system of representatives of the equivalence classes. Moreover, for each $\pi \in \mathrm{Sch}'_n$, we have

$$\sum_{\sigma \sim \pi} \mathrm{wt}(\sigma) = \mathrm{wt}'(\pi)$$

because the weight of a peak (U, V) and a horizontal step H starting at height n are, respectively, $n+1$ and $a-n$, whose sum is $a+1$. Summing the above equation over $\pi \in \mathrm{Sch}'_n$ gives the desired identity. ☐

Definition 10.5. A *type R_I Laguerre history* of length n is a labeled Schröder path from $(0,0)$ to $(n,0)$ in which each vertical step starting at height h is labeled by an integer in $\{1,2,\dots,h\}$. The set of type R_I Laguerre histories of length n is denoted by LH_n.

By the definition of a type R_I Laguerre history we have

$$\sum_{\pi \in \mathrm{Sch}'_n} \mathrm{wt}'(\pi) = \sum_{\pi \in \mathrm{LH}_n} (a+1)^{H(\pi)}, \tag{10.1}$$

where $H(\pi)$ is the number of horizontal steps in π. By Lemma 10.4 and (10.1), to show Proposition 10.3 it suffices to show the following proposition.

Proposition 10.6. *We have*

$$\sum_{\pi \in \mathrm{LH}_n} (a+1)^{H(\pi)} = (a+1)_n.$$

To show the above proposition we give a bijection $\phi : \mathrm{LH}_n \to \mathfrak{S}_n$, where $\mathfrak{S}_n$ is the set of permutations on $[n]$.

Let $\pi \in \mathrm{LH}_n$. Then $\phi(\pi)$ is the permutation in $\mathfrak{S}_n$ constructed as follows. The basic idea is to create a cycle for each horizontal step of π and the vertical steps following immediately after that.

- First, consider the leftmost horizontal step. Suppose there are k vertical steps, labeled $v_1, \dots, v_k$, following this horizontal step. If the horizontal step is between the lines $x = i-1$ and $x = i$, create a cycle starting with i. Then for $j = 1, 2, \dots, k$, add at the end of the cycle the v_jth smallest integer in $[i]$ that have not been used. This creates a cycle of length $k+1$ with largest integer i.
- For each of the remaining horizontal steps, from left to right, repeat the above process.

For example, let π be the type R_I Laguerre history in Figure 13. Then the corresponding permutation $\phi(\pi)$, in cycle notation, is given by

$$\phi(\pi) = (4,2,3)(8)(9,7,1)(10)(12)(13,5,11,6).$$

It is easy to check that the map ϕ is a bijection such that if $\phi(\pi) = w$, then $H(\pi)$ is equal to the number of cycles in w. This proves Proposition 10.6.

10.3. **Meixner polynomials.**

For the Meixner polynomials, the situation is similar to the Laguerre polynomials. The moments for the classical Meixner polynomials are

$$\mu_n = \sum_{j=1}^{n} S(n,j)(b)_j \left(\frac{c}{1-c}\right)^j, \tag{10.2}$$

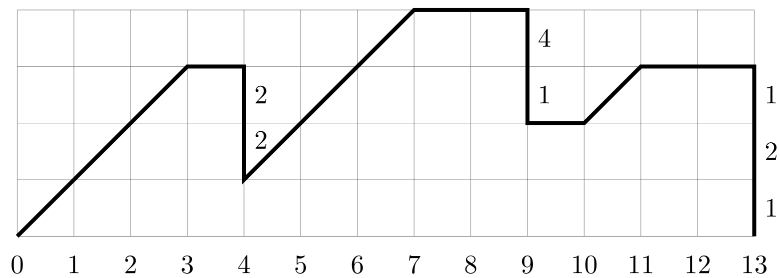

FIGURE 13. A type R_I Laguerre history of length 13.

which involves set partitions and permutations as the fundamental combinatorial objects.

The type R_I Meixner polynomials in Section 8.4 retain these moments, but with different paths and weights. In this section we develop a *type R_I Meixner history* to prove (10.2). To simplify matters we reformulate the formula for the moments. Let $d = c/(1-c)$, so that

$$b_n = n - dn + bd - d, \qquad a_n = nd, \qquad \lambda_n = bdn - dn^2. \tag{10.3}$$

Proposition 10.7. *Let b_n, a_n, and λ_n be given by* (10.3). *Then*

$$\mu_n = \sum_{j=1}^{n} S(n,j)(b)_j d^j.$$

In this subsection we give a combinatorial proof of Proposition 10.7.

For $\pi \in \mathrm{MS}_n$ define $\mathrm{wt}(\pi)$ to be the weight of π with respect to the weights in Proposition 10.7. Then a combinatorial restatement of Proposition 10.7 is Proposition 10.8

Proposition 10.8. *We have*

$$\sum_{\pi \in \mathrm{MS}_n} \mathrm{wt}(\pi) = \sum_{j=1}^{n} S(n,j)(b)_j d^j.$$

Let MS'_n denote the set of Motzkin-Schröder paths $\pi \in \mathrm{MS}_n$ that contain no peaks (U, V). For $\pi \in \mathrm{MS}'_n$, define $\mathrm{wt}'(\pi)$ to be the weight of π with respect to

$$b_n = n + bd, \qquad a_n = nd, \qquad \lambda_n = bdn - dn^2.$$

Then the following lemma is proved by the same argument as in the proof of Lemma 10.4.

Lemma 10.9. *We have*

$$\sum_{\pi \in \mathrm{MS}_n} \mathrm{wt}(\pi) = \sum_{\pi \in \mathrm{MS}'_n} \mathrm{wt}'(\pi).$$

Let MS''_n denote the set of Motzkin-Schröder paths $\pi \in \mathrm{MS}'_n$ that contain no down steps. Note that $\mathrm{MS}''_n \subset \mathrm{Sch}_n$. For $\pi \in \mathrm{MS}'_n$, define $\mathrm{wt}''(\pi)$ to be the product of the weight of each step, where the weight of a step starting at height n is given by

- 1 if the step is an up step,
- nd if the step is a vertical step,
- $bd + n$ if the step is a horizontal step not followed by a vertical step, and
- $bd + b$ if the step is a horizontal step followed by a vertical step.

By defining a relation $\pi \sim \sigma$ if π is obtained from σ by a sequence of replacing a pair (H, V) with a down step D or vice versa, we similarly obtain the following lemma.

Lemma 10.10. *We have*

$$\sum_{\pi\in \mathrm{MS}'_n} \mathrm{wt}'(\pi) = \sum_{\pi\in \mathrm{MS}''_n} \mathrm{wt}''(\pi).$$

Definition 10.11. A *type R_I Meixner history* of length n is a path $\pi \in \mathrm{MS}'_n$ together with a labeling such that

- each vertical step starting at height h is labeled by an integer in $\{1,2,\dots,h\}$,
- each horizontal step followed by a vertical step is not labeled or is labeled by 0, and
- each horizontal step not followed by a vertical step and starting at height h is not labeled or is labeled by an integer in $\{1,2,\dots,h\}$.

Let MH_n denote the set of type R_I Meixner histories of length n.

For $\pi\in \mathrm{MH}_n$, define $\mathrm{wt}(\pi)$ to be the product of the weight of each step defined as follows:

- An up step has weight 1.
- A vertical step has weight d.
- A non-labeled horizontal step has weight bd.
- A horizontal step labeled 0 has weight b.
- A horizontal step labeled i, for $i\ge 1$, has weight 1.

By definition it is clear that

$$\sum_{\pi\in \mathrm{MS}''_n} \mathrm{wt}''(\pi) = \sum_{\pi\in \mathrm{MH}_n} \mathrm{wt}(\pi). \tag{10.4}$$

By Lemmas 10.9 and 10.10 and (10.4), Proposition 10.8 is equivalent to

$$\sum_{\pi\in \mathrm{MH}_n} \mathrm{wt}(\pi) = \sum_{j=1}^{n} S(n,j)(b)_j d^j. \tag{10.5}$$

Let S_n denote the set of pairs (P,σ) of a set partition $P=\{B_1,\dots,B_k\}$ of $[n]$ and a permutation σ of the blocks $B_1,\dots,B_k$ of P. For $(P,\sigma)\in S_n$, define

$$\mathrm{wt}(P,\sigma) = b^{\mathrm{cyc}(\sigma)} d^{|P|},$$

where $\mathrm{cyc}(\sigma)$ is the number of cycles in σ and $|P|$ is the number of blocks in P. Since $\sum_{\pi\in\mathfrak{S}_n} b^{\mathrm{cyc}(\pi)} = (b)_n$, we have

$$\sum_{j=1}^{n} S(n,j)(b)_j d^j = \sum_{(P,\sigma)\in S_n} \mathrm{wt}(P,\sigma). \tag{10.6}$$

Using (10.6) we can rewrite (10.5) as

$$\sum_{\pi\in \mathrm{MH}_n} \mathrm{wt}(\pi) = \sum_{(P,\sigma)\in S_n} \mathrm{wt}(P,\sigma). \tag{10.7}$$

To prove (10.7) we find a weight-preserving bijection $\psi : \mathrm{MH}_n \to S_n$.

Let $\pi\in \mathrm{MH}_n$. Then there are n non-vertical steps in π, say $A_1,\dots,A_n$ from left to right. Note that the ending point of A_i has x-coordinate i. We create *available blocks* and *cycles of blocks* as follows. We will use the convention that once an available block is used as an element of a cycle the block is no longer available. Moreover, if there are several available blocks, these are ordered by their smallest elements.

- Initially there are no available blocks and no cycles of blocks.
- For $i=1,2,\dots,n$, do the following procedure:
  - If A_i is an up step, create a new available block containing a single element i.

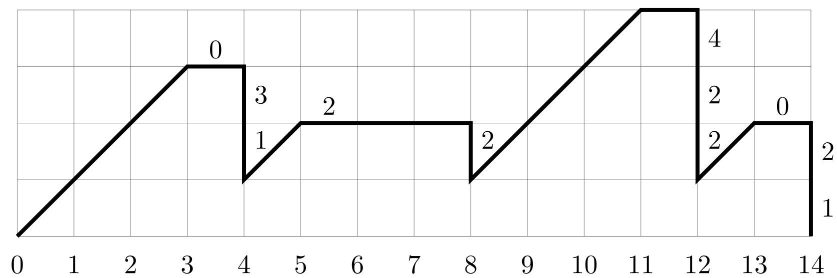

FIGURE 14. A type R_I Meixner history of length 14.

- If A_i is a non-labeled horizontal step and if it is not followed by a vertical step, then create a new available block containing a single element i and make a cycle consisting only of this block.
- If A_i is a horizontal step labeled by $j \geq 1$, then insert the integer i in the j-th available block.
- If A_i is a non-labeled horizontal step followed by a vertical step, then create a new available block containing a single element i and make a cycle starting with this block. Suppose that A_i is followed by k vertical steps labeled $r_1, r_2, \dots, r_k$. Then add the r_1-th available block at the end of the cycle, add the r_2-th available block at the end of the cycle, and so on. After this process we obtain a cycle consisting of $k+1$ blocks.
- If A_i is a horizontal step labeled by 0, then A_i must be followed by a vertical step. Suppose that A_i is followed by k vertical steps labeled $r_1, r_2, \dots, r_k$. First insert i in the r_1-th available block and then create a new cycle starting with this block. Then, as in the previous case, add the r_2-th available block at the end of the cycle, add the r_3-th available block at the end of the cycle, and so on. After this process we obtain a cycle consisting of k blocks.

For example, if π is the Meixner history in Figure 14, then the cycles of $\psi(\pi)$ are created as in Table 1 and we get

$$\psi(\pi) = (\{3,4\},\{1\})(\{7\})(\{8\},\{5,6\})(\{12\},\{11\},\{9\},\{10\})(\{13,14\},\{2\}).$$

It is straightforward to check that the map $\psi : \mathrm{MH}_n \to S_n$ is a weight-preserving bijection, which shows (10.7).

Finally we note that the nth moment of the Meixner polynomials has the following formula due to de Médicis [4, Theorem 2]:

$$\mu_n = (1-c)^{-n} \sum_{\pi \in \mathfrak{S}_n} b^{\mathrm{cyc}(\pi)} c^{\mathrm{nexc}(\pi)}, \tag{10.8}$$

where $\mathrm{nexc}(\pi)$ is the number of non-excedances of π, i.e., the number of integers $i \in [n]$ with $\pi(i) < i$. Since the Meixner polynomials and the type R_I Meixner polynomials have the same nth moment, combining Proposition 10.7 and (10.8) yields the following corollary.

Corollary 10.12. *We have*

$$\sum_{\pi \in \mathfrak{S}_n} b^{\mathrm{cyc}(\pi)} c^{\mathrm{nexc}(\pi)} = \sum_{j=1}^{n} S(n,j)(b)_j c^j (1-c)^{n-j}.$$

non-vertical steps	available blocks	new cycles of blocks
A_1	$\{1\}$	
A_2	$\{1\},\{2\}$	
A_3	$\{1\},\{2\},\{3\}$	
A_4	$\{1\},\{2\},\{3\}$	$(\{3,4\},\{1\})$
A_5	$\{2\},\{5\}$	
A_6	$\{2\},\{5,6\}$	
A_7	$\{2\},\{5,6\}$	$(\{7\})$
A_8	$\{2\},\{5,6\}$	$(\{8\},\{5,6\})$
A_9	$\{2\},\{9\}$	
A_{10}	$\{2\},\{9\},\{10\}$	
A_{11}	$\{2\},\{9\},\{10\},\{11\}$	
A_{12}	$\{2\},\{9\},\{10\},\{11\}$	$(\{12\},\{11\},\{9\},\{10\})$
A_{13}	$\{2\},\{13\}$	
A_{14}	$\{2\},\{13\}$	$(\{13,14\},\{2\})$

TABLE 1. The process of the map ψ for each non-vertical step A_i.

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