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# A generalization of Krull-Webster's theory to higher order convex functions: multiple $\Gamma$-type functions

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**Abstract**

We provide uniqueness and existence results for the eventually $p$-convex and eventually $p$-concave solutions to the difference equation $\Delta f = g$ on the open half-line $(0,\infty)$, where $p$ is a given nonnegative integer and $g$ is a given function satisfying the asymptotic property that the sequence $n \mapsto \Delta^p g(n)$ converges to zero. These solutions, that we call $\log\Gamma_p$-type functions, include various special functions such as the polygamma functions, the logarithm of the Barnes G-function, and the Hurwitz zeta function. Our results generalize to any nonnegative integer $p$ the special case when $p = 1$ obtained by Krull and Webster, who both generalized Bohr-Mollerup-Artin's characterization of the gamma function.

We also follow and generalize Webster's approach and provide for $\log\Gamma_p$-type functions analogues of Euler's infinite product, Weierstrass' infinite product, Gauss' limit, Gauss' multiplication formula, Legendre's duplication formula, Euler's constant, Stirling's constant, Stirling's formula, Wallis's product formula, and Raabe's formula for the gamma function. We also introduce and discuss analogues of Binet's function, Burnside's formula, Fontana-Mascheroni's series, Euler's reflection formula, and Gauss' digamma theorem.

Lastly, we apply our results to several special functions, including the Hurwitz zeta function and the generalized Stieltjes constants, and show through these examples how powerful is our theory to produce formulas and identities almost systematically.

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# Contents

## List of symbols

| | |
|---|---|
| $A$ | Glaisher-Kinkelin's constant |
| $b_n^r(x)$ | $D_x^r\binom{x}{n+r}$ |
| $\mathcal{C}^k$ | set of $k$ times continuously differentiable functions on $\mathbb{R}_+$ |
| $\mathcal{C}^k(I)$ | set of $k$ times continuously differentiable functions on $I$ |
| $D$ | ordinary derivative operator |
| $\mathcal{D}_S^p$ | $\{g\colon \mathbb{R}_+ \to \mathbb{R} : \Delta^p g(t) \to 0 \text{ as } t \to_S \infty\}$ |
| $\mathcal{D}_S^\infty$ | $\bigcup_{p\geqslant 0} \mathcal{D}_S^p$ |
| $\widetilde{\mathcal{D}}_{\mathbb{N}}^{-1}$ | $\{g\colon \mathbb{R}_+ \to \mathbb{R} : \text{the sequence } n \mapsto g(n) \text{ is summable}\}$ |
| $f[x_0,\ldots,x_p]$ | divided difference of $f$ at the points $x_0,\ldots,x_p$ |
| $f_n^p[g]$ | function defined in (2) |
| $G_n$ | $n$th Gregory coefficient $G_n = \int_0^1 \binom{t}{n}\, dt$ |
| $\overline{G}_n$ | $\overline{G}_n = 1 - \sum_{j=1}^n \lvert G_j\rvert$ |
| $H_x$ | harmonic number function |
| $I$ | arbitrary real interval whose interior is nonempty |
| $J^q[g]$ | Binet-like function defined in (38) |
| $\mathcal{K}^p$ | $\mathcal{K}_+^p \cup \mathcal{K}_-^p$ |
| $\mathcal{K}_+^p$ | set of functions $f\colon \mathbb{R}_+ \to \mathbb{R}$ that are eventually $p$-convex |
| $\mathcal{K}_-^p$ | set of functions $f\colon \mathbb{R}_+ \to \mathbb{R}$ that are eventually $p$-concave |
| $\mathcal{K}^p(I)$ | $\mathcal{K}_+^p(I) \cup \mathcal{K}_-^p(I)$ |
| $\mathcal{K}_+^p(I)$ | set of functions $f\colon I \to \mathbb{R}$ that are $p$-convex |
| $\mathcal{K}_-^p(I)$ | set of functions $f\colon I \to \mathbb{R}$ that are $p$-concave |
| $\mathcal{K}^\infty$ | $\bigcap_{p\geqslant 0} \mathcal{K}^p$ |
| $\mathrm{Log}\Gamma_p$ | set of $\log \Gamma_p$-type functions (see Subsection 5.2) |
| $\mathbb{N}$, $\mathbb{N}^*$ | $\mathbb{N} = \{0,1,2,\ldots\}$, $\mathbb{N}^* = \{1,2,\ldots\}$ |
| $P_p[f]$ | interpolating polynomial of degree $\leqslant p$ of $f$ |
| $\mathbb{R}_+$ | open half-line $(0,\infty)$ |
| $\mathcal{R}_S^p$ | $\{g\colon \mathbb{R}_+ \to \mathbb{R} : \text{for each } x>0,\ \rho_t^p[g](x) \to 0 \text{ as } t \to_S \infty\}$ |
| $R_{p,m,n}$ | remainder in Gregory's summation formula (44) |
| $\mathbf{ran}(\Sigma)$ | range of the map $\Sigma$ |
| $S$ | $S = \mathbb{N}$ or $\mathbb{R}$ |
| $x^{\underline{k}}$ | $x(x-1)\cdots(x-k+1)$ |
| $x \to_S \infty$ | $x$ tends to infinity, assuming only values in $S \in \{\mathbb{N},\mathbb{R}\}$ |
| $x_+$ | $\max\{0,x\}$ |
| $\gamma$ | Euler's constant |
| $\gamma[g]$ | generalized Euler's constant associated with $g$ |
| $\Gamma_p$ | function $\Gamma_p$ / set of $\Gamma_p$-type functions (see Subsection 5.2) |
| $\Delta$ | forward difference operator |
| $\rho_a^p[f]$ | function defined in (5) |
| $\sigma[g]$, $\overline{\sigma}[g]$ | $\sigma[g] = \int_0^1 \Sigma g(t+1)\, dt$, $\overline{\sigma}[g] = \int_0^1 \Sigma g(t)\, dt$ |
| $\Sigma$ | map defined in (23) |

| | |
|---|---|
| $\psi$, $\psi_\nu$ | digamma function, polygamma functions |

# 1 Introduction

Let $\mathbb{R}_+$ denote the open half-line $(0,\infty)$ and let $\Delta$ denote the forward difference operator on the space of functions from $\mathbb{R}_+$ to $\mathbb{R}$. In this paper, we are interested in the classical functional equation $\Delta f = g$ on $\mathbb{R}_+$, which can be written explicitly as

$$f(x+1) - f(x) \;=\; g(x), \qquad x > 0,$$

where $g\colon \mathbb{R}_+ \to \mathbb{R}$ is a given function. This equation appears naturally in the theory of the Euler gamma function, with $f(x) = \ln\Gamma(x)$ and $g(x) = \ln x$, but also in the study of many other special functions such as the Barnes G-function and the Hurwitz zeta function.

For any function $g\colon \mathbb{R}_+ \to \mathbb{R}$, the difference equation $\Delta f = g$ has infinitely many solutions, and each of them can be uniquely determined by prescribing its values in the interval $(0,1]$. Recall also that any two solutions differ by a 1-periodic function, i.e., a periodic function of period 1.

For certain functions $g$, however, special solutions can be determined by their local properties or their asymptotic behaviors. On this issue, a seminal result is the very nice characterization of the gamma function by Bohr and Mollerup [20]. They showed that all log-convex solutions $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ to the equation

$$f(x+1) \;=\; x\, f(x), \qquad x > 0, \tag{1}$$

are of the form $f(x) = c\,\Gamma(x)$, where $c > 0$. Thus, the gamma function is a kind of *principal solution* to its equation (Nörlund [69, Chapter 5] calls it the "Hauptlösung"). The additive, but equivalent, version of this result, obtained by taking the logarithm of both sides of (1), can be stated as follows. For $g(x) = \ln x$, all convex solutions $f\colon \mathbb{R}_+ \to \mathbb{R}$ to the difference equation $\Delta f = g$ are of the form $f(x) = c + \ln\Gamma(x)$, where $c \in \mathbb{R}$. Recall also that the proof of Bohr and Mollerup's result was simplified later by Artin [9] (see also Artin [10]) and, as observed by Webster [80], this result "has then become known as the Bohr-Mollerup-Artin Theorem, and was adopted by Bourbaki [21] as the starting point for his exposition of the gamma function."

A noteworthy generalization of Bohr-Mollerup-Artin's theorem was provided by Krull [46,47] and then independently by Webster [79,80]. Recall that a function $g\colon \mathbb{R}_+ \to \mathbb{R}$ is said to be eventually convex (resp. eventually concave) if it is convex (resp. concave) in a neighborhood of infinity. Krull [46] essentially showed that for any eventually concave function $g\colon \mathbb{R}_+ \to \mathbb{R}$ having the asymptotic property that, for each $h > 0$,

$$g(x+h) - g(x) \;\to\; 0 \qquad \text{as } x \to \infty,$$

there exists exactly one (up to an additive constant) eventually convex solution $f\colon \mathbb{R}_+ \to \mathbb{R}$ to the equation $\Delta f = g$ (and dually, if $g$ is eventually convex, then $f$ is eventually concave). He also provided an explicit expression for this solution as a pointwise limit of functions, namely

$$f(x) \;=\; f(1) + \lim_{n\to\infty} f_n^1[g](x), \qquad x > 0,$$

where

$$f_n^1[g](x) \;=\; -\,g(x) + \sum_{k=1}^{n-1}(g(k) - g(x+k)) + x\,g(n).$$

Much later, and independently, Webster [79, 80] established the multiplicative version of Krull's result.

In this paper, we generalize Krull-Webster's result by relaxing the asymptotic condition imposed on function $g$ into the much weaker requirement that the sequence $n \mapsto \Delta^p g(n)$ converges to zero for some nonnegative integer $p$. This relaxation leads us to replacing the convexity and concavity properties by the $p$-convexity and $p$-concavity properties (i.e., convexity and concavity of order $p$; see Definition 2.1 below). More precisely, we establish the uniqueness and existence theorems below (Theorems 1.1 and 1.2), as they were stated separately by Webster in the case when $p = 1$.

We let $\mathbb{N}$ denote the set of nonnegative integers and we let $\mathbb{N}^*$ denote the set of strictly positive integers. For any $p \in \mathbb{N}$, any $n \in \mathbb{N}^*$, and any $g\colon \mathbb{R}_+ \to \mathbb{R}$, we define the function $f_n^p[g]\colon \mathbb{R}_+ \to \mathbb{R}$ by the equation

$$f_n^p[g](x) \;=\; -\,g(x) + \sum_{k=1}^{n-1}(g(k) - g(x+k)) + \sum_{j=1}^{p}\tbinom{x}{j}\,\Delta^{j-1}g(n)\,. \tag{2}$$

**Theorem 1.1** (Uniqueness). *Let $p \in \mathbb{N}$ and let the function $g\colon \mathbb{R}_+ \to \mathbb{R}$ have the property that the sequence $n \mapsto \Delta^p g(n)$ converges to zero. Suppose that $f\colon \mathbb{R}_+ \to \mathbb{R}$ is an eventually $p$-convex or eventually $p$-concave function satisfying the difference equation $\Delta f = g$. Then $f$ is uniquely determined (up to an additive constant) by $g$ through the equation*

$$f(x) \;=\; f(1) + \lim_{n\to\infty} f_n^p[g](x), \qquad x > 0.$$

**Theorem 1.2** (Existence). *Let $p \in \mathbb{N}$ and suppose that the function $g\colon \mathbb{R}_+ \to \mathbb{R}$ is eventually $p$-convex or eventually $p$-concave and has the asymptotic property that the sequence $n \mapsto \Delta^p g(n)$ converges to zero. Then there exists a unique (up to an additive constant) eventually $p$-convex or eventually $p$-concave solution $f\colon \mathbb{R}_+ \to \mathbb{R}$ to the difference equation $\Delta f = g$. Moreover,*

$$f(x) \;=\; f(1) + \lim_{n\to\infty} f_n^p[g](x), \qquad x > 0, \tag{3}$$

*and $f$ is $p$-convex (resp. $p$-concave) on any unbounded subinterval of $\mathbb{R}_+$ on which $g$ is $p$-concave (resp. $p$-convex).*

We observe that Theorem 1.2 was first proved in the case when $p = 0$ by John [42]. As mentioned above, it was also established in the case when $p = 1$ by Krull [46] and Webster [80]. More recently, the case when $p = 2$ was investigated by Rassias and Trif [72], but the asymptotic condition they imposed on function $g$ is much stronger than ours and hence defines a very specific subclass of functions. (We discuss Rassias and Trif's result in Appendix A.) We also observe that attempts to establish Theorem 1.2 for any value of $p$ were made by Kuczma [50, Theorem 1] (see also Kuczma [52, pp. 118–121]) and then by Ardjomande [8]. However, the representation formulas they provide for the solutions are rather intricate. Thus, to the best of our knowledge, both Theorems 1.1 and 1.2, as stated above in their full generality and simplicity, were previously unknown.

For any solution $f$ arising from Theorem 1.2 when $p = 1$, Webster [80] calls the function $\exp \circ f$ a $\Gamma$*-type function*. In fact, $\exp \circ f$ reduces to the gamma function $\Gamma$ when $\exp \circ g$ is the identity function, which simply means that the gamma function restricted to $\mathbb{R}_+$ is itself a $\Gamma$-type function. In this particular case, the limit given in (3) reduces to the following Gauss well-known limit for the gamma function

$$\Gamma(x) \;=\; \lim_{n\to\infty} \frac{n!\, n^x}{x(x+1)\,\cdots\,(x+n)}\,. \tag{4}$$

Similarly, for any fixed $p \in \mathbb{N}$ and any solution $f$ arising from Theorem 1.2, we call the function $\exp \circ f$ a $\Gamma_p$*-type function*, and we naturally call the function $f$ a $\log \Gamma_p$*-type function*. When the value of $p$ is not specified, we call these functions *multiple* $\Gamma$*-type function* and *multiple* $\log \Gamma$*-type function*, respectively. This terminology will be defined more formally and justified in Subsection 5.2.

Interestingly, Webster established for $\Gamma$-type functions analogues of *Legendre's duplication formula*, *Gauss' multiplication formula*, *Stirling's formula*, *Euler's constant*, and *Weierstrass' infinite product* for the gamma function. In this paper, we also establish for multiple $\Gamma$-type functions and multiple $\log \Gamma$-type functions analogues of all the formulas above as well as analogues of *Stirling's constant*, *Euler's infinite product*, *Wallis's product formula*, and *Raabe's formula* for the gamma function. We also introduce analogues of *Binet's function*, *Burnside's formula*, and *Fontana-Mascheroni's series*, and discuss analogues of *Euler's reflection formula* and *Gauss' digamma theorem*. Thus, for each multiple $\Gamma$-type function, it is no longer surprising for instance that an analogue of Euler' infinite product must hold, almost rendering a formal proof unnecessary! All these results, together with the uniqueness and existence theorems above, show that our theory provides a very general and unified framework to study the properties of a large variety of functions. Thus, for each of these functions we can retrieve known formulas and establish new ones.

**Example 1.3** (The Hurwitz zeta function, see Subsection 9.6). Consider the Hurwitz zeta function $s \mapsto \zeta(s,a)$, defined when $\Re(a) > 0$ as an analytic continuation to $\mathbb{C}\setminus\{1\}$ of the series $\sum_{k=0}^{\infty}(a+k)^{-s}$. This function is known to satisfy the difference equation

$$\zeta(s,a+1)-\zeta(s,a) \;=\; -a^{-s}.$$

Also, it is not difficult to see that, for any $s\in\mathbb{R}\setminus\{1\}$, the restriction of the map $x\mapsto\zeta(s,x)$ to $\mathbb{R}_+$ is a $\log\Gamma_{p(s)}$-type function, where

$$p(s) \;=\; \max\{0,\lfloor 1-s\rfloor\}.$$

Theorem 1.2 then tells us that all eventually $p(s)$-convex or eventually $p(s)$-concave solutions $f_s\colon\mathbb{R}_+\to\mathbb{R}$ to the difference equation

$$f_s(x+1)-f_s(x) \;=\; -x^{-s}$$

are of the form $f_s(x) = c_s + \zeta(s,x)$, where $c_s\in\mathbb{R}$. Moreover, equation (3) provides the following analogue of *Gauss' limit* for the gamma function

$$\zeta(s,x) \;=\; \zeta(s)+x^{-s}+\lim_{n\to\infty}\left(\sum_{k=1}^{n-1}\left((x+k)^{-s}-k^{-s}\right)-\sum_{j=1}^{p(s)}\binom{x}{j}\Delta_n^{j-1}n^{-s}\right),$$

where $s\mapsto\zeta(s)$ is the Riemann zeta function. Using one of our new results (namely, Theorem 6.5), we are also able to derive the following analogue of *Stirling's formula*

$$\zeta(s,x)-\frac{x^{1-s}}{s-1}-\sum_{j=1}^{p(s)}G_j\,\Delta_x^{j-1}x^{-s} \;\to\; 0 \qquad \text{as } x\to\infty,$$

where $G_n = \int_0^1\binom{t}{n}\,dt$ is the $n$th Gregory coefficient. For instance, setting $s=-\frac{3}{2}$ in this asymptotic formula, we obtain

$$\zeta\left(-\tfrac{3}{2},x\right)+\tfrac{2}{5}\,x^{5/2}-\tfrac{7}{12}\,x^{3/2}+\tfrac{1}{12}\,(x+1)^{3/2} \;\to\; 0 \qquad \text{as } x\to\infty. \quad \square$$

**Example 1.4** (Barnes's G-function, see Subsection 9.5). The Barnes G-function $G\colon\mathbb{R}_+\to\mathbb{R}_+$ is the unique solution to the equation

$$f(x+1) \;=\; \Gamma(x)f(x)$$

whose logarithm is eventually 2-convex and vanishes at $x=1$. Thus defined, this function is a $\Gamma_2$-type function. In particular, formula (3) provides the following analogue of *Gauss' limit* for the gamma function

$$G(x) \;=\; \lim_{n\to\infty}\frac{\Gamma(1)\Gamma(2)\,\cdots\,\Gamma(n)}{\Gamma(x)\Gamma(x+1)\,\cdots\,\Gamma(x+n)}\,n!^x\,n^{\binom{x}{2}}.$$

Our results also enable us to derive various unusual formulas. For instance, we have the following analogue of *Euler's infinite product*

$$G(x) \;=\; \frac{1}{\Gamma(x)}\prod_{k=1}^{\infty}\frac{\Gamma(k)}{\Gamma(x+k)}\,k^x(1+1/k)^{\binom{x}{2}}$$

and the following analogue of *Weierstrass' infinite product*

$$G(x) \;=\; \frac{e^{(-\gamma-1)\binom{x}{2}}}{\Gamma(x)}\prod_{k=1}^{\infty}\frac{\Gamma(k)}{\Gamma(x+k)}\,k^x e^{\psi'(k)\binom{x}{2}},$$

where $\gamma$ is Euler's constant and $\psi$ is the digamma function. We also have the following surprising analogues of *Wallis's product formula*

$$\lim_{n\to\infty}\frac{\Gamma(1)\Gamma(3)\,\cdots\,\Gamma(2n-1)}{\Gamma(2)\Gamma(4)\,\cdots\,\Gamma(2n)}\left(\frac{2n}{e}\right)^n \;=\; \frac{1}{\sqrt{2}}$$

and

$$\lim_{n\to\infty}\frac{G(1)G(3)\,\cdots\,G(2n-1)}{G(2)G(4)\,\cdots\,G(2n)}\,\frac{n^{n^2-\frac{1}{2}n-\frac{1}{24}}\,2^{n^2-\frac{7}{24}}\,\pi^{\frac{1}{2}n}}{e^{\frac{3}{2}n^2-\frac{1}{2}n-\frac{1}{24}}} \;=\; A^{\frac{1}{2}}\,,$$

where $A$ is Glaisher-Kinkelin's constant defined by the equation $\zeta'(-1)=\frac{1}{12}-\ln A$. □

Throughout this paper we will use the basic function $g(x)=\ln x$ as the guiding example. However, many other functions, including the examples above, will be discussed in Section 9.

This paper is outlined as follows. In Section 2, we present some definitions and preliminary results on higher order convexities as well as on Newton interpolation theory. In Section 3, we establish Theorems 1.1 and 1.2 and provide conditions for the sequence $n\mapsto f_n^p[g](x)$ to converge uniformly on any bounded subset of $\mathbb{R}_+$. We also examine the particular case when the sequence $n\mapsto g(n)$ is summable, and we provide historical remarks on some improvements of Krull-Webster's theory. In Section 4, we investigate some properties of the set of functions $g(x)$ defined by the asymptotic condition stated in Theorems 1.1 and 1.2. We also investigate the subset of those functions that are eventually $p$-convex or eventually $p$-concave. In Section 5, we introduce, investigate, and characterize the multiple $\log\Gamma$-type functions. In Section 6, we show how Stirling's formula, Stirling's constant, and Euler's constant can be generalized to the multiple $\log\Gamma$-type functions and we introduce analogues of Binet's function, Burnside's formula, and Fontana-Mascheroni's series. We also show how the so-called Gregory summation formula, with an integral form of the remainder, can be very easily derived in this setting. In Section 7, we discuss conditions for the solutions arising from Theorem 1.2 (i.e., the $\log\Gamma_p$-type functions) to be differentiable and we show how these solutions can also be obtained by first differentiating both sides of the difference equation $\Delta f = g$. In Section 8, we explore further properties of the multiple $\log\Gamma$-type functions. Specifically, we provide analogues of Euler's infinite product, Weierstrass' infinite product, Raabe's formula, Gauss' multiplication formula, and Wallis's product formula. We also discuss analogues of Euler's reflection formula and Gauss' digamma theorem, and we define and solve a generalized version of a functional equation proposed by Webster. In Sections 9 and 10, we apply our results to a number of multiple $\Gamma$-type functions and multiple $\log\Gamma$-type functions, many of whose are well-known special functions related to the gamma function.

We use the following notation throughout. The symbol $I$ denotes an (arbitrary) interval of the real line whose interior is nonempty. For any points $x_0, x_1, \ldots, x_{p+1} \in I$ and any function $f\colon I \to \mathbb{R}$, the symbol $f[x_0, x_1, \ldots, x_{p+1}]$ stands for the divided difference of $f$ at the points $x_0, x_1, \ldots, x_{p+1}$. The symbol $S$ represents either $\mathbb{N}$ or $\mathbb{R}$. For any $S \in \{\mathbb{N}, \mathbb{R}\}$, the notation $x \to_S \infty$ means that $x$ tends to infinity, assuming only values in $S$. For any $x \in \mathbb{R}$ and any $k \in \mathbb{N}$, we set $x_+ = \max\{0, x\}$ and

$$x^{\underline{k}} \;=\; x(x-1)\cdots(x-k+1) \;=\; \frac{\Gamma(x+1)}{\Gamma(x-k+1)}\,.$$

For any $k \in \mathbb{N}$ and any nonempty open real interval $I$, we let $\mathcal{C}^k(I)$ denote the set of $k$ times continuously differentiable functions on $I$, and we set $\mathcal{C}^k = \mathcal{C}^k(\mathbb{R}_+)$. We also let $\Delta$ and $D$ denote the usual difference and derivative operators, respectively. We sometimes add a subscript to specify the variable on which the operator acts, e.g., writing $\Delta_n$ and $D_x$.

Recall that the digamma function $\psi$ is defined on $\mathbb{R}_+$ by the equation $\psi(x) = D\ln\Gamma(x)$. The polygamma functions $\psi_\nu$ ($\nu \in \mathbb{Z}$) are defined on $\mathbb{R}_+$ as follows. If $\nu \in \mathbb{N}$, then $\psi_\nu(x) = D^\nu\psi(x)$. In particular, $\psi_0 = \psi$ is the digamma function. If $\nu \in \mathbb{Z}\setminus\mathbb{N}$, then we have $\psi_{-1}(x) = \ln\Gamma(x)$ and

$$\psi_{\nu-1}(x) \;=\; \int_0^x \psi_\nu(t)\,dt \;=\; \int_0^x \frac{(x-t)^{-\nu-1}}{(-\nu-1)!}\,\ln\Gamma(t)\,dt.$$

Recall also that the harmonic number function $x \mapsto H_x$ is defined on $(-1,\infty)$ by the series

$$H_x \;=\; \sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{x+k}\right).$$

Both functions are strongly related: we have $H_{x-1} = \psi(x) + \gamma$ on $\mathbb{R}_+$, where $\gamma$ is Euler's constant (also called Euler-Mascheroni constant).

For any $a > 0$, any $p \in \mathbb{N}$, and any $g\colon \mathbb{R}_+ \to \mathbb{R}$, we define the function $\rho_a^p[g]\colon [0,\infty) \to \mathbb{R}$ by the equation

$$\rho_a^p[g](x) \;=\; g(x+a) - \sum_{j=0}^{p-1}\binom{x}{j}\,\Delta^j g(a)\,. \tag{5}$$

For any $p \in \mathbb{N}$ and any $S \in \{\mathbb{N}, \mathbb{R}\}$, we let $\mathcal{R}^p_S$ be the set of functions $g\colon \mathbb{R}_+ \to \mathbb{R}$ having the asymptotic property that, for each $x > 0$,

$$\rho^p_t[g](x) \;\to\; 0 \qquad \text{as } t \to_S \infty.$$

We also let $\mathcal{D}^p_S$ be the set of functions $g\colon \mathbb{R}_+ \to \mathbb{R}$ having the asymptotic property that

$$\Delta^p g(t) \;\to\; 0 \qquad \text{as } t \to_S \infty.$$

We immediately observe that the inclusion $\mathcal{D}^p_S \subset \mathcal{D}^{p+1}_S$ holds for every $p \in \mathbb{N}$. We will see in Subsection 4.1 that so does the inclusion $\mathcal{R}^p_S \subset \mathcal{R}^{p+1}_S$.

# 2 Preliminaries

This section is devoted to some basic definitions and results that are needed in this paper. We essentially focus on higher order convexity and Newton interpolation theory.

## 2.1 Higher order convexity and concavity

Let us recall the definition of higher order convexity and concavity properties and present some related results. For background see, e.g., [50], [53, Chapter 15], [70], and [73, pp. 237–240].

**Definition 2.1.** A function $f\colon I \to \mathbb{R}$ is said to be *convex of order* $p$ or simply $p$-*convex* for some integer $p \geqslant -1$ if for any system $x_0 < x_1 < \cdots < x_{p+1}$ of $p+2$ points in $I$ it holds that

$$f[x_0, x_1, \ldots, x_{p+1}] \;\geqslant\; 0.$$

The function $f$ is said to be *concave of order* $p$ or simply $p$-*concave* if $-f$ is $p$-convex.

Thus defined, a function $f\colon I \to \mathbb{R}$ is 1-convex (resp. 1-concave) if it is an ordinary convex (resp. concave) function, while it is a 0-convex (resp. 0-concave) if it is an increasing (resp. decreasing) function.

For any integer $p \geqslant -1$, we let $\mathcal{K}^p_+(I)$ (resp. $\mathcal{K}^p_-(I)$) denote the set of $p$-convex (resp. $p$-concave) functions $f\colon I \to \mathbb{R}$ and we let $\mathcal{K}^p_+$ (resp. $\mathcal{K}^p_-$) denote the set of functions $f\colon \mathbb{R}_+ \to \mathbb{R}$ that are eventually $p$-convex (resp. eventually $p$-concave), i.e., $p$-convex (resp. $p$-concave) in a neighborhood of infinity. We also set

$$\mathcal{K}^p(I) \;=\; \mathcal{K}^p_+(I) \cup \mathcal{K}^p_-(I) \qquad \text{and} \qquad \mathcal{K}^p \;=\; \mathcal{K}^p_+ \cup \mathcal{K}^p_-.$$

The following lemma provides some known connections between higher order convexity and higher order differentiability (see, e.g., [53, Chapter 15]).

**Lemma 2.2.** *Suppose that* $I$ *is an nonempty open real interval and let* $p \in \mathbb{N}^*$*. Then the following assertions hold.*

*(a) We have* $\mathcal{K}^p(I) \subset \mathcal{C}^{p-1}(I)$*.*

*(b) If* $f \in \mathcal{K}^p_+(I)$*, then* $\Delta^j f \in \mathcal{K}^{p-j}_+(I)$ *for* $j = 0, \ldots, p+1$*.*

*(c) If* $f \in \mathcal{C}^j(I) \cap \mathcal{K}^p_+(I)$ *for some* $j \in \{0, \ldots, p+1\}$*, then* $f^{(j)} \in \mathcal{K}^{p-j}_+(I)$*.*

*(d) If* $f \in \mathcal{C}^p(I)$*, then* $f \in \mathcal{K}^p_+(I)$ *if and only if* $f^{(p)} \in \mathcal{K}^0_+(I)$*.*

*(e) If* $f \in \mathcal{C}^p(I)$*, then* $f \in \mathcal{K}^{p-1}_+(I)$ *if and only if* $f^{(p)} \in \mathcal{K}^{-1}_+(I)$*.*

*(f) We have* $f \in \mathcal{K}^p_+(I)$ *if and only if* $f \in \mathcal{C}^{p-1}(I)$ *and* $f^{(p-1)} \in \mathcal{K}^1_+(I)$*.*

*(g) If* $f \in \mathcal{C}^1(I)$ *and* $f' \in \mathcal{K}^{p-1}_+(I)$*, then* $f \in \mathcal{K}^p_+(I)$*.*

We also have the following important lemma. It is interesting in its own right and will be very useful in the subsequent sections. A variant of this result can be found in Kuczma [53, Lemma 15.7.2]. Recall first that for any $f\colon I \to \mathbb{R}$, any $p \in \mathbb{N}$, and any $x \in I$ such that $x + p \in I$, we have

$$\Delta^p f(x) \; = \; p!\, f[x, x+1, \ldots, x+p], \tag{6}$$

where $\Delta$ stands for the standard forward difference operator.

**Lemma 2.3.** *Let* $p \in \mathbb{N}$*. A function* $f\colon I \to \mathbb{R}$ *is* $p$*-convex (resp.* $p$*-concave) if and only if the map* $(z_0, \ldots, z_p) \in I^{p+1} \mapsto f[z_0, \ldots, z_p]$ *is increasing (resp. decreasing) in each place. In particular, if* $f$ *is* $p$*-convex (resp.* $p$*-concave) and if* $\Delta^p f$ *is defined on* $I$*, then* $\Delta^p f$ *is increasing (resp. decreasing) on* $I$*.*

*Proof.* Using the definition of p-convexity and the standard recurrence relation for divided differences, we can see that $f$ is p-convex if and only if, for any pairwise distinct $x_0, \ldots, x_p \in I$, we have

$$\frac{f[x_1, x_2 \ldots, x_p] - f[x_0, x_2 \ldots, x_p]}{x_1 - x_0} \;\geqslant\; 0.$$

Equivalently, for any pairwise distinct $x_0, \ldots, x_p \in I$, we have

$$x_1 > x_0 \quad \Rightarrow \quad f[x_1, x_2 \ldots, x_p] - f[x_0, x_2 \ldots, x_p] \;\geqslant\; 0.$$

The latter condition exactly means that the map $(z_0, \ldots, z_p) \mapsto f[z_0, \ldots, z_p]$ is increasing in the first place. Since this map is known to be symmetric, it must be increasing in each place. The second part of the lemma follows from (6). □

## 2.2 Newton interpolation

For any integer $p \in \mathbb{N}$, any $p$ points $x_0, \ldots, x_{p-1} \in \mathbb{R}_+$, and any function $f\colon \mathbb{R}_+ \to \mathbb{R}$, we let the map

$$x \;\mapsto\; P_{p-1}[f](x_0, \ldots, x_{p-1}; x)$$

denote the unique interpolating polynomial of $f$ with nodes at $x_0, \ldots, x_{p-1}$. Recall that this polynomial has degree at most $p-1$. (The zero polynomial can be assumed to have degree $-1$.) For instance, using the classical Newton interpolation formula we obtain the following identity: for any $a > 0$,

$$P_{p-1}[f](a, a+1, \ldots, a+p-1; x) \;=\; \sum_{j=0}^{p-1} \tbinom{x-a}{j}\, \Delta^j f(a). \tag{7}$$

Also, the corresponding interpolation error at $x$ is

$$f(x) - \sum_{j=0}^{p-1} \tbinom{x-a}{j}\, \Delta^j f(a) \;=\; (x-a)^{\underline{p}}\, f[a, a+1, \ldots, a+p-1, x] \tag{8}$$

(see, e.g., [71, Section 8.2.2] and [77, Section 2.1.3]). The right side of (8) is actually the remainder of the $(p-1)$th degree Newton expansion of $f(x)$ about $x = a$ (see, e.g., [34, Section 5.3]). Note also that formula (8), which actually generalizes (6) on $\mathbb{R}_+$, is a pure identity and is therefore valid without any restriction on the form of $f(x)$. When $f \in \mathcal{C}^p$, the right side of (8) also takes the form $\binom{x-a}{p} f^{(p)}(\xi)$ for some real number $\xi$ satisfying

$$\min\{a, x\} \;<\; \xi \;<\; \max\{a+p-1, x\}.$$

Using (7) and (8) we see that, for any $a > 0$, any $p \in \mathbb{N}$, and any $g\colon \mathbb{R}_+ \to \mathbb{R}$, the quantity $\rho_a^p[g](x)$ defined in (5) is precisely the interpolation error at $a + x$ when considering the interpolating polynomial of $g$ with nodes at $a, a+1, \ldots, a+p-1$. We then immediately derive the following identities:

$$\rho_a^p[g](x) \;=\; g(x+a) - P_{p-1}[g](a, a+1, \ldots, a+p-1; a+x)\,, \tag{9}$$
$$\rho_a^p[g](x) \;=\; x^{\underline{p}}\, g[a, a+1, \ldots, a+p-1, a+x]\,. \tag{10}$$

We now provide a key technical lemma that will be used repeatedly in this paper to obtain various convergence results.

**Lemma 2.4.** *Let $p \in \mathbb{N}$, $f \in \mathcal{K}^p$, and $a > 0$ be so that $f$ is $p$-convex or $p$-concave on $[a, \infty)$. Then, for any $x \geqslant 0$, we have*

$$0 \;\leqslant\; \pm\, \varepsilon_p(x)\, \rho_a^{p+1}[f](x) \;\leqslant\; \pm \left|\tbinom{x-1}{p}\right| \sum_{j=0}^{\lceil x \rceil - 1} \Delta^{p+1} f(a+j),$$

*where* $\varepsilon_p(x) \in \{-1, 0, 1\}$ *is the sign of* $x^{\underline{p+1}}$ *and* $\pm$ *stands for* $1$ *or* $-1$ *according to whether* $f \in \mathcal{K}^p_+$ *or* $f \in \mathcal{K}^p_-$. *Moreover, if* $x \in \{0, 1, \ldots, p\}$ *(i.e.* $\varepsilon_p(x) = 0$*), then* $\rho_a^{p+1}[f](x) = 0$.

*Proof.* Let us first establish the inequalities. Negating $f$ if necessary, we may assume that it lies in $\mathcal{K}^p_+$. We may also assume that $x \notin \{0, 1, \ldots, p\}$, which means that $\varepsilon_p(x) \neq 0$. By (10) we have

$$\varepsilon_p(x)\,\rho_a^{p+1}[f](x) \;=\; \varepsilon_p(x)\,x^{\underline{p+1}}\,f[a, a+1, \ldots, a+p, a+x] \;\geqslant 0$$

and hence, using Lemma 2.3, identity (6), and the standard recurrence relation for divided differences, we obtain

$$\begin{aligned}
0 \;&\leqslant\; \varepsilon_p(x)\,\rho_a^{p+1}[f](x) \\
&=\; \varepsilon_p(x)\,x^{\underline{p+1}}\,f[a, a+1, \ldots, a+p, a+x] \\
&=\; \varepsilon_p(x)(x-1)^{\underline{p}}\,(f[a+x, a+1, \ldots, a+p] - f[a, a+1, \ldots, a+p]) \\
&\leqslant\; \varepsilon_p(x)\tbinom{x-1}{p}\,(\Delta^p f(a+x) - \Delta^p f(a)) \\
&\leqslant\; \varepsilon_p(x)\tbinom{x-1}{p}\,(\Delta^p f(a+\lceil x\rceil) - \Delta^p f(a)),
\end{aligned}$$

which proves the inequalities. Now, when $x \in \{0, 1, \ldots, p\}$, then $\binom{x}{j} = 0$ whenever $j > x$ and hence $\rho_a^{p+1}[g](x) = 0$ by (5). $\square$

*Remark* 2.5. It is not difficult to see that, in Lemma 2.4, the upper delimiter $\lceil x\rceil - 1$ of the sum could be replaced with $\lceil \mathbf{median}\{x, 1, x-p+1\}\rceil - 1$ whenever $p \geqslant 1$. Although this alternative delimiter would make the second inequality a little tighter, it would not have a great impact on our subsequent results.

# 3 Uniqueness and existence results

In this section we establish Theorems 1.1 and 1.2 and show that, under the assumptions of these theorems, the sequence $n \mapsto f_n^p[g](x)$ converges uniformly on any bounded subset of $\mathbb{R}_+$. We also discuss the particular case where the sequence $n \mapsto g(n)$ is summable. Lastly, we provide historical notes on Krull-Webster's theory and some of its improvements.

## 3.1 Main results

We start this section by establishing a slightly improved version of our uniqueness Theorem 1.1. We state this new version in Theorem 3.1 below and provide a very short proof. Let us first note that any solution $f\colon \mathbb{R}_+ \to \mathbb{R}$ to the equation

$\Delta f = g$ satisfies the equations

$$\begin{aligned} f(n) &= f(1) + \sum_{k=1}^{n-1} g(k), \qquad n \in \mathbb{N}^*; & (11) \\ f(x+n) &= f(x) + \sum_{k=0}^{n-1} g(x+k), \qquad n \in \mathbb{N}^*. & (12) \end{aligned}$$

Also, using (2), (5), (11), and (12), we can easily derive the identity

$$f(x) = f(1) + f_n^p[g](x) + \rho_n^{p+1}[f](x). \tag{13}$$

**Theorem 3.1** (Uniqueness). *Let $p \in \mathbb{N}$ and $g \in \mathcal{D}_{\mathbb{S}}^p$. Suppose that $f\colon \mathbb{R}_+ \to \mathbb{R}$ is a solution to the equation $\Delta f = g$ that lies in $\mathcal{K}^p$. Then, the following assertions hold.*

*(a) We have $f \in \mathcal{R}_{\mathbb{S}}^{p+1}$.*

*(b) For each $x > 0$, the sequence $n \mapsto f_n^p[g](x)$ converges and we have*

$$f(x) = f(1) + \lim_{n\to\infty} f_n^p[g](x)\,, \qquad x > 0.$$

*(c) For any nonempty bounded subset $E$ of $\mathbb{R}_+$, the sequence $n \mapsto f_n^p[g]$ converges uniformly on $E$ to $f - f(1)$.*

*Proof.* We clearly have $f \in \mathcal{D}_{\mathbb{S}}^{p+1}$. Assertion (a) then follows from Lemma 2.4. Assertion (b) follows from assertion (a) and identity (13). Using again identity (13) and Lemma 2.4, for large integer $n$ we obtain

$$\begin{aligned} \sup_{x\in E} |f_n^p[g](x) - f(x) + f(1)| &= \sup_{x\in E} \left|\rho_n^{p+1}[f](x)\right| \\ &\leqslant \sup_{x\in E} \left|\tbinom{x-1}{p}\right| \sup_{x\in E} \sum_{j=0}^{\lceil x\rceil - 1} \left|\Delta^{p+1} f(n+j)\right|. \end{aligned}$$

This establishes assertion (c). □

**Example 3.2.** Using Theorem 3.1 with $g(x) = \ln x$ and $p = 1$, it follows that all solutions $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ to the equation $f(x+1) = x\,f(x)$ for which $\ln f$ lie in $\mathcal{K}^1$ are of the form $f(x) = c\,\Gamma(x)$, where $c > 0$. We thus simply retrieve Bohr-Mollerup-Artin's theorem as expected, as well as Gauss' limit (4). □

Using the definition of $\rho_a^p[g](x)$, we can easily derive the following two identities:

$$\begin{aligned} \rho_a^p[g](p) &= \Delta^p g(a)\,; & (14) \\ \rho_a^p[g](x) - \rho_a^{p+1}[g](x) &= \tbinom{x}{p}\,\rho_a^p[g](p)\,. & (15) \end{aligned}$$

Identity (14) shows that the inclusion $\mathcal{R}_S^p \subset \mathcal{D}_S^p$ holds for any $p \in \mathbb{N}$. We will see in Subsection 4.1 that the converse inclusion does not hold. Now, the following straightforward identities will also be useful as we continue:

$$\begin{aligned} f_{n+1}^p[g](x) - f_n^p[g](x) &= -\rho_n^{p+1}[g](x)\,; && (16)\\ f_n^p[g](x+1) - f_n^p[g](x) &= g(x) - \rho_n^p[g](x)\,. && (17) \end{aligned}$$

For any integers $1 \leqslant m \leqslant n$, from (16) we obtain

$$f_n^p[g](x) \;=\; f_m^p[g](x) - \sum_{k=m}^{n-1} \rho_k^{p+1}[g](x)\,, \tag{18}$$

which shows that, for any $x > 0$, the convergence of the sequence $n \mapsto f_n^p[g](x)$ is equivalent to the summability of the sequence $n \mapsto \rho_n^{p+1}[g](x)$.

We now establish a slightly improved version of our existence Theorem 1.2. We first present a technical lemma, which follows straightforwardly from Lemma 2.4.

**Lemma 3.3.** *Let $p \in \mathbb{N}$, $g \in \mathcal{K}^p$, and $m \in \mathbb{N}^*$ be so that $g$ is $p$-convex or $p$-concave on $[m, \infty)$. Then, for any $x \geqslant 0$ and any integer $n \geqslant m$, we have*

$$\left|\sum_{k=m}^{n-1} \rho_k^{p+1}[g](x)\right| \;\leqslant\; \left|\binom{x-1}{p}\right| \sum_{j=0}^{\lceil x\rceil - 1} |\Delta^p g(n+j) - \Delta^p g(m+j)|.$$

**Theorem 3.4** (Existence). *Let $p \in \mathbb{N}$ and $g \in \mathcal{D}_S^p \cap \mathcal{K}^p$. Then, the following assertions hold.*

*(a) We have $g \in \mathcal{R}_S^p$.*

*(b) For each $x > 0$, the sequence $n \mapsto f_n^p[g](x)$ converges and the function $f\colon \mathbb{R}_+ \to \mathbb{R}$ defined by*

$$f(x) \;=\; \lim_{n\to\infty} f_n^p[g](x)\,, \qquad x > 0,$$

*is a solution to the equation $\Delta f = g$ that is $p$-concave (resp. $p$-convex) on any unbounded subinterval $I$ of $\mathbb{R}_+$ on which $g$ is $p$-convex (resp. $p$-concave). Moreover, we have $f(1) = 0$ and, for every $n \in I \cap \mathbb{N}^*$,*

$$|f_n^p[g](x) - f(x)| \;\leqslant\; \lceil x\rceil \left|\binom{x-1}{p}\right| |\Delta^p g(n)|\,, \qquad x > 0.$$

*(c) For any nonempty bounded subset $E$ of $\mathbb{R}_+$, the sequence $n \mapsto f_n^p[g]$ converges uniformly on $E$ to $f$.*

*Proof.* We have $g \in \mathcal{D}^p_{\mathbb{S}} \subset \mathcal{D}^{p+1}_{\mathbb{S}}$. By Lemma 2.4, it follows immediately that $g$ lies in $\mathcal{R}^{p+1}_{\mathbb{S}}$, and hence also in $\mathcal{R}^p_{\mathbb{S}}$ by (14) and (15). This establishes assertion (a). Now, suppose for instance that $g$ lies in $\mathcal{K}^p_+$. Let $I$ be any unbounded subinterval of $\mathbb{R}_+$ on which $g$ is $p$-convex and let $m \in I \cap \mathbb{N}^*$. For any $x > 0$, the sequence $k \mapsto \rho^{p+1}_k[g](x)$ for $k \geqslant m$ does not change in sign by Lemma 2.4. Thus, since $g$ lies in $\mathcal{D}^p_{\mathbb{N}}$, for any $x > 0$ the sequence

$$n \ \mapsto \ \sum_{k=m}^{n-1} \rho^{p+1}_k[g](x)$$

converges by Lemma 3.3. By (18) it follows that the sequence $n \mapsto f^p_n[g](x)$ converges. Denoting the limiting function by $f$, by (17) and assertion (a) we must have $\Delta f = g$. Moreover, we also have $f(1) = 0$ by Theorem 3.1.

It is also easy to see that every $f^p_n[g]$ is $p$-concave on $I$. (Note that the second sum in (2) is a polynomial of degree $p$ in $x$, hence it is both $p$-convex and $p$-concave.) Since $f$ is a pointwise limit of functions $p$-concave on $I$, it too is $p$-concave on $I$.

The inequality then follows from Eq. (13), Lemma 2.4, and the observation that the restriction of the sequence $n \mapsto \Delta^p g(n)$ to $I \cap \mathbb{N}^*$ increases to zero by Lemma 2.3. This proves assertion (b). Assertion (c) immediately follows from assertion (b). $\square$

Theorems 3.1 and 3.4 show that the assumption $g \in \mathcal{D}^p_{\mathbb{N}} \cap \mathcal{K}^p$ constitutes a sufficient condition to ensure both the uniqueness (up to an additive constant) and existence of solutions to the equation $\Delta f = g$ that lie in $\mathcal{K}^p$. Nevertheless, we can show that this condition is actually not quite necessary. We discuss and elaborate on this natural question in Appendix B.

We now present an important property of the sequence $n \mapsto f^p_n[g](x)$. Considering the straightforward identity

$$f^{p+1}_n[g](x) - f^p_n[g](x) \ = \ \tbinom{x}{p+1}\, \Delta^p g(n),$$

we immediately see that if the sequence $n \mapsto f^{p+1}_n[g](x) - f^p_n[g](x)$ approaches zero for some $x \in \mathbb{R}_+ \setminus \{0, 1, \ldots, p\}$, then necessarily $g \in \mathcal{D}^p_{\mathbb{N}}$. More importantly, this identity also shows that if $g \in \mathcal{D}^p_{\mathbb{N}}$ and if the sequence $n \mapsto f^p_n[g](x)$ converges, then so does the sequence $n \mapsto f^{p+1}_n[g](x)$ and both sequences converge to the same limit. Since we have $\mathcal{D}^p_{\mathbb{N}} \subset \mathcal{D}^{p+1}_{\mathbb{N}}$ for any $p \in \mathbb{N}$, we immediately obtain the following important proposition.

**Proposition 3.5.** *Let $p \in \mathbb{N}$. If $g \in \mathcal{D}^p_{\mathbb{N}}$ and if the sequence $n \mapsto f^p_n[g](x)$ converges, then for any integer $q \geqslant p$ the sequence*

$$n \ \mapsto \ |f^p_n[g](x) - f^q_n[g](x)|$$

*converges to zero. Moreover, the convergence is uniform on any nonempty bounded subset of $\mathbb{R}_+$.*

### 3.2 The case when the sequence $g(n)$ is summable

Let $\widetilde{\mathcal{D}}^{-1}_{\mathbb{N}}$ be the set of functions $g\colon \mathbb{R}_+ \to \mathbb{R}$ having the asymptotic property that the sequence $n \mapsto \sum_{k=1}^{n-1} g(k)$ converges. We immediately observe that $\widetilde{\mathcal{D}}^{-1}_{\mathbb{N}} \subset \mathcal{D}^0_{\mathbb{N}}$. In this context, our uniqueness and existence results reduce to the following two theorems.

**Theorem 3.6** (Uniqueness). *Let $g \in \widetilde{\mathcal{D}}^{-1}_{\mathbb{N}}$ and suppose that $f\colon \mathbb{R}_+ \to \mathbb{R}$ is a solution to the equation $\Delta f = g$ that lies in $\mathcal{K}^0$. Then, the following assertions hold.*

*(a) $f(x)$ has a finite limit as $x \to \infty$, denote it by $f(\infty)$.*

*(b) For each $x > 0$, the sequence $n \mapsto \sum_{k=0}^{n-1} g(x+k)$ converges and we have*

$$f(x) \;=\; f(\infty) - \sum_{k=0}^{\infty} g(x+k)\,, \qquad x > 0.$$

*(c) The sequence $n \mapsto \sum_{k=0}^{n-1} g(x+k)$ converges uniformly on $\mathbb{R}_+$ to $f(\infty) - f(x)$.*

*Proof.* The sequence $n \mapsto f(n)$ converges by (11). Assuming for instance that $f \in \mathcal{K}^0_+$, for any $x > 0$ we obtain

$$f(\lfloor x \rfloor + n) \;\leqslant\; f(x+n) \;\leqslant\; f(\lceil x \rceil + n) \quad \text{for large integer } n.$$

Letting $n \to_{\mathbb{N}} \infty$ in these inequalities and using the squeeze theorem, we get assertion (a). Assertion (b) follows from assertion (a) and identity (12). Now, for large integer $n$, by (12) we have

$$\sup_{x \in \mathbb{R}_+} \left| \sum_{k=n}^{\infty} g(x+k) \right| \;=\; \sup_{x \in \mathbb{R}_+} |f(x+n) - f(\infty)| \;\leqslant\; |f(n) - f(\infty)|.$$

This proves assertion (c). ☐

**Theorem 3.7** (Existence). *Let $g \in \widetilde{\mathcal{D}}^{-1}_{\mathbb{N}} \cap \mathcal{K}^0$. Then, the following assertions hold.*

*(a) We have $g \in \mathcal{R}^0_{\mathbb{N}}$.*

*(b) For each $x > 0$, the sequence $n \mapsto \sum_{k=0}^{n-1} g(x+k)$ converges and the function $f\colon \mathbb{R}_+ \to \mathbb{R}$ defined by*

$$f(x) \;=\; -\sum_{k=0}^{\infty} g(x+k)\,, \qquad x > 0,$$

*is a solution to the equation* $\Delta f = g$ *that is decreasing (resp. increasing) on any unbounded subinterval* $I$ *of* $\mathbb{R}_+$ *on which* $g$ *is increasing (resp. decreasing). Moreover, we have* $f(\infty) = 0$ *and, for every* $n \in I \cap \mathbb{N}^*$,

$$\left|\sum_{k=n}^{\infty} g(x+k)\right| \leqslant |f(n)|, \qquad x > 0.$$

(c) *The sequence* $n \mapsto \sum_{k=0}^{n-1} g(x+k)$ *converges uniformly on* $\mathbb{R}_+$ *to* $-f(x)$.

*Proof.* This follows straightforwardly from Theorems 3.4 and 3.6. ☐

## 3.3 Historical notes

As mentioned in the Introduction, the uniqueness and existence result in the case when $p = 1$ was established in the pioneering work of Krull [46,47] and then independently by Webster [79, 80] as a generalization of Bohr-Mollerup-Artin's theorem. We observe that it was also partially rediscovered by Dinghas [29]. In addition, we note that Krull's result was presented and somewhat revisited by Kuczma [48] (see also Kuczma [51] and Kuczma [52, pp. 114–118]) as well as by Anastassiadis [6, pp. 69–73]. To our knowledge, the only attempts to establish uniqueness and existence results for any value of $p$ were made by Kuczma [52, pp. 118–121] and Ardjomande [8]. Independently of these latter results, a special investigation of the case when $p = 2$, which involves the Barnes G-function, was made by Rassias and Trif [72] (see our Appendix A).

We also observe that Gronau and Matkowski [37, 38] improved the multiplicative version of Krull's result by replacing the log-convexity property with the much weaker condition of geometrical convexity (see also Guan [39] for a recent application of this result), thus providing another characterization of the gamma function, which was later improved by Alzer and Matkowski [3] and Matkowski [58]. (For further characterizations of the gamma function and generalizations, see also Anastassiadis [6] and Muldoon [67].)

Many other variants and improvements of Krull's result can actually be found in the literature. For instance, Anastassiadis [5] (see also [6, p. 71]) generalized Krull's result by modifying the asymptotic condition. Rohde [74] also generalized that result by modifying the convexity property. Gronau [35] proposed a variant of Krull's result and applied it to characterize the Euler beta and gamma functions and study certain spirals (see also Gronau [36]). Merkle and Ribeiro Merkle [60] proposed to combine Krull's result with differentiation techniques to characterize the Barnes G-function. Himmel and Matkowski [41] also proposed improvements of Krull's result to characterize the beta and gamma functions.

# 4 Interpretations of the asymptotic conditions

In this section, we provide interpretations of the asymptotic conditions that define the sets $\mathcal{R}^p_S$ and $\mathcal{D}^p_S$ and we investigate some properties of these sets. We also describe the sets $\mathcal{R}^p_S \cap \mathcal{K}^p$ and $\mathcal{D}^p_S \cap \mathcal{K}^p$ and show that they actually coincide. Moreover, we show that $\mathcal{C}^p \cap \mathcal{D}^p_S \cap \mathcal{K}^p$ is exactly the set of functions $g \in \mathcal{C}^p$ for which $g^{(p)}$ eventually increases or decreases to zero.

## 4.1 Asymptotic conditions

Using (9) and (10), we can immediately state the following characterization of the set $\mathcal{R}^p_S$ in terms of interpolating polynomials. Using (9), (10), and (14), we can obtain a similar characterization for the set $\mathcal{D}^p_S$.

**Proposition 4.1.** *Let* $p \in \mathbb{N}$. *A function* $g\colon \mathbb{R}_+ \to \mathbb{R}$ *lies in* $\mathcal{R}^p_S$ *if and only if, for each* $x > 0$*,*

$$g[a, a+1, \ldots, a+p-1, a+x] \;\to\; 0 \qquad \text{as } a \to_S \infty. \tag{19}$$

*When* $S = \mathbb{R}$ *(resp.* $S = \mathbb{N}$*), condition* (19) *means that* $g$ *asymptotically coincides with its interpolating polynomial whose nodes are any* $p$ *points equally spaced by* $1$ *(resp. any* $p$ *consecutive integers).*

It is clear that the sets $\mathcal{R}^p_S$ and $\mathcal{D}^p_S$ are closed under linear combinations; hence they are linear spaces. Moreover, using (14) and (15) we see that

$$\mathcal{R}^p_S \;=\; \mathcal{R}^{p+1}_S \cap \mathcal{D}^p_S\,. \tag{20}$$

In particular, the sets $\mathcal{R}^0_S, \mathcal{R}^1_S, \ldots$ are increasingly nested. As already observed, this property also holds trivially for the sets $\mathcal{D}^0_S, \mathcal{D}^1_S, \ldots$. Now, identity (9) shows that the polynomial function $x \mapsto x^p$ lies in $\mathcal{R}^{p+1}_S \setminus \mathcal{R}^p_S$ and we can easily see that it lies also in $\mathcal{D}^{p+1}_S \setminus \mathcal{D}^p_S$. Thus, we have proved that

$$\mathcal{R}^p_S \;\subsetneq\; \mathcal{R}^{p+1}_S \quad \text{and} \quad \mathcal{D}^p_S \;\subsetneq\; \mathcal{D}^{p+1}_S.$$

We also have $\mathcal{R}^p_S \subsetneq \mathcal{D}^p_S$ for any $p \in \mathbb{N}^*$. Indeed, the 1-periodic function $g(x) = \sin(2\pi x)$ lies in $\mathcal{D}^p_S \setminus \mathcal{R}^p_S$ for any $p \in \mathbb{N}^*$. On the other hand, we have

$$\mathcal{R}^0_{\mathbb{R}} \;=\; \mathcal{D}^0_{\mathbb{R}} \;\subsetneq\; \mathcal{R}^0_{\mathbb{N}} \;\subsetneq\; \mathcal{D}^0_{\mathbb{N}}\,.$$

For instance, we can easily construct a continuous (or even smooth) function $g\colon \mathbb{R}_+ \to \mathbb{R}$ such that for any $n \in \mathbb{N}^*$, we have $g = 0$ on the interval $[n-1, n-\frac{1}{n}]$ and $g(n - \frac{1}{2n}) = 1$. Such a function has the property that, for each $x > 0$, $g(x+n) \to 0$ as $n \to_{\mathbb{N}} \infty$. However, since it does not vanish at infinity, it must lie in $\mathcal{R}^0_{\mathbb{N}} \setminus \mathcal{R}^0_{\mathbb{R}}$.

It is clear that if a function $f\colon \mathbb{R}_+ \to \mathbb{R}$ lies in $\mathcal{D}^{p+1}_{\mathbb{S}}$, then $\Delta f$ lies in $\mathcal{D}^p_{\mathbb{S}}$. Also, if $f$ lies in $\mathcal{R}^{p+1}_{\mathbb{S}}$, then $\Delta f$ lies in $\mathcal{R}^p_{\mathbb{S}}$ by (20). This latter observation follows also from the second of the straightforward identities

$$\begin{aligned} \rho^p_{a+1}[f](x) - \rho^p_a[f](x) &= \rho^p_a[\Delta f](x)\,; && (21)\\ \rho^{p+1}_a[f](x+1) - \rho^{p+1}_a[f](x) &= \rho^p_a[\Delta f](x)\,. && (22) \end{aligned}$$

Thus, we have proved the following proposition.

**Proposition 4.2.** *Let $j, p \in \mathbb{N}$ be such that $j \leqslant p$. Then the following assertions hold.*

*(a) If $f \in \mathcal{R}^p_{\mathbb{S}}$, then $\Delta^j f \in \mathcal{R}^{p-j}_{\mathbb{S}}$.*

*(b) $f \in \mathcal{D}^p_{\mathbb{S}}$ if and only if $\Delta^j f \in \mathcal{D}^{p-j}_{\mathbb{S}}$.*

We will see in Corollary 4.7 that, if $f \in \mathcal{K}^{p-1}$, then the implication in assertion (a) of Proposition 4.2 becomes an equivalence.

It is easy to see that, for any $p \in \mathbb{N}$, the space $\mathcal{R}^p_{\mathbb{S}}$ contains every function that behaves asymptotically like a polynomial of degree less than or equal to $p-1$; that is, every function $g\colon \mathbb{R}_+ \to \mathbb{R}$ such that $g(x) - P(x) \to 0$ as $x \to \infty$ for some polynomial $P$ of degree less than or equal to $p-1$. More generally, if $g-h \in \mathcal{R}^p_{\mathbb{S}}$ and $h \in \mathcal{R}^p_{\mathbb{S}}$, then clearly $g \in \mathcal{R}^p_{\mathbb{S}}$. To give another illustration of this latter property, we observe for instance that both functions $\ln x$ and $H_x - \ln x$ lie in $\mathcal{R}^1_{\mathbb{R}}$ and hence so does the function $H_x$ (which, a priori, is a not completely trivial result).

It is clear that the spaces $\mathcal{R}^\infty_{\mathbb{S}} = \cup_{p\geqslant 0}\mathcal{R}^p_{\mathbb{S}}$ and $\mathcal{D}^\infty_{\mathbb{S}} = \cup_{p\geqslant 0}\mathcal{D}^p_{\mathbb{S}}$ contain a very large variety of functions, including not only all the functions that have polynomial behaviors at infinity as discussed above, and in particular all the rational functions, but also many other functions. We observe, however, that they do not contain any strictly increasing exponential function. For instance, if $g(x) = 2^x$, then $\Delta^p g(x) = 2^x$ for any $p \in \mathbb{N}$, and this function does not vanish at infinity. Actually, such exponential functions grow asymptotically much faster than polynomial functions and may remain eventually p-convex even after adding nonconstant 1-periodic functions. For instance, both functions $2^x$ and $2^x + \sin(2\pi x)$ are eventually p-convex for any $p \in \mathbb{N}$.

*Remark* 4.3. Using (5) and (20), we also obtain $\mathcal{R}^p_{\mathbb{S}} = \mathcal{R}^\infty_{\mathbb{S}} \cap \mathcal{D}^p_{\mathbb{S}}$ for any $p \in \mathbb{N}$.

## 4.2 Eventually p-convex or p-concave functions

Let us now investigate the sets $\mathcal{K}^p$, $\mathcal{R}^p_{\mathbb{S}} \cap \mathcal{K}^p$, and $\mathcal{D}^p_{\mathbb{S}} \cap \mathcal{K}^p$. It is easy to see that none of these sets is a linear space. For instance, both functions $f(x) = x + \sin x$ and $g(x) = x$ lie in $\mathcal{K}^0$ but $f - g$ does not. We also have $\Delta f \notin \mathcal{K}^0$, which shows that $\mathcal{K}^p$ is not closed under the operator $\Delta$. Finally, we can see that both functions $f(x) = 2\ln x + \frac{\sin x}{x^2}$ and $g(x) = 2\ln x$ lie in $\mathcal{R}^1_{\mathbb{S}} \cap \mathcal{K}^1$ (use, e.g., Proposition 4.5 and Theorem 4.9 below) but $f - g$ does not.

The following proposition shows that, just as the sets $\mathcal{C}^0, \mathcal{C}^1, \mathcal{C}^2, \ldots$ are decreasingly nested, so are the sets $\mathcal{K}^{-1}, \mathcal{K}^0, \mathcal{K}^1, \ldots$ and thus we can naturally let $\mathcal{K}^\infty$ denote the intersection set $\cap_{p\geqslant 0}\mathcal{K}^p$.

**Proposition 4.4.** *For any integer* $p \geqslant -1$*, we have* $\mathcal{K}^{p+1} \subsetneq \mathcal{K}^p$*.*

*Proof.* Let $f \in \mathcal{K}^{p+1}$ for some integer $p \geqslant -1$. Suppose without loss of generality that $f \in \mathcal{K}^{p+1}_+$ and let $I$ be an unbounded subinterval of $\mathbb{R}_+$ on which $f$ is $(p+1)$-convex. By Lemma 2.3, it follows that the restriction of the map $(z_0, \ldots, z_{p+1}) \mapsto f[z_0, \ldots, z_{p+1}]$ to $I^{p+2}$ is increasing in each place. If $f \notin \mathcal{K}^p_-$, then there are pairwise distinct points $x_0, \ldots, x_{p+1} \in I$ such that $f[x_0, \ldots, x_{p+1}] > 0$. But then, $f$ is $p$-convex on the interval $(\max_i x_i, \infty)$, that is, $f \in \mathcal{K}^p_+$. To see that the strict inclusion holds, we just observe that the function $f(x) = x^{p+1} + \sin x$ lies in $\mathcal{K}^p \setminus \mathcal{K}^{p+1}$. $\square$

Interestingly, Proposition 4.4 shows that the assumption $g \in \mathcal{K}^p$, which occurs in many statements (e.g., in Theorem 3.4), can be given equivalently by $g \in \cup_{q\geqslant p}\mathcal{K}^q$.

We also have the following two important propositions.

**Proposition 4.5.** *For any* $p \in \mathbb{N}$*, we have* $\mathcal{R}^p_{\mathbb{R}} \cap \mathcal{K}^p = \mathcal{D}^p_{\mathbb{R}} \cap \mathcal{K}^p = \mathcal{R}^p_{\mathbb{N}} \cap \mathcal{K}^p = \mathcal{D}^p_{\mathbb{N}} \cap \mathcal{K}^p$*.*

*Proof.* We already know that $\mathcal{R}^p_{\mathbb{S}} \subset \mathcal{D}^p_{\mathbb{S}}$ and $\mathcal{D}^p_{\mathbb{R}} \subset \mathcal{D}^p_{\mathbb{N}}$. Also, $\mathcal{D}^p_{\mathbb{S}} \cap \mathcal{K}^p \subset \mathcal{R}^p_{\mathbb{S}}$ by Theorem 3.4. It remains to show that $\mathcal{D}^p_{\mathbb{N}} \cap \mathcal{K}^p \subset \mathcal{D}^p_{\mathbb{R}}$. Let $g \in \mathcal{D}^p_{\mathbb{N}} \cap \mathcal{K}^p$. Suppose for instance that $g \in \mathcal{K}^p_+$ and let $a > 0$ be so that $g$ is $p$-convex on $[a, \infty)$. By Lemma 2.3, $\Delta^p g$ is increasing on $[a, \infty)$. Thus, for any $x \geqslant a + 1$, we have

$$\Delta^p g(\lfloor x \rfloor) \ \leqslant\ \Delta^p g(x) \ \leqslant\ \Delta^p g(\lceil x \rceil).$$

Letting $x \to \infty$ and using the squeeze theorem, we obtain that $g \in \mathcal{D}^p_{\mathbb{R}}$. $\square$

**Proposition 4.6.** *If* $f \in \mathcal{K}^p$ *for some* $p \in \mathbb{N}$*, then the following assertions are equivalent:*

(i) $f \in \mathcal{R}^{p+1}_{\mathbb{S}}$, (ii) $f \in \mathcal{D}^{p+1}_{\mathbb{S}}$, (iii) $\Delta f \in \mathcal{R}^p_{\mathbb{S}}$, (iv) $\Delta f \in \mathcal{D}^p_{\mathbb{S}}$.

*Proof.* We clearly have (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (iv). By Proposition 4.2, we also have (i) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (iv). Finally, by Theorem 3.1, we have (iv) $\Rightarrow$ (i). $\square$

Combining Lemma 2.2(b) with Propositions 4.2 and 4.6, we obtain the following corollary, which naturally complements Proposition 4.2.

**Corollary 4.7.** *Let* $j, p \in \mathbb{N}$ *be such that* $j \leqslant p$*. If* $f \in \mathcal{K}^{p-1}$*, then we have* $f \in \mathcal{R}^p_{\mathbb{S}}$ *if and only if* $\Delta^j f \in \mathcal{R}^{p-j}_{\mathbb{S}}$*.*

Due to Proposition 4.5, we will henceforth write $\mathcal{D}^p \cap \mathcal{K}^p$ instead of $\mathcal{D}^p_{\mathbb{S}} \cap \mathcal{K}^p$. We also observe that when $g$ lies in $\mathcal{D}^p \cap \mathcal{K}^p$, then by (10) and Lemma 2.3 the maps $t \mapsto \rho^p_t[g](x)$ and $t \mapsto \Delta^p g(t)$ eventually increase or decrease to zero. It is not known whether these latter conditions characterize the set $\mathcal{D}^p \cap \mathcal{K}^p$. However, when $g$ lies in $\mathcal{C}^p$, we have the nice characterization given in Theorem 4.9 below, which immediately follows from the next proposition.

**Proposition 4.8.** *Let $p, r \in \mathbb{N}$ be such that $r \leqslant p$ and let $g \in \mathcal{C}^r$. Then the following assertions hold.*

*(a)* $g \in \mathcal{K}^p_+$ *if and only if* $g^{(r)} \in \mathcal{K}^{p-r}_+$. *More precisely, for any unbounded open interval $I$ of $\mathbb{R}_+$, $g$ is $p$-convex on $I$ if and only if $g^{(r)}$ is $(p-r)$-convex on $I$.*

*(b)* $g \in \mathcal{D}^p \cap \mathcal{K}^p_+$ *if and only if* $g^{(r)} \in \mathcal{D}^{p-r} \cap \mathcal{K}^{p-r}_+$.

*Proof.* Assertion (a) follows from Lemma 2.2(c) and Lemma 2.2(g). To see that assertion (b) holds, it is enough to show that, for any $p \geqslant 1$, we have $g \in \mathcal{D}^p \cap \mathcal{K}^p_+$ if and only if $g' \in \mathcal{D}^{p-1} \cap \mathcal{K}^{p-1}_+$. Suppose first that $g \in \mathcal{D}^p \cap \mathcal{K}^p_+$. Then $g' \in \mathcal{K}^{p-1}_+$ by assertion (a). Let $x > 1$ be so that $g$ is p-convex on $[x, \infty)$. By the mean value theorem, there exist $\xi^1_x, \xi^2_x \in (0,1)$ such that

$$\Delta^p g(x-1) \;=\; \Delta^{p-1} g'(x-1+\xi^1_x) \;\leqslant\; \Delta^{p-1} g'(x) \;\leqslant\; \Delta^{p-1} g'(x+\xi^2_x) \;=\; \Delta^p g(x).$$

Letting $x \to \infty$, we see that $g' \in \mathcal{D}^{p-1}_{\mathbb{R}}$. Conversely, suppose that $g' \in \mathcal{D}^{p-1} \cap \mathcal{K}^{p-1}_+$. Then $g \in \mathcal{K}^p_+$ by assertion (a). Let $x > 1$ be so that $g'$ is $(p-1)$-convex on $[x, \infty)$ and let $t \in (x, x+1)$. Then we have

$$\Delta^{p-1} g'(x) \;\leqslant\; \Delta^{p-1} g'(t) \;\leqslant\; \Delta^{p-1} g'(x+1).$$

Integrating on $t \in (x, x+1)$, we obtain

$$\Delta^{p-1} g'(x) \;\leqslant\; \Delta^p g(x) \;\leqslant\; \Delta^{p-1} g'(x+1).$$

Letting $x \to \infty$, we see that $g \in \mathcal{D}^p_{\mathbb{R}}$. ☐

**Theorem 4.9.** *Let $p \in \mathbb{N}$ and $g \in \mathcal{C}^p$. Then $g \in \mathcal{D}^p \cap \mathcal{K}^p_+$ (resp. $g \in \mathcal{D}^p \cap \mathcal{K}^p_-$) if and only if $g^{(p)}$ eventually increases (resp. decreases) to zero.*

*Remark* 4.10. The function $g(x) = \frac{1}{x} \sin x^2$ vanishes at infinity but its derivative does not. Theorem 4.9 shows that if $g \in \mathcal{C}^q \cap \mathcal{D}^p \cap \mathcal{K}^q$ for some $p, q \in \mathbb{N}$ such that $p \leqslant q$, then all the functions $g^{(p)}, g^{(p+1)}, \ldots, g^{(q)}$ vanish at infinity.

Proposition 4.8 does not provide any information on the derivative $g'$ when $g$ lies in $\mathcal{C}^1 \cap \mathcal{D}^0 \cap \mathcal{K}^0$. The following proposition deals with this issue.

**Proposition 4.11.** *If $g \in \mathcal{C}^1 \cap \mathcal{D}^0 \cap \mathcal{K}^0_-$ is such that $g' \in \mathcal{K}^0$, then $g' \in \mathcal{C}^0 \cap \widetilde{\mathcal{D}}^{-1}_{\mathbb{N}} \cap \mathcal{K}^0_+$.*

*Proof.* Let $x > 1$ be so that $g$ is decreasing and $g'$ is monotone on $I_x = [x-1, \infty)$. By the mean value theorem, there exist $\xi_x^1, \xi_x^2 \in (0,1)$ such that $\Delta g(x-1) = g'(x-1+\xi_x^1)$ and $\Delta g(x) = g'(x+\xi_x^2)$. Thus, we have

$$\Delta g(x-1) \leqslant g'(x) \leqslant \Delta g(x) \quad \text{or} \quad \Delta g(x-1) \geqslant g'(x) \geqslant \Delta g(x)$$

according to whether $g'$ is increasing or decreasing. In both cases, we see that $g'(x)$ approaches zero as $x \to \infty$. Since $g'$ is nonpositive on $I_x$, it must be increasing on $I_x$; hence $g' \in \mathcal{K}^0_+$. For any integers $m, n$ such that $x \leqslant m \leqslant n$, we then have

$$g(n-1) - g(m-1) = \sum_{k=m}^{n-1} \Delta g(k-1) \leqslant \sum_{k=m}^{n-1} g'(k) \leqslant 0.$$

Letting $n \to_{\mathbb{N}} \infty$, we can see that $g' \in \widetilde{\mathcal{D}}^{-1}_{\mathbb{N}}$. $\square$

*Remark* 4.12. The assumption that $g' \in \mathcal{K}^0$ in Proposition 4.11 cannot be ignored. Indeed, one can show that the function $g(x) = \frac{1}{x^3}(x + \sin x)$ lies in $\mathcal{C}^1 \cap \mathcal{D}^0 \cap \mathcal{K}^0_-$ whereas its derivative $g'$ does not lie in $\mathcal{K}^0$.

Combining Lemma 2.2(b) with Theorem 3.4 and Proposition 4.2(b), we can very easily obtain the following two corollaries, in which the symbols $\mathcal{R}$ and $\mathcal{D}$ can be used interchangeably.

**Corollary 4.13.** *Let $g \in \mathcal{K}^p_+$ (resp. $g \in \mathcal{K}^p_-$) for some $p \in \mathbb{N}$. Then $g \in \mathcal{D}^p_{\mathbb{S}}$ if and only if there exists a solution $f \colon \mathbb{R}_+ \to \mathbb{R}$ to the equation $\Delta f = g$ that lies in $\mathcal{D}^{p+1}_{\mathbb{S}} \cap \mathcal{K}^p_-$ (resp. $\mathcal{D}^{p+1}_{\mathbb{S}} \cap \mathcal{K}^p_+$).*

**Corollary 4.14.** *For any $p \in \mathbb{N}$, we have $\mathcal{D}^p \cap \mathcal{K}^p_+ \subset \mathcal{K}^{p-1}_-$ and $\mathcal{D}^p \cap \mathcal{K}^p_- \subset \mathcal{K}^{p-1}_+$. More precisely, if $g \in \mathcal{D}^p$ and is $p$-convex (resp. $p$-concave) on an unbounded interval of $\mathbb{R}_+$, then on this interval it is also $(p-1)$-concave (resp. $(p-1)$-convex).*

We end this section by providing a characterization of the set $\mathcal{R}^p \cap \mathcal{K}^p = \mathcal{D}^p \cap \mathcal{K}^p$ in terms of interpolating polynomials.

**Proposition 4.15.** *Let $g \in \mathcal{K}^p$ for some $p \in \mathbb{N}$. Then we have $g \in \mathcal{R}^p_{\mathbb{S}}$ if and only if for any $x_0, \ldots, x_p > 0$, $g[a+x_0, \ldots, a+x_p] \to 0$ as $a \to_{\mathbb{S}} \infty$. This latter condition means that $g$ asymptotically coincides with its interpolating polynomial with any $p$ nodes.*

*Proof.* (Necessity) Suppose for instance that $g$ lies in $\mathcal{K}^p_+$ and let $s \in \mathbb{S}$, $s > 0$, be so that $g$ is $p$-convex on $I_s = [s, \infty)$. By Lemma 2.3, the map $(z_0, \ldots, z_p) \in I_s^{p+1} \mapsto g[z_0, \ldots, z_p]$ is increasing in each place. Since $g \in \mathcal{R}^p_{\mathbb{S}}$, this map is also nonpositive on $I_s^{p+1}$; indeed, for $s \leqslant z_0 \leqslant \cdots \leqslant z_p \leqslant a$, $a \in \mathbb{S}$, and $x > 0$, we have

$$g[z_0, \ldots, z_p] \leqslant g[a, a+1, \ldots, a+p-1, a+x],$$

where the right side increases to zero as $a \to_S \infty$ by (10). Now, for any $x_0, \ldots, x_p > 0$ and any $a \geqslant s + p$, we have

$$g[a-p, \ldots, a-1, a+x_p] \;\leqslant\; g[a+x_0, \ldots, a+x_p] \;\leqslant\; 0,$$

where the left side increases to zero as $a \to_S \infty$.

(Sufficiency) This immediately follows from Proposition 4.1. $\square$

# 5 Multiple $\log\Gamma$-type functions

In this section we define and investigate the map, denote it by $\Sigma$, that carries any function $g \in \cup_{p\geqslant 0}(\mathcal{D}^p \cap \mathcal{K}^p)$ into the unique solution $f$ to the equation $\Delta f = g$ that arises from the existence Theorem 3.4. We also investigate certain properties of these solutions, that we call multiple $\log\Gamma$-type functions.

## 5.1 The map $\Sigma$

We define the *asymptotic degree* of a function $f \in \mathcal{K}^0$ to be the integer value

$$\deg f \;=\; \min\{q \in \mathbb{N} : f \in \mathcal{D}^q_{\mathbb{R}}\} - 1.$$

For instance, if $f$ is a polynomial of degree $p$ for some $p \in \mathbb{N}$, then $\deg f = p$. If $f(x) = 0$ or $f(x) = \ln(1 + 1/x)$, then $\deg f = -1$. If $f(x) = x + \sin x$ or $f(x) = 2^x$, then $\deg f = \infty$. It is easy to see that the identity $\deg f = 1 + \deg \Delta f$ holds whenever $\deg f \geqslant 0$. However, it is no longer true when $\deg f = -1$. Note also that $\deg f$ should not to be confused with the limiting value of $x\Delta f(x)/f(x)$ as $x \to \infty$, which is related to the notion of elasticity of a function.

Now, define the map

$$\Sigma : \bigcup_{p\geqslant 0}(\mathcal{D}^p \cap \mathcal{K}^p) \;\to\; \mathrm{ran}(\Sigma)$$

by the condition

$$g \in \mathcal{D}^p \cap \mathcal{K}^p \quad\Rightarrow\quad \Sigma g(x) \;=\; \lim_{n\to\infty} f^p_n[g](x)\,, \tag{23}$$

where $\mathrm{ran}(\Sigma)$ denotes the range of $\Sigma$.

This map is well defined; indeed, if $g \in (\mathcal{D}^p \cap \mathcal{K}^p) \cap (\mathcal{D}^q \cap \mathcal{K}^q)$ for some $0 \leqslant p < q$, then by Proposition 3.5 the sequences $n \mapsto f^p_n[g](x)$ and $n \mapsto f^q_n[g](x)$ have the same limit. Thus, in view of Proposition 4.4, we can see that condition (23) holds for $p = 1 + \deg g$.

Just as the indefinite integral of a function $g$ is the class of functions whose derivative is $g$, the indefinite sum of a function $g$ is the class of functions whose difference is $g$ (see, e.g., [34, p. 48]). The map $\Sigma$ now enables one to refine the latter definition as follows.

**Definition 5.1.** The *principal indefinite sum* of a function $g \in \cup_{p \geqslant 0}(\mathcal{D}^p \cap \mathcal{K}^p)$ is the class of functions $c + \Sigma g$, where $c \in \mathbb{R}$.

We will sometimes add a subscript to specify the variable on which the map $\Sigma$ acts. For instance, $\Sigma_x\, g(2x)$ stands for the function obtained by applying $\Sigma$ to the function $x \mapsto g(2x)$ while $\Sigma g(2x)$ stands for the value of the function $\Sigma g$ at $2x$.

Let us now examine some immediate properties of the map $\Sigma$. Theorems 3.1 and 3.4 and Proposition 4.6 show that, for any $p \in \mathbb{N}$ and any $g \in \mathcal{D}^p \cap \mathcal{K}^p$, the function $\Sigma g$ has the following features:

- it lies in $\mathcal{D}^{p+1} \cap \mathcal{K}^p = \mathcal{R}^{p+1} \cap \mathcal{K}^p$, and
- it is the unique solution to the equation $\Delta f = g$ that lies in $\mathcal{K}^p$ and vanishes at 1.

We also have that $\deg \Sigma g = 1 + \deg g$ whenever $\deg g \geqslant 0$; but this property no longer holds if $\deg g = -1$. Finally, using (11) we immediately see that the restriction of $\Sigma g$ to $\mathbb{N}^*$ is

$$\Sigma g(n) \;=\; \sum_{j=1}^{n-1} g(j)\,, \qquad n \in \mathbb{N}^*. \tag{24}$$

Now, it is clear that the map $\Sigma$ is one-to-one and it is even a bijection since we have restricted its codomain to its range. We then have the following immediate result.

**Proposition 5.2.** *The map $\Sigma$ is a bijection and its inverse is the restriction of the difference operator $\Delta$ to $\mathrm{ran}(\Sigma)$.*

*Remark* 5.3. Quite surprisingly, we observe that if $g \in \mathcal{D}^p \cap \mathcal{K}^p$ for some $p \in \mathbb{N}$, then $\Sigma g$ need not lie in $\mathcal{K}^{p+1}$ (and hence the converse of Lemma 2.2(b) does not hold). For instance, for any $c \in \mathbb{R}$, the function $f(x) = c + 2^{-x}(1 + \frac{1}{3}\sin x)$ lies in $\mathcal{K}^0_- \setminus \mathcal{K}^1$. Indeed, $2^x f'(x)$ is $2\pi$-periodic and negative while $2^x f''(x)$ is $2\pi$-periodic and change in sign from $x = \frac{\pi}{6}$ to $x = \pi$. However, the function $g = \Delta f$ lies in $\mathcal{D}^0 \cap \mathcal{K}^0_+$ for $2^x \Delta f'(x)$ is $2\pi$-periodic and positive.

## 5.2 Multiple $\log \Gamma$-type functions

Barnes [11–13] introduced a sequence of functions $\Gamma_1, \Gamma_2, \ldots$, called *multiple gamma functions*, that generalize the Euler gamma function. The restrictions of these functions to $\mathbb{R}_+$ are characterized by the equations

$$\begin{aligned} \Gamma_{p+1}(x+1) &= \frac{\Gamma_{p+1}(x)}{\Gamma_p(x)}\,, \\ \Gamma_1(x) = \Gamma(x), \quad \Gamma_p(1) &= 1, \qquad x > 0,\ p \in \mathbb{N}^*, \end{aligned}$$

together with the convexity condition

$$(-1)^{p+1}D^{p+1}\ln\Gamma_p(x) \;\geqslant\; 0, \qquad x>0.$$

For recent references, see, e.g., Adamchik [1, 2] and Srivastava and Choi [76].

Thus defined, this sequence satisfies the conditions

$$\ln\Gamma_{p+1} \;=\; -\Sigma\ln\Gamma_p \quad\text{and}\quad \deg\ln\Gamma_p \;=\; p.$$

Also, it can be naturally extended to the case when $p=0$ by setting $\Gamma_0(x)=1/x$.

When $g\in\mathcal{D}^p\cap\mathcal{K}^p$ and $\deg g=p-1$ for some $p\in\mathbb{N}$, we say that $\exp\circ\Sigma g$ (resp. $\Sigma g$) is a $\Gamma_p$*-type function* (resp. a $\log\Gamma_p$*-type function*). When $p\geqslant 1$, $\exp\circ\Sigma g$ reduces to the function $\Gamma_p$ when $\exp\circ g$ is precisely the function $1/\Gamma_{p-1}$, which simply shows that the function $\Gamma_p$ restricted to $\mathbb{R}_+$ is itself a $\Gamma_p$-type function. We also let $\Gamma_p$ (resp. $\mathrm{Log}\Gamma_p$) denote the set of $\Gamma_p$-type functions (resp. $\log\Gamma_p$-type functions). Thus, by definition we have

$$\mathrm{ran}(\Sigma) \;=\; \bigcup_{p\geqslant 0}\mathrm{ran}(\Sigma|_{\mathcal{D}^p\cap\mathcal{K}^p}) \;=\; \bigcup_{p\geqslant 0}\mathrm{Log}\Gamma_p\,.$$

Finally, we say that a function $f\colon\mathbb{R}_+\to\mathbb{R}$ is a *multiple* $\Gamma$*-type function* (resp. *multiple* $\log\Gamma$*-type function*) if it lies in $\cup_{p\geqslant 0}\Gamma_p$ (resp. $\cup_{p\geqslant 0}\mathrm{Log}\Gamma_p$).

Thus defined, the set of $\log\Gamma_p$-type functions can be characterized as follows.

**Proposition 5.4.** *For any function* $f\colon\mathbb{R}_+\to\mathbb{R}$ *and any* $p\in\mathbb{N}$*, the following assertions are equivalent.*

*(i)* $f\in\mathrm{Log}\Gamma_p$ .

*(ii)* $f(1)=0$, $f\in\mathcal{K}^p$, $\Delta f\in\mathcal{D}^p\cap\mathcal{K}^p$, *and* $\deg\Delta f=p-1$.

*(iii)* $f=\Sigma\Delta f$, $\Delta f\in\mathcal{D}^p\cap\mathcal{K}^p$, *and* $\deg\Delta f=p-1$.

*(iv)* $f\in\mathrm{ran}(\Sigma)$ *and* $\deg\Delta f=p-1$.

*(v) If* $p\geqslant 1$, *then* $f\in\mathrm{ran}(\Sigma)$ *and* $\deg f=p$ .
*If* $p=0$, *then* $f\in\mathrm{ran}(\Sigma)$ *and* $\deg f\in\{-1,0\}$.

It follows from Proposition 5.4 that a function $f\colon\mathbb{R}_+\to\mathbb{R}$ lies in $\mathrm{ran}(\Sigma)$ if and only if there exists $p\in\mathbb{N}$ such that $f(1)=0$, $f\in\mathcal{K}^p$, and $\Delta f\in\mathcal{D}^p\cap\mathcal{K}^p$.

We also have the following result, which was proved by Webster [80, Theorem 5.1] in the special case when $p=1$.

**Proposition 5.5.** *Let* $g_1,g_2,g\in\mathcal{D}^p\cap\mathcal{K}^p$ *for some* $p\in\mathbb{N}$*, let* $a\geqslant 0$*, and let* $h\colon\mathbb{R}_+\to\mathbb{R}$ *be defined by the equation* $h(x)=g(x+a)$ *for* $x>0$*. Then*

*(a)* $g_1+g_2\in\mathcal{D}^p\cap\mathcal{K}^p$ *and* $\Sigma(g_1+g_2)=\Sigma g_1+\Sigma g_2$*;*

(b) *if* $g_1 - g_2 \in \mathcal{D}^p \cap \mathcal{K}^p$*, then* $\Sigma(g_1 - g_2) = \Sigma g_1 - \Sigma g_2$*;*

(c) $h \in \mathcal{D}^p \cap \mathcal{K}^p$ *and* $\Sigma h(x) = \Sigma_x\, g(x+a) = \Sigma g(x+a) - \Sigma g(a+1)$.

*Proof.* Assertions (a) and (b) are immediate. To see that (c) holds, define a function $j\colon \mathbb{R}_+ \to \mathbb{R}$ by the equation $j(x) = \Sigma g(x+a) - \Sigma g(a+1)$ for $x > 0$. Then $j$ is a solution to the equation $\Delta j = h$ that lies in $\mathcal{K}^p$ and satisfies $j(1) = 0$. Hence, $\Sigma h = j$, as required. $\square$

**Example 5.6** (see [80])**.** Consider the function $g\colon \mathbb{R}_+ \to \mathbb{R}$ defined by $g(x) = \ln \frac{x}{x+a}$ for some $a > 0$. Then we have $g \in \mathcal{D}^0 \cap \mathcal{K}^0$ (and also $g \in \mathcal{D}^1 \cap \mathcal{K}^1$) and Proposition 5.5 shows that

$$\Sigma g(x) \;=\; \ln \frac{\Gamma(x)\Gamma(a+1)}{\Gamma(x+a)}\,.$$

Also, since $g$ is concave on $\mathbb{R}_+$, we have that $\Sigma g$ is convex on $\mathbb{R}_+$. As Webster [80, p. 615] observed, this is "a not completely trivial result, but one immediate from the approach adopted here." $\square$

**Example 5.7** (A rational function)**.** The function

$$g(x) \;=\; \frac{x^4+1}{x^3+x} \;=\; x + \frac{1}{x} - 2\,\Re\left(\frac{1}{x+i}\right)$$

clearly lies in $\mathcal{D}^2 \cap \mathcal{K}^2$. Using Proposition 5.5, we then have

$$\Sigma g(x) \;=\; c + \tbinom{x}{2} + \psi(x) - 2\,\Re\,\psi(x+i)$$

for some $c \in \mathbb{R}$, where $\psi(x+i) = D_x \ln\Gamma(x+i)$. Indeed, the function

$$h(x) \;=\; \Re\left(\frac{1}{x+i}\right) \;=\; \frac{x}{x^2+1}$$

lies in $\mathcal{D}^0 \cap \mathcal{K}^0$ while the function $f(x) = \Re\,\psi(x+i)$ lies in $\mathcal{K}^0$ and satisfies $\Delta f = h$. $\square$

## 5.3 Integration of multiple $\log\Gamma$-type functions

The uniform convergence of the sequence $n \mapsto f_n^p[g]$ shows that the function $\Sigma g$ is continuous whenever so is $g$. More generally, we also have the following result.

**Proposition 5.8.** *Let* $g \in \mathcal{C}^0 \cap \mathcal{D}^p \cap \mathcal{K}^p$ *for some* $p \in \mathbb{N}$*. Then the following assertions hold.*

(a) $\Sigma g \in \mathcal{C}^0 \cap \mathcal{D}^{p+1} \cap \mathcal{K}^p$.

*(b)* $\Sigma g$ *is integrable at* $0$ *if and only if so is* $g$.

*(c) Let* $n \in \mathbb{N}^*$ *be so that* $g$ *is* $p$*-convex or* $p$*-concave on* $[n, \infty)$*. For any* $0 \leqslant a \leqslant x$*, the following inequality holds*

$$\left|\int_a^x (f_n^p[g](t) - \Sigma g(t))\, dt\right| \;\leqslant\; \int_a^x \lceil t \rceil \left|\binom{t-1}{p}\right| dt\;\; |\Delta^p g(n)|\,.$$

*Moreover, the following assertions hold.*

*(c1) The sequence* $n \mapsto \int_a^x (f_n^p[g](t) - \Sigma g(t))\, dt$ *converges to zero.*

*(c2) The sequence* $n \mapsto \int_a^x (f_n^p[g](t) + g(t))\, dt$ *converges to*

$$\int_a^x (\Sigma g(t) + g(t))\, dt \;=\; \int_a^x \Sigma g(t+1)\, dt.$$

*(c3) For any* $m \in \mathbb{N}^*$*, the sequence* $n \mapsto \int_a^x (f_n^p[g](t) - f_m^p[g](t))\, dt$ *converges to*

$$\int_a^x (\Sigma g(t) - f_m^p[g](t))\, dt.$$

*Proof.* Assertion (a) follows from the uniform convergence of the sequence $n \mapsto f_n^p[g]$. Assertion (b) follows from assertion (a) and the identity $\Sigma g(x+1) - \Sigma g(x) = g(x)$. Now, using (13) we see that the function $\Sigma g - f_n^p[g] = \rho_n^{p+1}[\Sigma g]$ is integrable at 0 and hence on $(a, x)$. The inequality of assertion (c) then follows from Theorem 3.4(b); and hence assertion (c1) also holds. Assertion (c2) follows from assertion (c1) and the identity $\Sigma g(x+1) - \Sigma g(x) = g(x)$. Finally, using (18) we see that the function $f_m^p[g] - f_n^p[g]$ is integrable on $(a, x)$ and hence assertion (c3) follows from assertion (c1). $\square$

# 6 Asymptotic analysis

In this section we essentially provide for multiple $\log\Gamma$-type functions analogues of *Stirling's formula*, *Stirling's constant*, and *Euler's constant*. We also revisit *Gregory's summation formula* and show how it can be derived almost trivially in this context.

## 6.1 Generalized Stirling's formula and related results

The asymptotic behavior of the gamma function for large values of its argument can be summarized as follows: for any $a \geqslant 0$, we have the following asymptotic equivalences

$$\Gamma(x+a) \;\sim\; x^a\,\Gamma(x) \qquad \text{as } x \to \infty; \tag{25}$$

$$\Gamma(x) \;\sim\; \sqrt{2\pi}\, e^{-x} x^{x-\frac{1}{2}} \qquad \text{as } x \to \infty. \tag{26}$$

In this subsection we provide and discuss analogues of these formulas for the multiple $\log\Gamma$-type functions. We start with a technical but fundamental lemma. Recall first that, for any $n\in\mathbb{N}$, the $n$th Gregory coefficient (also called the $n$th Bernoulli number of the second kind) is the number $G_n$ defined by the equation (see, e.g., [17–19, 61])

$$G_n ~=~ \int_0^1 \binom{t}{n}\, dt.$$

The first few values of $G_n$ are: $1, \frac{1}{2}, -\frac{1}{12}, \frac{1}{24}, -\frac{19}{720}, \ldots$. These numbers are decreasing in absolute value and satisfy the equations

$$\sum_{n=1}^{\infty}|G_n| ~=~ 1 \qquad\text{and}\qquad G_n ~=~ (-1)^{n-1}|G_n| \quad\text{for } n\geqslant 1. \tag{27}$$

To simplify the notation, for any $n\in\mathbb{N}$ we set

$$\overline{G}_n ~=~ 1-\sum_{j=1}^{n}|G_j|.$$

By (27) we have $\overline{G}_n>0$ for all $n\in\mathbb{N}$. Also, from the straightforward identity

$$(-1)^n\binom{t-1}{n} ~=~ 1-\sum_{j=1}^{n}(-1)^{j-1}\binom{t}{j},$$

we easily derive

$$\int_0^1\left|\binom{t-1}{n}\right|\, dt ~=~ (-1)^n\int_0^1\binom{t-1}{n}\, dt ~=~ \left|\int_0^1\binom{t-1}{n}\, dt\right| ~=~ \overline{G}_n\,. \tag{28}$$

**Lemma 6.1.** *Let $g\in\mathcal{D}^p\cap\mathcal{K}^p$ for some $p\in\mathbb{N}$ (hence $g\in\mathcal{K}^{p-1}$ if $p\geqslant 1$) and let $a\geqslant 0$.*

*(a) Let $x>0$ be so that $g$ is $p$-convex or $p$-concave on $[x,\infty)$. Then*

$$|\rho_x^{p+1}[\Sigma g](a)| ~\leqslant~ \lceil a\rceil\left|\binom{a-1}{p}\right||\Delta^p g(x)|, \tag{29}$$

*and if $g\in\mathcal{C}^0$,*

$$\left|\int_0^1\rho_x^{p+1}[\Sigma g](t)\, dt\right| ~\leqslant~ \overline{G}_p\,|\Delta^p g(x)|. \tag{30}$$

*(b) Suppose that $p\geqslant 1$ and let $x>0$ be so that $g$ is $(p-1)$-convex or $(p-1)$-concave on $[x,\infty)$. Then*

$$|\rho_x^{p}[g](a)| ~\leqslant~ \lceil a\rceil\left|\binom{a-1}{p-1}\right||\Delta^p g(x)|, \tag{31}$$

*and if $g\in\mathcal{C}^0$,*

$$\left|\int_0^1\rho_x^{p}[g](t)\, dt\right| ~\leqslant~ \overline{G}_{p-1}\,|\Delta^p g(x)|. \tag{32}$$

*Proof.* Assuming for instance that $g$ is eventually $p$-convex, the function $\Delta^p g$ increases to zero on $[x,\infty)$ by Lemma 2.3. Using Lemma 2.4, we then obtain

$$|\rho_x^{p+1}[\Sigma g](a)| \;\leqslant\; \left|\binom{a-1}{p}\right| \sum_{j=0}^{\lceil a\rceil -1} |\Delta^p g(x+j)|$$

from which we immediately derive (29). Now, observing that the function $t \mapsto \rho_x^{p+1}[\Sigma g](t)$ does not change in sign on $(0,1)$ by Lemma 2.4, and then integrating both sides of (29) on $a \in (0,1)$, we obtain

$$\left|\int_0^1 \rho_x^{p+1}[\Sigma g](t)\, dt\right| \;=\; \int_0^1 |\rho_x^{p+1}[\Sigma g](t)|\, dt \;\leqslant\; \int_0^1 \left|\binom{t-1}{p}\right| dt\; |\Delta^p g(x)|,$$

which, using (28), gives (30). We prove (31) and (32) similarly. $\square$

*A first asymptotic result.* If $g \in \mathcal{D}^p \cap \mathcal{K}^p$ for some $p \in \mathbb{N}$, then for any $a \in \mathbb{N}$ the $a$th degree Newton expansion of $\Sigma g(x+a)$ is given by

$$\Sigma g(x+a) \;=\; \sum_{j=0}^{a} \binom{a}{j} \Delta^j \Sigma g(x)\,,$$

or equivalently,

$$\Sigma g(x+a) - \Sigma g(x) - \sum_{j\geqslant 1} \binom{a}{j} \Delta^{j-1} g(x) \;=\; 0.$$

If the index variable $j$ in the latter sum is bounded above by $p$, then clearly the resulting left-hand expression need no longer be zero but it approaches zero as $x \to \infty$ (because $g \in \mathcal{D}^p_{\mathbb{R}}$). The following theorem shows that this latter property still holds when $a$ is any nonnegative real number, thus providing the asymptotic behavior of the difference $\Sigma g(x+a) - \Sigma g(x)$ for large values of $x$. We omit the proof of this theorem for it follows immediately from (29) and (31). We also observe that the first convergence result (33) was established by Webster [80, Theorem 6.1] in the case when $p = 1$. The second one (34) simply expresses the fact that $g$ lies in $\mathcal{R}^p_{\mathbb{R}}$ by Proposition 4.5.

**Theorem 6.2.** *Let $g \in \mathcal{D}^p \cap \mathcal{K}^p$ for some $p \in \mathbb{N}$ and let $a \geqslant 0$. Let also $x > 0$ be so that $g$ is $p$-convex or $p$-concave on $[x,\infty)$. Then*

$$\left|\Sigma g(x+a) - \Sigma g(x) - \sum_{j=1}^{p} \binom{a}{j} \Delta^{j-1} g(x)\right| \;\leqslant\; \lceil a\rceil \left|\binom{a-1}{p}\right| |\Delta^p g(x)|\;,$$

*with equality if $a \in \{1,2,\ldots,p\}$. In particular,*

$$\Sigma g(x+a) - \Sigma g(x) - \sum_{j=1}^{p} \binom{a}{j} \Delta^{j-1} g(x) \;\to\; 0 \qquad \text{as } x \to \infty\,. \tag{33}$$

*If* $p \geqslant 1$ *and if* $x > 0$ *is so that* $g$ *is* $(p-1)$*-convex or* $(p-1)$*-concave on* $[x, \infty)$*, then*

$$\left|g(x+a) - \sum_{j=0}^{p-1} \tbinom{a}{j} \Delta^j g(x)\right| \;\leqslant\; \lceil a \rceil \left|\tbinom{a-1}{p-1}\right| |\Delta^p g(x)| \,,$$

*with equality if* $a \in \{1, 2, \ldots, p-1\}$*. In particular,*

$$g(x+a) - \sum_{j=0}^{p-1} \tbinom{a}{j} \Delta^j g(x) \;\to\; 0 \qquad \text{as } x \to \infty\,. \tag{34}$$

**Example 6.3.** Let us apply Theorem 6.2 to the function $g(x) = \ln x$. For this function we have $p = 1 + \deg g = 1$ and $\Sigma g(x) = \ln \Gamma(x)$. Thus, for any $x > 0$ and any $a \geqslant 0$ we obtain

$$\left(1 + \frac{1}{x}\right)^{-\lceil a \rceil |a-1|} \;\leqslant\; \frac{\Gamma(x+a)}{\Gamma(x)\, x^a} \;\leqslant\; \left(1 + \frac{1}{x}\right)^{\lceil a \rceil |a-1|}, \tag{35}$$

with equalities if $a = 1$. Thus, we retrieve the asymptotic equivalence (25). Interestingly, Wendel [81] provided the following tighter inequalities

$$\left(1 + \frac{a}{x}\right)^{a-1} \;\leqslant\; \frac{\Gamma(x+a)}{\Gamma(x)\, x^a} \;\leqslant\; 1 \qquad \text{if } 0 \leqslant a \leqslant 1.$$

Considering higher values of $p$ may provide inequalities that are tighter than (35). For instance, taking $p = 2$, we obtain

$$\begin{aligned} \left(1 + \frac{1}{x}\right)^{\binom{a}{2} - 2\lceil a \rceil |\binom{a-1}{2}|} \left(1 + \frac{2}{x}\right)^{\lceil a \rceil |\binom{a-1}{2}|} \;&\leqslant\; \frac{\Gamma(x+a)}{\Gamma(x)\, x^a} \\ \leqslant\; \left(1 + \frac{1}{x}\right)^{\binom{a}{2} + 2\lceil a \rceil |\binom{a-1}{2}|} \left(1 + \frac{2}{x}\right)^{-\lceil a \rceil |\binom{a-1}{2}|}&. \end{aligned}$$

Thus, we can see that the central function in the inequalities above can always be "sandwiched" by finite products of powers of rational functions. Similar inequalities for this function can be found, e.g., in [76, pp. 106–107]. □

*The asymptotic constant and Binet-like function.* With any function $g$ lying in $\cup_{p \geqslant 0}(\mathcal{C}^0 \cap \mathcal{D}^p \cap \mathcal{K}^p)$ we associate the number

$$\sigma[g] \;=\; \int_0^1 \Sigma g(t+1)\, dt \;=\; \int_0^1 (\Sigma g(t) + g(t))\, dt\,. \tag{36}$$

We then observe that the following identity holds for any $x > 0$,

$$\int_x^{x+1} \Sigma g(t)\, dt \;=\; \sigma[g] + \int_1^x g(t)\, dt. \tag{37}$$

Indeed, both sides have the same derivative and the same value at $x = 1$.

It would be convenient to name the constant $\sigma[g]$ for we will make an intensive use of it throughout the rest of this paper. In view of Theorem 6.5 below, we will call it the *asymptotic constant* associated with the function $g$, although a more appropriate name for this constant could also be proposed and used in subsequent papers.

Just as in Lemma 6.1, we have assumed the continuity of function $g$ to ensure that the integrals in (36) and (37) be defined. Of course, this assumption could have been relaxed into weaker properties such as local integrability of both $g$ and $\Sigma g$. However, for the sake of simplicity we will henceforth assume the continuity of any function whenever we need to integrate it on a compact interval; see also Remark 11.1.

We also have the following proposition, which follows immediately from Proposition 5.5 and identity (37).

**Proposition 6.4.** *Let* $g \in \cup_{p\geqslant 0}(\mathcal{C}^0 \cap \mathcal{D}^p \cap \mathcal{K}^p)$*, let* $a \geqslant 0$*, and let* $h\colon \mathbb{R}_+ \to \mathbb{R}$ *be defined by the equation* $h(x) = g(x+a)$ *for* $x > 0$*. Then*

$$\sigma[h] \;=\; \sigma[g] + \int_1^{1+a} g(t)\,dt - \Sigma g(a+1).$$

Now, for any $q \in \mathbb{N}$ and any $g \in \mathcal{C}^0$, we define the function $J^q[g]\colon \mathbb{R}_+ \to \mathbb{R}$ by the equations

$$J^q[g](x) \;=\; -\int_0^1 \rho_x^q[g](t)\,dt \;=\; \sum_{j=0}^{q-1} G_j \Delta^j g(x) - \int_x^{x+1} g(t)\,dt\,. \tag{38}$$

When $g(x) = \ln \Gamma(x)$ and $q = 2$, this function reduces to *Binet's function* $J(x)$ related to $\ln \Gamma(x)$ (see, e.g., [28, p. 224]). That is,

$$J^2[\ln \Gamma](x) \;=\; J(x) \;=\; \ln \Gamma(x) - \frac{1}{2}\ln(2\pi) + x - \left(x - \frac{1}{2}\right)\ln x.$$

We will say that the function $J^q[g]$ is the *Binet-like function* associated with the function $g$ and the parameter $q$. As we will see in the rest of this paper, many subsequent definitions and results will be expressed in terms of the Binet-like function.

Using (37), we can also see that for any $q \in \mathbb{N}$ and any $g \in \cup_{p\geqslant 0}(\mathcal{C}^0 \cap \mathcal{D}^p \cap \mathcal{K}^p)$ we have

$$J^{q+1}[\Sigma g](x) \;=\; \Sigma g(x) - \int_1^x g(t)\,dt + \sum_{j=1}^{q} G_j \Delta^{j-1} g(x) - \sigma[g]\,. \tag{39}$$

In particular, we have

$$\sigma[g] \;=\; -J^1[\Sigma g](1) \quad\text{and}\quad \Delta J^{q+1}[\Sigma g] = J^{q+1}[g].$$

*Generalized Stirling's formula.* The following important theorem enables one to investigate, for any function $g$ lying in $\cup_{p\geqslant 0}(\mathcal{C}^0\cap\mathcal{D}^p\cap\mathcal{K}^p)$, the asymptotic behavior of the function $\Sigma g$ for large values of its argument. In particular, the convergence result (40) gives for $\Sigma g$ an analogue of Stirling's formula. We call it the *generalized Stirling formula*. Combining (33) with (40) then immediately provides the asymptotic behavior of $\Sigma g(x+a)$ for any $a\geqslant 0$. We also observe that alternative formulations of (40) in the case when $p=1$ were established by Krull [46, p. 368] and later by Webster [80, Theorem 6.3].

**Theorem 6.5** (Generalized Stirling's formula)**.** *Let $g\in\mathcal{C}^0\cap\mathcal{D}^p\cap\mathcal{K}^p$ for some $p\in\mathbb{N}$ and let $x>0$ be so that $g$ is $p$-convex or $p$-concave on $[x,\infty)$. Then*

$$\left|J^{p+1}[\Sigma g](x)\right| \;\leqslant\; \overline{G}_p\,|\Delta^p g(x)|.$$

*In particular, the function $J^{p+1}[\Sigma g]$ lies in $\mathcal{D}^0_{\mathbb{R}}$, that is,*

$$\Sigma g(x)-\int_1^x g(t)\,dt+\sum_{j=1}^{p}G_j\Delta^{j-1}g(x) \;\to\; \sigma[g] \qquad \text{as } x\to\infty\,. \tag{40}$$

*If $p\geqslant 1$ and if $x>0$ is so that $g$ is $(p-1)$-convex or $(p-1)$-concave on $[x,\infty)$, then*

$$|J^p[g](x)| \;\leqslant\; \overline{G}_{p-1}\,|\Delta^p g(x)|.$$

*In particular, the function $J^p[g]$ lies in $\mathcal{D}^0_{\mathbb{R}}$, that is,*

$$-\int_x^{x+1} g(t)\,dt+\sum_{j=0}^{p-1}G_j\Delta^j g(x) \;\to\; 0 \qquad \text{as } x\to\infty\,. \tag{41}$$

*Proof.* The first inequality is obtained from (30), (37), and (39). The first convergence result (40) immediately follows. The second inequality and its associated convergence result (41) is obtained similarly using (32) and (38). ☐

*Remark* 6.6. We can readily see that (34) can be obtained by applying the operator $\Delta_x$ to (33). More generally, the first part of Theorem 6.2 can be obtained by replacing $g$ by $\Sigma g$ and $p$ by $p+1$ in the second part. The same observation applies to Theorem 6.5 (cf. the identity $\Delta J^{p+1}[\Sigma g]=J^{p+1}[g]$).

**Example 6.7.** Applying Theorem 6.5 to the function $g(x)=\ln x$ with $p=1$, we immediately obtain the following inequalities for any $x>0$

$$\left(1+\frac{1}{x}\right)^{-\frac{1}{2}} \;\leqslant\; \frac{\Gamma(x)}{\sqrt{2\pi}\,e^{-x}\,x^{x-\frac{1}{2}}} \;\leqslant\; \left(1+\frac{1}{x}\right)^{\frac{1}{2}};$$

$$\left(1+\frac{1}{x}\right)^{x} \;\leqslant\; e \;\leqslant\; \left(1+\frac{1}{x}\right)^{x+2}.$$

Thus, we retrieve the well-known asymptotic equivalences (including (26))

$$\begin{aligned} \Gamma(x) &\sim \sqrt{2\pi}\, e^{-x}\, x^{x-\frac{1}{2}} && \text{as } x \to \infty\,; \\ x! \;=\; \Gamma(x+1) &\sim \sqrt{2\pi x}\, e^{-x}\, x^{x} && \text{as } x \to \infty\,; \\ \left(1+\tfrac{1}{x}\right)^{x} &\sim e && \text{as } x \to \infty\,. \end{aligned}$$

Just as in Example 6.3, tighter inequalities can be obtained by considering higher values of $p$. For instance, for $p = 2$, we obtain

$$\left(1+\frac{1}{x}\right)^{-\frac{3}{4}}\left(1+\frac{2}{x}\right)^{\frac{5}{12}} \;\leqslant\; \frac{\Gamma(x)}{\sqrt{2\pi}\, e^{-x}\, x^{x-\frac{1}{2}}} \;\leqslant\; \left(1+\frac{1}{x}\right)^{\frac{11}{12}}\left(1+\frac{2}{x}\right)^{-\frac{5}{12}}.$$

For $p = 3$, we obtain

$$\begin{aligned} \left(1+\frac{1}{x}\right)^{-\frac{23}{24}}\left(1+\frac{2}{x}\right)^{\frac{13}{12}}\left(1+\frac{3}{x}\right)^{-\frac{3}{8}} &\;\leqslant\; \frac{\Gamma(x)}{\sqrt{2\pi}\, e^{-x}\, x^{x-\frac{1}{2}}} \\ &\;\leqslant\; \left(1+\frac{1}{x}\right)^{\frac{31}{24}}\left(1+\frac{2}{x}\right)^{-\frac{7}{6}}\left(1+\frac{3}{x}\right)^{\frac{3}{8}}. \end{aligned}$$

Thus, we see that the central function in these inequalities can always be bracketed by finite products of radical functions. □

Example 6.7 illustrates the possibility of obtaining closer bounds for the Binet-like function $J^{p+1}[\Sigma g](x)$ by considering any value of $p$ that is higher than $1 + \deg g$. Actually, it is not difficult to see that this feature applies to every continuous multiple $\log\Gamma$-type function. We discuss this issue in Appendix C and show that the inequalities actually get tighter and tighter as $p$ increases.

*Improvements of Stirling's formula.* The following estimate of the gamma function is due to Gosper [33]

$$\Gamma(x) \;\sim\; \sqrt{2\pi}\, e^{-x}\, x^{x-\frac{1}{2}}\left(1+\frac{1}{6x}\right)^{\frac{1}{2}} \qquad \text{as } x \to \infty,$$

and is more accurate than Stirling's formula. On the basis of this alternative approximation, Mortici [64] provided the following narrow inequalities

$$\left(1+\frac{\alpha}{2x}\right)^{\frac{1}{2}} \;<\; \frac{\Gamma(x)}{\sqrt{2\pi}\, e^{-x}\, x^{x-\frac{1}{2}}} \;<\; \left(1+\frac{\beta}{2x}\right)^{\frac{1}{2}}, \qquad \text{for } x \geqslant 2,$$

where $\alpha = \frac{1}{3} \approx 0.333$ and $\beta = (391/30)^{1/3} - 2 \approx 0.353$. We actually observe that the quest for finer and finer bounds and approximations for the gamma function has gained an increasing interest during this last decade (see [23, 25, 26, 31, 56, 63–66, 82, 83] and the references therein). We believe that some of these investigations could be generalized to various $\Gamma_p$-type functions. New results along this line would be most welcome.

*Series expressions for* $\Sigma g$ *and* $\sigma[g]$. The following result provides series expressions for $\Sigma g(x)$ and $\sigma[g]$ in terms of Gregory's coefficients (see also Proposition C.2 in Appendix C).

**Proposition 6.8.** *Let* $g \in \mathcal{C}^0 \cap \mathcal{D}^p \cap \mathcal{K}^\infty$ *for some* $p \in \mathbb{N}$. *Let* $x > 0$ *be so that for every integer* $q \geqslant p$ *the function* $g$ *is* $q$*-convex or* $q$*-concave on* $[x, \infty)$. *Suppose also that the sequence* $q \mapsto \Delta^q g(x)$ *is bounded. Then the sequence* $n \mapsto G_n \Delta^{n-1} g(x)$ *is summable and we have* $J^\infty[\Sigma g](x) = 0$, *i.e.,*

$$\Sigma g(x) \;=\; \sigma[g] + \int_1^x g(t)\,dt - \sum_{n=1}^{\infty} G_n \Delta^{n-1} g(x).$$

*In particular, if the assumptions above are satisfied for* $x = 1$, *then we have*

$$\sigma[g] \;=\; \sum_{n=1}^{\infty} G_n \Delta^{n-1} g(1). \tag{42}$$

*Proof.* Since the sequence $n \mapsto \overline{G}_n$ converges to zero, by (30) so does the sequence

$$q \;\mapsto\; \int_0^1 \rho_x^{q+1}[\Sigma g](t)\,dt.$$

We then obtain the result using (37). □

**Example 6.9.** Applying Proposition 6.8 to the function $g(x) = \ln(x)$, we obtain the following infinite product representation of the gamma function

$$\Gamma(x) \;=\; \sqrt{2\pi}\, e^{-x}\, x^{x-\frac{1}{2}} \prod_{n=2}^{\infty} e^{-G_n \Delta^{n-1} \ln(x)},$$

that is,

$$\begin{aligned} \Gamma(x) \;&=\; \sqrt{2\pi}\, e^{-x}\, x^{x-\frac{1}{2}} \left(\frac{x+1}{x}\right)^{\frac{1}{12}} \left(\frac{(x+2)x}{(x+1)^2}\right)^{-\frac{1}{24}} \\ &\quad \times \left(\frac{(x+3)(x+1)^3}{(x+2)^3 x}\right)^{\frac{19}{720}} \cdots . \end{aligned}$$

A similar representation of the gamma function can be found in Feng and Wang [31]. □

*Fontana-Mascheroni's series.* When $g(x) = \frac{1}{x}$ and $p = 0$, identity (42) reduces to the well-known formula

$$\gamma \;=\; \sum_{n=1}^{\infty} \frac{|G_n|}{n},$$

and the latter series is called *Fontana-Mascheroni's series* (see, e.g., [17]). Thus, the series representation of the asymptotic constant $\sigma[g]$ given in (42) provides the analogue of Fontana-Mascheroni's series for any function $g$ satisfying the assumptions of Proposition 6.8.

The following proposition provides a way to construct a function $g(x)$ that has a prescribed asymptotic constant $\sigma[g]$ given in the form (42).

**Proposition 6.10.** *Let $S = \sum_{n=1}^{\infty} G_n s_n$ for some sequence $n \mapsto s_n$ and let $g\colon \mathbb{R}_+ \to \mathbb{R}$ be such that*

$$g(n) \;=\; \sum_{k=1}^{n} \tbinom{n-1}{k-1}\, s_k\,, \qquad n \in \mathbb{N}^*.$$

*If $g$ satisfies the assumptions of Proposition 6.8, then the following assertions hold.*

*(a)* $S = \sigma[g]$.

*(b)* $\Sigma g(n) = \sum_{k=1}^{n-1} \binom{n-1}{k} s_k$ *for any* $n \in \mathbb{N}^*$.

*(c)* $s_n = \Delta^{n-1} g(1) = \Delta^n \Sigma g(1)$ *for any* $n \in \mathbb{N}^*$.

*Proof.* Using the classical inversion formula [34, p. 192], we obtain

$$s_{n+1} \;=\; \sum_{k=0}^{n} (-1)^{n-k} \tbinom{n}{k}\, g(k+1) \;=\; \Delta^n g(1).$$

This establishes assertion (c) and then assertion (a) by Proposition 6.8. Assertion (b) is straightforward using (24). ☐

**Example 6.11.** Let us apply Proposition 6.10 to the series

$$S \;=\; \sum_{n=1}^{\infty} \frac{|G_n|}{n^2}\,.$$

Let $g\colon \mathbb{R}_+ \to \mathbb{R}$ be a function such that

$$g(n) \;=\; \sum_{k=1}^{n} (-1)^{k-1} \tbinom{n-1}{k-1} \frac{1}{k^2} \;=\; \frac{1}{n} H_n\,, \qquad n \in \mathbb{N}^*.$$

We naturally take $g(x) = \frac{1}{x} H_x$ , from which we derive

$$\Sigma g(x) \;=\; \frac{\pi^2}{12} - \frac{1}{2}\,\psi_1(x) + \frac{1}{2}\, H_{x-1}^2\,.$$

Thus, we have $S = \sigma[g]$. Combining this with the definition of $\sigma[g]$, we derive the surprising identity

$$\int_0^1 H_t^2\, dt \;=\; 1 - \frac{\pi^2}{6} + 2 \sum_{n=1}^{\infty} \frac{|G_n|}{n^2}\,,$$

which is worth comparing with

$$\int_0^1 H_t\, dt \;=\; \gamma \;=\; \sum_{n=1}^{\infty} \frac{|G_n|}{n}\,.$$

To give another example, consider the series

$$S \;=\; \sum_{n=1}^{\infty} \frac{|G_n|}{n+a}\,,$$

where $a > 0$. Proposition 6.10 shows that we can take

$$g(x) \;=\; B(x, a+1) \quad \text{and} \quad \Sigma g(x) \;=\; \frac{1}{a} - B(x,a),$$

where $(x,y) \mapsto B(x,y)$ is the beta function. We then derive the identity

$$\sum_{n=1}^{\infty} \frac{|G_n|}{n+a} \;=\; \frac{1}{a} - \int_0^1 B(x+1,a)\, dx.$$

Using the definition of the beta function as an integral, this identity also reads

$$\sum_{n=1}^{\infty} \frac{|G_n|}{n+a} \;=\; \frac{1}{a} + \int_0^1 \frac{x^a}{\ln(1-x)}\, dx.$$

Setting $a = \frac{1}{2}$ for instance, we obtain

$$\sum_{n=1}^{\infty} \frac{|G_n|}{2n+1} \;=\; 1 + \frac{1}{2}\int_0^1 \frac{\sqrt{x}}{\ln(1-x)}\, dx$$

and the decimal expansion of the latter integral is Sloane's A094691 [75]. ☐

*Asymptotic behaviors and trends.* The following corollary, which immediately follows from (37) and (40), particularizes the generalized Stirling formula when the function $g$ lies in $\mathcal{C}^0 \cap \mathcal{D}^0 \cap \mathcal{K}^0$.

**Corollary 6.12.** *Let* $g \in \mathcal{C}^0 \cap \mathcal{D}^0 \cap \mathcal{K}^0$. *Then*

$$\Sigma g(x) - \int_x^{x+1} \Sigma g(t)\, dt \;\to\; 0 \qquad \text{as } x \to \infty\,.$$

*Equivalently,*

$$\Sigma g(x) - \int_1^x g(t)\, dt \;\to\; \sigma[g] \qquad \text{as } x \to \infty\,.$$

It is clear that the integral (37) planes and cancels out the cyclic variations of any 1-periodic additive component of $\Sigma g$ in the sense that the function

$$x \;\mapsto\; \int_x^{x+1} \omega(t)\,dt$$

is constant for any 1-periodic function $\omega\colon \mathbb{R}_+ \to \mathbb{R}$. In fact, the integral (37) can be interpreted as the trend of the function $\Sigma g$, just as a moving average enables one to decompose a time series into its trend and its seasonal variation. Thus, Corollary 6.12 shows that $\Sigma g(x)$ coincides asymptotically with its trend (as we could expect from a function lying in $\mathcal{C}^0 \cap \mathcal{D}^1 \cap \mathcal{K}^0$) and behaves asymptotically like the antiderivative of $g$.

The centered version of integral (37), namely

$$\int_{x-\frac{1}{2}}^{x+\frac{1}{2}} \Sigma g(t)\,dt \;=\; \sigma[g] + \int_1^{x-\frac{1}{2}} g(t)\,dt, \qquad x > \tfrac{1}{2}\,,$$

naturally provides a more accurate trend of $\Sigma g$. The following corollary shows that $\Sigma g(x)$ coincides asymptotically with this latter trend whenever $g$ lies in $\mathcal{C}^0 \cap \mathcal{D}^0 \cap \mathcal{K}^0$ or in $\mathcal{C}^0 \cap \mathcal{D}^1 \cap \mathcal{K}^1$. It is not difficult to see that in general this result no longer holds when $g$ lies in $\mathcal{C}^0 \cap \mathcal{D}^2 \cap \mathcal{K}^2$. The logarithm of the Barnes G-function (see Subsection 9.5) could serve as an example here.

**Corollary 6.13.** *Let* $p \in \{0,1\}$*, let* $g \in \mathcal{C}^0 \cap \mathcal{D}^p \cap \mathcal{K}^p$*, and let* $x > 0$ *be so that* $g$ *is* $p$*-convex or* $p$*-concave on* $[x,\infty)$*. Then*

$$\left|\Sigma g\left(x+\frac{1}{2}\right) - \int_x^{x+1} \Sigma g(t)\,dt\right| \;\leqslant\; \left|J^{p+1}[\Sigma g](x)\right| \;\leqslant\; \overline{G}_p\,|\Delta^p g(x)|.$$

*In particular,*

$$\Sigma g(x) - \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} \Sigma g(t)\,dt \;\to\; 0 \qquad \text{as } x \to \infty\,,$$

*or equivalently,*

$$\Sigma g(x) - \int_1^{x-\frac{1}{2}} g(t)\,dt \;\to\; \sigma[g] \qquad \text{as } x \to \infty\,.$$

*Proof.* Using Theorem 6.5, we see that it is enough to prove the first inequality. Let

$$h(x) \;=\; \Sigma g\left(x+\frac{1}{2}\right) - \int_x^{x+1} \Sigma g(t)\,dt.$$

Consider first the case when $p = 0$ and suppose for instance that $g$ lies in $\mathcal{K}^0_+$;

hence $\Sigma g$ is decreasing on $[x, \infty)$. Then it is geometrically clear that

$$|h(x)| \leqslant \begin{cases} \displaystyle\int_x^{x+\frac{1}{2}} \Sigma g(t)\,dt - \frac{1}{2}\Sigma g\left(x+\frac{1}{2}\right), & \text{if } h(x) \leqslant 0, \\ \displaystyle\frac{1}{2}\Sigma g\left(x+\frac{1}{2}\right) - \int_{x+\frac{1}{2}}^{x+1} \Sigma g(t)\,dt, & \text{if } h(x) \geqslant 0, \end{cases}$$

and that both quantities are less than or equal to $J^1[\Sigma g](x)$.

Now, suppose that $p = 1$ and for instance that $g$ lies in $\mathcal{K}^1_+$; hence $\Sigma g$ is concave on $[x, \infty)$. Using the Hermite-Hadamard inequality and then the trapezoidal rule on the intervals $[x, x+\frac{1}{2}]$ and $[x+\frac{1}{2}, x+1]$, we obtain the following chain of inequalities:

$$0 \leqslant h(x) \leqslant \int_x^{x+1} \Sigma g(t)\,dt - \frac{1}{2}\,\Sigma g(x+1) - \frac{1}{2}\,\Sigma g(x)$$

and the latter quantity is exactly $-J^2[\Sigma g](x)$. $\square$

Applying Corollary 6.13 to the function $g(x) = \ln x$, we obtain *Burnside's formula* [24] (see also [63])

$$\Gamma(x) \sim \sqrt{2\pi}\left(\frac{x-\frac{1}{2}}{e}\right)^{x-\frac{1}{2}} \qquad \text{as } x \to \infty\,. \tag{43}$$

Thus, Corollary 6.13 gives an analogue of Burnside's formula for any continuous $\Gamma_p$-type function when $p \in \{0, 1\}$. It also shows that this formula provides a better approximation than the generalized Stirling formula for all functions $g$ lying in $\mathcal{C}^0 \cap \mathcal{D}^p \cap \mathcal{K}^p$ with $p \in \{0, 1\}$.

## 6.2 Generalized Stirling's constant

The number $\sqrt{2\pi}$ arising in Example 6.7 is called Stirling's constant (see, e.g., [32]). For certain multiple $\Gamma$-type functions, analogues of Stirling's constant can be easily defined as follows.

**Definition 6.14** (Generalized Stirling's constant)**.** For any function $g$ lying in $\cup_{p\geqslant 0}(\mathcal{C}^0 \cap \mathcal{D}^p \cap \mathcal{K}^p)$ and integrable at $0$, we define the number

$$\overline{\sigma}[g] = \sigma[g] - \int_0^1 g(t)\,dt = \int_0^1 \Sigma g(t)\,dt.$$

We say that the number $\exp(\overline{\sigma}[g])$ is the *generalized Stirling constant* associated with $g$.

Note that, contrary to the generalized Stirling constant, the asymptotic constant $\sigma[g]$ exists for any function $g$ lying in $\cup_{p\geqslant 0}(\mathcal{C}^0\cap\mathcal{D}^p\cap\mathcal{K}^p)$, even if it is not integrable at 0. This shows that the asymptotic constant is the "good" constant to consider in this new theory. When $g$ is integrable at 0, then (40) can take the form

$$\Sigma g(x)-\int_0^x g(t)\,dt+\sum_{j=1}^{p}G_j\Delta^{j-1}g(x)\ \to\ \overline{\sigma}[g]\qquad\text{as } x\to\infty\,.$$

## 6.3 The Gregory summation formula revisited

Let $g\in\mathcal{C}^0$, $q\in\mathbb{N}$, and let $1\leqslant m\leqslant n$ be integers. Integrating both sides of (18) on $x\in(0,1)$, we obtain

$$\int_m^n g(t)\,dt\ =\ \sum_{k=m}^{n-1}g(k)+\sum_{j=1}^{q}G_j(\Delta^{j-1}g(n)-\Delta^{j-1}g(m))+R_{q,m,n}\,,\tag{44}$$

where

$$R_{q,m,n}\ =\ \int_0^1\sum_{k=m}^{n-1}\rho_k^{q+1}[g](t)\,dt\ =\ \int_0^1(f_m^q[g](t)-f_n^q[g](t))\,dt.$$

Identity (44) is nothing other than *Gregory's summation formula* (see, e.g., [14, 43, 62]) with an integral form of the remainder. Note that, just like identity (8), equation (44) is a pure identity and therefore holds without any restriction on the form of $g(x)$. Actually, this identity can be simply written in terms of the Binet-like function as

$$\sum_{k=m}^{n-1}J^{q+1}[g](k)\ =\ -R_{q,m,n}\,,$$

or equivalently, if $g\in\cup_{p\geqslant 0}(\mathcal{C}^0\cap\mathcal{D}^p\cap\mathcal{K}^p)$,

$$J^{q+1}[\Sigma g](n)-J^{q+1}[\Sigma g](m)\ =\ -R_{q,m,n}\,.$$

*Remark* 6.15. We observe that Jordan [43, p. 285] established that

$$\text{“}\ R_{q,m,n}\ =\ G_{q+1}(n-m)\,\Delta^{q+1}g(\xi)\ \text{”}$$

for some $\xi\in(m,n)$. However, taking for instance $g(x)=x^2$ and $(q,m,n)=(0,1,2)$ shows that this form of the remainder is not correct. However, we conjecture that Jordan's formula can be corrected by assuming that $\xi\in(m-1,n-1)$.

The following lemma, which is an immediate consequence of Lemma 2.4, provides an upper bound for $|R_{p,m,n}|$ when $g$ is p-convex or p-concave on $[m,\infty)$. We then see that, under this latter assumption, Gregory's formula (44) provides a quadrature formula for the numerical computation of the integral of $g$ over the interval $(m,n)$.

**Lemma 6.16.** *Let $g \in \mathcal{C}^0 \cap \mathcal{K}^p$ for some $p \in \mathbb{N}$ and let $m \in \mathbb{N}^*$ be so that $g$ is p-convex or p-concave on $[m,\infty)$. Then, for any integer $n \geqslant m$, we have*

$$|R_{p,m,n}| \;=\; \left|\sum_{k=m}^{n-1}\int_0^1 \rho_k^{p+1}[g](t)\,dt\right| \;\leqslant\; \overline{G}_p\,|\Delta^p g(n) - \Delta^p g(m)|. \tag{45}$$

**Example 6.17.** Let us compute the integral

$$I \;=\; \int_{\pi}^{2\pi} \ln x\,dx \;=\; 4.809854526737\ldots$$

numerically using Gregory's summation formula (44). Using an appropriate linear change of variable, we obtain $I = \int_1^n g(t)\,dt$, where

$$g(t) \;=\; \tfrac{\pi}{n-1}\,\ln\left(\tfrac{\pi}{n-1}\,(t-1)+\pi\right).$$

Taking $n = 20$ and $q = 10$ for instance, we obtain

$$I \;\approx\; \sum_{k=1}^{19} g(k) + \sum_{j=1}^{10} G_j(\Delta^{j-1}g(20) - \Delta^{j-1}g(1)) \;=\; 4.809854526746\ldots$$

and (45) gives $|R_{10,1,20}| \leqslant 5.9\times 10^{-11}$. □

In the following result, we give sufficient conditions on function $g$ for the sequence $q \mapsto R_{q,m,n}$ to converge to zero. Gregory's formula (44) then takes a special form.

**Proposition 6.18.** *Let $g \in \mathcal{C}^0$, $p \in \mathbb{N}$, and let $1 \leqslant m \leqslant n$ be integers. Suppose that, for every integer $q \geqslant p$, the function $g$ is q-convex or q-concave on $[m,\infty)$. Suppose also that the sequence $q \mapsto \Delta^q g(n) - \Delta^q g(m)$ is bounded. Then we have*

$$\int_m^n g(t)\,dt \;=\; \sum_{k=m}^{n-1} g(k) + \sum_{j=1}^{\infty} G_j(\Delta^{j-1}g(n) - \Delta^{j-1}g(m)),$$

*or equivalently,*

$$\sum_{k=m}^{n-1} J^{\infty}[g](k) \;=\; 0.$$

*If $g \in \cup_{p\geqslant 0}(\mathcal{C}^0 \cap \mathcal{D}^p \cap \mathcal{K}^p)$, then this latter condition simply means that $J^{\infty}[\Sigma g](n) = J^{\infty}[\Sigma g](m)$.*

*Proof.* The sequence $q \mapsto R_{q,m,n}$ converges to zero by (45). The result then immediately follows from Gregory's formula (44). $\square$

**Example 6.19.** Taking $g(x) = \ln x$ and $m = p = 1$ in Proposition 6.18, we obtain the following identity for any $n \in \mathbb{N}^*$

$$\ln n! \;=\; 1 - n + \left(n + \frac{1}{2}\right)\ln n + \frac{1}{12}\ln\left(\frac{n+1}{2n}\right) - \frac{1}{24}\ln\left(\frac{4n(n+2)}{3(n+1)^2}\right) + \cdots \quad \square$$

Gregory's formula (44) is sometimes presented in a more general form in the literature. We provide this general form in the following proposition using our integral expression for the remainder. Lemma 6.16 then can be easily adapted to this general form.

**Proposition 6.20** (General form of Gregory's formula)**.** *Let* $a \in \mathbb{R}$*,* $n, q \in \mathbb{N}$*,* $h > 0$*, and* $f \in \mathcal{C}^0([a, \infty))$*. Then*

$$\begin{aligned}\frac{1}{h}\int_a^{a+nh} f(t)\,dt \;&=\; \sum_{k=0}^{n-1} f(a+kh)\\ &\quad + \sum_{j=1}^{q} G_j\left((\Delta_{[h]}^{j-1} f)(a+nh) - (\Delta_{[h]}^{j-1} f)(a)\right) + R_{q,a,n}^{h}\,,\end{aligned}$$

*where*

$$R_{q,a,n}^{h} \;=\; \int_0^1 \sum_{k=1}^{n} \rho_k^{p+1}[g](t)\,dt \quad \textit{and} \quad g(x) \;=\; f(a+(x-1)h).$$

*Here,* $\Delta_{[h]}$ *denotes the forward difference operator with step* $h > 0$*.*

*Proof.* This formula can be obtained immediately from (44) replacing $n$ by $n+1$ and then setting $m = 1$ and $g(x) = f(a+(x-1)h)$. $\square$

Gregory's formula is often compared with the corresponding *Euler-Maclaurin summation formula.* We now recall the latter in its general form (see, e.g., [76, p. 220]) as we will use it a few times in this paper. We also note that Euler-Maclaurin's formula is more advantageous than Gregory's formula if we deal with functions whose derivatives are less complicated than their differences. However, there are functions for which Euler-Maclaurin's formula leads to divergent series while the corresponding Gregory's formula-based series (see Proposition 6.18) are convergent. For instance, this may be due to the fact that $D^n \frac{1}{x}$ increases indefinitely with $n$ while $\Delta^n \frac{1}{x}$ tends to zero if $n$ increases. (Here, we paraphrase from Jordan [43, p. 285].)

**Proposition 6.21** (Euler-Maclaurin formula). *Let* $N \in \mathbb{N}^*$, $f \in \mathcal{C}^1([a,b])$, *and* $h = (b-a)/N$, *for some real numbers* $a < b$. *Then we have*

$$\begin{aligned} h\sum_{k=0}^{N} f(a+kh) &= \int_a^b f(x)\,dx + \frac{h}{2}\,(f(a)+f(b)) \\ &\quad + h^2\int_0^N B_1(s-\lfloor s\rfloor)\,Df(a+sh)\,ds\,. \end{aligned}$$

*If, in addition,* $f \in \mathcal{C}^{2q}([a,b])$ *for some* $q \in \mathbb{N}^*$, *then*

$$\begin{aligned} h\sum_{k=0}^{N} f(a+kh) &= \int_a^b f(x)\,dx + \frac{h}{2}\,(f(a)+f(b)) \\ &\quad + \sum_{j=1}^{q} h^{2j}\,\frac{B_{2j}}{(2j)!}\,\big(D^{2j-1}f(b) - D^{2j-1}f(a)\big) + R\,, \end{aligned}$$

*where*

$$R \;=\; -\,h^{2q+1}\int_0^N \frac{B_{2q}(s-\lfloor s\rfloor)}{(2q)!}\,D^{2q}f(a+sh)\,ds$$

*and*

$$|R| \;\leqslant\; h^{2q}\,\frac{|B_{2q}|}{(2q)!}\int_a^b |D^{2q}f(x)|\,dx.$$

*Here* $f \in \mathcal{C}^k([a,b])$ *means that* $f \in \mathcal{C}^k(I)$ *for some open interval* $I$ *containing* $[a,b]$.

## 6.4 Generalized Euler's constant

Suppose that $g \in \mathcal{C}^0 \cap \mathcal{D}^p \cap \mathcal{K}^p$ for some $p \in \mathbb{N}$. Let also $m \in \mathbb{N}^*$ be so that $g$ is p-convex or p-concave on $[m,\infty)$. By Lemma 6.16 the sequence $n \mapsto R_{p,m,n}$ converges and we have (see also Proposition 5.8)

$$\begin{aligned} R_{p,m,\infty} &= \lim_{n\to\infty}\int_0^1 (f^p_m[g](t) - f^p_n[g](t))\,dt \;=\; \sum_{k=m}^{\infty}\int_0^1 \rho_k^{p+1}[g](t)\,dt \\ &= \int_0^1 \sum_{k=m}^{\infty}\rho_k^{p+1}[g](t)\,dt \;=\; \int_0^1 (f^p_m[g](t) - \Sigma g(t))\,dt \\ &= -\int_0^1 \rho_m^{p+1}[\Sigma g](t)\,dt \;=\; J^{p+1}[\Sigma g](m), \end{aligned}$$

where the fifth equality follows from (13). Also, (45) reduces to

$$|R_{p,m,\infty}| \;=\; |J^{p+1}[\Sigma g](m)| \;\leqslant\; \overline{G}_p\,|\Delta^p g(m)|\,, \tag{46}$$

which is also an immediate consequence of Theorem 6.5.

Let us now provide a geometric interpretation of the remainder $R_{p,m,\infty}$. Suppose for instance that $g$ is $p$-concave on $[m,\infty)$ and that $p$ is even; the other cases are similar. Then by (9) and Lemma 2.4, for any integer $k \in [m,\infty)$ and any $t \in (0,1)$, we have

$$0 \;\geqslant\; \rho_k^{p+1}[g](t) \;=\; g(k+t) - P_p[g](k, k+1, \ldots, k+p; k+t),$$

which means that, on the interval $[k, k+1]$, the graph of $g$ lies under (or on) that of its interpolating polynomial with nodes at $k, k+1, \ldots, k+p$. Also, the (signed) surface area between both graphs is

$$-\int_0^1 \rho_k^{p+1}[g](t)\, dt \;=\; J^{p+1}[g](k)\,.$$

Summing this area for $k = m, \ldots, n-1$ and letting $n \to_{\mathbb{N}} \infty$, we obtain the cumulated (signed) surface area

$$-\sum_{k=m}^{\infty} \int_0^1 \rho_k^{p+1}[g](t)\, dt \;=\; \sum_{k=m}^{\infty} J^{p+1}[g](k) \;=\; -\, R_{p,m,\infty}. \tag{47}$$

This interpretation is particularly visual when $p = 0$ or $p = 1$. For instance, when $p = 1$, the graph of $g$ on $[m,\infty)$ lies either over or under the polygonal line through the points $(k, g(k))$ for all integers $k \geqslant m$. The value $-R_{p,m,\infty}$ is then the signed area between the graph of $g$ and this polygonal line and corresponds to the remainder in the trapezoidal rule on $[m,\infty)$.

In view of this interpretation, we now propose the following definition.

**Definition 6.22** (Generalized Euler's constant)**.** The *generalized Euler constant* associated with a function $g \in \cup_{p \geqslant 0}(\mathcal{C}^0 \cap \mathcal{D}^p \cap \mathcal{K}^p)$ is the number

$$\gamma[g] \;=\; -\, R_{p,1,\infty} \;=\; \sum_{k=1}^{\infty} J^{p+1}[g](k) \;=\; -\, J^{p+1}[\Sigma g](1)\,,$$

where $p = 1 + \deg g$.

For instance, if $g \in \mathcal{C}^0 \cap \mathcal{D}^0 \cap \mathcal{K}^0$, we have

$$\begin{aligned}
\gamma[g] \;&=\; \lim_{n \to \infty} \left( \sum_{k=1}^{n-1} g(k) - \int_1^n g(t)\, dt \right) \\
&=\; \sum_{k=1}^{\infty} \left( g(k) - \int_k^{k+1} g(t)\, dt \right),
\end{aligned} \tag{48}$$

which represents the remainder in the rectangle method on $[1,\infty)$.

Similarly, if $g \in \mathcal{C}^0 \cap \mathcal{D}^1 \cap \mathcal{K}^1$ and $\deg g = 0$, we get

$$\begin{aligned}\gamma[g] &= \lim_{n\to\infty}\left(\sum_{k=1}^{n-1} g(k) - \int_1^n g(t)\,dt + \frac{1}{2}\,g(n) - \frac{1}{2}\,g(1)\right)\\ &= \sum_{k=1}^{\infty}\left(g(k) - \int_k^{k+1} g(t)\,dt + \frac{1}{2}\,\Delta g(k)\right),\end{aligned}$$

which represents the remainder in the trapezoidal rule on $[1,\infty)$. If $g \in \mathcal{C}^1$, then this latter expression also reduces to (use integration by parts)

$$\gamma[g] = \sum_{k=1}^{\infty}\int_k^{k+1}\left(t-k-\frac{1}{2}\right)g'(t)\,dt = \int_1^{\infty}\left(\{t\}-\frac{1}{2}\right)g'(t)\,dt, \tag{49}$$

where $\{t\} = t - \lfloor t \rfloor$.

Thus defined, the number $\gamma[g]$ generalizes to any function lying in $\cup_{p\geqslant 0}(\mathcal{C}^0 \cap \mathcal{D}^p \cap \mathcal{K}^p)$ not only the classical Euler constant $\gamma$ but also the generalized Euler constant $\gamma[g]$ associated with a positive and strictly decreasing function $g$ (see, e.g., [7, 32]), as defined in (48). Moreover, as we will see in Subsection 8.2, this number plays a central role in the Weierstrassian form of $\Sigma g$ (which also justifies the choice $m = 1$ in the definition of $\gamma[g]$).

If $g$ is $p$-convex or $p$-concave on $[1,\infty)$, then by (46) we also have the inequality

$$|\gamma[g]| \leqslant \overline{G}_p\,|\Delta^p g(1)|. \tag{50}$$

*A conversion formula between* $\gamma[g]$ *and* $\sigma[g]$. The following proposition, which immediately follows from the identity $\gamma[g] = -J^{p+1}[\Sigma g](1)$, shows how the numbers $\gamma[g]$ and $\sigma[g]$ are related and provides an alternative way to compute the value of $\gamma[g]$.

**Proposition 6.23.** *For any function $g$ lying in $\cup_{p\geqslant 0}(\mathcal{C}^0 \cap \mathcal{D}^p \cap \mathcal{K}^p)$, we have*

$$\sigma[g] = \gamma[g] + \sum_{j=1}^{p} G_j \Delta^{j-1} g(1),$$

*where* $p = 1 + \deg g$.

*An analogue of Liu's formula for $\Gamma$-type functions.* Using (49) together with Proposition 6.21 with $a = h = 1$ and $b = N = n$ (first-order version of the Euler-Maclaurin formula), we obtain the following statement. For any $n \in \mathbb{N}^*$ and any $g \in \mathcal{C}^1 \cap \mathcal{D}^1 \cap \mathcal{K}^1$, with $\deg g = 0$, we have

$$\begin{aligned}\sum_{k=1}^{n} g(k) &= \gamma[g] + \int_1^n g(t)\,dt + \frac{1}{2}(g(1)+g(n))\\ &\quad + \int_n^{\infty}\left(\frac{1}{2}-\{t\}\right)g'(t)\,dt,\end{aligned}$$

that is, using Proposition 6.23,

$$\sum_{k=1}^{n} g(k) \;=\; \sigma[g] + \int_1^n g(t)\,dt + \frac{1}{2}\,g(n) + \int_n^\infty \left(\frac{1}{2} - \{t\}\right) g'(t)\,dt, \tag{51}$$

or equivalently,

$$J^2[\Sigma g](n) \;=\; \int_n^\infty \left(\frac{1}{2} - \{t\}\right) g'(t)\,dt,$$

where $\{t\} = t - \lfloor t \rfloor$. Applying this result to $g(n) = \ln n$, we obtain Liu's exact formula [55] (see also [63])

$$n! \;=\; \sqrt{2\pi n}\,\left(\frac{n}{e}\right)^n \exp\left(\int_n^\infty \frac{\frac{1}{2} - \{t\}}{t}\,dt\right), \qquad n \in \mathbb{N}^*.$$

A generalization of (51) to any function $g$ lying in $\cup_{p\geqslant 0}(\mathcal{C}^0 \cap \mathcal{D}^p \cap \mathcal{K}^p)$ would be welcome.

*An integral form of* $\gamma[g]$*.* The following proposition shows that the integral representation of the Euler constant

$$\gamma \;=\; \int_1^\infty \left(\frac{1}{\lfloor t \rfloor} - \frac{1}{t}\right) dt$$

can be generalized to the constant $\gamma[g]$ for any function $g$ lying in $\cup_{p\geqslant 0}(\mathcal{C}^0 \cap \mathcal{D}^p \cap \mathcal{K}^p)$. This result is a straightforward consequence of (47).

**Proposition 6.24.** *For any* $g \in \mathcal{C}^0 \cap \mathcal{D}^p \cap \mathcal{K}^p$*, where* $p = 1 + \deg g$*, we have*

$$\gamma[g] \;=\; \int_1^\infty \left(\sum_{j=0}^{p} G_j \Delta^j g(\lfloor t \rfloor) - g(t)\right) dt.$$

*In particular, when* $\deg g = -1$*, we have* $\gamma[g] = \int_1^\infty (g(\lfloor t \rfloor) - g(t))\,dt$*.*

## 6.5 Further asymptotic results

We end this section by establishing two additional asymptotic results. The first one concerns only the case when the sequence $n \mapsto g(n)$ is summable. The second one is much more general and concerns all the continuous multiple $\log\Gamma$-type functions. We also discuss the search for simple conditions on function $g\colon \mathbb{R}_+ \to \mathbb{R}$ to ensure the existence of $\Sigma g$.

*The case when* $g(n)$ *is summable.* In the special case when $g \in \widetilde{\mathcal{D}}^{-1}_{\mathbb{N}} \cap \mathcal{K}^0$, the generalized Stirling formula and the constants $\gamma[g]$ and $\sigma[g]$ take very special forms. We present them in the following proposition, which immediately follows from Theorem 3.7, Eq. (48), and Proposition 6.23.

**Proposition 6.25.** *If* $g \in \widetilde{\mathcal{D}}^{-1}_{\mathbb{N}} \cap \mathcal{K}^0$*, then we have*

$$\Sigma g(x) \;\to\; \sum_{k=1}^{\infty} g(k) \qquad \text{as } x \to \infty.$$

*If, in addition we have* $g \in \mathcal{C}^0$*, then* $g$ *is integrable at infinity and*

$$\sigma[g] \;=\; \gamma[g] \;=\; \sum_{k=1}^{\infty} g(k) - \int_1^{\infty} g(t)\,dt.$$

*A general asymptotic result.* The following general result gives a sufficient condition on a function $g$ lying in $\cup_{p\geqslant 0}(\mathcal{C}^0\cap\mathcal{D}^p\cap\mathcal{K}^p)$ for $\Sigma g$ to be asymptotically equivalent to its (possibly shifted) trend.

**Proposition 6.26.** *Let* $g \in \mathcal{C}^0 \cap \mathcal{D}^p \cap \mathcal{K}^p$ *for some* $p \in \mathbb{N}$ *and let* $a \geqslant 0$*. If* $p \geqslant 1$*, we assume that the function*

$$x \;\mapsto\; \frac{\int_x^{x+1} \Sigma g(t)\,dt}{\Sigma g(x+a)} \tag{52}$$

*(that is defined in a neighborhood of infinity) is eventually monotone. Then we have*

$$\Sigma g(x+a) \;\sim\; \int_x^{x+1} \Sigma g(t)\,dt \qquad \text{as } x \to \infty. \tag{53}$$

*Proof.* Assume first that $g$ lies in $\mathcal{C}^0\cap\mathcal{D}^0\cap\mathcal{K}^0$. Let us prove that for any $c \in \mathbb{R}$ we have

$$\frac{c + \int_x^{x+1} \Sigma g(t)\,dt}{c + \Sigma g(x+a)} \;\to\; 1 \qquad \text{as } x \to \infty. \tag{54}$$

Suppose that $g$ lies in $\widetilde{\mathcal{D}}^{-1}_{\mathbb{N}}$ and for instance that $\Sigma g$ is eventually increasing. Then for sufficiently large $x$ we have

$$\frac{c+\Sigma g(x)}{|c+\Sigma g(x+a)|} \;\leqslant\; \frac{c+\int_x^{x+1} \Sigma g(t)\,dt}{|c+\Sigma g(x+a)|} \;\leqslant\; \frac{c+\Sigma g(x+1)}{|c+\Sigma g(x+a)|}$$

and (54) then follows from Corollary 6.25. Suppose now that $g$ does not lie in $\widetilde{\mathcal{D}}^{-1}_{\mathbb{N}}$, which implies that $g$ is not integrable at infinity. It follows that the integral (37) tends to infinity as $x \to \infty$, and hence so does the function $\Sigma g(x)$. In this case, by (33) and (40) we obtain

$$\Sigma g(x+a) - \int_x^{x+1} \Sigma g(t)\,dt \;\to\; 0 \qquad \text{as } x \to \infty$$

and then (54) follows immediately.

Suppose now that $g$ lies in $\mathcal{C}^0 \cap \mathcal{D}^p \cap \mathcal{K}^p$, with $\deg g = p - 1$, for some $p \in \mathbb{N}^*$. We first observe that

$$\Delta_x^p \int_x^{x+1} \Sigma g(t)\, dt \;=\; \int_x^{x+1} \Delta^p \Sigma g(t)\, dt$$

and that $\Delta^p \Sigma g = c_p + \Sigma \Delta^p g$ for some $c_p \in \mathbb{R}$. Since $\Delta^p g$ lies in $\mathcal{C}^0 \cap \mathcal{D}^0 \cap \mathcal{K}^0$, by (54) we have

$$\frac{\Delta_x^p \int_x^{x+1} \Sigma g(t)\, dt}{\Delta_x^p \Sigma g(x+a)} \;=\; \frac{c_p + \int_x^{x+1} \Sigma \Delta^p g(t)\, dt}{c_p + \Sigma \Delta^p g(x+a)} \;\to\; 1 \qquad \text{as } x \to \infty.$$

Let us now show that the sequence

$$n \;\mapsto\; \frac{\Delta_n^{p-1} \int_n^{n+1} \Sigma g(t)\, dt}{\Delta_n^{p-1} \Sigma g(n+a)}$$

(which exists for large values of $n$) converges to 1. By minimality of $p$, the function $\Delta^{p-1}\Sigma g$ lies in $\mathcal{D}^2_{\mathbb{N}} \setminus \mathcal{D}^1_{\mathbb{N}}$ and hence the sequence $n \mapsto \Delta^{p-1}\Sigma g(n+a)$ tends to infinity. Using the discrete version of L'Hospital's rule, we then obtain

$$\lim_{n\to\infty} \frac{\Delta_n^{p-1} \int_n^{n+1} \Sigma g(t)\, dt}{\Delta_n^{p-1} \Sigma g(n+a)} \;=\; \lim_{n\to\infty} \frac{\Delta_n^{p} \int_n^{n+1} \Sigma g(t)\, dt}{\Delta_n^{p} \Sigma g(n+a)} \;=\; 1.$$

Iterating this process, we finally see that condition (53) holds for the integer values of $x$, and then also for the real values of $x$ using the eventual monotonicity of the function specified by (52) together with the squeeze theorem. $\square$

*Remark* 6.27. In Proposition 6.26 we made the assumption that the function specified by (52) is eventually monotone. We conjecture that this assumption is actually satisfied whenever $g$ lies in $\cup_{p\geqslant 1}(\mathcal{C}^0 \cap \mathcal{D}^p \cap \mathcal{K}^p)$.

*The quest for a characterization of the domain of definition of the map* $\Sigma$. Recall that the domain of definition of the map $\Sigma$ is the set $\cup_{p\geqslant 0}(\mathcal{D}^p \cap \mathcal{K}^p)$. In this respect, it would be useful to have a very simple test to check whether a given function $g$ lies in this set. The following result shows that both conditions $g \in \mathcal{K}^0$ and

$$\lim_{n\to\infty} \frac{|g(n+1)|}{|g(n)|} \;\leqslant\; 1 \tag{55}$$

are necessary. However, they are not sufficient. For instance, for any $q \in \mathbb{N}$ the function $g_q(x) = x^{q+1} + \sin x$ lies in $\mathcal{K}^q \setminus \mathcal{K}^{q+1}$ and satisfies condition (55). However, it does not lie in $\mathcal{D}^\infty_{\mathbb{N}}$.

**Proposition 6.28.** *Let* $g \in \mathcal{K}^0$. *If* $g$ *lies in* $\mathcal{D}^p \cap \mathcal{K}^p$ *for some* $p \in \mathbb{N}$, *then condition* (55) *holds. Conversely, if condition* (55) *holds, then* $g$ *lies in* $\widetilde{\mathcal{D}}^{-1}_{\mathbb{N}}$ *or we have* $\Delta g(x)/g(x) \to 0$ *as* $x \to \infty$.

*Proof.* Assume that $g$ lies in $\mathcal{D}^p \cap \mathcal{K}^p$ for $p = 1 + \deg g$. If $p = 0$, then the function $x \mapsto |g(x)|$ eventually decreases to zero and hence condition (55) holds. Now suppose that $p \geqslant 1$. Then the function $\Delta^p g$ lies in $\mathcal{D}^0 \cap \mathcal{K}^0$ and there are two exclusive cases to consider.

(a) Suppose that the sequence $n \mapsto \Delta^{p-1} g(n)$ tends to infinity. Using the discrete version of L'Hospital's rule, we then obtain

$$\lim_{n\to\infty} \frac{\Delta^{p-1} g(n+1)}{\Delta^{p-1} g(n)} \;=\; \lim_{n\to\infty} \frac{\Delta^{p} g(n+1)}{\Delta^{p} g(n)} \;=\; \ell$$

for some $\ell \in \mathbb{R}$ satisfying $|\ell| \leqslant 1$. Iterating this process, we see that condition (55) holds.

(b) Suppose that the sequence $n \mapsto \Delta^{p-1} g(n)$ has a nonzero limit. If $p = 1$, then condition (55) holds trivially. If $p \geqslant 2$, then the sequence $n \mapsto \Delta^{p-2} g(n)$ tends to infinity and we can use the discrete version of L'Hospital's rule and iterate the process as in the previous case.

Conversely, suppose that $g \in \mathcal{K}^0$ and that condition (55) holds. If the inequality is strict, then $g$ is summable by the ratio test and hence $g$ lies in $\widetilde{\mathcal{D}}^{-1}_{\mathbb{N}}$. Otherwise, if the inequality is an equality, then we have $|g(n+1)| \sim |g(n)|$ as $n \to_{\mathbb{N}} \infty$. Since $g$ lies in $\mathcal{K}^0$ and hence eventually no longer changes in sign (i.e., $g$ lies in $\mathcal{K}^{-1}$), we also have $g(x+1) \sim g(x)$ as $x \to \infty$, that is, $\Delta g(x)/g(x) \to 0$ as $x \to \infty$. ☐

It is easy to see that the function $g\colon \mathbb{R}_+ \to \mathbb{R}$ lies in $\mathcal{D}^\infty_{\mathbb{N}}$ if and only if there exists $p \in \mathbb{N}$ for which the sequence $n \mapsto \Delta^p g(n)$ converges. This observation follows from the immediate identity

$$\Delta^p g(n) \;=\; \Delta^p g(1) + \sum_{k=1}^{n-1} \Delta^{p+1} g(k)\,, \qquad n \in \mathbb{N}^*,\; p \in \mathbb{N}.$$

In particular, if we assume that $g \in \mathcal{K}^\infty$, then $g$ does not lie to $\mathcal{D}^\infty_{\mathbb{N}}$ if and only if for every $p \in \mathbb{N}$ the sequence $n \mapsto \Delta^p g(n)$ tends to infinity. It is easy to see that condition (55) fails to hold for many functions $g$ lying in $\mathcal{K}^\infty \setminus \mathcal{D}^\infty_{\mathbb{N}}$. Examples of such functions include $g(x) = 2^x$ and $g(x) = \Gamma(x)$. It seems reasonable to think that this observation actually follows from a general rule. We then formulate the following conjecture.

*Conjecture.* Let $g \in \mathcal{K}^\infty$. Then $g$ lies in $\mathcal{D}^\infty_{\mathbb{N}}$ if and only if condition (55) holds.

# 7 Derivatives of multiple $\log\Gamma$-type functions

In this section we discuss certain differentiability properties of $\Sigma g$ when $g$ lies in $\mathcal{C}^r \cap \mathcal{D}^p \cap \mathcal{K}^p$ for some $p, r \in \mathbb{N}$. In particular, when $r \leqslant p$ we show that $\Sigma g$

also lies in $\mathcal{C}^r$ and that $D^r\Sigma g(x)$ can be computed as the limit of the sequence $n \mapsto D^r f_n^p[g](x)$. We also discuss how the functions $(\Sigma g)^{(r)}$ and $\Sigma g^{(r)}$ are related and show how $\Sigma g$ can be computed by first computing $\Sigma g^{(r)}$. Finally, we provide an alternative uniqueness result for differentiable solutions to the equation $\Delta f = g$.

## 7.1 On differentiability of multiple $\log\Gamma$-type functions

We investigate the differentiability of the function $\Sigma g$ when $g$ is of class $\mathcal{C}^r$ for some $r \in \mathbb{N}$. The central result is given in Theorem 7.2 below. We first consider a technical lemma. For any $n, r \in \mathbb{N}$, we set

$$b_n^r(x) \;=\; D_x^r\tbinom{x}{n+r}.$$

**Lemma 7.1.** *For any integers $0 \leqslant r \leqslant p$, any $g \in \mathcal{C}^r$, and any $a, x > 0$ we have $\rho_a^{p+1}[g] \in \mathcal{C}^r$ and*

$$\begin{aligned}
D_x^r\,\rho_a^{p+1}[g](x) \;&=\; \rho_a^{p+1-r}[g^{(r)}](x)\\
&\quad -\sum_{i=1}^{r}\sum_{j=0}^{p-r} b_j^{r+1-i}(x)\int_0^1 \rho_a^{p-j-r+1}[\Delta^{j+r-i}g^{(i)}](t)\,dt.
\end{aligned}$$

*Proof.* It is clear that $\rho_a^{p+1}[g] \in \mathcal{C}^r$ and we can assume that $1 \leqslant r \leqslant p$. Using (5) it is then easy to see that

$$\begin{aligned}
D_x^r\,\rho_a^{p+1}[g](x) \;&=\; g^{(r)}(x+a) - \sum_{j=0}^{p-r} b_j^r(x)\Delta^{j+r}g(a)\,;\\
D_x^{r-1}\rho_a^{p}[g'](x) \;&=\; g^{(r)}(x+a) - \sum_{n=0}^{p-r} b_n^{r-1}(x)\Delta^{n+r-1}g'(a).
\end{aligned}$$

Subtracting the second equation from the first one, we obtain

$$\begin{aligned}
D_x^r\,\rho_a^{p+1}[g](x) - D_x^{r-1}\rho_a^{p}[g'](x) \;&=\; \sum_{n=0}^{p-r} b_n^{r-1}(x)\Delta^{n+r-1}g'(a)\\
&\quad -\sum_{j=0}^{p-r} b_j^r(x)\Delta_a^j\left(\Delta^r g(a) - \sum_{n=0}^{p-j-r} G_n\Delta^{n+r-1}g'(a)\right)\\
&\quad -\sum_{j=0}^{p-r} b_j^r(x)\Delta_a^j\sum_{n=0}^{p-j-r} G_n\Delta^{n+r-1}g'(a),
\end{aligned}$$

where the expression in parentheses reduces to

$$\int_0^1 \rho_a^{p-j-r+1}[\Delta^{r-1}g'](t)\,dt.$$

Now, we have

$$\begin{aligned}
&\sum_{n=0}^{p-r} b_n^{r-1}(x)\Delta^{n+r-1}g'(a) - \sum_{j=0}^{p-r} b_j^r(x)\Delta_a^j \sum_{n=0}^{p-j-r} G_n\Delta^{n+r-1}g'(a)\\
&\quad = \sum_{n=0}^{p-r} b_n^{r-1}(x)\Delta^{n+r-1}g'(a) - \sum_{j=0}^{p-r} b_j^r(x)\sum_{n=j}^{p-r} G_{n-j}\Delta^{n+r-1}g'(a)\\
&\quad = \sum_{n=0}^{p-r} \Delta^{n+r-1}g'(a)\left(b_n^{r-1}(x) - \sum_{j=0}^{n} G_{n-j}b_j^r(x)\right).
\end{aligned}$$

The latter expression in parentheses is identically zero. Indeed, the sum therein is the convolution of the sequences $n \mapsto G_n$ and $n \mapsto b_n^r(x)$, whose ordinary generating functions are

$$\int_0^1 \left(\sum_{n\geqslant 0} \binom{t}{n} z^n\right) dt \;=\; \int_0^1 (1+z)^t\, dt \;=\; \frac{z}{\ln(1+z)}$$

and

$$D_x^r\left(\sum_{n\geqslant 0} \binom{x}{n+r} z^n\right) \;=\; D_x^r\left(\frac{1}{z^r}(1+z)^x\right) \;=\; \frac{1}{z^r}\,(1+z)^x(\ln(1+z))^r,$$

respectively. Thus, the ordinary generating function for the convolution is

$$\frac{1}{z^{r-1}}(1+z)^x(\ln(1+z))^{r-1}$$

and hence it defines the sequence $n \mapsto b_n^{r-1}(x)$.

Now, collecting the remaining nonzero terms and using (21) we obtain

$$D_x^r\rho_a^{p+1}[g](x) \;=\; D_x^{r-1}\rho_a^p[g'](x) - \sum_{j=0}^{p-r} b_j^r(x)\int_0^1 \rho_a^{p-j-r+1}[\Delta^{j+r-1}g'](t)\,dt.$$

Finally, using a simple induction on $r$, we obtain the claimed formula. $\square$

**Theorem 7.2** (Differentiability of multiple $\log\Gamma$-type functions)**.** *Let $g \in \mathcal{C}^r \cap \mathcal{D}^p \cap \mathcal{K}^p$ for some $r, p \in \mathbb{N}$. If $r > p$, we further assume that the derivatives $g^{(p+1)}, g^{(p+2)}, \ldots, g^{(r)}$ lie in $\mathcal{K}^0$. Then the following assertions hold.*

*(a)* $\Sigma g \in \mathcal{C}^r \cap \mathcal{D}^{p+1} \cap \mathcal{K}^p$.

*(b) For each $x > 0$, the sequence $n \mapsto D^r f_n^p[g](x)$ converges and we have*

$$D^r\Sigma g(x) \;=\; \lim_{n\to\infty} D^r f_n^p[g](x), \qquad x > 0.$$

(c) *For any nonempty bounded subset* $E$ *of* $\mathbb{R}_+$, *the sequence* $n \mapsto D^r f_n^p[g]$ *converges uniformly on* $E$ *to* $D^r \Sigma g$.

*Proof.* Let us first assume that $r \leqslant p$. The result holds for $r = 0$ by Theorem 3.4 and Proposition 5.8. So we can assume that $1 \leqslant r \leqslant p$. Let $x > 0$ and let $m \in \mathbb{N}^*$ be so that $g$ is $p$-convex or $p$-concave on $[m, \infty)$. For every $i \in \{1, \ldots, r\}$ and every $j \in \{0, \ldots, p-r\}$, by Lemma 2.2(b), Lemma 3.3, Proposition 4.8, and Lemma 6.16, both sequences

$$n \mapsto \sum_{k=m}^{n-1} \rho_k^{p+1-r}[g^{(r)}](x) \quad \text{and} \quad n \mapsto \sum_{k=m}^{n-1} \int_0^1 \rho_k^{p-j-r+1}[\Delta^{j+r-i} g^{(i)}](t)\, dt$$

converge and, for any integer $n \geqslant m$, we have

$$\left| \sum_{k=n}^{\infty} \rho_k^{p+1-r}[g^{(r)}](x) \right| \;\leqslant\; \lceil x \rceil \left| \binom{x-1}{p-r} \right| \left| \Delta^{p-r} g^{(r)}(n) \right|$$

and

$$\left| \sum_{k=n}^{\infty} \int_0^1 \rho_k^{p-j-r+1}[\Delta^{j+r-i} g^{(i)}](t)\, dt \right| \;\leqslant\; \overline{G}_{p-j-r} \left| \Delta^{p-i} g^{(i)}(n) \right|.$$

Combining these inequalities with Lemma 7.1, it follows that for any bounded subset $E$ of $\mathbb{R}_+$ the sequence

$$n \;\mapsto\; \sup_{x \in E} \left| \sum_{k=n}^{\infty} D_x^r\, \rho_k^{p+1}[g](x) \right|$$

converges to zero. Using the classical result on differentiability of uniformly convergent sequences, it follows that the function

$$\sum_{k=m}^{\infty} \rho_k^{p+1}[g](x) \;=\; f_m^p[g](x) - \Sigma g(x)$$

lies in $\mathcal{C}^r$ (and hence so does $\Sigma g$) and that

$$\sum_{k=m}^{\infty} D^r \rho_k^{p+1}[g](x) \;=\; D^r \sum_{k=m}^{\infty} \rho_k^{p+1}[g](x) \;=\; D^r f_m^p[g](x) - D^r \Sigma g(x).$$

This proves the theorem when $r \leqslant p$.

Let us now assume that $r > p$. By Proposition 4.8, the function $g^{(p)}$ lies in $\mathcal{C}^{r-p} \cap \mathcal{D}^0 \cap \mathcal{K}^0$. By Proposition 4.11, for any $i \in \{p+1, \ldots, r\}$, the function $g^{(i)}$ lies in $\mathcal{C}^{r-i} \cap \widetilde{\mathcal{D}}_{\mathbb{N}}^{-1} \cap \mathcal{K}^0$. By Theorem 3.7, it follows that the sequence

$$n \;\mapsto\; -\sum_{k=0}^{n-1} g^{(i)}(x+k) \;=\; D^i f_n^p[g](x)$$

converges uniformly on $\mathbb{R}_+$. Again, we conclude the proof by using the classical result on differentiability of uniformly convergent sequences. $\square$

*Remark* 7.3. If $g \in \mathcal{C}^r \cap \mathcal{D}^p \cap \mathcal{K}^p$ for some integers $0 \leqslant r \leqslant p$, then the function $\Sigma g$ lies in $\mathcal{C}^r$ by Theorem 7.2. Actually, this result can also be established by elementary means. Indeed, by Proposition 5.8 we have $\Sigma g^{(r)} \in \mathcal{C}^0$. Hence, there exists $F \in \mathcal{C}^r$ such that $F^{(r)} = \Sigma g^{(r)}$. Since $\Sigma g^{(r)}$ also lies in $\mathcal{K}^{p-r}$, we have $F \in \mathcal{K}^p$ by Proposition 4.8. Now, we also have $D^r \Delta F = \Delta F^{(r)} = \Delta\Sigma g^{(r)} = g^{(r)}$, which shows that $\Delta(F+P) = g$ for some polynomial $P$ of degree at most $r$. Since $F+P$ lies in $\mathcal{K}^p$, by the uniqueness theorem we must have $F+P = \Sigma g + c$ for some $c \in \mathbb{R}$. Hence $\Sigma g$ lies in $\mathcal{C}^r$.

**Proposition 7.4.** *Let $g \in \mathcal{C}^r \cap \mathcal{D}^p \cap \mathcal{K}^p$ for some integers $p \in \mathbb{N}$ and $r \in \mathbb{N}^*$. If $r > p$, we further assume that the derivatives $g^{(p+1)}, g^{(p+2)}, \ldots, g^{(r)}$ lie in $\mathcal{K}^0$. Then for any $x > 0$ we have*

$$(\Sigma g)^{(r)}(x) - \Sigma g^{(r)}(x) \;=\; (\Sigma g)^{(r)}(1) \;=\; g^{(r-1)}(1) - \sigma[g^{(r)}]. \tag{56}$$

*If $r > p$, then this value reduces to $-\sum_{k=1}^{\infty} g^{(r)}(k)$.*

*Proof.* By Propositions 4.8 and 4.11, we have $g^{(r)} \in \mathcal{D}_{\mathbb{N}}^{(p-r)_+}$. By Theorem 7.2, we have $\Sigma g \in \mathcal{C}^r \cap \mathcal{D}^{p+1} \cap \mathcal{K}^p$. Also, by the existence Theorem 3.4, both functions $\varphi_1 = (\Sigma g)^{(r)}$ and $\varphi_2 = \Sigma g^{(r)}$ are solutions in $\mathcal{K}^{(p-r)_+}$ to the equation $\Delta\varphi = g^{(r)}$. By the uniqueness Theorem 3.1, we have $(\Sigma g)^{(r)} - \Sigma g^{(r)} = c$ for some $c \in \mathbb{R}$. For any $x > 0$, using (37) we then get

$$\begin{aligned} g^{(r-1)}(1) - \sigma[g^{(r)}] &= g^{(r-1)}(x) - \int_x^{x+1} \Sigma g^{(r)}(t)\, dt \\ &= c + g^{(r-1)}(x) - \int_x^{x+1} (\Sigma g)^{(r)}(t)\, dt \\ &= c + g^{(r-1)}(x) - (\Sigma g)^{(r-1)}(x+1) + (\Sigma g)^{(r-1)}(x), \end{aligned}$$

which reduces to the constant $c$. If $r > p$, then we have $g^{(r)} \in \mathcal{C}^0 \cap \widetilde{\mathcal{D}}_{\mathbb{N}}^{-1} \cap \mathcal{K}^0$ by Proposition 4.11. The last part of the proof then follows from Proposition 6.25. ☐

**Example 7.5.** The function $g(x) = \frac{1}{x}$ lies in $\mathcal{C}^\infty \cap \mathcal{D}^0 \cap \mathcal{K}^\infty$ and all its derivatives lie in $\mathcal{K}^0$. By Theorem 7.2, the function

$$\Sigma g(x) \;=\; \sum_{k=0}^{\infty} \left( \frac{1}{k+1} - \frac{1}{x+k} \right) \;=\; \psi(x) + \gamma$$

lies in $\mathcal{C}^\infty \cap \mathcal{D}^1 \cap \mathcal{K}^\infty$. Thus, the series can be differentiated term by term and hence, for any $r \in \mathbb{N}^*$, we have

$$(\Sigma g)^{(r)}(x) \;=\; -\sum_{k=0}^{\infty} (-1)^r r! (x+k)^{-r-1} \;=\; \psi_r(x). \quad \square$$

Combining Theorems 6.2 and 6.5 with Proposition 7.4, we obtain the following corollary, which includes the generalized Stirling formula for $(\Sigma g)^{(r)}$.

**Corollary 7.6.** *Let $g \in \mathcal{C}^r \cap \mathcal{D}^p \cap \mathcal{K}^p$ for some $p, r \in \mathbb{N}$ and let $x > 0$ be so that $g$ is $p$-convex or $p$-concave on $[x, \infty)$. If $r > p$, we further assume that the derivatives $g^{(p+1)}, g^{(p+2)}, \ldots, g^{(r)}$ lie in $\mathcal{K}^0$. Then the following assertions hold.*

*(a) For any $a \geqslant 0$, we have*

$$\left|(\Sigma g)^{(r)}(x+a) - (\Sigma g)^{(r)}(x) - \sum_{j=1}^{(p-r)_+} \binom{a}{j} \Delta^{j-1} g^{(r)}(x)\right| \leqslant \lceil a \rceil \left|\binom{a-1}{(p-r)_+}\right| \left|\Delta^{(p-r)_+} g^{(r)}(x)\right|$$

*In particular, the left-hand expression tends to zero as $x \to \infty$.*

*(b) If $r \geqslant 1$, we have*

$$\left|(\Sigma g)^{(r)}(x) - g^{(r-1)}(x) + \sum_{j=1}^{(p-r)_+} G_j \Delta^{j-1} g^{(r)}(x)\right| \leqslant \overline{G}_{(p-r)_+} |\Delta^{(p-r)_+} g^{(r)}(x)|.$$

*In particular, the left-hand expression tends to zero as $x \to \infty$.*

*Moreover, if $r > p$, then $(\Sigma g)^{(r)}(x) \to 0$ as $x \to \infty$.*

It turns out that the convergence results in Corollary 7.6 can be obtained just by taking the $r$th derivative of (33) and (40), respectively. In particular, the function $J^{p+1}[\Sigma g](x)$ and its derivatives vanish at infinity.

*Derivatives of $\Sigma g(x)$ at $x = 1$.* Proposition 7.4 enables us to compute the value of $(\Sigma g)^{(r)}(1)$ whenever $g^{(r)}$ exists. For instance, for $g(x) = \ln x$ we obtain

$$\psi(1) = (\ln \Gamma)'(1) = -\sigma[g'] = -\gamma$$

and, for any integer $r \geqslant 2$,

$$\begin{aligned}\psi^{(r-1)}(1) = (\ln \Gamma)^{(r)}(1) &= (-1)^r (r-2)! - \sigma[g^{(r)}] \\ &= (-1)^r (r-1)!\, \zeta(r).\end{aligned}$$

If the function $\Sigma g$ is real analytic at 1, then the following Taylor series expansion

$$\Sigma g(x+1) = \sum_{k=1}^{\infty} (\Sigma g)^{(k)}(1) \frac{x^k}{k!} \tag{57}$$

holds in some neighborhood of $x = 0$. For instance, for $g(x) = \ln x$ we obtain

$$\ln \Gamma(x+1) \;=\; -\gamma x + \sum_{k=2}^{\infty} (-1)^k \, \frac{\zeta(k)}{k} \, x^k \,, \qquad |x| < 1.$$

*Exponential generating function for the sequence* $\sigma[g^{(n)}]$. Suppose that $g$ lies in $\mathcal{C}^\infty \cap \mathcal{D}^p \cap \mathcal{K}^p$ for some $p \in \mathbb{N}$ and that $g^{(k)}$ lies in $\mathcal{K}^0$ for any $k \in \mathbb{N}$. Identity (56) enables us to write formally the following power series

$$\sum_{k=0}^{\infty} \sigma[g^{(k)}] \, \frac{x^k}{k!} \;=\; \sigma[g] + \int_1^{x+1} g(t)\, dt - \Sigma g(x+1)\,. \tag{58}$$

Thus, the right side of (58) is precisely the exponential generating function $\mathrm{egf}_\sigma[g](x)$ for the sequence $n \mapsto \sigma[g^{(n)}]$. We then have

$$\mathrm{egf}_\sigma[g](x) \;=\; -\, J^1[\Sigma g](x+1)\,, \qquad x > 0,$$

and hence

$$\sigma[g^{(k)}] \;=\; (\mathrm{egf}_\sigma[g])^{(k)}(0) \;=\; -\, (J^1[\Sigma g])^{(k)}(1)\,, \qquad k \in \mathbb{N}.$$

For instance, if $g(x) = \ln x$, then we have $\sigma[g] = -1 + \frac{1}{2}\ln(2\pi)$, $\sigma[g'] = \gamma$, and for any integer $k \geqslant 2$

$$\sigma[g^{(k)}] \;=\; (-1)^k (k-2)! \, (1 - (k-1)\zeta(k))\,.$$

Similarly, if the sequence $n \mapsto \overline{\sigma}[g^{(n)}]$ is defined, then the corresponding exponential generating function $\mathrm{egf}_{\overline{\sigma}}[g](x)$ is

$$\sum_{k=0}^{\infty} \overline{\sigma}[g^{(k)}] \, \frac{x^k}{k!} \;=\; \overline{\sigma}[g] + \int_0^{x} g(t)\, dt - \Sigma g(x+1).$$

Now, if $p = 1 + \deg g$, then by Propositions 4.8, 6.23, and 6.25 we also have

$$\sigma[g^{(k)}] \;=\; \gamma[g^{(k)}] + \sum_{j=1}^{(p-k)_+} G_j \Delta^{j-1} g^{(k)}(1)\,, \qquad k \in \mathbb{N}$$

and hence the exponential generating function for the sequence $n \mapsto \gamma[g^{(n)}]$ is the function

$$\mathrm{egf}_\gamma[g](x) \;=\; \mathrm{egf}_\sigma[g](x) - \sum_{j=1}^{p} G_j \sum_{k=0}^{p-j} \frac{x^k}{k!} \, \Delta^{j-1} g^{(k)}(1).$$

*Analogues of Euler's series representation of* $\gamma$. Integrating both sides of (57) on $(0,1)$ (assuming that the series can be integrated term by term), we obtain the identity

$$\sigma[g] \;=\; \sum_{k=1}^{\infty} \frac{(\Sigma g)^{(k)}(1)}{(k+1)!}\,. \tag{59}$$

Similarly, integrating both sides of (58) on $(0,1)$ (assuming that the series can be integrated term by term), we obtain the identity

$$\sum_{k=0}^{\infty}\frac{\sigma[g^{(k)}]}{(k+1)!} ~=~ \int_1^2(2-t)\,g(t)\,dt. \tag{60}$$

Taking for instance $g(x)=1/x$ in (59), we immediately retrieve Euler's series representation of $\gamma$ (see, e.g., [76, p. 272])

$$\gamma ~=~ \sum_{k=2}^{\infty}(-1)^k\,\frac{\zeta(k)}{k}\,.$$

This formula can also be obtained by taking $g(x)=1/x$ in (60) and using the straightforward identity

$$\sigma[g^{(k)}] ~=~ (-1)^k k!\left(\zeta(k+1)-\frac{1}{k}\right), \qquad k\in\mathbb{N}^*.$$

Considering different functions $g(x)$ in (59) and (60) enables one to derive various interesting identities.

**Example 7.7.** Taking $g(x)=\psi(x)$ in (60) and using the straightforward identity

$$\sigma[g^{(k)}] ~=~ (-1)^{k-1}(k-1)(k-1)!\,\zeta(k) \qquad k\in\mathbb{N},~ k\geqslant 2,$$

we obtain

$$\sum_{k=2}^{\infty}(-1)^k\frac{k-1}{k(k+1)}\,\zeta(k) ~=~ 2-\ln(2\pi)\,.$$

Similarly, taking $g(x)=\ln x$ and then $g(x)=\ln\Gamma(x)$ in (59) and (60) we obtain the identities

$$\begin{aligned}
\sum_{k=2}^{\infty}(-1)^k\frac{1}{k(k+1)}\,\zeta(k) &~=~ \frac{1}{2}\,\gamma-1+\frac{1}{2}\ln(2\pi)\,,\\
\sum_{k=2}^{\infty}(-1)^k\,\frac{1}{(k+1)(k+2)}\,\zeta(k) &~=~ \frac{1}{2}+\frac{1}{6}\,\gamma-2\ln A\,,\\
\sum_{k=2}^{\infty}(-1)^k\frac{k-1}{k(k+1)(k+2)}\,\zeta(k) &~=~ \frac{5}{4}-\frac{1}{4}\ln(2\pi)-3\ln A\,,
\end{aligned}$$

where $A$ is Glaisher-Kinkelin's constant; see also [76, Section 3.4]. □

## 7.2 Finding solutions from derivatives

Given $r\in\mathbb{N}^*$ and a function $g\in\mathcal{C}^r$, a solution $f\in\mathcal{C}^r$ to the equation $\Delta f=g$ can sometimes be found by first searching for an appropriate solution $\varphi\in\mathcal{C}^0$

to the equation $\Delta\varphi = g^{(r)}$ and then calculating $f$ as an $r$th antiderivative of $\varphi$. To our knowledge, this approach was investigated thoroughly by Krull [47] and then by Dufresnoy and Pisot [30]. Here we present a general theory based on this idea.

We first observe that if $\varphi \in \mathcal{C}^0$ is a solution to the equation $\Delta\varphi = g^{(r)}$, then the map

$$x \;\mapsto\; \int_x^{x+1} \varphi(t)\,dt - g^{(r-1)}(x)$$

has a zero derivative and hence it is constant on $\mathbb{R}_+$. In particular, it has a finite right limit at $x = 0$. Recall also that the *Bernoulli numbers* $B_0, B_1, B_2, \ldots$ are defined implicitly by the single equation (see, e.g., [34, p. 284])

$$\sum_{j=0}^{m} \binom{m+1}{j} B_j \;=\; 0^m, \qquad \text{integer} \;\; m \geqslant 0.$$

**Theorem 7.8.** *Let $r \in \mathbb{N}^*$, $a > 0$, $g \in \mathcal{C}^r$, and let $\varphi \colon \mathbb{R}_+ \to \mathbb{R}$ be a continuous solution to the equation $\Delta\varphi = g^{(r)}$. Then there exists a solution $f \in \mathcal{C}^r$ to the equation $\Delta f = g$ such that $f^{(r)} = \varphi$ if and only if*

$$\int_a^{a+1} \varphi(t)\,dt \;=\; g^{(r-1)}(a). \tag{61}$$

*If any of these equivalent conditions holds, then $f$ is uniquely determined (up to an additive constant) by*

$$f(x) \;=\; f(a) + \sum_{k=1}^{r-1} c_k \,\frac{(x-a)^k}{k!} + \int_a^x \frac{(x-t)^{r-1}}{(r-1)!}\,\varphi(t)\,dt, \tag{62}$$

*where, for $k = 1, \ldots, r-1$,*

$$c_k \;=\; \sum_{j=0}^{r-k-1} \frac{B_j}{j!} \left( g^{(j+k-1)}(a) - \int_a^{a+1} \frac{(a+1-t)^{r-j-k}}{(r-j-k)!}\,\varphi(t)\,dt \right). \tag{63}$$

*Proof.* Condition (61) is clearly necessary. Indeed, we have

$$\int_a^{a+1} \varphi(t)\,dt \;=\; f^{(r-1)}(a+1) - f^{(r-1)}(a) \;=\; g^{(r-1)}(a).$$

Let us show that it is sufficient. Since $\varphi$ is continuous, there exists $f \in \mathcal{C}^r$ such that $f^{(r)} = \varphi$. Taylor's theorem then provides the expansion formula (62) with arbitrary parameters $c_k = f^{(k)}(a)$ for $k = 1, \ldots, r-1$. Now we need to determine the parameters $c_1, \ldots, c_k$ for $f$ to be a solution to the equation $\Delta f = g$. To this extent, we need the following claim.

*Claim.* The function $f$ satisfies the equation $\Delta f = g$ if and only if $f^{(r)}$ satisfies the equation $\Delta f^{(r)} = g^{(r)}$ and $\Delta f^{(j)}(a) = g^{(j)}(a)$ for $j = 0, \ldots, r-1$.

*Proof of the claim.* The condition is clearly necessary. To see that it is sufficient, we simply show by decreasing induction on $j$ that $\Delta f^{(j)} = g^{(j)}$. Clearly, this is true for $j = r$. Suppose that it is true for some integer $j$ satisfying $1 \leqslant j \leqslant r$. For any $x > 0$ we have

$$\begin{aligned}\Delta f^{(j-1)}(x) - \Delta f^{(j-1)}(a) &= \int_a^x \Delta f^{(j)}(t)\,dt \;=\; \int_a^x g^{(j)}(t)\,dt \\ &= g^{(j-1)}(x) - g^{(j-1)}(a) \;=\; g^{(j-1)}(x) - \Delta f^{(j-1)}(a),\end{aligned}$$

which shows that the result still holds for $j - 1$. □

By the claim, $f$ satisfies the equation $\Delta f = g$ if and only if $\Delta f^{(j)}(a) = g^{(j)}(a)$ for $j = 0, \ldots, r-1$. When $j = r-1$, the latter condition is nothing other than condition (61) and hence it is satisfied. Applying Taylor's theorem to $f^{(j)}$, we obtain

$$f^{(j)}(a+1) - f^{(j)}(a) \;=\; \sum_{k=1}^{r-j-1} \frac{1}{k!}\, f^{(j+k)}(a) + \int_a^{a+1} \frac{(a+1-t)^{r-j-1}}{(r-j-1)!}\,\varphi(t)\,dt\,,$$

and hence we see that the remaining $r-1$ conditions are

$$\sum_{k=1}^{r-j-1} \frac{1}{k!}\, c_{j+k} \;=\; d_j, \qquad j = 0, \ldots, r-2,$$

where

$$\begin{aligned} d_j &= g^{(j)}(a) - \int_a^{a+1} \frac{(a+1-t)^{r-j-1}}{(r-j-1)!}\,\varphi(t)\,dt, \qquad j = 0, \ldots, r-2,\\ c_k &= f^{(k)}(a), \qquad k = 1, \ldots, r-1.\end{aligned}$$

It is not difficult to see that these $r-1$ conditions form a consistent triangular system of $r-1$ linear equations in the $r-1$ unknowns $c_1, \ldots, c_{r-1}$. This establishes the uniqueness of $f$ up to an additive constant.

Let us now show that formula (63) holds. For $k = 1, \ldots, r-1$, we have

$$\sum_{j=0}^{r-k-1} \frac{B_j}{j!}\, d_{j+k-1} \;=\; \sum_{j=0}^{r-k-1} \frac{B_j}{j!} \sum_{i=1}^{r-j-k} \frac{1}{i!}\, c_{i+j+k-1}.$$

Replacing $i$ by $i-j-k+1$ and then permuting the resulting sums, the latter expression reduces to

$$\begin{aligned}\sum_{j=0}^{r-k-1} \frac{B_j}{j!} \sum_{i=j+k}^{r-1} \frac{1}{(i-j-k+1)!}\, c_i &= \sum_{i=k}^{r-1} \frac{c_i}{(i-k+1)!} \sum_{j=0}^{i-k} \tbinom{i-k+1}{j} B_j \\ &= \sum_{i=k}^{r-1} \frac{c_i}{(i-k+1)!}\, 0^{i-k} \;=\; c_k.\end{aligned}$$

This completes the proof of the theorem. □

Adding an appropriate constant to $\varphi$ if necessary in Theorem 7.8, we can always assume that condition (61) holds. More precisely, the function $\varphi^\star = \varphi + C$, where

$$C \;=\; g^{(r-1)}(a) - \int_a^{a+1} \varphi(t)\,dt,$$

satisfies $\int_a^{a+1} \varphi^\star(t)\,dt \;=\; g^{(r-1)}(a)$. In fact, this is exactly what we did in Proposition 7.4, where (56) represents the equation

$$f^{(r)}(x) - \varphi(x) \;=\; g^{(r-1)}(1) - \int_1^2 \varphi(t)\,dt.$$

**Example 7.9.** Let $g \in \mathcal{C}^0$, let $G \in \mathcal{C}^1$ be defined by the equation

$$G(x) \;=\; \int_1^x g(t)\,dt,$$

and let $f \in \mathcal{C}^0$ be any solution to the equation $\Delta f = g$. Then the function $F \in \mathcal{C}^1$ defined by the equation

$$F(x) \;=\; \int_1^x f(t)\,dt - (x-1)\int_1^2 f(t)\,dt,$$

is a solution to the equation $\Delta F = G$. Moreover, if $f \in \mathcal{K}^p$ for some $p \in \mathbb{N}$, then $F \in \mathcal{K}^{p+1}$ by Lemma 2.2(g). For similar results, see [47, p. 254] and [50, Section 2]. $\square$

Now, using Theorem 7.2, Proposition 7.4, and Theorem 7.8, we can easily derive the following useful corollary.

**Corollary 7.10.** *Let $g \in \mathcal{C}^r \cap \mathcal{D}^p \cap \mathcal{K}^p$ for some $p \in \mathbb{N}$ and some $r \in \mathbb{N}^*$. If $r > p$, we further assume that the derivatives $g^{(p+1)}, g^{(p+2)}, \ldots, g^{(r)}$ lie in $\mathcal{K}^0$. Then $\Sigma g \in \mathcal{C}^r \cap \mathcal{D}^{p+1} \cap \mathcal{K}^p$ and*

$$(\Sigma g)^{(r)} - \Sigma g^{(r)} \;=\; g^{(r-1)}(1) - \sigma[g^{(r)}].$$

*(This value reduces to $-\sum_{k=1}^{\infty} g^{(r)}(k)$ if $r > p$.) Moreover, for any $a > 0$, we have $\Sigma g = f_a - f_a(1)$, where $f_a \in \mathcal{C}^r$ is defined by*

$$f_a(x) \;=\; \sum_{k=1}^{r-1} c_k(a)\,\frac{(x-a)^k}{k!} + \int_a^x \frac{(x-t)^{r-1}}{(r-1)!}\,(\Sigma g)^{(r)}(t)\,dt$$

*and, for $k = 1, \ldots, r-1$,*

$$c_k(a) \;=\; \sum_{j=0}^{r-k-1} \frac{B_j}{j!}\left(g^{(j+k-1)}(a) - \int_a^{a+1} \frac{(a+1-t)^{r-j-k}}{(r-j-k)!}\,(\Sigma g)^{(r)}(t)\,dt\right).$$

Corollary 7.10 has an important practical value. It provides an explicit integral expression for $\Sigma g$ from an explicit expression for $\Sigma g^{(r)}$. The following two examples illustrate the use of this result.

**Example 7.11.** The function $g(x) = \int_0^x (x-t)\ln t\, dt$ lies in $\mathcal{C}^\infty \cap \mathcal{D}^3 \cap \mathcal{K}^\infty$. Choosing $r = 2$ and $a = 0$ (as a limiting value) in Corollary 7.10, we get

$$\begin{aligned} g''(x) &= \ln x\,, \\ \Sigma g''(x) &= \ln \Gamma(x)\,, \\ (\Sigma g)''(x) &= \ln \Gamma(x) - \tfrac{1}{2}\ln(2\pi), \end{aligned}$$

and

$$\Sigma g(x) \;=\; -(\ln A)\, x - \frac{1}{4}\ln(2\pi)\, x^2 + \int_0^x (x-t)\ln\Gamma(t)\, dt,$$

where $A$ is Glaisher-Kinkelin's constant and the integral is the polygamma function $\psi_{-3}(x)$. Using Theorem 6.5, we also obtain the following asymptotic behavior of $\Sigma g$

$$\begin{aligned} &\Sigma g(x) + \frac{1}{72}(22x^3 - 27x^2 + 9x) - \frac{1}{48}x^2(8x-15)\ln x \\ &- \frac{1}{12}(x+1)^2\ln(x+1) + \frac{1}{48}(x+2)^2\ln(x+2) \;\to\; \frac{\zeta(3)}{8\pi^2} \qquad \text{as } x \to \infty. \quad \square \end{aligned}$$

**Example 7.12.** The function $g(x) = \arctan(x)$ lies in $\mathcal{C}^\infty \cap \mathcal{D}^1 \cap \mathcal{K}^\infty$. Choosing $r = 1$ and $a = 0$ (as a limiting value) in Corollary 7.10, we get

$$\begin{aligned} g'(x) &= (x^2+1)^{-1} \;=\; -\Im(x+i)^{-1}, \\ \Sigma g'(x) &= \Im\psi(1+i) - \Im\psi(x+i), \\ (\Sigma g)'(x) &= \tfrac{\pi}{2} - \Im\psi(x+i), \end{aligned}$$

and

$$\begin{aligned} \Sigma g(x) &= \frac{\pi}{2}(x-1) + \Im\ln\Gamma(1+i) - \Im\ln\Gamma(x+i) \\ &= c + \frac{\pi}{2}x + \frac{i}{2}\ln\frac{\Gamma(x+i)}{\Gamma(x-i)} \end{aligned}$$

for some $c \in \mathbb{R}$. Using Theorem 6.5, we also obtain the inequality

$$\begin{aligned} &\left|\Sigma g(x) - \left(x - \frac{1}{2}\right)\arctan(x) + \frac{1}{2}\ln(x^2+1) - 1 + \frac{\pi}{4} - \Im\ln\Gamma(1+i)\right| \\ &\qquad\qquad \leqslant\; \frac{1}{2}\arctan\frac{1}{x^2+x+1} \end{aligned}$$

and hence the left side approaches zero as $x \to \infty$. $\square$

## 7.3 An alternative uniqueness result

The following theorem provides a uniqueness result for differentiable solutions to the equation $\Delta f = g$. These solutions can be computed from their derivatives using Theorem 7.8.

**Fact 7.13.** *A periodic function* $\omega\colon \mathbb{R}_+ \to \mathbb{R}$ *is constant if and only if it lies in* $\mathcal{K}^0$.

**Theorem 7.14.** *Let* $r \in \mathbb{N}^*$ *and* $g \in \mathcal{C}^r$, *and assume that there exists* $\varphi \in \mathcal{C}^r$ *such that* $\Delta\varphi = g$ *and* $\varphi^{(r)} \in \mathcal{R}^0_{\mathbb{N}}$. *Then, the following assertions hold.*

*(i) For each* $x > 0$, *the series*

$$\sum_{k=0}^{\infty} g^{(r)}(x+k)$$

*converges.*

*(ii) For any* $f \in \mathcal{C}^r \cap \mathcal{K}^{r-1}$ *such that* $\Delta f = g$, *we have* $f = c + \varphi$ *for some* $c \in \mathbb{R}$ *and*

$$f^{(r)}(x) \;=\; -\sum_{k=0}^{\infty} g^{(r)}(x+k)\,.$$

*Proof.* Assertion (i) follows immediately from (12) and we clearly have

$$\varphi^{(r)}(x) \;=\; -\sum_{k=0}^{\infty} g^{(r)}(x+k)\,, \qquad x > 0.$$

Now, let $f \in \mathcal{C}^r \cap \mathcal{K}^{r-1}$ be such that $\Delta f = g$. Negating $f$, $\varphi$, and $g$ if necessary, we can assume that $f \in \mathcal{K}^{r-1}_+$, which implies that $f^{(r)}$ is eventually nonnegative by Lemma 2.2(e). To complete the proof, by Proposition 4.4 and Fact 7.13 it is enough to show that the 1-periodic function $\omega = f - \varphi$ lies in $\mathcal{K}^{r-1}$, i.e., it satisfies $\omega^{(r)} \geqslant 0$ on $\mathbb{R}_+$. Suppose on the contrary that $\omega^{(r)}(z) < 0$ for some $z > 0$. Since $\omega$ is 1-periodic, we have

$$0 \;\leqslant\; f^{(r)}(z+m) \;<\; \varphi^{(r)}(z+m), \qquad \text{for large integer } m.$$

In particular, we have

$$0 \;<\; -\,\omega^{(r)}(z) \;=\; -\,\omega^{(r)}(z+m) \;\leqslant\; \varphi^{(r)}(z+m)$$

for large integer $m$, which contradicts the assumption that $\varphi^{(r)} \in \mathcal{R}^0_{\mathbb{N}}$. This proves assertion (ii). ☐

**Example 7.15.** The assumptions of Theorem 7.14 hold if $g(x) = \ln x$, $\varphi(x) = \ln\Gamma(x)$, and $r = 2$. It follows from Theorem 7.14 that all solutions to the equation $\Delta f = g$ that lie in $\mathcal{C}^2 \cap \mathcal{K}^1$ are of the form $f(x) = c + \ln\Gamma(x)$, where $c \in \mathbb{R}$. We thus retrieve the Bohr-Mollerup-Artin Theorem with the additional assumption that $f$ lies in $\mathcal{C}^2$. ☐

# 8 Further results

Keeping in mind the objective of generalizing Webster's formulas to multiple $\log\Gamma$-type functions, we now explore further questions related to our main results. In particular, we provide for multiple $\log\Gamma$-type functions analogues of *Euler's infinite product*, *Weierstrass' infinite product*, *Raabe's formula*, *Gauss' multiplication formula*, and *Wallis's product formula*.

## 8.1 Series representation and Eulerian form

Let $g \in \mathcal{D}^p \cap \mathcal{K}^p$ for some $p \in \mathbb{N}$. As we already observed in the Introduction, the representation of $\Sigma g$ as the pointwise limit of the sequence $n \mapsto f_n^p[g]$ is the analogue of Gauss' limit for the gamma function. Using identity (18), we can see that this form of $\Sigma g$ can be easily translated into a series, namely

$$\Sigma g(x) \;=\; f_1^p[g](x) - \sum_{k=1}^{\infty} \rho_k^{p+1}[g](x), \qquad x > 0.$$

We also observe that, when $g(x) = \ln x$ and $p = 1$, the multiplicative version of this series representation reduces to the classical Euler product form of the gamma function (see, e.g., [76, p. 3]), as given in Example 8.2 below. Thus, for any multiple $\log\Gamma$-type function, the series representation above is the analogue of the Eulerian form of the gamma function. Rewriting this identity explicitly and using the uniform convergence of the sequence $n \mapsto f_n^p[g]$ (cf. Theorem 3.4), we immediately obtain the following result.

**Theorem 8.1** (Eulerian form). *Let $g \in \mathcal{D}^p \cap \mathcal{K}^p$ for some $p \in \mathbb{N}$. Then*

$$\Sigma g(x) \;=\; -\,g(x) + \sum_{j=1}^{p} \binom{x}{j} \Delta^{j-1} g(1) - \sum_{k=1}^{\infty} \left( g(x+k) - \sum_{j=0}^{p} \binom{x}{j} \Delta^{j} g(k) \right)$$

*and the series converges uniformly on any bounded subset of $\mathbb{R}_+$.*

Recall also that the uniform convergence enables one to integrate the series above term by term on any bounded interval (see Proposition 5.8). We can also differentiate the series term by term as shown in Theorem 7.2.

**Example 8.2.** Considering the function $g(x) = \ln x$ for which $p = 1 + \deg g = 1$, we obtain the following infinite product representations for any $x > 0$:

$$\Gamma(x) \;=\; \frac{1}{x} \prod_{k=1}^{\infty} \frac{(1+1/k)^x}{1+x/k}, \qquad e^{\psi(x)} \;=\; e^{-\frac{1}{x}} \prod_{k=1}^{\infty} (1+1/k)\, e^{-\frac{1}{x+k}},$$

and

$$e^{\int_0^x \ln\Gamma(t)\,dt} \;=\; \frac{e^x}{x^x} \prod_{k=1}^{\infty} \frac{e^x (1+1/k)^{x^2/2}}{(1+x/k)^{x+k}}. \quad \square$$

## 8.2 Weierstrassian form

We now show that the classical Weierstrass factorization of the gamma function (see Example 8.6 below) can be generalized to any $\log\Gamma_p$-type function that is of class $\mathcal{C}^p$. The following two theorems deal separately with the cases $p = 0$ and $p \geqslant 1$. Note that the case $p = 1$ was previously established by John [42, Theorem B'] and in the multiplicative notation by Webster [80, Theorem 7.1].

**Theorem 8.3** (Weierstrassian form). *For any $g \in \mathcal{C}^0 \cap \mathcal{D}^0 \cap \mathcal{K}^0$, we have $\gamma[g] = \sigma[g]$ and*

$$\Sigma g(x) \;=\; \sigma[g] - g(x) - \sum_{k=1}^{\infty}\left(g(x+k) - \int_k^{k+1} g(t)\,dt\right)$$

*and the series converges uniformly on any bounded subset of $\mathbb{R}_+$.*

*Proof.* The result immediately follows from Proposition 6.23, Eq. (48), and Theorem 8.1. ☐

**Lemma 8.4.** *Let $g \in \mathcal{C}^1 \cap \mathcal{D}^p \cap \mathcal{K}^p$ for some $p \in \mathbb{N}^*$. Then*

$$\Delta g(x) - \sum_{j=0}^{p-2} G_j \Delta^j g'(x) \;\to\; 0 \qquad \text{as } x \to \infty.$$

*If, in addition, $g \in \mathcal{C}^{p-1}$, then $\Delta^{p-1} g(x) - g^{(p-1)}(x) \to 0$ as $x \to \infty$.*

*Proof.* The first convergence result follows immediately from (41). Let us now assume that $g \in \mathcal{C}^{p-1}$. For every $i \in \{0, \ldots, p-2\}$, the function $g_i = \Delta^i g^{(p-2-i)}$ lies in $\mathcal{C}^1 \cap \mathcal{D}^2 \cap \mathcal{K}^2$ and hence, using the first result, we see that $\Delta g_i(x) - g_i'(x) \to 0$ as $x \to \infty$. Summing these limits for $i = 0, \ldots, p-2$, we obtain the claimed limit. ☐

**Theorem 8.5** (Weierstrassian form). *Let $g \in \mathcal{C}^p \cap \mathcal{D}^p \cap \mathcal{K}^p$ with $\deg g = p - 1$ for some $p \in \mathbb{N}^*$. Then we have*

$$\gamma[g^{(p)}] \;=\; \sigma[g^{(p)}] \;=\; g^{(p-1)}(1) - (\Sigma g)^{(p)}(1)$$

*and*

$$\begin{aligned}\Sigma g(x) \;=\; & \sum_{j=1}^{p-1} \tbinom{x}{j}\, \Delta^{j-1} g(1) + \tbinom{x}{p} (\Sigma g)^{(p)}(1) \\ & -g(x) - \sum_{k=1}^{\infty}\left(g(x+k) - \sum_{j=0}^{p-1} \tbinom{x}{j}\, \Delta^j g(k) - \tbinom{x}{p} g^{(p)}(k)\right)\end{aligned}$$

*and the series converges uniformly on any bounded subset of $\mathbb{R}_+$.*

*Proof.* The identities involving the constants follow from Propositions 4.8, 6.23, and 7.4. Now, using (48) we get

$$\gamma[g^{(p)}] \;=\; \sum_{k=1}^{\infty}(g^{(p)}(k)-\Delta g^{(p-1)}(k)).$$

Using Theorem 8.1, we then obtain

$$\begin{aligned}\Sigma g(x) \;=\;& \sum_{j=1}^{p-1}\tbinom{x}{j}\,\Delta^{j-1}g(1)+\tbinom{x}{p}\left(g^{(p-1)}(1)-\gamma[g^{(p)}]\right)\\ &-g(x)-\lim_{n\to\infty}\sum_{k=1}^{n-1}\left(g(x+k)-\sum_{j=0}^{p-1}\tbinom{x}{j}\,\Delta^{j}g(k)-\tbinom{x}{p}\,g^{(p)}(k)\right)\\ &+\lim_{n\to\infty}\tbinom{x}{p}\left(\Delta^{p-1}g(n)-g^{(p-1)}(n)\right),\end{aligned}$$

where the latter limit is zero by Lemma 8.4. Also, the uniform convergence is ensured by Theorem 8.1. ☐

It is important to note that, just as the series given in Theorem 8.1, the series given in Theorems 8.3 and 8.5 also represent the limit of the sequence $n\mapsto f^p_n[g](x)$. Thus, by Theorem 7.2, those series can be integrated and differentiated term by term.

**Example 8.6.** Considering the function $g(x)=\ln x$ for which $p=1+\deg g=1$, we retrieve the following Weierstrassian form of the gamma function in an effortless way

$$\Gamma(x) \;=\; \frac{e^{-\gamma x}}{x}\prod_{k=1}^{\infty}\frac{e^{\frac{x}{k}}}{1+\frac{x}{k}}\,,\qquad x>0.\quad \square$$

*Remark* 8.7. Under the assumptions of Lemma 8.4, by Propositions 4.5 and 4.8 we have $g'\in\mathcal{R}^{p-1}_{\mathbb{R}}$, i.e., for any $a\geqslant 0$

$$g'(x+a)-\sum_{j=0}^{p-2}\tbinom{a}{j}\Delta^j g'(x) \;\to\; 0\qquad \text{as } x\to\infty.$$

Combining this with Lemma 8.4, we can derive surprising limits. For instance, if $p\in\{1,2,3\}$, then $\Delta g(x)-g'(x+\frac{1}{2})\to 0$ as $x\to\infty$.

## 8.3 Raabe's formula

Recall that Raabe's formula yields, for any $x>0$, a simple explicit expression for the integral of the log-gamma function over the interval $(x,x+1)$. That is,

$$\int_x^{x+1}\ln\Gamma(t)\,dt \;=\; \frac{1}{2}\,\ln(2\pi)+x\ln x-x\,,\qquad x>0. \tag{64}$$

In particular, setting $x=1$, we obtain the identity

$$\int_1^2 \ln\Gamma(t)\,dt ~=~ -1+\frac{1}{2}\,\ln(2\pi)\,,$$

which is precisely the value of $\sigma[g]$ when $g(x)=\ln x$. For recent references on Raabe's formula, see, e.g., [27] and see [76, p. 29].

Clearly, identities (36) and (37) enable us to define for any continuous multiple $\log\Gamma$-type function $\Sigma g$ the analogue of Raabe's formula. Thus defined, this new formula can be obtained simply by computing the value $\sigma[g]$, or even the value $\overline{\sigma}[g]$ (see Definition 6.14) when $g$ is integrable at $0$.

In general, the value of $\sigma[g]$ can be computed using Proposition 5.8(c2). Specifically, if $g\in\mathcal{C}^0\cap\mathcal{D}^p\cap\mathcal{K}^p$ for some $p\in\mathbb{N}$, we have

$$\begin{aligned}\sigma[g] ~&=~ \lim_{n\to\infty}\int_0^1 (f_n^p[g](t)+g(t))\,dt\\ &=~ \lim_{n\to\infty}\left(\sum_{k=1}^{n-1}g(k)-\int_1^n g(t)\,dt+\sum_{j=1}^{p}G_j\Delta^{j-1}g(n)\right).\end{aligned}\tag{65}$$

which is nothing other than the restriction of the generalized Stirling formula (40) to the natural integers. Equivalently, this value can be obtained by integrating on the interval $(0,1)$ the series representation of $\Sigma g+g$ given in Theorem 8.1. That is,

$$\sigma[g] ~=~ \sum_{j=1}^{p}G_j\,\Delta^{j-1}g(1)-\sum_{k=1}^{\infty}\left(\int_k^{k+1}g(t)\,dt-\sum_{j=0}^{p}G_j\,\Delta^j g(k)\right).\tag{66}$$

Note also that, under certain assumptions, the series above converges to zero as $p\to_{\mathbb{N}}\infty$; see Proposition 6.8.

**Example 8.8.** If $g(x)=\frac{1}{x}$, we obtain

$$\sigma[g] ~=~ \sum_{k=1}^{\infty}\left(\frac{1}{k}-\ln\left(1+\frac{1}{k}\right)\right),$$

which is the Euler constant $\gamma$. Identity (37) then immediately provides the following analogue of Raabe's formula

$$\int_x^{x+1}\psi(t)\,dt ~=~ \ln x\,,\qquad x>0.\quad\square$$

## 8.4 Gauss' multiplication formula

Webster [80, Theorem 5.2] showed how an analogue of Gauss' multiplication formula can be constructed for any $\Gamma$-type function. His proof is very easy and essentially uses the uniqueness theorem. We now show that this formula can be further extended to multiple Γ-type functions. As usual, we use the additive notation.

**Theorem 8.9** (Gauss' multiplication formula)**.** *Let* $p \in \mathbb{N}$*,* $m \in \mathbb{N}^*$*, and* $g \in \mathcal{D}^p \cap \mathcal{K}^p$*. Define also the functions* $g_m, h_m \colon \mathbb{R}_+ \to \mathbb{R}$ *by the equations* $g_m(x) = g(\frac{x}{m})$ *and* $h_m(x) = g(x) - g_m(x)$ *for* $x > 0$*. Then* $g_m \in \mathcal{D}^p \cap \mathcal{K}^p$ *and, for* $x > 0$*,*

$$\sum_{j=0}^{m-1} (\Sigma g)\left(\frac{x+j}{m}\right) ~=~ \sum_{j=1}^{m-1} (\Sigma g)\left(\frac{j}{m}\right) + \Sigma g_m(x). \tag{67}$$

*If* $h_m \in \mathcal{D}^p \cap \mathcal{K}^p$*, then for* $x > 0$*,*

$$\sum_{j=0}^{m-1} (\Sigma g)\left(\frac{x+j}{m}\right) + \Sigma h_m(x) ~=~ \sum_{j=1}^{m-1} (\Sigma g)\left(\frac{j}{m}\right) + \Sigma g(x).$$

*Proof.* We clearly have $g_m \in \mathcal{K}^p$. Also, it is easy to see that $g_m \in \mathcal{D}^p \cap \mathcal{K}^p$ (we can use Proposition 4.15 for instance). Now, we can readily check that the function $f \colon \mathbb{R}_+ \to \mathbb{R}$ defined by

$$f(x) ~=~ \sum_{j=0}^{m-1} (\Sigma g)\left(\frac{x+j}{m}\right) - \sum_{j=1}^{m-1} (\Sigma g)\left(\frac{j}{m}\right)$$

is a solution to the equation $\Delta f = g_m$ that lies in $\mathcal{K}^p$ and such that $f(1) = 0$. By the uniqueness Theorem 3.1, it follows that $f = \Sigma g_m$. Now, if $h_m \in \mathcal{D}^p \cap \mathcal{K}^p$, then we also have $f = \Sigma g - \Sigma h_m$. ☐

**Corollary 8.10.** *Let* $g \in \mathcal{C}^0 \cap \mathcal{D}^p \cap \mathcal{K}^p$ *for some* $p \in \mathbb{N}$*. Define also the functions* $g_m \colon \mathbb{R}_+ \to \mathbb{R}$ $(m \in \mathbb{N}^*)$ *by the equation* $g_m(x) = g(\frac{x}{m})$*. Then we have*

$$\lim_{m\to\infty} \frac{(\Sigma g_m)(mx) - (\Sigma g_m)(m)}{m} ~=~ \int_1^x g(t)\, dt\,, \qquad x > 0.$$

*Moreover, if* $g$ *is integrable at* $0$*, then*

$$\lim_{m\to\infty} \frac{1}{m}\, (\Sigma g_m)(mx) ~=~ \int_0^x g(t)\, dt\,, \qquad x > 0.$$

*Proof.* Replacing $x$ by $mx$ in (67) and dividing through by $m$, we obtain two Riemann sums that converge, letting $m \to_{\mathbb{N}} \infty$, to the integrals of $\Sigma g(x+t)$ and $\Sigma g(t)$ over $t \in (0,1)$. Combining the resulting equation with (37) gives the result. ☐

To use Theorem 8.9 to its full capacity, a closed-form expression for the right-hand sum of identity (67) would be welcome. The following proposition brings a partial answer to this natural question. Recall first that $B_k$ denotes the kth Bernoulli number (see Subsection 7.2). Also, for $n, r \in \mathbb{N}$, $b_n^r(x)$ stands for the function $D_x^r\binom{x}{n+r}$. Finally, $\Delta_{[h]}$ denotes the forward difference operator with step $h$.

**Lemma 8.11.** *For any* $m, q \in \mathbb{N}^*$, *we have*

$$\begin{aligned}\sum_{i=1}^{m}\binom{i/m}{q} &= m\,G_q + \binom{1}{q} + \sum_{i=1}^{q+1}\frac{B_i}{i!\,m^{i-1}}\left(b_{q-i+1}^{i-1}(1) - b_{q-i+1}^{i-1}(0)\right)\\ &= m\,G_q + \binom{1}{q} - \sum_{i=1}^{q+1} G_i\left((\Delta_{[\frac{1}{m}]}^{i-1} b_q^0)(1) - (\Delta_{[\frac{1}{m}]}^{i-1} b_q^0)(0)\right).\end{aligned}$$

*Also, for* $i = 1, \ldots, q+1$, *we have*

$$b_{q-i+1}^{i-1}(1) - b_{q-i+1}^{i-1}(0) = \frac{1}{q!}\sum_{k=i}^{q}(-1)^{k-q}\begin{bmatrix} q\\ k\end{bmatrix}\frac{k!}{(k-i+1)!},$$

*where* $\begin{bmatrix} q\\ k\end{bmatrix}$ *is the number of ways to arrange* $q$ *objects into* $k$ *cycles (Stirling number of the first kind).*

*Proof.* The first formula results from a straightforward application of the Euler-Maclaurin formula (Proposition 6.21) with $a = 0$, $b = 1$, and $N = m$. We prove the second formula similarly using the general form of Gregory's formula (Proposition 6.20) with $a = 0$, $n = m$, and $h = \frac{1}{m}$. The last part follows from the classical linear decomposition of binomial coefficients into ordinary powers (see, e.g., [34, p. 263]). $\square$

**Proposition 8.12.** *Let* $p \in \mathbb{N}$, $m \in \mathbb{N}^*$, $g \in \mathcal{C}^0 \cap \mathcal{D}^p \cap \mathcal{K}^p$, *and set*

$$c_{m,j} = \sum_{i=1}^{m}\binom{i/m}{j}, \qquad \textit{for } j = 1, \ldots, p.$$

*Then*

$$\begin{aligned}\sum_{j=1}^{m}(\Sigma g)\left(\frac{j}{m}\right) &= m\,\sigma[g] - \sigma[g_m]\\ &\quad + \lim_{n\to\infty}\left(m\int_1^n g(t)\,dt - \int_1^{mn+1} g_m(t)\,dt\right.\\ &\quad \left. + \sum_{j=1}^{p}\left((c_{m,j} - m\,G_j)\Delta^{j-1}g(n) + G_j\,\Delta^{j-1}g_m(mn+1)\right)\right),\end{aligned}$$

*where the numbers* $c_{m,j}$ *can be computed using Lemma 8.11 and the function* $g_m\colon \mathbb{R}_+ \to \mathbb{R}$ *is defined by the equation* $g_m(x) = g(\frac{x}{m})$.

*Proof.* We have

$$\sum_{j=1}^{m}(\Sigma g)\left(\frac{j}{m}\right) = \lim_{n\to\infty}\sum_{j=1}^{m} f_n^p[g]\left(\frac{j}{m}\right) = -\sum_{j=1}^{m} g\left(\frac{j}{m}\right)$$
$$+\lim_{n\to\infty}\left(\sum_{k=1}^{n-1}\left(m\,g(k)-\sum_{j=1}^{m} g\left(\frac{j}{m}+k\right)\right)+\sum_{j=1}^{p} c_{m,j}\,\Delta^{j-1}g(n)\right).$$

Also,

$$\sum_{k=1}^{n-1}\sum_{j=1}^{m} g\left(\frac{j}{m}+k\right) = \sum_{k=1}^{n-1}\Delta_k\sum_{j=1}^{km} g\left(\frac{j}{m}\right) = \sum_{j=1}^{mn} g\left(\frac{j}{m}\right)-\sum_{j=1}^{m} g\left(\frac{j}{m}\right).$$

Thus, we have

$$\sum_{j=1}^{m}(\Sigma g)\left(\frac{j}{m}\right) = \lim_{n\to\infty}\left(m\sum_{k=1}^{n-1} g(k)-\sum_{k=1}^{mn} g\left(\frac{k}{m}\right)+\sum_{j=1}^{p} c_{m,j}\,\Delta^{j-1}g(n)\right).$$

The claimed formula then follows from identity (65). □

**Example 8.13.** Let us apply Theorem 8.9 to the function $g(x) = \ln x$. We have $g_m(x) = \ln x - \ln m$ and $\Sigma g_m(x) = \ln\Gamma(x) - (x-1)\ln m$. Hence, we retrieve the following *Gauss multiplication formula*

$$\prod_{j=0}^{m-1}\Gamma\left(\frac{x+j}{m}\right) = \frac{\Gamma(x)}{m^{x-1}}\prod_{j=1}^{m-1}\Gamma\left(\frac{j}{m}\right), \qquad x>0,$$

and it can be proved using Proposition 8.12 that the right-hand product is

$$m^{-\frac{1}{2}}(2\pi)^{\frac{m-1}{2}}.$$

When $m = 2$, this identity reduces to *Legendre's duplication formula*

$$\Gamma\left(\frac{x}{2}\right)\Gamma\left(\frac{x+1}{2}\right) = \frac{\Gamma(x)}{2^{x-1}}\sqrt{\pi}, \qquad x>0. \quad \square$$

The following result provides an asymptotic expansion of the left-hand sum of identity (67). This expansion can be used for instance to estimate the integral (37) (and hence also the asymptotic constant $\sigma[g]$), e.g, using Richardson's extrapolation method. As a byproduct, this result also provides an asymptotic expansion of $\Sigma g$ (or even of the difference between $\Sigma g$ and its trend) in terms of the higher derivatives of $g$. We omit the proof for it is a straightforward application of Euler-Maclaurin's formula (Proposition 6.21) with $a = 0$, $b = 1$, and $N = m$.

**Proposition 8.14.** *Let* $g \in \mathcal{C}^{2q} \cap \mathcal{D}^p \cap \mathcal{K}^p$ *for some* $p \in \mathbb{N}$ *and some* $q \in \mathbb{N}^*$. *Then, for any* $m \in \mathbb{N}^*$ *and any* $x > 0$, *we have*

$$\begin{aligned}\frac{1}{m}\sum_{j=0}^{m-1}(\Sigma g)\left(x+\frac{j}{m}\right) &= \int_x^{x+1}\Sigma g(t)\,dt \\ &\quad -\frac{1}{2m}\,g(x)+\sum_{k=1}^{q}\frac{1}{m^{2k}}\,\frac{B_{2k}}{(2k)!}\,D^{2k-1}g(x)+R_{m,q}(x)\,,\end{aligned}$$

*with*

$$|R_{m,q}(x)| \;\leqslant\; \frac{1}{m^{2q}}\,\frac{|B_{2q}|}{(2q)!}\int_x^{x+1}|D^{2q}\Sigma g(t)|\,dt.$$

*In particular,*

$$\Sigma g(x) \;=\; \sigma[g]+\int_1^x g(t)\,dt-\frac{1}{2}\,g(x)+\sum_{k=1}^{q}\frac{B_{2k}}{(2k)!}\,D^{2k-1}g(x)+R_{1,q}(x).$$

**Example 8.15.** Taking $g(x) = \ln x$ in the second part of Proposition 8.14, for any $q \in \mathbb{N}^*$ we obtain the following asymptotic expansion as $x \to \infty$ (see, e.g., [76, p. 7])

$$\ln\Gamma(x) \;=\; \frac{1}{2}\ln(2\pi)-x+\left(x-\frac{1}{2}\right)\ln x+\sum_{k=1}^{q}\frac{(-1)^{k-1}\,B_{k+1}}{k(k+1)\,x^k}+O\left(\frac{1}{x^{q+1}}\right).$$

*Remark* 8.16. A similar asymptotic expansion of the left-hand sum in (67) can be obtained using the general form of GregoryâĂŹs formula (see Proposition 6.20). Setting $m = 1$ in this expansion, we then retrieve the Gregory formula-based series expression of $\Sigma g$ given in Proposition 6.8.

## 8.5 Wallis's product formula

One of the different versions of Wallis's formula is given by the following limit (see, e.g., [32, p. 21])

$$\lim_{n\to\infty}\frac{1\cdot 3\,\cdots\,(2n-1)}{2\cdot 4\,\cdots\,(2n)}\,\sqrt{n} \;=\; \frac{1}{\sqrt{\pi}}\,. \tag{68}$$

The following proposition gives an analogue of this formula in the additive notation for any function $g$ lying in $\cup_{p\geqslant 0}(\mathcal{C}^0 \cap \mathcal{D}^p \cap \mathcal{K}^p)$.

**Proposition 8.17.** *Let* $g \in \mathcal{C}^0 \cap \mathcal{D}^p \cap \mathcal{K}^p$ *for some* $p \in \mathbb{N}$. *Let* $\tilde{g}\colon \mathbb{R}_+ \to \mathbb{R}$ *be the function defined by the equation* $\tilde{g}(x) = 2\,g(2x)$. *Let also* $h\colon \mathbb{N}^* \to \mathbb{R}$ *be the sequence defined by the equation*

$$\begin{aligned}h(n) &= \sigma[\tilde{g}]-\sigma[g]+\int_1^2(g(2n+t)-g(t))\,dt \\ &\quad +\sum_{j=1}^{p}G_j\left(\Delta^{j-1}g(2n+1)-\Delta^{j-1}\tilde{g}(n+1)\right).\end{aligned}$$

*Then we have*

$$\lim_{n\to\infty}\left(h(n)+\sum_{k=1}^{2n}(-1)^{k-1}g(k)\right) \;=\; 0. \tag{69}$$

*Proof.* It is clear that the function $\tilde{g}$ also lies in $\mathcal{C}^0\cap\mathcal{D}^p\cap\mathcal{K}^p$. We then have

$$\sum_{k=1}^{2n}(-1)^{k-1}g(k) \;=\; \sum_{k=1}^{2n}g(k)-\sum_{k=1}^{n}\tilde{g}(k) \;=\; (\Sigma g)(2n+1)-(\Sigma\tilde{g})(n+1).$$

Using (65), we then obtain the claimed formula. $\square$

Formula (69) actually holds for infinitely many sequences $n\mapsto h(n)$. Indeed, if it holds for a sequence $h(n)$, then it also holds for the sequence $h(n)+n^{-q}$ for any $q\in\mathbb{N}^*$. Thus, to obtain an elegant analogue of Wallis's formula, it is advisable to choose $h$ among the simplest functions. For instance, we could consider the sequence obtained from the series expansion for $h(n)$ about infinity after removing all the summands that vanish at infinity.

**Example 8.18.** If $g(x)=\ln x$, then we have

$$\begin{aligned} h(n) \;&=\; 2n\ln\frac{2n+2}{2n+1}-\frac{1}{2}\ln(2n+1)+\ln(n+1)-1+\frac{1}{2}\ln(2\pi)\\ &=\; \frac{1}{2}\ln(\pi n)+O\left(\frac{1}{n^2}\right). \end{aligned}$$

Replacing $h(n)$ with $\frac{1}{2}\ln(\pi n)$ in (69) as recommended, we retrieve Wallis's formula. If $g(x)=H_x$ is the harmonic number function, then we have

$$\begin{aligned} h(n) \;&=\; \frac{1}{2}H_{2n+1}+\frac{1}{2}\ln 2+\ln(n+1)-\psi(2n+3)\\ &=\; \frac{1}{2}(\gamma+\ln n)+O\left(\frac{1}{n}\right). \end{aligned}$$

We then obtain the following analogue of Wallis's formula

$$\lim_{n\to\infty}\left(-\ln n+2\sum_{k=1}^{2n}(-1)^k H_k\right) \;=\; \gamma\,,$$

which provides an alternative definition of Euler's constant $\gamma$. To give an additional example, if $g(x)=H_x^{(2)}=\zeta(2)-\zeta(2,x+1)$ is the harmonic number function of order 2, we obtain the following analogue of Wallis's formula

$$\lim_{n\to\infty}\sum_{k=1}^{2n}(-1)^k H_k^{(2)} \;=\; \frac{\pi^2}{24}\,. \quad \square$$

## 8.6 Euler's reflection formula

Recall that the identity $\Gamma(z)\Gamma(1-z) = \pi\csc(\pi x)$ holds for any $z \in \mathbb{C} \setminus \mathbb{Z}$. This identity, known as Euler's reflection formula (see, e.g., [76, p. 3]), can be proved for instance by using the Weierstrassian form of the gamma function.

Motivated by this and similar examples, it is then natural to wonder if an analogue of Euler's reflection formula holds for any multiple $\log\Gamma$-type function, at least on the interval $(0,1)$. Unfortunately, we do not have any answer to this interesting question. Thus, results along this line would be most welcome.

Actually, reflection formulas may take various forms. For instance, for the digamma function $\psi$ we have

$$\psi(x) - \psi(1-x) \;=\; -\pi\cot(\pi x)\,, \tag{70}$$

while for the Barnes G-function, we have

$$\ln G(1+x) - \ln G(1-x) \;=\; x\ln(2\pi) - \int_0^x \pi t\cot(\pi t)\,dt\,. \tag{71}$$

We also observe that the right sides of some reflection formulas are 1-periodic.

Now, given a function $g$ in $\cup_{p\geqslant 0}(\mathcal{D}^p \cap \mathcal{K}^p)$, the discussion above suggests searching for an expression for either

$$\Sigma g(x) \pm \Sigma g(1-x) \quad \text{or} \quad \Sigma g(1+x) \pm \Sigma g(1-x)$$

on the interval $(0,1)$ by means of the Eulerian form or the Weierstrassian form of $\Sigma g$. If the resulting expression is rather simple, then we have found a reflection formula for $\Sigma g$ on $(0,1)$ and we may try to find an extension of this formula to a more general domain by analytic continuation. For instance, using the Eulerian form of the digamma function

$$\psi(x) \;=\; -\gamma - \frac{1}{x} + \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{x+k}\right),$$

we obtain

$$\psi(x) - \psi(1-x) \;=\; -\frac{1}{x} + \frac{1}{1-x} + \sum_{k=1}^{\infty}\left(-\frac{1}{x+k} + \frac{1}{1-x+k}\right)$$

and we can show (see, e.g., [15, p. 4] and [34, Eq. (6.88)]) that the latter expression reduces to the 1-periodic function $-\pi\cot(\pi x)$, thus retrieving the reflection formula (70) on the interval $(0,1)$, which can then be extended to the domain $\mathbb{C} \setminus \mathbb{Z}$.

Regarding reflection formulas involving 1-periodic functions, we can make the following interesting observation. Let $f, g\colon \mathbb{C} \setminus \mathbb{Z} \to \mathbb{C}$ be two complex functions and suppose that $\Delta f = g$ on $\mathbb{C} \setminus \mathbb{Z}$. Define the functions $h_+, h_-\colon \mathbb{C} \setminus \mathbb{Z} \to \mathbb{C}$ by $h_\pm(z) = f(z) \pm f(1-z)$. Then we have $\Delta_z h_\pm(z) = g(z) \mp g(-z)$ and hence the function $h_+$ (resp. $h_-$) is 1-periodic if and only if $g$ is even (resp. odd).

## 8.7 Gauss' digamma theorem

The following formula, due to Gauss, enables one to compute the values of the digamma function $\psi$ for rational arguments. If $a, b \in \mathbb{N}^*$ with $a < b$, then we have

$$\psi\left(\frac{a}{b}\right) = -\gamma - \ln(2b) - \frac{\pi}{2}\cot\frac{a\pi}{b} + 2\sum_{j=1}^{\lfloor (b-1)/2 \rfloor} \cos\left(2j\pi\frac{a}{b}\right)\ln\left(\sin\frac{j\pi}{b}\right) \quad (72)$$

(see, e.g., [45, p. 95] and [76, p. 30]). This formula can be extended to all integers $a, b \in \mathbb{N}^*$ by means of the difference equation $\psi(x+1) - \psi(x) = \frac{1}{x}$.

For instance, we have

$$\psi\left(\frac{3}{4}\right) = -\gamma + \frac{\pi}{2} - 3\ln 2.$$

It is natural to wonder if an analogue of formula (72) holds for any multiple $\log\Gamma$-type function. Finding an analogue as beautiful as this formula seems to be hard. However, we have the following partial result.

**Proposition 8.19.** *Let* $g \in \mathcal{D}^0 \cap \mathcal{K}^0$ *and let* $a, b \in \mathbb{N}^*$ *with* $a < b$. *Then*

$$\Sigma g\left(\frac{a}{b}\right) = \frac{1}{b}\sum_{j=0}^{b-1}\left(1 - \omega_b^{-aj}\right) S_j^b[g],$$

*where*

$$\omega_b = e^{\frac{2\pi i}{b}} \quad \text{and} \quad S_j^b[g] = \sum_{k=1}^{\infty} \omega_b^{jk}\, g\left(\frac{k}{b}\right).$$

*Proof.* By definition of the map $\Sigma$, we have

$$\begin{aligned}\Sigma g\left(\frac{a}{b}\right) &= \lim_{n\to\infty}\left(\sum_{k=1}^{n-1} g\left(\frac{bk}{b}\right) - \sum_{k=0}^{n-1} g\left(\frac{bk+a}{b}\right)\right)\\ &= \lim_{n\to\infty}\sum_{k=1}^{bn-1}\left(u_b(k) - u_b(k-a)\right) g\left(\frac{k}{b}\right),\end{aligned}$$

where $u_b(k) = 1$, if $b$ divides $k$, and $u_b(k) = 0$, otherwise; that is,

$$u_b(k) = \frac{1}{b}\sum_{j=0}^{b-1}\omega_b^{jk}.$$

This completes the proof. ∎

Proposition 8.19 provides a first step in the search for an explicit expression for $\Sigma g(\frac{a}{b})$. Depending upon the function $g$, more computations may be necessary to obtain a useful expression. In this respect, the derivation of formula (72) by means of Proposition 8.19 can be found in [57, p. 13].

**Example 8.20.** Let us apply Proposition 8.19 to the function $g_s(x) = -x^{-s}$, where $s > 1$. This function lies in $\mathcal{D}^0 \cap \mathcal{K}^0$ and we have $\Sigma g_s(x) = \zeta(s,x) - \zeta(s)$; see Example 1.3. Let $a, b \in \mathbb{N}^*$ with $a < b$. For $j = 0, \ldots, b-1$, we then have

$$S_j^b[g_s] \;=\; -\, b^s \operatorname{Li}_s(\omega_b^j),$$

where $\operatorname{Li}_s(z)$ is the polylogarithm function. Using Proposition 8.19, we then obtain

$$\begin{aligned}
\zeta\left(s, \frac{a}{b}\right) &= \zeta(s) - b^{s-1} \sum_{j=0}^{b-1} \left(1 - \omega_b^{-aj}\right) \operatorname{Li}_s(\omega_b^j) \\
&= b^{s-1} \sum_{j=0}^{b-1} \omega_b^{-aj} \operatorname{Li}_s(\omega_b^j).
\end{aligned}$$

The inverse conversion formula is simply given by

$$\operatorname{Li}_s(\omega_b^j) \;=\; b^{-s} \sum_{k=1}^{b} \omega_b^{jk}\, \zeta\left(s, \frac{k}{b}\right), \qquad j = 1, \ldots, b-1. \quad \square$$

## 8.8 Webster's functional equation

In the framework of Γ-type functions, Webster [80, Section 8] investigated the multiplicative version of the functional equation

$$f(x) + f(x + \tfrac{1}{2}) \;=\; h(x), \qquad x > 0,$$

and, more generally, of the functional equation

$$\sum_{j=0}^{m-1} f\left(x + \frac{j}{m}\right) \;=\; h(x), \qquad x > 0, \tag{73}$$

for any $m \in \mathbb{N}^*$, where $h$ is a given function satisfying certain conditions. On this subject, we present the following result, a variant of which was established by Webster [80, Theorem 8.1] in the case when $p = 1$.

**Theorem 8.21** (Webster's functional equation)**.** *Let* $p \in \mathbb{N}$*,* $m \in \mathbb{N}^*$*, and* $h \in \cup_{q \geqslant 0}(\mathcal{D}^q \cap \mathcal{K}^q)$ *be such that* $\Delta h \in \mathcal{D}^p \cap \mathcal{K}^p_+$ *(resp.* $\Delta h \in \mathcal{D}^p \cap \mathcal{K}^p_-$*). Then there is a unique solution to equation* (73) *lying in* $\mathcal{K}^p$*, namely*

$$f(x) \;=\; (\Sigma h)\left(x + \frac{1}{m}\right) - (\Sigma h)(x).$$

*Moreover, this solution lies in* $\mathcal{K}^p_-$ *(resp.* $\mathcal{K}^p_+$*).*

*Proof.* Let $g\colon \mathbb{R}_+ \to \mathbb{R}$ and $g_m\colon \mathbb{R}_+ \to \mathbb{R}$ be defined by the equations

$$g(x) \;=\; h\left(x+\frac{1}{m}\right)-h(x) \quad\text{and}\quad g_m(x) \;=\; g\left(\frac{x}{m}\right),$$

respectively. It is easy to see that $g_m$ lies in $\mathcal{D}^p \cap \mathcal{K}^p_+$ (resp. $\mathcal{D}^p \cap \mathcal{K}^p_-$) and hence so does $g$. Let $f\colon \mathbb{R}_+ \to \mathbb{R}$ be a solution to equation (73). Then necessarily

$$g(x) \;=\; \sum_{j=0}^{m-1} \Delta_j f\left(x+\frac{j}{m}\right) \;=\; \Delta f(x).$$

If $f$ lies in $\mathcal{K}^p$, then by the uniqueness and existence theorems and Proposition 5.5, there exists $c \in \mathbb{R}$ such that

$$f(x) \;=\; c + (\Sigma h)\left(x+\frac{1}{m}\right) - (\Sigma h)(x). \tag{74}$$

But the function $f$ specified by (74) satisfies (73) if and only if $c = 0$. ☐

Theorem 8.21 can be somewhat generalized by considering the functional equation

$$\sum_{j=0}^{m-1} f(x+aj) \;=\; h(x), \qquad x>0,$$

for some $a > 0$. Indeed, if we define the function $g\colon \mathbb{R}_+ \to \mathbb{R}$ by the equation $g(x) = h(amx + a) - h(amx)$, we see that any solution $f\colon \mathbb{R}_+ \to \mathbb{R}$ to the equation above satisfies $g(x) = \Delta_x f(amx)$. It then remains to add appropriate assumptions on function $h$ to ensure the uniqueness of the solution.

**Example 8.22.** We can show that the unique convex or decreasing solution to the functional equation

$$f(x)f(x+a)\,x^p \;=\; 1, \qquad x>0,\ a>0,\ p>0,$$

is the function

$$f(x) \;=\; \left(\frac{\Gamma(\frac{x}{2a})}{\sqrt{2a}\,\Gamma(\frac{x+a}{2a})}\right)^p.$$

This result was established by Thielman [78] (see also [4]). The special case when $p = 1$ was previously shown by Mayer [59]. ☐

# 9 Application to some special functions

We now apply our results to certain multiple Γ-type functions and multiple log Γ-type functions that are known to be well-studied special functions, namely: the gamma function, the digamma function, the polygamma functions, the Barnes

G-function, the Hurwitz zeta function and its higher derivatives, the generalized Stieltjes constants, and the Catalan number function. We also introduce and investigate the principal indefinite sum of the Hurwitz zeta function. For recent background on some of these functions, see, e.g., Srivastava and Choi [76]. Further examples will be briefly discussed in Section 10.

All these examples illustrate how powerful are some of our results to produce formulas and identities methodically. Although many of these formulas and identities are already known, they had never been derived from such a general and unified setting.

We begin this section with gathering our most relevant and useful results to perform a systematic treatment of these special functions.

## 9.1 A toolbox for multiple $\log\Gamma$-type functions

Let $g \in \mathcal{C}^r \cap \mathcal{D}^p \cap \mathcal{K}^p$ for some $p, r \in \mathbb{N}$. Based on the results of this paper, we can now describe the steps to follow in order to investigate certain properties of the function $\Sigma g$. Note that the function $g$ can also be chosen from a given multiple $\log\Gamma$-type function $F$ by taking $g = \Delta F$.

*ID card.* Given a function $g \in \mathcal{C}^r \cap \mathcal{D}^p \cap \mathcal{K}^p$ for some $p, r \in \mathbb{N}$, we determine the asymptotic degree of $g$ and, whenever possible, a simple expression for $\Sigma g$.

*Characterization.* A characterization of the function $\Sigma g$ as a solution to the difference equation $\Delta f = g$ immediately follows from the uniqueness Theorem 3.1. This characterization states that if $f\colon \mathbb{R}_+ \to \mathbb{R}$ is a solution to the equation $\Delta f = g$, then it lies in $\mathcal{K}^p$ if and only if $f = c + \Sigma g$ for some $c \in \mathbb{R}$.

*Asymptotic constant, generalized Stirling's and Euler's constants, Raabe's formula.* Expressions for $\sigma[g]$, $\overline{\sigma}[g]$, and $\gamma[g]$ can be obtained from Eqs. (36), (65), (66), Definition 6.14, and Propositions 6.23 and 6.25. Recall also that $\gamma[g]$ is subject to inequality (50) and that $\overline{\sigma}[g]$ is defined if and only if $g$ is integrable at 0. An integral form of $\gamma[g]$ is given in Proposition 6.24. Finally, the analogue of Raabe's formula is identity (37).

*Restriction to the natural integers.* The restriction of $\Sigma g$ to $\mathbb{N}^*$ is given in (24). Series representations are given in Propositions 6.8 and 6.18. If $\deg g = 0$, we also have the representation given in (51).

*Derivatives of $\Sigma g(x)$ at $x = 1$.* A formula for the derivatives of $\Sigma g(x)$ at $x = 1$ is given in Proposition 7.4. If $\Sigma g(x)$ is real analytic at $x = 1$, then we can also write the Taylor series expansion of $\Sigma g(x + 1)$ about $x = 0$. Also, the exponential generating function for the sequence $n \mapsto \sigma[g^{(n)}]$ is given in (58).

*Asymptotic analysis.* The asymptotic behavior of $\Sigma g$ is summarized in Theorems 6.2 and 6.5 and Proposition 6.25. The Binet-like function $J^{p+1}[\Sigma g](x)$ is given in (38). As shown in Corollary 7.6, the convergence formulas stated in

Theorems 6.2 and 6.5 can also be differentiated to derive the asymptotic behavior of the derivatives of $\Sigma g$. In particular, the Binet-like function $J^{p+1}[\Sigma g](x)$ and its derivatives vanish at infinity. Further asymptotic results, including the analogue of Burnside's formula, are given in Corollary 6.13 and Proposition 6.26.

*Eulerian and Weierstrassian forms (series and infinite product representations).* The Eulerian and Weierstrassian forms are given in Theorems 8.1, 8.3, and 8.5. These series can be integrated and differentiated term by term. Also, the analogue of Gauss' limit for the gamma function is given by the definition of $\Sigma g$ as the limit of the sequence $n \mapsto f_n^p[g]$.

*Alternative series expression and Fontana-Mascheroni's series.* These series representations are given in Proposition 6.8.

*Alternative representation.* An alternative expression (e.g., an integral representation) for $\Sigma g$ can sometimes be obtained from Theorem 7.8 and Corollary 7.10 by first searching for an appropriate solution to the equation $\Delta\varphi = g^{(r)}$.

*Gauss' multiplication formula.* A general multiplication formula is given in both Theorem 8.9 and its companion Proposition 8.12. It should be noted, however, that this formula leads to an interesting identity only when a rather simple expression for $\Sigma g_m$, where $g_m(x) = g(\frac{x}{m})$, is available. In addition, an asymptotic expansion of $\Sigma g$ is given in Proposition 8.14.

*Wallis's and reflection formulas.* These formulas are discussed in Subsections 8.5 and 8.6.

*Webster's functional equation.* This part is described in Theorem 8.21.

## 9.2 The gamma function

As the gamma function was Webster's motivating example in his introduction of the $\Gamma$-type functions, it is natural to test our results on this function first. Note that Webster also mentioned the q-gamma functions as noteworthy examples of $\Gamma$-type functions. Recall that for any $0 < q < 1$ the q-gamma function $\Gamma_q$ is defined by the equation

$$\ln \Gamma_q(x) \;=\; \Sigma_x \frac{1-q^x}{1-q}\,, \qquad x > 0.$$

The following investigation of the gamma function does not reveal quite new formulas. However, it clearly demonstrates how our results can be used to carry out this investigation in a systematic way.

*ID card.*

| $g(x)$ | Membership | $\deg g$ | $\Sigma g(x)$ |
|---|---|---|---|
| $\ln x$ | $\mathcal{C}^\infty \cap \mathcal{D}^1 \cap \mathcal{K}^\infty$ | $0$ | $\ln \Gamma(x)$ |

*Characterization.* A characterization of the gamma function is given in Bohr-Mollerup-Artin's theorem (see Example 3.2).

*Asymptotic constant, generalized Stirling's and Euler's constants, Raabe's formula.*

| $\exp(\overline{\sigma}[g])$ | $\sigma[g]$ | $\gamma[g]$ |
|---|---|---|
| $\sqrt{2\pi}$ | $-1+\frac{1}{2}\ln(2\pi)$ | $\gamma[g]=\sigma[g]$ |

We have the inequality $|\sigma[g]| \leqslant \frac{1}{2}\ln 2$ and the following representations

$$
\begin{aligned}
\sigma[g] &= \lim_{n\to\infty}\big(\ln n! + n - 1 - \big(n+\tfrac{1}{2}\big)\ln n\big)\,;\\
\sigma[g] &= \textstyle\sum_{k=1}^{\infty}\big(1-\big(k+\tfrac{1}{2}\big)\ln\big(1+\tfrac{1}{k}\big)\big)\,;\\
\sigma[g] &= \textstyle\int_1^{\infty}\big(\tfrac{1}{2}\ln(\lfloor t\rfloor^2+\lfloor t\rfloor)-\ln t\big)\,dt\,;\\
\sigma[g] &= \textstyle\int_1^{\infty}\frac{t-\lfloor t\rfloor-1/2}{t}\,dt\,;\\
\sigma[g] &= \textstyle\int_0^1 \ln\Gamma(t+1)\,dt.
\end{aligned}
$$

Also, Raabe's formula is given in (64).

*Restriction to the natural integers.* For any $n\in\mathbb{N}$ we have $\Gamma(n+1)=n!$. Gregory's formula states that for any $n\in\mathbb{N}^*$ and any $q\in\mathbb{N}$ we have

$$
\ln n! \;=\; 1-n+(n+1)\ln n-\sum_{j=1}^{q}G_j\left(\Delta^{j-1}g(n)-\Delta^{j-1}g(1)\right)-R_{q,n}\,,
$$

with

$$
|R_{q,n}| \;\leqslant\; \overline{G}_q\,|\Delta^q g(n)-\Delta^q g(1)|.
$$

Moreover, Proposition 6.8 gives the following series representation

$$
\ln n! \;=\; \frac{1}{2}\ln(2\pi)-n+(n+1)\ln n-\sum_{k=0}^{\infty}G_{k+1}\Delta^k g(n)\,,\quad n\in\mathbb{N}^*. \tag{75}
$$

Finally, recall Liu's formula (see Subsection 6.4)

$$
\ln n! \;=\; \frac{1}{2}\ln(2\pi)+\left(n+\frac{1}{2}\right)\ln n-n+\int_n^{\infty}\frac{\frac{1}{2}-\{t\}}{t}\,dt\,.
$$

*Derivatives of* $\Sigma g(x)$ *at* $x=1$. We have $\psi(1)=(\ln\Gamma)'(1)=-\sigma[g']=-\gamma$ and, for any integer $k\geqslant 2$,

$$
\begin{aligned}
\psi_{k-1}(1) \;=\; (\ln\Gamma)^{(k)}(1) &= (-1)^k(k-2)!-\sigma[g^{(k)}]\\
&= (-1)^k(k-1)!\,\zeta(k)\,,
\end{aligned}
$$

and hence

$$
\sigma[g^{(k)}] \;=\; (-1)^k(k-2)!\,(1-(k-1)\zeta(k))\,.
$$

The Taylor series expansion of $\ln\Gamma(x+1)$ about $x=0$ is

$$\ln\Gamma(x+1) \;=\; -\gamma x+\sum_{k=2}^{\infty}\frac{\zeta(k)}{k}\,(-x)^k\,,\qquad |x|<1.$$

Integrating this equation on $(0,1)$, we obtain

$$\sum_{k=2}^{\infty}(-1)^k\frac{1}{k(k+1)}\,\zeta(k) \;=\; \frac{1}{2}\,\gamma-1+\frac{1}{2}\ln(2\pi)\,.$$

Also, the exponential generating function for the sequence $n\mapsto\sigma[g^{(n)}]$ is

$$\mathrm{egf}_{\sigma}[g](x) \;=\; \sigma[g]-x+(x+1)\ln(x+1)-\ln\Gamma(x+1).$$

*Asymptotic analysis.* For every $a\geqslant 0$, we have

$$\begin{aligned}\Gamma(x+a) \;\sim\; x^a\,\Gamma(x) \;\sim\; \sqrt{2\pi}\,e^{-x}x^{x+a-\frac{1}{2}} &\qquad \text{as } x\to\infty\,;\\ \ln\Gamma(x+a) \;\sim\; x\ln x-x &\qquad \text{as } x\to\infty\,.\end{aligned}$$

We also have the results given in Examples 6.3 and 6.7.

Considering Binet's function

$$J(x) \;=\; \ln\Gamma(x)-\tfrac{1}{2}\ln(2\pi)+x-\left(x-\tfrac{1}{2}\right)\ln x\,,$$

for any $x>0$ we also have the inequalities

$$\left|\ln\Gamma\left(x+\tfrac{1}{2}\right)-\tfrac{1}{2}\ln(2\pi)+x-x\ln x\right| \;\leqslant\; |J(x)| \;\leqslant\; \tfrac{1}{2}\left|\ln\left(1+\tfrac{1}{x}\right)\right|,$$

which confirm that Burnside's formula (43) provides a better approximation of $\ln\Gamma(x)$ than Stirling's formula.

Since all the derivatives of $J(x)$ vanish at infinity, for any $k\in\mathbb{N}^*$ we get

$$\psi(x)-\ln x \;\to\; 0 \quad\text{and}\quad \psi_k(x) \;\to\; 0 \qquad \text{as } x\to\infty.$$

*Eulerian and Weierstrassian forms.* For any $x>0$, we have

$$\Gamma(x) \;=\; \frac{1}{x}\prod_{k=1}^{\infty}\frac{(1+\frac{1}{k})^x}{1+\frac{x}{k}} \;=\; \frac{e^{-\gamma x}}{x}\prod_{k=1}^{\infty}\frac{e^{\frac{x}{k}}}{1+\frac{x}{k}}$$

and the corresponding series can be integrated and differentiated term by term (see Examples 8.2 and 8.6). These identities can also be written as follows

$$\Gamma(x) \;=\; \lim_{n\to\infty}\frac{n!\,n^x}{x(x+1)\,\cdots\,(x+n)} \;=\; \lim_{n\to\infty}\frac{n!\,e^{x\psi(n)}}{x(x+1)\,\cdots\,(x+n)}\,.$$

The inequality in Theorem 3.4 gives

$$\left(1+\frac{1}{n}\right)^{-\lceil x\rceil\,|x-1|} \;\leqslant\; \frac{\Gamma(x)}{\frac{(n-1)!\,n^x}{x(x+1)\cdots(x+n-1)}} \;\leqslant\; \left(1+\frac{1}{n}\right)^{\lceil x\rceil\,|x-1|}.$$

*Alternative series expression and Fontana-Mascheroni's series.* Identity (75) is also valid for a real argument: for any $x>0$ we have

$$\begin{aligned}\ln\Gamma(x) &= \frac{1}{2}\ln(2\pi)-x+x\ln x-\sum_{n=0}^{\infty}G_{n+1}\Delta^n g(x)\\ &= \frac{1}{2}\ln(2\pi)-x+x\ln x-\sum_{n=0}^{\infty}|G_{n+1}|\sum_{k=0}^{n}(-1)^k\tbinom{n}{k}\ln(x+k)\end{aligned}$$

(see Example 6.9). Setting $x=1$ in this identity yields the analogue of Fontana-Mascheroni series:

$$\sum_{n=0}^{\infty}|G_{n+1}|\sum_{k=0}^{n}(-1)^k\tbinom{n}{k}\ln(k+1) \;=\; -1+\frac{1}{2}\ln(2\pi).$$

*Alternative representation.* Considering the antiderivative of the solution $\varphi=\psi$ to the equation $\Delta\varphi=g'$, we obtain

$$\ln\Gamma(x) \;=\; \psi_{-1}(x) \;=\; \int_1^x \psi(t)\,dt.$$

*Gauss' multiplication formula.* As described in Example 8.13, for any $m\in\mathbb{N}^*$ and any $x>0$, we have

$$\prod_{j=0}^{m-1}\Gamma\left(\frac{x+j}{m}\right) \;=\; (2\pi)^{\frac{m-1}{2}}\,m^{\frac{1}{2}-x}\,\Gamma(x).$$

Also, Corollary 8.10 provides the following formula for any $x>0$

$$\Gamma(mx)^{\frac{1}{m}} \;\sim\; e^{-x}x^x m^x \qquad \text{as } m\to_{\mathbb{N}}\infty,$$

which also follows from Stirling's formula. Moreover, Proposition 8.14 yields the following asymptotic expansion as $x\to\infty$. For any $m,q\in\mathbb{N}^*$ we have

$$\begin{aligned}\frac{1}{m}\sum_{j=0}^{m-1}\ln\Gamma\left(x+\frac{j}{m}\right) \;&=\; \frac{1}{2}\ln(2\pi)+x\ln x-x-\frac{1}{2m}\ln x\\ &\quad+\sum_{k=1}^{q}(-1)^{k-1}\,\frac{B_{k+1}}{k(k+1)}\,\frac{1}{x^k\,m^{k+1}}+O\left(\frac{1}{x^{q+1}}\right).\end{aligned}$$

Setting $m = 1$ in this formula, we obtain (see, e.g., [76, p. 7])

$$\ln\Gamma(x) \;=\; \frac{1}{2}\ln(2\pi) - x + \left(x - \frac{1}{2}\right)\ln x + \sum_{k=1}^{q} \frac{(-1)^{k-1}\,B_{k+1}}{k(k+1)\,x^k} + O\left(\frac{1}{x^{q+1}}\right).$$

Thus, we have

$$\ln\Gamma(x) \;=\; \frac{1}{2}\ln(2\pi) - x + \left(x - \frac{1}{2}\right)\ln x + \frac{1}{12x} - \frac{1}{360x^3} + O\left(\frac{1}{x^5}\right),$$

which is consistent with the analogue of Stirling's formula

$$-\log\Gamma(x) + \frac{1}{2}\ln(2\pi) - x + \left(x - \frac{1}{2}\right)\ln x \;\to\; 0 \qquad \text{as } x \to \infty.$$

*Wallis's product formula.* The original Wallis formula is presented in (68).

*Reflection formula.* For any $x \in (0,1)$, we have $\Gamma(x)\Gamma(1-x) = \pi\csc(\pi x)$.

*Webster's functional equation.* For any $m \in \mathbb{N}^*$, there is a unique solution $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ to the equation $\prod_{j=0}^{m-1} f(x + \frac{j}{m}) = x$ such that $\ln f$ is eventually monotone, namely

$$f(x) \;=\; \frac{\Gamma(x + \frac{1}{m})}{\Gamma(x)}.$$

More generally, for any $m \in \mathbb{N}^*$ and any $a > 0$, there is a unique solution $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ to the equation $\prod_{j=0}^{m-1} f(x + aj) = x$ such that $\ln f$ is eventually monotone, namely

$$f(x) \;=\; (am)^{\frac{1}{m}}\,\frac{\Gamma(\frac{x}{am} + \frac{1}{m})}{\Gamma(\frac{x}{am})}.$$

## 9.3 The digamma and harmonic number functions

Let us now see what we get if we apply our results to the digamma function $x \mapsto \psi(x)$ and the harmonic number function $x \mapsto H_x$. Recall first that the identity $H_{x-1} = \psi(x) + \gamma$ holds for any $x > 0$.

*ID card.*

| $g(x)$ | Membership | $\deg g$ | $\Sigma g(x)$ |
|---|---|---|---|
| $1/x$ | $\mathcal{C}^\infty \cap \mathcal{D}^0 \cap \mathcal{K}^\infty$ | $-1$ | $H_{x-1} = \psi(x) + \gamma$ |

*Characterization.* The digamma function can be characterized as follows:

> *All eventually monotone solutions* $f\colon \mathbb{R}_+ \to \mathbb{R}$ *to the equation* $f(x+1) - f(x) = 1/x$ *are of the form* $f(x) = c + \psi(x)$*, where* $c \in \mathbb{R}$.

Interestingly, this characterization enables us to establish almost instantly the following identities for every $x > 0$,

$$H_{x-1} \;=\; \psi(x) + \gamma \;=\; \int_0^1 \frac{1 - t^{x-1}}{1-t}\, dt\,.$$

Indeed, each of the three expressions above vanishes at $x = 1$ and is an eventually increasing solution to the equation $f(x+1) - f(x) = 1/x$. Hence, they must coincide on $\mathbb{R}_+$. We can prove the following Gauss representation (see, e.g., [76, p. 26]) similarly

$$\psi(x) \;=\; \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-xt}}{1-e^{-1}}\right) dt\,, \qquad x > 0.$$

Kairies [44] obtained a variant of the characterization of the digamma function above by replacing the eventual monotonicity with the convexity property. This variant is also immediate from our results since $g$ also lies in $\mathcal{D}^1 \cap \mathcal{K}^1$.

*Asymptotic constant, generalized Stirling's and Euler's constants, Raabe's formula.*

| $\overline{\sigma}[g]$ | $\sigma[g]$ | $\gamma[g]$ |
|---|---|---|
| $\infty$ | $\gamma$ | $\gamma$ |

We have the following representations

$$\gamma \;=\; \lim_{n\to\infty}\left(\sum_{k=1}^n \frac{1}{k} - \ln n\right) \;=\; \sum_{k=1}^\infty \left(\frac{1}{k} - \ln\left(1+\frac{1}{k}\right)\right);$$

$$\gamma \;=\; \int_1^\infty \left(\frac{1}{\lfloor t\rfloor} - \frac{1}{t}\right) dt \;=\; \int_0^1 H_t\, dt.$$

Also, the analogue of Raabe's formula is

$$\int_x^{x+1} \psi(t)\, dt \;=\; \ln x\,, \qquad x > 0.$$

We also have for any $q \in \mathbb{N}$ and any $x > 0$

$$J^{q+1}[\Sigma g](x) \;=\; \psi(x) - \ln x + \sum_{j=1}^q |G_j|\, B(x, j),$$

where $(x, y) \mapsto B(x, y)$ is the beta function.

*Restriction to the natural integers.* For any $n \in \mathbb{N}$ we have $H_n = \sum_{k=1}^n \frac{1}{k}$. Gregory's formula states that for any $n \in \mathbb{N}^*$ and any $q \in \mathbb{N}$ we have

$$H_{n-1} \;=\; \ln n - \sum_{j=1}^q |G_j| \left(B(n, j) - \frac{1}{j}\right) - R_{q,n}\,,$$

with

$$|R_{q,n}| \leqslant \overline{G}_q \left| B(n, q+1) - \frac{1}{q} \right| .$$

*Derivatives of* $\Sigma g(x)$ *at* $x = 1$. We have $\psi(1) = -\gamma$ and, for any $k \in \mathbb{N}^*$,

$$\psi_k(1) = (-1)^{k-1}(k-1)! - \sigma[g^{(k)}] = (-1)^{k-1} k!\, \zeta(k+1)$$

and hence

$$\sigma[g^{(k)}] = (-1)^{k-1}(k-1)!\, (1 - k\, \zeta(k+1)) .$$

The Taylor series expansion of $\psi(x+1)$ about $x = 0$ is

$$H_x = \psi(x+1) + \gamma = \sum_{k=1}^{\infty} (-1)^{k-1} \zeta(k+1)\, x^k , \qquad |x| < 1.$$

Integrating this equation on $(0, 1)$, we retrieve Euler's series representation of $\gamma$

$$\gamma = \sum_{k=2}^{\infty} (-1)^k \frac{\zeta(k)}{k} .$$

Also, the exponential generating function for the sequence $n \mapsto \sigma[g^{(n)}]$ is

$$\mathbf{egf}_{\sigma}[g](x) = \ln(x+1) - \psi(x+1).$$

*Asymptotic analysis.* For any $a \geqslant 0$ and any $x > 0$, we have

$$|\psi(x+a) - \psi(x)| \leqslant \frac{\lceil a \rceil}{x} \quad \text{and} \quad |\psi(x) - \ln x| \leqslant \frac{1}{x} .$$

Considering the value $p = 1$ in Theorem 6.5, we see that the latter inequality can be refined into

$$-\frac{x+2}{2x(x+1)} \leqslant \psi(x) - \ln x \leqslant -\frac{1}{2(x+1)} .$$

We also have $\psi(x+a) - \psi(x) \to 0$, $\psi(x) - \ln x \to 0$, and $\psi(x+a) \sim \ln x$ as $x \to \infty$. Since all the derivatives of $J^1[\Sigma g]$ vanish at infinity, so do the functions $\psi_k$ for any $k \in \mathbb{N}^*$. Finally, for any $x > 0$ we also have the inequalities

$$\left| \psi\left(x + \tfrac{1}{2}\right) - \ln x \right| \leqslant |J^1[\Sigma g](x)| \leqslant \tfrac{1}{x} ,$$

which shows that the analogue of Burnside's formula

$$\psi(x) - \ln(x - \tfrac{1}{2}) \to 0 , \qquad \text{as } x \to \infty,$$

provides a better approximation of $\psi$ than generalized Stirling's formula.

*Eulerian and Weierstrassian forms.* For any $x > 0$, we have

$$\begin{aligned}\psi(x) &= -\gamma - \frac{1}{x} + \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{x+k}\right)\\ &= -\frac{1}{x} + \sum_{k=1}^{\infty}\left(\ln\left(1+\frac{1}{k}\right) - \frac{1}{x+k}\right)\end{aligned}$$

and these series can be integrated and differentiated term by term. In particular, we retrieve the product form of $e^{\psi(x)}$ obtained in Example 8.2. Also, the inequality in Theorem 3.4 gives

$$\left|\psi(x) + \gamma + \frac{1}{x} - \sum_{k=1}^{n-1}\left(\frac{1}{k} - \frac{1}{x+k}\right)\right| \leqslant \frac{\lceil x \rceil}{n}, \qquad x > 0,\ n \in \mathbb{N}^*.$$

*Alternative series expression and Fontana-Mascheroni's series.* Proposition 6.8 yields the following series representation

$$\psi(x) = \ln x - \sum_{n=1}^{\infty}|G_n|\, B(x,n) = \ln x - \sum_{n=1}^{\infty}\frac{|G_n|}{n\binom{x+n-1}{n}}, \qquad x > 0.$$

Setting $x = 1$ in this identity, we retrieve Fontana-Mascheroni series:

$$\gamma = \sum_{n=1}^{\infty}\frac{|G_n|}{n}.$$

Setting $x = 2$, we get

$$1 - \ln 2 = \sum_{n=1}^{\infty}\frac{|G_n|}{n+1},$$

which is consistent with the last identity given in Example 6.11.

*Alternative representation.* We have $H_{x-1} = H_x - \frac{1}{x} = \psi(x) + \gamma$.

*Gauss' multiplication formula.* For any $m \in \mathbb{N}^*$ and any $x > 0$, we have (see, e.g., [15, p. 5])

$$\sum_{j=0}^{m-1}\psi\left(\frac{x+j}{m}\right) = m(\psi(x) - \ln m)$$

and

$$\sum_{j=0}^{m-1} H_{(x+j)/m} = m(H_{x+m-1} - \ln m).$$

Also, Corollary 8.10 provides the following formula for any $x > 0$

$$\lim_{m\to\infty}(H_{mx-1} - H_{m-1}) = \ln x.$$

Moreover, Proposition 8.14 yields the following asymptotic expansion as $x \to \infty$. For any $m, q \in \mathbb{N}^*$ we have

$$\frac{1}{m}\sum_{j=0}^{m-1} \psi\left(x+\frac{j}{m}\right) \;=\; \ln x + \sum_{k=1}^{q} \frac{(-1)^{k-1}\, B_k}{k\,(mx)^k} + O\left(\frac{1}{x^{q+1}}\right).$$

Setting $m = 1$ in this formula, we obtain (see, e.g., [76, p. 36])

$$\psi(x) \;=\; \ln x + \sum_{k=1}^{q} \frac{(-1)^{k-1}\, B_k}{k\, x^k} + O\left(\frac{1}{x^{q+1}}\right).$$

*Wallis's product formula.* The analogue of Wallis's formula is the classical identity

$$\sum_{k=1}^{\infty} (-1)^{k-1}\frac{1}{k} \;=\; \ln 2\,.$$

Interestingly, the analogue of Wallis's formula for the function $g(x) = \psi(x)$ gives

$$\lim_{n\to\infty}\left(-\ln(4n) + 2\sum_{k=1}^{2n}(-1)^k\psi(k)\right) \;=\; \gamma\,,$$

which provides yet another formula to define Euler's constant $\gamma$. This latter formula is obtained by first considering the duplication formula $2\psi(2x) = \psi(x) + \psi(x+\frac{1}{2}) + 2\ln 2$.

*Reflection formula.* For any $x \in (0,1)$, we have $\psi(x) - \psi(1-x) = -\pi\cot(\pi x)$.

*Webster's functional equation.* For any $m \in \mathbb{N}^*$, there is a unique eventually monotone solution $f\colon \mathbb{R}_+ \to \mathbb{R}$ to the equation $\sum_{j=0}^{m-1} f(x+\frac{j}{m}) = \frac{1}{x}$, namely

$$f(x) \;=\; \psi\left(x+\frac{1}{m}\right) - \psi(x)\,.$$

More generally, for any $m \in \mathbb{N}^*$ and any $a > 0$, there is a unique eventually monotone solution $f\colon \mathbb{R}_+ \to \mathbb{R}$ to the equation $\sum_{j=0}^{m-1} f(x+aj) = \frac{1}{x}$, namely

$$f(x) \;=\; \frac{1}{am}\,\psi\left(\frac{x}{am}+\frac{1}{m}\right) - \frac{1}{am}\,\psi\left(\frac{x}{am}\right).$$

**Example 9.1.** Suppose we wish to prove that $\ln\Gamma(x) \sim x\,\psi(x) - x$ as $x \to \infty$. Considering the function $g(x) = \psi(x) + 1/x$, we have $\deg g = 0$ and $\Sigma g(x) = x\,\psi(x) - x + 1 + \gamma$. Then, the generalized Stirling formula yields

$$(x\,\psi(x) - x + 1) - \ln\Gamma(x) - \ln x + \tfrac{1}{2}\,\psi(x) \;\to\; \tfrac{1}{2} - \tfrac{1}{2}\,\ln(2\pi) \quad \text{as } x \to \infty.$$

Dividing through by $\ln\Gamma(x)$, we obtain the claim asymptotic equivalence. We can also derive the equivalence

$$\ln\Gamma(x) \ \sim \ (x - \tfrac{1}{2})\,\psi(x) - x \qquad \text{as } x \to \infty$$

from taking the derivative of the generalized Stirling formula applied to $g(x) = \ln\Gamma(x)$. Finally, we also have the equivalence

$$\ln\Gamma(x) \ \sim \ x\ln x - x \qquad \text{as } x \to \infty,$$

which is nothing other than Proposition 6.26 with $g(x) = \ln x$. $\square$

## 9.4 The polygamma functions

We now investigate the polygamma functions $\psi_\nu$ ($\nu \in \mathbb{Z}$). Our results will prove to be particularly useful when $\nu < -1$ since, in this case, the function $\psi_\nu$ has a strictly positive asymptotic degree.

For any $\nu \in \mathbb{Z}$, we set $g_\nu = \Delta\psi_\nu$; hence $g'_\nu = g_{\nu+1}$ and $\psi'_\nu = \psi_{\nu+1}$. We then have $\Sigma g_\nu(x) = \psi_\nu(x) - \psi_\nu(1)$. (The cases $\nu = 0$ and $\nu = -1$ correspond to the functions $\psi$ and $\ln\Gamma$, respectively, and have been already considered above.) Let us deal with the cases $\nu \in \mathbb{N}^*$ and $\nu \in \mathbb{Z} \setminus \mathbb{N}$ separately. In the latter case, we often consider the value $\nu = -2$ for simplicity and brevity.

### 9.4.1 Case $\nu \in \mathbb{N}^*$

*ID card.*

| $g_\nu(x)$ | Membership | $\deg g_\nu$ | $\Sigma g_\nu(x)$ |
|---|---|---|---|
| $(-1)^\nu \nu!\, x^{-\nu-1}$ | $\mathcal{C}^\infty \cap \widetilde{\mathcal{D}}^{-1}_{\mathbb{N}} \cap \mathcal{K}^\infty$ | $-1$ | $\psi_\nu(x) - \psi_\nu(1)$ |

Recall that $\psi_\nu(1) = (-1)^{\nu+1}\nu!\,\zeta(\nu+1)$ (cf. derivatives of $\psi(x)$ at $x = 1$).

*Characterization.* The function $\psi_\nu$ can be characterized as follows:

> *All eventually monotone solutions $f\colon \mathbb{R}_+ \to \mathbb{R}$ to the equation $f(x+1) - f(x) = g_\nu(x)$ are of the form $f(x) = c_\nu + \psi_\nu(x)$, where $c_\nu \in \mathbb{R}$.*

This characterization enables us to prove almost immediately the following identity

$$\psi_\nu(x) \ = \ (-1)^{\nu-1}\int_0^\infty \frac{t^\nu\, e^{-xt}}{1 - e^{-1}}\, dt\,, \qquad x > 0.$$

*Asymptotic constant, generalized Stirling's and Euler's constants, Raabe's formula.*

| $\overline{\sigma}[g_\nu]$ | $\sigma[g_\nu]$ | $\gamma[g_\nu]$ |
|---|---|---|
| $\infty$ | $g_{\nu-1}(1) - \psi_\nu(1)$ | $\gamma[g_\nu] = \sigma[g_\nu]$ |

We have the inequality $|\sigma[g_\nu]| \leqslant \nu!$ and the following representations

$$\begin{aligned}
\sigma[g_\nu] &= \lim_{n\to\infty} \left(\sum_{k=1}^{n} g_\nu(k) + g_{\nu-1}(1)\right);\\
\sigma[g_\nu] &= \sum_{k=1}^{\infty} \left(g_\nu(k) + g_{\nu-1}(k) - g_{\nu-1}(k+1)\right);\\
\sigma[g_\nu] &= (-1)^\nu \nu! \int_1^\infty \left(\lfloor t\rfloor^{-\nu-1} - t^{-\nu-1}\right) dt;\\
\sigma[g_\nu] &= \int_0^1 (\psi_\nu(t+1) - \psi_\nu(1))\, dt.
\end{aligned}$$

Also, the analogue of Raabe's formula is

$$\int_x^{x+1} \psi_\nu(t)\, dt \;=\; g_{\nu-1}(x)\,, \qquad x > 0.$$

We also have for any $q \in \mathbb{N}$ and any $x > 0$

$$J^{q+1}[\Sigma g_\nu](x) \;=\; \psi_\nu(x) - g_{\nu-1}(x) + \sum_{j=1}^{q} G_j\, \Delta^{j-1} g_\nu(x).$$

*Restriction to the natural integers.* For any $n \in \mathbb{N}^*$, we have

$$\psi_\nu(n) - \psi_\nu(1) \;=\; (-1)^\nu \nu! \sum_{k=1}^{n-1} k^{-\nu-1}\,.$$

Gregory's formula states that for any $n \in \mathbb{N}^*$ and any $q \in \mathbb{N}$ we have

$$\begin{aligned}
\sum_{k=1}^{n-1} g_\nu(k) \;&=\; g_{\nu-1}(n) - g_{\nu-1}(1)\\
&\qquad - \sum_{j=1}^{q} G_j \left(\Delta^{j-1} g_\nu(n) - \Delta^{j-1} g_\nu(1)\right) - R_{q,n}\,,
\end{aligned}$$

with

$$|R_{q,n}| \;\leqslant\; \overline{G}_q\, |\Delta^q g_\nu(n) - \Delta^q g_\nu(1)|\,.$$

*Derivatives of* $\Sigma g_\nu(x)$ *at* $x = 1$. We have $\psi_\nu(1) = (-1)^{\nu+1}\nu!\,\zeta(\nu+1)$ and, for any $k \in \mathbb{N}^*$,

$$\psi_{\nu+k}(1) \;=\; g_{\nu+k-1}(1) - \sigma[g_\nu^{(k)}] \;=\; (-1)^{\nu+k-1}(\nu+k)!\,\zeta(\nu+k+1)$$

and hence

$$\sigma[g_\nu^{(k)}] \;=\; g_{\nu+k-1}(1) + (-1)^{\nu+k}(\nu+k)!\,\zeta(\nu+k+1)\,.$$

The exponential generating function for the sequence $n \mapsto \sigma[g_\nu^{(n)}]$ is

$$\mathrm{egf}_\sigma[g_\nu](x) \;=\; g_{\nu-1}(x+1) - \psi_\nu(x+1).$$

*Asymptotic analysis.* For any $a \geqslant 0$ and any $x > 0$, we have

$$|\psi_\nu(x+a) - \psi_\nu(x)| \;\leqslant\; \lceil a \rceil \, |g_\nu(x)| \quad \text{and} \quad |\psi_\nu(x) - g_{\nu-1}(x)| \;\leqslant\; |g_\nu(x)|\,.$$

Considering the value $p = 1$ in Theorem 6.5, we see that the latter inequality can be refined into

$$\left|\psi_\nu(x) - g_{\nu-1}(x) + \frac{1}{2}\, g_\nu(x)\right| \;\leqslant\; \frac{1}{2}\,|\Delta g_\nu(x)|.$$

We also have, $\psi_\nu(x) \to 0$ and $\psi_\nu(x+a) \sim g_{\nu-1}(x)$ as $x \to \infty$. Finally, for any $x > 0$ we have the inequalities

$$\left|\psi_\nu\left(x + \tfrac{1}{2}\right) - g_{\nu-1}(x)\right| \;\leqslant\; |J^1[\Sigma g_\nu](x)| \;\leqslant\; |g_\nu(x)|\,,$$

which shows that the analogue of Burnside's formula

$$\psi_\nu(x) - g_{\nu-1}(x - \tfrac{1}{2}) \;\to\; 0\,, \qquad \text{as } x \to \infty,$$

provides a better approximation of $\psi_\nu$ than generalized Stirling's formula.

*Eulerian and Weierstrassian forms.* For any $x > 0$, we have

$$\psi_\nu(x) \;=\; -\sum_{k=0}^{\infty} g_\nu(x+k).$$

and this series can be integrated and differentiated term by term.

*Alternative series expression and Fontana-Mascheroni's series.* Proposition 6.8 gives the following series representation: for any $x > 0$ we have

$$\begin{aligned}
\psi_\nu(x) \;&=\; g_{\nu-1}(x) - \sum_{n=0}^{\infty} G_{n+1}\, \Delta^n g_\nu(x) \\
&=\; g_{\nu-1}(x) - \sum_{n=0}^{\infty} |G_{n+1}| \sum_{k=0}^{n} (-1)^k \tbinom{n}{k} g_\nu(x+k)\,.
\end{aligned}$$

Setting $x = 1$ in this identity yields the analogue of Fontana-Mascheroni series. For instance, taking $\nu = 1$, we derive the identity

$$\sum_{n=1}^{\infty} |G_n|\, \frac{H_n}{n} \;=\; \frac{\pi^2}{6} - 1\,.$$

Taking $\nu = 2$, we obtain

$$\sum_{n=1}^{\infty} |G_n|\, \frac{\psi_1(n+1) - H_n^2}{n} \;=\; 1 - 2\,\zeta(3) + \gamma\, \frac{\pi^2}{6}\,.$$

*Gauss' multiplication formula.* Differentiating the multiplication formula for $\psi$, we obtain the following formula. For any $m \in \mathbb{N}^*$ and any $x > 0$, we have

$$\sum_{j=0}^{m-1} \psi_\nu\left(\frac{x+j}{m}\right) \;=\; m^{\nu+1}\, \psi_\nu(x).$$

Also, Corollary 8.10 provides the following limit

$$\lim_{m\to\infty} m^\nu \psi_\nu(mx) \;=\; g_{\nu-1}(x), \qquad x > 0.$$

Moreover, Proposition 8.14 yields the following asymptotic expansion as $x \to \infty$. For any $m, q \in \mathbb{N}^*$ we have

$$\frac{1}{m}\sum_{j=0}^{m-1} \psi_\nu\left(x+\frac{j}{m}\right) \;=\; \sum_{k=0}^{q} \frac{B_k}{m^k\, k!}\, g_{\nu+k-1}(x) + O(g_{\nu+q}(x))\,.$$

Setting $m = 1$ in this formula, we obtain

$$\psi_\nu(x) \;=\; \sum_{k=0}^{q} \frac{B_k}{k!}\, g_{\nu+k-1}(x) + O(g_{\nu+q}(x))\,.$$

*Wallis's product formula.* We have

$$\sum_{k=1}^{\infty} (-1)^{k-1} g_\nu(k) \;=\; (-1)^\nu (1-2^{-\nu})\, \nu!\, \zeta(\nu+1),$$

that is,

$$\sum_{k=1}^{\infty} (-1)^{k-1} g_\nu(k) \;=\; (-1)^\nu\, \nu!\, \eta(\nu+1),$$

where $\eta$ is Dirichlet's eta function.

*Reflection formula.* Differentiating the reflection formula for $\psi$, we obtain the following formula. For any $x \in (0,1)$, we have

$$\psi_\nu(x) - (-1)^\nu \psi_\nu(1-x) \;=\; -\pi\, D^\nu \cot(\pi x).$$

*Webster's functional equation.* For any $m \in \mathbb{N}^*$, there is a unique eventually monotone solution $f\colon \mathbb{R}_+ \to \mathbb{R}$ to the equation $\sum_{j=0}^{m-1} f(x+\frac{j}{m}) = g_\nu(x)$, namely

$$f(x) \;=\; \psi_\nu\left(x+\frac{1}{m}\right) - \psi_\nu(x)\,.$$

### 9.4.2 Case $\nu \in \mathbb{Z}\setminus\mathbb{N}$

*ID card.*

| $g_\nu(x)$ | Membership | $\deg g_\nu$ | $\Sigma g_\nu(x)$ |
|---|---|---|---|
| see below | $\mathcal{C}^\infty\cap\mathcal{D}^{-\nu}\cap\mathcal{K}^\infty$ | $-\nu-1$ | $\psi_\nu(x)-\psi_\nu(1)$ |

Using (37), we obtain the following recursive way to compute $g_\nu$. For any integer $\nu\leqslant -1$,

$$\begin{aligned} g_{\nu-1}(x) &= \int_x^{x+1}\psi_\nu(t)\,dt = \int_0^x g_\nu(t)\,dt+\int_0^1\psi_\nu(t)\,dt \\ &= \int_0^x g_\nu(t)\,dt+\psi_{\nu-1}(1). \end{aligned}$$

Solving this recurrence equation, we obtain $g_{-1}(x)=\ln x$ and for any integer $\nu\leqslant -2$,

$$g_\nu(x) = \int_0^x\frac{(x-t)^{-\nu-2}}{(-\nu-2)!}\ln t\,dt+\sum_{j=0}^{-\nu-2}\psi_{\nu+j}(1)\,\frac{x^j}{j!}\,.$$

For instance, $g_{-2}(x)=x\ln x-x+\frac{1}{2}\ln(2\pi)$ and

$$g_{-3}(x) = \frac{1}{2}\,x^2\ln x-\frac{3}{4}\,x^2+\left(\frac{1}{2}\,x+\frac{1}{4}\right)\ln(2\pi)+\ln A.$$

*Characterization.* The function $\psi_\nu$ can be characterized as follows:

> *All solutions* $f\colon\mathbb{R}_+\to\mathbb{R}$ *to the equation* $f(x+1)-f(x)=g_\nu(x)$ *that lie in* $\mathcal{K}^{-\nu}$ *are of the form* $f(x)=c+\psi_\nu(x)$, *where* $c\in\mathbb{R}$.

*Asymptotic constant, generalized Stirling's and Euler's constants, Raabe's formula.*

| $\overline{\sigma}[g_\nu]$ | $\sigma[g_\nu]$ | $\gamma[g_\nu]$ |
|---|---|---|
| $\psi_{\nu-1}(1)-\psi_\nu(1)$ | $g_{\nu-1}(1)-\psi_\nu(1)$ | $\sigma[g_\nu]-\sum_{j=1}^{-\nu}G_j\Delta^{j-1}g_\nu(1)$ |

| $\overline{\sigma}[g_{-2}]$ | $\sigma[g_{-2}]$ | $\gamma[g_{-2}]$ |
|---|---|---|
| $\ln A-\frac{1}{4}\ln(2\pi)$ | $\ln A+\frac{1}{4}\ln(2\pi)-\frac{3}{4}$ | $\ln A+\frac{1}{6}\ln 2-\frac{1}{3}$ |

We have

$$\begin{aligned} |\gamma[g_\nu]| &\leqslant \overline{G}_{-\nu}\,|\Delta^{-\nu}g_\nu(1)|\,, \\ \gamma[g_\nu] &= \int_1^\infty\left(\sum_{j=0}^{-\nu}G_j\,\Delta^j g_\nu(\lfloor t\rfloor)-g_\nu(t)\right)dt\,, \\ \sigma[g_\nu] &= \int_0^1(\psi_\nu(t+1)-\psi_\nu(1))\,dt\,, \end{aligned}$$

and

$$\begin{aligned}
\sigma[g_\nu] &= \lim_{n\to\infty}\left(\sum_{k=1}^{n-1} g_\nu(k) + g_{\nu-1}(1) - g_{\nu-1}(n) + \sum_{j=1}^{-\nu} G_j\,\Delta^{j-1} g_\nu(n)\right),\\
\sigma[g_\nu] &= \sum_{j=1}^{-\nu} G_j\,\Delta^{j-1} g_\nu(1)\\
&\quad + \sum_{k=1}^{\infty}\left(\sum_{j=0}^{-\nu} G_j\,\Delta^{j} g_\nu(k) + g_{\nu-1}(k) - g_{\nu-1}(k+1)\right).
\end{aligned}$$

Also, the analogue of Raabe's formula is

$$\int_x^{x+1}\psi_\nu(t)\,dt \;=\; g_{\nu-1}(x)\,,\qquad x>0.$$

We also have for any $q\in\mathbb{N}$ and any $x>0$

$$J^{q+1}[\Sigma g_\nu](x) \;=\; \psi_\nu(x) - g_{\nu-1}(x) + \sum_{j=1}^{q} G_j\,\Delta^{j-1} g_\nu(x).$$

For instance,

$$\begin{aligned}
J^3[\Sigma g_{-2}](x) &= \psi_{-2}(x) - \frac{1}{12}(x+1)\ln(x+1) + \frac{1}{12}(3x-1)^2\\
&\quad - \frac{1}{12}x(6x-7)\ln x - \frac{1}{2}x\ln(2\pi) - \ln A.
\end{aligned}$$

*Derivatives of* $\Sigma g_\nu(x)$ *at* $x=1$. For any $k\in\mathbb{N}^*$ we have

$$\psi_{k-2}(1) \;=\; (\Sigma g_{-2})^{(k)}(1) \;=\; g_{-2}^{(k-1)}(1) - \sigma[g_{-2}^{(k)}].$$

We have $\sigma[g'_{-2}] = \sigma[\ln] = -1 + \frac{1}{2}\ln(2\pi)$, $\sigma[g''_{-2}] = \gamma$, and for any integer $k \geqslant 3$,

$$\sigma[g_{-2}^{(k)}] \;=\; (-1)^{k-1}(k-3)!(1-(k-2)\,\zeta(k-1)).$$

The exponential generating function for the sequence $n\mapsto\sigma[g_\nu^{(n)}]$ is

$$\mathbf{egf}_\sigma[g_\nu](x) \;=\; g_{\nu-1}(x+1) - \psi_\nu(x+1).$$

Integrating this equation for $\nu=-2$ on $(0,1)$ (i.e., we use (60)), we obtain after some algebra

$$\sum_{k=2}^{\infty}(-1)^k\,\frac{\zeta(k)}{k(k+1)(k+2)} \;=\; \frac{1}{6}\gamma - \frac{3}{4} + \frac{1}{4}\ln(2\pi) + \ln A\,.$$

*Asymptotic analysis.* For every $a \geqslant 0$, we have

$$\psi_\nu(x+a) - \psi_\nu(x) - \sum_{j=0}^{-\nu-1} \tbinom{a}{j+1}\, \Delta^j g_\nu(x) \quad\to\quad 0 \qquad \text{as } x\to\infty\,;$$

$$g_\nu(x+a) - \sum_{j=0}^{-\nu-1} \tbinom{a}{j}\, \Delta^j g_\nu(x) \quad\to\quad 0 \qquad \text{as } x\to\infty.$$

For instance, when $\nu = -2$ the first limit reduces to

$$\int_x^{x+a} \ln\Gamma(t)\, dt - a\ln\left(\sqrt{2\pi}\,\frac{x^x}{e^x}\right) - \tbinom{a}{2}\ln\left(\frac{(x+1)^{x+1}}{e\, x^x}\right) \;\to\; 0 \quad \text{as } x\to\infty,$$

with equality if $a \in \{0,1,2\}$. Also, for any $x > 0$, we have

$$\left|J^{-\nu+1}[\Sigma g_\nu](x)\right| \;\leqslant\; \overline{G}_{-\nu}\left|\Delta^{-\nu} g_\nu(x)\right|$$

and

$$\psi_\nu(x) - g_{\nu-1}(x) + \sum_{j=1}^{-\nu} G_j \Delta^{j-1} g_\nu(x) \;\to\; 0 \qquad \text{as } x\to\infty;$$

$$\Delta g_{\nu-1}(x) - \sum_{j=0}^{-\nu-1} G_j \Delta^j g_\nu(x) \;\to\; 0 \qquad \text{as } x\to\infty.$$

Also,

$$\psi_\nu(x+a) \;\sim\; g_{\nu-1}(x) \qquad \text{as } x\to\infty.$$

For instance, if $\nu = -2$, then the first limit above reduces to

$$\psi_{-2}(x) - \frac{1}{12}\,(x+1)\ln(x+1) + \frac{1}{12}\,(3x-1)^2 - \frac{1}{12}\,x(6x-7)\ln x - \frac{1}{2}\,x\ln(2\pi) \;\to\; \ln A\,.$$

*Eulerian and Weierstrassian forms.* For any $x > 0$, we have

$$\begin{aligned}\psi_\nu(x) - \psi_\nu(1) \;&=\; -g_\nu(x) + \sum_{j=0}^{-\nu-1} \tbinom{x}{j+1}\Delta^j g_\nu(1)\\ &\quad + \sum_{k=1}^{\infty}\left(-g_\nu(x+k) + \sum_{j=0}^{-\nu} \tbinom{x}{j}\, \Delta^j g_\nu(k)\right)\end{aligned}$$

and

$$\begin{aligned}\psi_\nu(x) - \psi_\nu(1) \;&=\; -g_\nu(x) + \sum_{j=0}^{-\nu-2} \tbinom{x}{j+1}\Delta^j g_\nu(1) - \gamma\tbinom{x}{-\nu}\\ &\quad + \sum_{k=1}^{\infty}\left(-g_\nu(x+k) + \sum_{j=0}^{-\nu-1} \tbinom{x}{j}\, \Delta^j g_\nu(k) + \tbinom{x}{-\nu}\frac{1}{k}\right).\end{aligned}$$

When $\nu = -2$, these identities reduce to

$$\psi_{-2}(x) \;=\; \ln\left(\frac{(2\pi)^{\frac{1}{2}x}(\frac{4}{e})^{\binom{x}{2}}}{x^x}\prod_{k=1}^{\infty}\frac{(1+2/k)^{(k+2)\binom{x}{2}}}{(1+x/k)^{x+k}\,(1+1/k)^{(k+1)x(x-2)}}\right)$$

and

$$\psi_{-2}(x) \;=\; \ln\left(\frac{(2\pi)^{\frac{1}{2}x}e^{-\gamma\binom{x}{2}}}{x^x}\prod_{k=1}^{\infty}\frac{e^{\frac{1}{k}\binom{x}{2}}\,(1+1/k)^{(k+1)x}}{(1+x/k)^{x+k}}\right).$$

Integrating both the Eulerian and Weierstrassian forms of $\psi_{-1}(x) = \ln\Gamma(x)$, we obtain the following representations (which are simpler than the previous ones since less terms are involved; see also Example 8.2)

$$\begin{aligned}\psi_{-2}(x) \;&=\; \ln\left(\frac{e^x}{x^x}\prod_{k=1}^{\infty}\frac{e^x(1+1/k)^{x^2/2}}{(1+x/k)^{x+k}}\right)\\ &=\; \ln\left(e^{-\gamma x^2/2}\,\frac{e^x}{x^x}\prod_{k=1}^{\infty}\frac{e^{x+x^2/(2k)}}{(1+x/k)^{x+k}}\right).\end{aligned}$$

We also have the analogue of Gauss' limit for the gamma function

$$\psi_{-2}(x) \;=\; x - x\ln x + \lim_{n\to\infty}\left(\sum_{k=1}^{n-1}\left(x-(x+k)\ln\left(1+\frac{x}{k}\right)\right)+\frac{x^2}{2}\ln n\right).$$

*Alternative series expression and Fontana-Mascheroni's series.* Here the formulas are the same as in the case when $\nu \in \mathbb{N}^*$.

*Gauss' multiplication formula.* For any $m \in \mathbb{N}^*$ and any $x > 0$, we have

$$\sum_{j=0}^{m-1}\psi_\nu\left(\frac{x+j}{m}\right) \;=\; \sum_{j=1}^{m-1}\psi_\nu\left(\frac{j}{m}\right)+\psi_\nu(1)+\Sigma_x\, g_\nu\left(\frac{x}{m}\right).$$

Let us expand this formula in the case when $\nu = -2$. First, we have

$$g_{-2}\left(\frac{x}{m}\right) \;=\; \frac{1}{m}\,g_{-2}(x) - x\frac{\ln m}{m} + \frac{m-1}{m}\,\psi_{-2}(1)$$

and hence

$$\Sigma_x\, g_{-2}\left(\frac{x}{m}\right) \;=\; \frac{1}{m}\,\psi_{-2}(x) - \binom{x}{2}\frac{\ln m}{m} + \left(\frac{m-1}{m}\,x - 1\right)\psi_{-2}(1).$$

Also, we have

$$\sum_{j=1}^{m-1}\psi_{-2}\left(\frac{j}{m}\right) \;=\; \left(1-\frac{1}{m}\right)\ln A - \frac{\ln m}{12\,m} + (m-1)\ln((2\pi)^{\frac{1}{4}}A).$$

Thus, we obtain the following multiplication formula for $\psi_{-2}$

$$\begin{aligned}\sum_{j=0}^{m-1} \psi_{-2}\left(\frac{x+j}{m}\right) &= \left(1-\frac{1}{m}\right)\ln((2\pi)^{\frac{x}{2}}A) + (m-1)\ln((2\pi)^{\frac{1}{4}}A)\\ &\qquad - \frac{1}{12m}(6x^2-6x+1)\ln m + \frac{1}{m}\psi_{-2}(x).\end{aligned}$$

This formula can also be derived by integrating the multiplication formula obtained from $g_{-1}(x) = \ln x$. Taking $m = 2$, we obtain the following analogue of Legendre's duplication formula

$$\begin{aligned}\psi_{-2}\left(\frac{x}{2}\right) + \psi_{-2}\left(\frac{x+1}{2}\right) &= \frac{1}{2}\ln((2\pi)^{\frac{x}{2}}A) + \ln((2\pi)^{\frac{1}{4}}A)\\ &\qquad - \frac{1}{24}(6x^2-6x+1)\ln 2 + \frac{1}{2}\psi_{-2}(x).\end{aligned}$$

Setting $x = 0$ in this latter identity, we obtain

$$\psi_{-2}\left(\frac{1}{2}\right) = \frac{5}{24}\ln 2 + \frac{3}{2}\ln A + \frac{1}{4}\ln\pi\,.$$

We also observe that Proposition 8.14 yields the same asymptotic expansions as in the case when $\nu \in \mathbb{N}^*$.

*Wallis's product formula.* We have for instance

$$\lim_{n\to\infty}\left(h(n) + \sum_{k=1}^{2n}(-1)^{k-1}g_{-2}(k)\right) = \frac{1}{12}\ln 2 - 3\ln A.$$

where $h(n) = \left(n+\frac{1}{4}\right)\ln n - n(1-\ln 2)$. Incidentally, the analogue of Wallis's formula for the function $g(x) = \psi_{-2}(x)$ is

$$\lim_{n\to\infty}\left(h(n) + \sum_{k=1}^{2n}(-1)^{k-1}\psi_{-2}(k)\right) = \ln A - \frac{1}{12}\ln 2\,,$$

where $h(n) = n^2\ln(2n) - \frac{3}{2}n^2 + \frac{1}{2}n\ln(2\pi) - \frac{1}{12}\ln n$. This latter formula is a little harder to obtain than the former one; it requires the computation of both functions $\Sigma\psi_{-2}(x)$ and $2\,\Sigma_x\psi_{-2}(2x)$ using Corollary 7.10 with $r = 2$. That is,

$$\begin{aligned}\Sigma\psi_{-2}(x) &= -\frac{1}{12}x(x-1)(2x-1) + \frac{1}{4}x(x+1)\ln(2\pi)\\ &\qquad + 2x\ln A + (x-1)\psi_{-2}(x) - 2\psi_{-3}(x)\end{aligned}$$

and

$$\begin{aligned}2\,\Sigma_x\psi_{-2}(2x) &= -\frac{1}{6}x(2x-1)(4x-1) + (4x+3)\ln A\\ &\qquad + \frac{1}{12}(-24x^2+48x+5)\ln 2 - 4\psi_{-2}(x)\\ &\qquad + 2x\psi_{-2}(2x) - 2\psi_{-2}\left(x+\frac{1}{2}\right) - 2\psi_{-3}(2x).\end{aligned}$$

*Reflection formula.* A reflection formula can be obtained by integrating the identity $\ln\Gamma(x)+\ln\Gamma(1-x)=\ln\pi-\ln\sin(\pi x)$. For example, for any $x\in(0,1)$, we have

$$\psi_{-2}(x)-\psi_{-2}(1-x) \;=\; x\,\ln\pi-\frac{1}{2}\ln(2\pi)-\int_0^x\ln\sin(\pi t)\,dt.$$

In particular, we obtain $\int_0^{1/2}\ln\sin(\pi t)\,dt=-\frac{1}{2}\ln 2$.

*Webster's functional equation.* For any $m\in\mathbb{N}^*$, there is a unique solution $f\colon\mathbb{R}_+\to\mathbb{R}$ to the equation $\sum_{j=0}^{m-1}f(x+\frac{j}{m})=g_\nu(x)$ that lies in $\mathcal{K}^{-\nu-1}$, namely

$$f(x) \;=\; \psi_\nu\left(x+\frac{1}{m}\right)-\psi_\nu(x)\,.$$

## 9.5 The Barnes G-function

The Barnes function $G\colon\mathbb{R}_+\to\mathbb{R}_+$ is the function $G=1/\Gamma_2$ (see Subsection 5.2). Hence, it can be defined by the equation $\ln G=\Sigma\ln\Gamma=\Sigma\psi_{-1}$.

*ID card.*

| $g(x)$ | Membership | $\deg g$ | $\Sigma g(x)$ |
|---|---|---|---|
| $\ln\Gamma(x)$ | $\mathcal{C}^\infty\cap\mathcal{D}^2\cap\mathcal{K}^\infty$ | 1 | $\ln G(x)$ |

*Characterization.* The function G can be characterized as follows:

> *All solutions $f\colon\mathbb{R}_+\to\mathbb{R}_+$ to the equation $f(x+1)=\Gamma(x)f(x)$ for which $\ln f$ lies in $\mathcal{K}^2$ are of the form $f(x)=c\,G(x)$, where $c>0$.*

*Asymptotic constant, generalized Stirling's and Euler's constants, Raabe's formula.*

| $\exp(\overline{\sigma}[g])$ | $\sigma[g]$ | $\gamma[g]$ |
|---|---|---|
| $e^{1/12}(2\pi)^{-1/4}A^{-2}$ | $\frac{1}{12}+\frac{1}{4}\ln(2\pi)-2\ln A$ | $\gamma[g]=\sigma[g]$ |

We have the inequality $|\gamma[g]|\leqslant\frac{5}{12}\ln 2$ as well as the following representations

$$\sigma[g] \;=\; \frac{1}{2}\ln(2\pi)+\lim_{n\to\infty}\left(\sum_{k=1}^{n}\ln\Gamma(k)-\psi_{-2}(n)-\frac{1}{2}\ln\Gamma(n)-\frac{1}{12}\ln n\right);$$

$$\sigma[g] \;=\; \sum_{k=1}^{\infty}\ln\left(\frac{\Gamma(k)\,e^k\sqrt{k}}{(1+\frac{1}{k})^{\frac{1}{12}}\,k^k\sqrt{2\pi}}\right);$$

$$\sigma[g] \;=\; \int_1^\infty\left(\ln\frac{\Gamma(\lfloor t\rfloor)}{\Gamma(t)}+\ln\frac{\lfloor t\rfloor^{7/12}}{\lfloor t+1\rfloor^{1/12}}\right)dt\,;$$

$$\sigma[g] \;=\; \int_0^1\ln G(t+1)\,dt.$$

Also, the analogue of Raabe's formula is

$$\int_x^{x+1} \ln G(t)\, dt \;=\; \overline{\sigma}[g] + \psi_{-2}(x)\,, \qquad x>0.$$

We also have for any $q \in \mathbb{N}$ and any $x>0$

$$\begin{aligned} J^{q+1}[\Sigma g](x) \;&=\; \ln G(x) - \psi_{-2}(x) + \frac{1}{4}\ln(2\pi) - \frac{1}{12} + 2\ln A \\ &\quad + \sum_{j=1}^{q} G_j\, \Delta^{j-1} g(x). \end{aligned}$$

For instance,

$$\begin{aligned} J^{3}[\Sigma g](x) \;&=\; \ln G(x) - \psi_{-2}(x) + \frac{1}{4}\ln(2\pi) - \frac{1}{12} \\ &\quad + 2\ln A + \frac{1}{2}\ln\Gamma(x) - \frac{1}{12}\ln x. \end{aligned}$$

Note that the functions $\ln G(x)$ and $\psi_{-2}(x)$ are strongly related (see (76) below) in the sense that we can easily express one of it in terms of the other.

*Restriction to the natural integers.* For any $n \in \mathbb{N}^*$ we have $G(n) = \prod_{k=0}^{n-2} k!$.

*Derivatives of* $\Sigma g(x)$ *at* $x=1$. For any $k \in \mathbb{N}^*$ we have

$$(\Sigma g)^{(k)}(1) \;=\; g^{(k-1)}(1) - \sigma[g^{(k)}].$$

We also have $\sigma[g'] = \sigma[\psi] = \frac{1}{2}(1-\ln(2\pi))$, $\sigma[g''] = \sigma[\psi_1] = 1$ and for any integer $k \geqslant 3$,

$$\sigma[g^{(k)}] \;=\; (-1)^k (k-2)(k-2)!\,\zeta(k-1).$$

The Taylor series expansion of $\ln G(x+1)$ about $x=0$ is (see, e.g., [76, p. 311])

$$\ln G(x+1) \;=\; \frac{1}{2}\,(\ln(2\pi)-1)\,x - \frac{\gamma+1}{2}\,x^2 - \sum_{k=2}^{\infty} \frac{\zeta(k)}{k+1}\,(-x)^{k+1}\,, \qquad |x|<1.$$

Integrating this equation on $(0,1)$, we obtain

$$\sum_{k=2}^{\infty} (-1)^k\, \frac{\zeta(k)}{(k+1)(k+2)} \;=\; \frac{1}{2} + \frac{1}{6}\,\gamma - 2\ln A.$$

Also, the exponential generating functions for the sequences $n \mapsto \sigma[g^{(n)}]$ and $n \mapsto \gamma[g^{(n)}]$ are

$$\mathrm{egf}_{\sigma}[g](x) \;=\; \ln G(x+1) - \psi_{-2}(x+1) + \frac{1}{4}\ln(2\pi) - \frac{1}{12} + 2\ln A$$

and

$$\mathrm{egf}_{\gamma}[g](x) \;=\; \mathrm{egf}_{\sigma}[g](x) + \frac{1}{2}\,\gamma\, x,$$

respectively. Integrating the first of these equations on $(0,1)$ (i.e., we use (60)), we obtain after some algebra

$$\sum_{k=2}^{\infty}(-1)^k\,\frac{k-1}{k(k+1)(k+2)}\,\zeta(k) ~=~ \frac{5}{4}-3\ln A-\frac{1}{4}\ln(2\pi)\,.$$

*Asymptotic analysis.* For any $x>0$ and any $a\geqslant 0$ we have

$$\left(1+\frac{1}{x}\right)^{-\lceil a\rceil\left|\binom{a-1}{2}\right|} ~\leqslant~ \frac{G(x+a)}{G(x)\,\Gamma(x)^a\,x^{\binom{a}{2}}} ~\leqslant~ \left(1+\frac{1}{x}\right)^{\lceil a\rceil\left|\binom{a-1}{2}\right|},$$

with equality if $a\in\{1,2\}$. Thus,

$$G(x+a) ~\sim~ G(x)\,\Gamma(x)^a\,x^{\binom{a}{2}} \qquad \text{as } x\to\infty,$$

In view of Wendel's inequalities for the gamma function (see Example 6.3), we conjecture that the inequalities above can be simplified and tightened by replacing the extreme functions by $(1+a/x)^{-\left|\binom{a-1}{2}\right|}$ and $(1+a/x)^{\left|\binom{a-1}{2}\right|}$.

We also have

$$|J^3[\Sigma g](x)| ~\leqslant~ \tfrac{5}{12}\,\ln\left(1+\tfrac{1}{x}\right), \qquad x>0,$$

that is, in the multiplicative notation,

$$\left(1+\frac{1}{x}\right)^{-5/12} ~\leqslant~ \frac{G(x)\,\Gamma(x)^{1/2}}{e^{\psi_{-2}(x)+\overline{\sigma}[g]}\,x^{1/12}} ~\leqslant~ \left(1+\frac{1}{x}\right)^{5/12}, \qquad x>0.$$

Thus, we obtain the following analogues of Stirling's formula

$$\begin{aligned} G(x) &~\sim~ \exp(\psi_{-2}(x)+\overline{\sigma}[g])\,\Gamma(x)^{-\frac{1}{2}}\,x^{\frac{1}{12}} \qquad \text{as } x\to\infty;\\ G(x+1) &~\sim~ \exp(\psi_{-2}(x)+\overline{\sigma}[g])\,\Gamma(x)^{\frac{1}{2}}\,x^{\frac{1}{12}} \qquad \text{as } x\to\infty. \end{aligned}$$

Using the definition of $G$ in terms of $\psi_{-2}(x)$ (see (76) below) as well as the Stirling formula for the gamma function, we obtain the following simpler form

$$G(x+1) ~\sim~ A^{-1}\,x^{\frac{1}{2}x^2-\frac{1}{12}}\,(2\pi)^{\frac{x}{2}}\,e^{-\frac{3}{4}x^2+\frac{1}{12}} \qquad \text{as } x\to\infty.$$

We also have, for any $a\geqslant 0$

$$\ln G(x+a) ~\sim~ \psi_{-2}(x) \qquad \text{as } x\to\infty.$$

Finally, recall that all the derivatives of $J^3[\Sigma g](x)$ vanish at infinity. For instance, the first derivative yields the convergence result

$$\ln\Gamma(x)-\left(x-\frac{1}{2}\right)\,\psi(x)+x ~\to~ \frac{1}{2}\,(1+\ln(2\pi)) \qquad \text{as } x\to\infty$$

while the second derivative gives $x\,\psi_1(x) \to 0$ as $x \to \infty$.

*Eulerian and Weierstrassian forms.* For any $x > 0$, we have

$$\begin{aligned} G(x) &= \frac{1}{\Gamma(x)} \prod_{k=1}^{\infty} \frac{\Gamma(k)}{\Gamma(x+k)}\, k^x (1+1/k)^{\binom{x}{2}} \\ &= \frac{e^{(-\gamma-1)\binom{x}{2}}}{\Gamma(x)} \prod_{k=1}^{\infty} \frac{\Gamma(k)}{\Gamma(x+k)}\, k^x e^{\psi_1(k)\binom{x}{2}}. \end{aligned}$$

Also, the analogue of Gauss' limit for the gamma function is

$$G(x) = \lim_{n\to\infty} \frac{\Gamma(1)\Gamma(2)\,\cdots\,\Gamma(n)}{\Gamma(x)\Gamma(x+1)\,\cdots\,\Gamma(x+n)}\, n!^x\, n^{\binom{x}{2}}.$$

*Alternative series expression and Fontana-Mascheroni's series.* Using Proposition 6.8, we also derive the following product representation: for any $x > 0$ we have

$$\begin{aligned} \ln G(x) &= \psi_{-2}(x) + \overline{\sigma}[g] - \frac{1}{2}\ln\Gamma(x) - \sum_{n=0}^{\infty} G_{n+2}\Delta^{n+1} g(x) \\ &= \psi_{-2}(x) + \overline{\sigma}[g] - \frac{1}{2}\ln\Gamma(x) - \sum_{n=0}^{\infty} |G_{n+2}| \sum_{k=0}^{n} (-1)^k \binom{n}{k} \ln(x+k). \end{aligned}$$

In the multiplicative notation:

$$\begin{aligned} G(x) &= \exp(\psi_{-2}(x) + \overline{\sigma}[g])\,\Gamma(x)^{-\frac{1}{2}}\, x^{\frac{1}{12}} \left(\frac{x+1}{x}\right)^{-\frac{1}{24}} \\ &\quad \times \left(\frac{(x+2)x}{(x+1)^2}\right)^{\frac{19}{720}} \left(\frac{(x+3)(x+1)^3}{(x+2)^3 x}\right)^{-\frac{3}{160}} \cdots \end{aligned}$$

Setting $x = 1$ in this identity yields the analogue of Fontana-Mascheroni series:

$$\overline{\sigma}[g] = -\frac{1}{2}\ln(2\pi) + \sum_{n=0}^{\infty} |G_{n+2}| \sum_{k=0}^{n} (-1)^k \binom{n}{k} \ln(k+1).$$

*Alternative representation.* Considering the antiderivative of the solution

$$\varphi(x) = (x-1)\psi(x) - x + \tfrac{1}{2} + \tfrac{1}{2}\ln(2\pi)$$

to the equation $\Delta\varphi = g' = \psi$, we obtain

$$\ln G(x) = -\binom{x}{2} + (x-1)\ln\Gamma(x) + \tfrac{1}{2}\ln(2\pi)\,x - \psi_{-2}(x). \tag{76}$$

This identity can also be proved directly using the characterization result; indeed, both sides vanish at $x = 1$ and are eventually 2-convex solutions to the equation $f(x+1) - f(x) = \ln\Gamma(x)$. Hence, they must coincide on $\mathbb{R}_+$.

*Gauss' multiplication formula.* For any $m \in \mathbb{N}^*$ and any $x > 0$, we have

$$\prod_{j=0}^{m-1} G\left(\frac{x+j}{m}\right) \;=\; e^{\Sigma_x \ln\Gamma(\frac{x}{m})} \prod_{j=1}^{m-1} G\left(\frac{j}{m}\right).$$

For instance, setting $m = 2$ in this identity, we obtain

$$\ln G\left(\frac{x+1}{2}\right) + \ln G\left(\frac{x}{2}\right) - \ln G\left(\frac{1}{2}\right) \;=\; \Sigma_x \ln\Gamma\left(\frac{x}{2}\right).$$

However, to make this multiplication formula interesting and usable, we need to find a simple expression for its right side. In particular, we need a closed-form expression for the function $\Sigma_x \ln\Gamma(\frac{x}{m})$.

Proposition 8.14 yields the following asymptotic expansion as $x \to \infty$. For any $m, q \in \mathbb{N}^*$ we have

$$\frac{1}{m}\sum_{j=0}^{m-1} \ln G\left(x + \frac{j}{m}\right) \;=\; \overline{\sigma}[g] + \sum_{k=0}^{q} \frac{B_k}{m^k\, k!}\, \psi_{k-2}(x) + O(\psi_{q-1}(x)).$$

Setting $m = 1$ in this formula, we obtain

$$\ln G(x) \;=\; \overline{\sigma}[g] + \sum_{k=0}^{q} \frac{B_k}{k!}\, \psi_{k-2}(x) + O(\psi_{q-1}(x)).$$

Thus, we have

$$\ln G(x) \;=\; \overline{\sigma}[g] + \psi_{-2}(x) - \frac{1}{2}\,\psi_{-1}(x) + \frac{1}{12}\,\psi(x) - \frac{1}{720}\,\psi_2(x) + O\left(\frac{1}{x^4}\right),$$

which is consistent with the analogue of Stirling's formula

$$-\ln G(x) + \overline{\sigma}[g] + \psi_{-2}(x) - \frac{1}{2}\,\psi_{-1}(x) + \frac{1}{12}\,\ln(x) \;\to\; 0 \qquad \text{as } x \to \infty.$$

*Wallis's product formula.* Using Legendre's duplication formula for the gamma function, we obtain

$$\begin{aligned}\Sigma_x \ln\Gamma(2x) \;&=\; \ln G(x) + \ln G(x+\tfrac{1}{2}) - \ln G(\tfrac{1}{2})\\ &\quad + (x^2+1)\ln 2 - \tfrac{x}{2}\ln(16\pi).\end{aligned}$$

Using this identity, we can derive the surprising analogue of Wallis's formula

$$\lim_{n\to\infty} \frac{\Gamma(1)\Gamma(3)\,\cdots\,\Gamma(2n-1)}{\Gamma(2)\Gamma(4)\,\cdots\,\Gamma(2n)} \left(\frac{2n}{e}\right)^{n} \;=\; \frac{1}{\sqrt{2}}.$$

Incidentally, the analogue of Wallis's formula for the function $g(x) = \ln G(x)$ is

$$\lim_{n\to\infty} \frac{G(1)G(3)\,\cdots\,G(2n-1)}{G(2)G(4)\,\cdots\,G(2n)}\, \frac{n^{n^2-\frac{1}{2}n-\frac{1}{24}}\, 2^{n^2-\frac{7}{24}}\, \pi^{\frac{1}{2}n}}{e^{\frac{3}{2}n^2-\frac{1}{2}n-\frac{1}{24}}} \;=\; A^{\frac{1}{2}},$$

which could be used to provide a new definition of the Glaisher-Kinkelin constant. This latter formula is a little harder to obtain than the former one; it requires the computation of both functions $\Sigma \ln G(x)$ and $2\,\Sigma_x \ln G(2x)$ using Corollary 7.10 with $r = 1$. That is,

$$\begin{aligned}\Sigma \ln G(x) &= -\frac{1}{8}\,x(x-1)(2x-5) + \frac{1}{4}\,x(x-3)\ln(2\pi) - x\ln A\\ &\quad + \frac{1}{2}\,(x-1)(x-2)\ln\Gamma(x) - \frac{1}{2}\,(2x-3)\,\psi_{-2}(x) + \psi_{-3}(x)\end{aligned}$$

and

$$\begin{aligned}2\,\Sigma_x \ln G(2x) &= -\frac{1}{4}\,x(2x-1)(4x-7) - 2x\ln A\\ &\quad + \frac{1}{2}\,(2x^2-3x-1)\ln 2 + x(x-2)\ln\pi\\ &\quad + \frac{1}{2}\ln\Gamma(x) + \frac{1}{2}(2x-1)(2x-3)\ln\Gamma(2x)\\ &\quad - 2(x-1)\,\psi_{-2}(2x) + \psi_{-3}(2x).\end{aligned}$$

*Reflection formula.* A reflection formula for the Barnes G-function is given in (71); see, e.g., [76, p. 45].

*Webster's functional equation.* For any $m \in \mathbb{N}^*$, there is a unique solution $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ to the equation $\prod_{j=0}^{m-1} f(x + \frac{j}{m}) = \Gamma(x)$ such that $\ln f$ lies in $\mathcal{K}^1$, namely

$$f(x) \;=\; \frac{G(x+\frac{1}{m})}{G(x)}.$$

## 9.6 The Hurwitz zeta function

For any $x > 0$, the function $s \mapsto \zeta(s,x)$ is defined as an analytic continuation to $\mathbb{C} \setminus \{1\}$ of the series (see, e.g., [76])

$$\sum_{k=0}^{\infty}(x+k)^{-s} \;=\; \frac{1}{\Gamma(s)}\int_0^{\infty}\frac{t^{s-1}e^{-xt}}{1-e^{-t}}\,dt, \qquad \Re(s) > 1.$$

It is known (see, e.g., [76, p. 160]) that this function satisfies the difference equation

$$\zeta(s,x+1) - \zeta(s,x) \;=\; -\,x^{-s}, \qquad x > 0. \tag{77}$$

For any fixed $s \in \mathbb{R} \setminus \{1\}$, define the function $g_s\colon \mathbb{R}_+ \to \mathbb{R}$ by $g_s(x) = -x^{-s}$. We then have $g_s \in \mathcal{K}^{\infty}$. If $s > 0$ and $s \neq 1$, then $g_s \in \mathcal{D}^0_{\mathbb{N}}$. If $s > 1$, then $g_s \in \widetilde{\mathcal{D}}^{-1}_{\mathbb{N}}$. If $-p < s < 1$ for some $p \in \mathbb{N}$, then $g_s \in \mathcal{D}^p_{\mathbb{N}}$, and hence we can consider $p = 1 + \deg g_s = \lfloor 1 - s \rfloor$. In all cases, we have $\Sigma g_s(x) = \zeta(s,x) - \zeta(s)$.

*ID card.*

| $g_s(x)$ | Membership | **deg** $g_s$ | $\Sigma g_s(x)$ |
|---|---|---|---|
| $-x^{-s}$ | $\mathcal{C}^\infty\cap\widetilde{\mathcal{D}}^{-1}_{\mathbb{N}}\cap\mathcal{K}^\infty$, if $s>1$,<br>$\mathcal{C}^\infty\cap\mathcal{D}^{\lfloor 1-s\rfloor}\cap\mathcal{K}^\infty$, if $s<1$. | $-1+\lfloor 1-s\rfloor_+$ | $\zeta(s,x)-\zeta(s)$ |

*Characterization.* The function $\zeta(s,x)$ can be characterized as follows:

> *All solutions* $f_s\colon\mathbb{R}_+\to\mathbb{R}$ *to the equation* $f_s(x+1)-f_s(x)=-x^{-s}$ *that lie in* $\mathcal{K}^{\lfloor 1-s\rfloor_+}$ *are of the form* $f_s(x)=c_s+\zeta(s,x)$*, where* $c_s\in\mathbb{R}$*.*

*Asymptotic constant, generalized Stirling's and Euler's constants, Raabe's formula.*

| $\overline{\sigma}[g_s]$ | $\sigma[g_s]$ | $\gamma[g_s]$ |
|---|---|---|
| $\infty$, if $s>1$,<br>$-\zeta(s)$, if $s<1$. | $\frac{1}{s-1}-\zeta(s)$ | $\sigma[g_s]-\sum_{j=1}^{\lfloor 1-s\rfloor_+}G_j\Delta^{j-1}g_s(1)$ |

We have the inequality

$$|\gamma[g_s]|\;\leqslant\;\overline{G}_{\lfloor 1-s\rfloor_+}\,|\Delta^{\lfloor 1-s\rfloor_+}g_s(1)|$$

as well as the following representations

$$\begin{aligned}
\sigma[g_s] &= \int_0^1(\zeta(s,t+1)-\zeta(s))\,dt\,,\\
\gamma[g_s] &= \int_1^\infty\left(\sum_{j=0}^{\lfloor 1-s\rfloor_+}G_j\Delta^j g_s(\lfloor t\rfloor)-g_s(t)\right)dt\,,\\
\sigma[g_s] &= \lim_{n\to\infty}\left(\frac{1-n^{1-s}}{s-1}-\sum_{k=1}^{n-1}k^{-s}-\sum_{j=1}^{\lfloor 1-s\rfloor_+}G_j\,\Delta^{j-1}g_s(n)\right),\\
\sigma[g_s] &= \sum_{j=1}^{\lfloor 1-s\rfloor_+}G_j\,\Delta^{j-1}g_s(1)\\
&\quad+\sum_{k=1}^{\infty}\left(\frac{k^{1-s}-(k+1)^{1-s}}{s-1}+\sum_{j=0}^{\lfloor 1-s\rfloor_+}G_j\,\Delta^j g_s(k)\right).
\end{aligned}$$

Also, the analogue of Raabe's formula is

$$\int_x^{x+1}\zeta(s,t)\,dt\;=\;\frac{x^{1-s}}{s-1}\,,\qquad x>0.$$

We also have for any $q \in \mathbb{N}$ and any $x > 0$

$$J^{q+1}[\Sigma g_s](x) \;=\; \zeta(s,x) - \frac{x^{1-s}}{s-1} + \sum_{j=1}^{q} G_j\, \Delta^{j-1} g(x).$$

*Restriction to the natural integers.* For any $n \in \mathbb{N}^*$ we have

$$\zeta(s,n) - \zeta(s) \;=\; -\sum_{k=1}^{n-1} k^{-s} \quad \text{and} \quad \zeta(s,n) \;=\; \sum_{k=n}^{\infty} k^{-s}.$$

Gregory's formula states that for any $n \in \mathbb{N}^*$ and any $q \in \mathbb{N}$ we have

$$\sum_{k=1}^{n-1} k^{-s} \;=\; \frac{1-n^{1-s}}{s-1} + \sum_{j=1}^{q} G_j \left(\Delta^{j-1} g_s(n) - \Delta^{j-1} g_s(1)\right) + R^q_{s,n}\,,$$

with

$$|R^q_{s,n}| \;\leqslant\; \overline{G}_q\, |\Delta^q g_s(n) - \Delta^q g_s(1)|.$$

Moreover, Proposition 6.8 gives the following series representation

$$\sum_{k=1}^{n-1} k^{-s} \;=\; \zeta(s) - \frac{x^{1-s}}{s-1} + \sum_{k=1}^{\infty} G_k \Delta^{k-1} g_s(n)\,, \quad n \in \mathbb{N}^*. \tag{78}$$

*Derivatives of* $\Sigma g_s(x)$ *at* $x = 1$. We have

$$(\Sigma g_s)^{(k)}(1) \;=\; (-s)^{\underline{k}}\, \zeta(s+k), \qquad k \in \mathbb{N}^*,$$

and

$$\sigma[g_s^{(k)}] \;=\; -\,(-s)^{\underline{k-1}}\,(1 + (-s-k+1)\,\zeta(s+k)), \qquad k \in \mathbb{N}.$$

The Taylor series expansion of $\zeta(s, x+1)$ about $x = 0$ is

$$\zeta(s, x+1) \;=\; \sum_{k=0}^{\infty} \tbinom{-s}{k}\, \zeta(s+k)\, x^k\,, \qquad |x| < 1.$$

Integrating this equation on $(0,1)$, we obtain the identity

$$\sum_{k=0}^{\infty} \tbinom{-s}{k} \frac{\zeta(s+k)}{k+1} \;=\; \frac{1}{s-1}\,, \qquad s < 2,\; s \notin \mathbb{Z}\,.$$

(When $s > 2$, the summand in the series above does not approach zero as $k$ increases.) Also, the exponential generating function for the sequence $n \mapsto \sigma[g_s^{(n)}]$ is

$$\mathrm{egf}_{\sigma}[g_s](x) \;=\; \frac{(x+1)^{1-s}}{s-1} - \zeta(s, x+1).$$

*Asymptotic analysis.* For any $a \geqslant 0$ and any $x > 0$, we have

$$\left|\zeta(s,x+a)-\zeta(s,x)-\sum_{j=1}^{\lfloor 1-s\rfloor_+}\tbinom{a}{j}\,\Delta^{j-1}g_s(x)\right| \;\leqslant\; \lceil a\rceil\left|\tbinom{a-1}{\lfloor 1-s\rfloor_+}\right|\left|\Delta^{\lfloor 1-s\rfloor_+}g_s(x)\right|.$$

In particular,

$$\zeta(s,x+a)-\zeta(s,x)-\sum_{j=1}^{\lfloor 1-s\rfloor_+}\tbinom{a}{j}\,\Delta^{j-1}g_s(x) \;\to\; 0 \qquad \text{as } x\to\infty,$$

with equality if $a \in \{1, 2, \ldots, \lfloor 1-s\rfloor_+\}$. Also, for any $x > 0$, we have

$$\left|\zeta(s,x)-\frac{x^{1-s}}{s-1}+\sum_{j=1}^{\lfloor 1-s\rfloor_+}G_j\Delta^{j-1}g_s(x)\right| \;\leqslant\; \overline{G}_{\lfloor 1-s\rfloor_+}\left|\Delta^{\lfloor 1-s\rfloor_+}g_s(x)\right|,$$

from which we derive the following analogue of Stirling's formula

$$\zeta(s,x)-\frac{x^{1-s}}{s-1}+\sum_{j=1}^{\lfloor 1-s\rfloor_+}G_j\Delta^{j-1}g_s(x) \;\to\; 0 \qquad \text{as } x\to\infty.$$

In particular, if $s > 1$, then $\zeta(s,x) \to 0$ as $x \to \infty$.

The results above enable us to investigate the asymptotic behavior of $\zeta(s,x)$ for large values of $x$. For instance, when $s = -\frac{3}{2}$ we obtain that for every $a \geqslant 0$ the expression

$$\zeta\left(-\tfrac{3}{2},x+a\right)+\tfrac{2}{5}\,x^{5/2}+\left(a-\tbinom{a}{2}-\tfrac{7}{12}\right)x^{3/2}+\left(\tbinom{a}{2}+\tfrac{1}{12}\right)(x+1)^{3/2}$$

approaches zero as $x \to \infty$.

We also have

$$\zeta(s,x+a) \;\sim\; \frac{x^{1-s}}{s-1} \qquad \text{as } x\to\infty.$$

Finally, if $s > -1$, then we have the analogue of Burnside's formula

$$\zeta(s,x)-\tfrac{1}{s-1}\,(x-\tfrac{1}{2})^{1-s} \;\to\; 0\,, \qquad \text{as } x\to\infty,$$

which provides a better approximation of $\zeta(s,x)$ than generalized Stirling's formula.

*Eulerian and Weierstrassian forms.* If $s > 1$, then for any $x > 0$, we simply have

$$\zeta(s,x) \;=\; \sum_{k=0}^{\infty}(x+k)^{-s}$$

and this series can be integrated and differentiated term by term. In particular, we observe that

$$\psi_\nu(x) \;=\; (-1)^{\nu+1}\nu!\,\zeta(\nu+1,x)\,, \qquad \nu\in\mathbb{N}^*,\; x>0.$$

If $s < 1$, then for any $x > 0$, we have

$$\begin{aligned}\zeta(s,x)-\zeta(s) &= -g_s(x)+\sum_{j=0}^{\lfloor -s\rfloor}\tbinom{x}{j+1}\Delta^j g_s(1)\\ &\quad +\sum_{k=1}^{\infty}\left(-g_s(x+k)+\sum_{j=0}^{\lfloor 1-s\rfloor}\tbinom{x}{j}\,\Delta^j g_s(k)\right)\end{aligned}$$

and the Weierstrassian form can be obtained similarly. Again, both series can be integrated and differentiated term by term.

For instance, we have

$$\begin{aligned}\zeta\left(-\tfrac{3}{2},x\right)-\zeta\left(-\tfrac{3}{2}\right) &= x^{\frac{3}{2}}+\lim_{n\to\infty}\left(\sum_{k=1}^{n-1}\left((x+k)^{\frac{3}{2}}-k^{\frac{3}{2}}\right)-x\,n^{\frac{3}{2}}-\tbinom{x}{2}\Delta_n n^{\frac{3}{2}}\right)\\ &= x^{\frac{3}{2}}-x-(2\sqrt{2}-1)\tbinom{x}{2}+\sum_{k=1}^{\infty}\left((x+k)^{\frac{3}{2}}-k^{\frac{3}{2}}-x\Delta_k k^{\frac{3}{2}}-\tbinom{x}{2}\Delta_k^2 k^{\frac{3}{2}}\right)\\ &= x^{\frac{3}{2}}-x+\tfrac{3}{4}\,\zeta\left(\tfrac{1}{2}\right)\tbinom{x}{2}+\sum_{k=1}^{\infty}\left((x+k)^{\frac{3}{2}}-k^{\frac{3}{2}}-x\Delta_k k^{\frac{3}{2}}-\tfrac{3}{4}\tbinom{x}{2}k^{-\frac{1}{2}}\right).\end{aligned}$$

*Alternative series expression and Fontana-Mascheroni's series.* Identity (78) is also valid for a real argument: for any $x > 0$ we have

$$\begin{aligned}\zeta(s,x) &= \frac{x^{1-s}}{s-1}-\sum_{n=0}^{\infty}G_{n+1}\Delta^n g_s(x)\\ &= \frac{x^{1-s}}{s-1}+\sum_{n=0}^{\infty}|G_{n+1}|\sum_{k=0}^{n}(-1)^k\tbinom{n}{k}(x+k)^{-s}.\end{aligned}$$

Setting $x = 1$ in this identity yields a known series expression for $\zeta(s)$ that is the analogue of Fontana-Mascheroni series

$$\zeta(s) = \frac{1}{s-1}+\sum_{n=0}^{\infty}|G_{n+1}|\sum_{k=0}^{n}(-1)^k\tbinom{n}{k}(k+1)^{-s}.$$

*Gauss' multiplication formula.* For any $m \in \mathbb{N}^*$ and any $x > 0$, we have

$$\sum_{j=0}^{m-1}\zeta\left(s,\frac{x+j}{m}\right) = \sum_{j=1}^{m-1}\zeta\left(s,\frac{j}{m}\right)+\zeta(s)+m^s(\zeta(s,x)-\zeta(s)).$$

Since $\sum_{j=1}^{m}\zeta(s,j/m) = m^s\zeta(s)$, the formula above actually reduces to

$$\sum_{j=0}^{m-1}\zeta\left(s,\frac{x+j}{m}\right) = m^s\zeta(s,x).$$

Also, Corollary 8.10 provides the following limits

$$\begin{aligned}\lim_{m\to\infty} m^{s-1}\zeta(s,mx) &= \frac{x^{1-s}}{s-1}, \qquad x>0,\ s<1,\\ \lim_{m\to\infty} m^{s-1}(\zeta(s,mx)-\zeta(s,m)) &= \frac{x^{1-s}-1}{s-1}, \qquad x>0,\ s\neq 1.\end{aligned}$$

Moreover, Proposition 8.14 yields the following asymptotic expansion as $x\to\infty$. For any $m,q\in\mathbb{N}^*$ we have

$$\begin{aligned}\frac{1}{m}\sum_{j=0}^{m-1}\zeta\left(s,x+\frac{j}{m}\right) &= \frac{x^{1-s}}{s-1}-\sum_{k=1}^{q}\binom{-s}{k-1}\frac{B_k}{k\,m^k\,x^{k+s-1}}+O\left(\frac{1}{x^{q+s}}\right)\\ &= \frac{1}{s-1}\sum_{k=0}^{q}\binom{1-s}{k}\frac{B_k}{m^k\,x^{k+s-1}}+O\left(\frac{1}{x^{q+s}}\right).\end{aligned}$$

Setting $m=1$ in this formula, we obtain

$$\zeta(s,x) = \frac{1}{s-1}\sum_{k=0}^{q}\binom{1-s}{k}\frac{B_k}{x^{k+s-1}}+O\left(\frac{1}{x^{q+s}}\right).$$

*Wallis's product formula.* If $s>1$, then we have

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^s} = (1-2^{1-s})\,\zeta(s) = \eta(s), \tag{79}$$

where $s\mapsto\eta(s)$ is Dirichlet's eta function. When $s<1$, the form of the formula strongly depends upon the value of $s$. When $s=-\frac{3}{2}$ for instance, we obtain

$$\lim_{n\to\infty}\left(h(n)+\sum_{k=1}^{2n}(-1)^k k^{\frac{3}{2}}\right) = (4\sqrt{2}-1)\,\zeta(-\tfrac{3}{2}).$$

where $h(n)=-\frac{8n+3}{4}\sqrt{\frac{n}{2}}$.

*Reflection formula.* If $s$ is an integer, then the extension to the domain $\mathbb{R}\setminus\mathbb{Z}$ of the function $\zeta(s,x)+(-1)^s\zeta(s,1-x)$ is 1-periodic. However, no closed-form expression for this function seems to be known.

*Webster's functional equation.* For any $m\in\mathbb{N}^*$, there is a unique solution $f_s\colon\mathbb{R}_+\to\mathbb{R}$ to the equation $\sum_{j=0}^{m-1}f_s(x+\frac{j}{m})=-x^{-s}$ that lies in $\mathcal{K}^{\lfloor -s\rfloor_+}$, namely

$$f_s(x) = \zeta\left(s,x+\frac{1}{m}\right)-\zeta(s,x).$$

**Example 9.2.** Consider the function $g(x)=x^2(x+1)^{-\frac{1}{2}}$. We then have $g(x)=(x+1)^{\frac{3}{2}}-2(x+1)^{\frac{1}{2}}+(x+1)^{-\frac{1}{2}}$ and hence

$$\Sigma g(x) = c-\zeta(-\tfrac{3}{2},x+1)+2\zeta(-\tfrac{1}{2},x+1)-\zeta(\tfrac{1}{2},x+1)$$

for some $c\in\mathbb{R}$. □

## 9.7 The generalized Stieltjes constants

Recall that the *generalized Stieltjes constants* are the numbers $\gamma_n(x)$ that occur in the Laurent series expansion of the Hurwitz zeta function

$$\zeta(s,x) \;=\; \frac{1}{s-1}+\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\,\gamma_n(x)(s-1)^n. \tag{80}$$

Here we naturally restrict the values of $x$ to the set $\mathbb{R}_+$. Recall also that the numbers $\gamma_n=\gamma_n(1)$, where $n\in\mathbb{N}$, are called the *Stieltjes constants.* For recent background on these constants, see, e.g., Blagouchine [16, 17] and Blagouchine and Coppo [19] (see also Nan-Yue and Williams [68]).

These constants are known to satisfy $\gamma_0(x)=-\psi(x)$ and $\gamma_0=\gamma$ as well as the following identities for every $q\in\mathbb{N}$

$$\begin{aligned}\gamma_q \;&=\; \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{(\ln k)^q}{k}-\frac{(\ln n)^{q+1}}{q+1}\right),\\ \gamma_q(x) \;&=\; \lim_{n\to\infty}\left(\sum_{k=0}^{n}\frac{(\ln(x+k))^q}{x+k}-\frac{(\ln(x+n))^{q+1}}{q+1}\right).\end{aligned}$$

Interestingly, the generalized Stieltjes constants also satisfy the difference equation

$$\gamma_q(x+1)-\gamma_q(x) \;=\; g_q(x),$$

where $g_q\colon\mathbb{R}_+\to\mathbb{R}$ is the function defined by the equation

$$g_q(x) \;=\; -\frac{1}{x}(\ln x)^q.$$

Thus, our theory is particularly suitable to investigate these constants. For any $q\in\mathbb{N}$, the function $g_q$ lies in $\mathcal{C}^\infty\cap\mathcal{D}^0\cap\mathcal{K}^\infty$ and is eventually increasing. By uniqueness of $\Sigma g_q$, it follows that $\Sigma g_q(x)=\gamma_q(x)-\gamma_q$.

*ID card.*

| $g_q(x)$ | Membership | deg $g_q$ | $\Sigma g_q(x)$ |
|---|---|---|---|
| $-\frac{1}{x}(\ln x)^q$ | $\mathcal{C}^\infty\cap\mathcal{D}^0\cap\mathcal{K}^\infty$ | $-1$ | $\gamma_q(x)-\gamma_q$ |

*Characterization.* The function $\gamma_q$ can be characterized as follows:

> *All eventually monotone solutions* $f\colon\mathbb{R}_+\to\mathbb{R}$ *to the equation* $f(x+1)-f(x)=g_q(x)$ *are of the form* $f(x)=c_q+\gamma_q(x)$*, where* $c_q\in\mathbb{R}$.

*Asymptotic constant, generalized Stirling's and Euler's constants, Raabe's formula.*

| $\overline{\sigma}[g_q]$ | $\sigma[g_q]$ | $\gamma[g_q]$ |
|---|---|---|
| $\infty$ | $-\gamma_q$ | $-\gamma_q$ |

We thus observe that the asymptotic constant $\sigma[g_q]$ is exactly the opposite of the Stieltjes constant $\gamma_q$. We also have the following representations

$$\begin{aligned}\gamma_q &= \sum_{k=1}^{\infty}\left(\frac{(\ln k)^q}{k}-\frac{(\ln(k+1))^{q+1}-(\ln(k))^{q+1}}{q+1}\right);\\ \gamma_q &= \int_1^{\infty}\left(\frac{(\ln\lfloor t\rfloor)^q}{\lfloor t\rfloor}-\frac{(\ln t)^q}{t}\right)dt.\end{aligned}$$

The analogue of Raabe's formula is

$$\int_x^{x+1}\gamma_q(t)\,dt \;=\; -\frac{(\ln x)^{q+1}}{q+1}, \qquad x>0.$$

We also have for any $r\in\mathbb{N}$ and any $x>0$

$$J^{r+1}[\Sigma g_q](x) \;=\; \gamma_q(x)+\frac{(\ln x)^{q+1}}{q+1}+\sum_{j=1}^{r}G_j\,\Delta^{j-1}g_q(x).$$

*Derivatives of* $\Sigma g_q(x)$ *at* $x=1$*.* We have

$$\gamma_q^{(k)}(1) \;=\; g_q^{(k-1)}(1)-\sigma[g_q^{(k)}] \;=\; -\sum_{j=1}^{\infty}g_q^{(k)}(j), \qquad k\in\mathbb{N}^*.$$

The exponential generating function for the sequence $n\mapsto\sigma[g_q^{(n)}]$ is

$$\mathbf{egf}_{\sigma}[g_q](x) \;=\; -\gamma_q(x+1)-\frac{1}{q+1}\,(\ln(x+1))^{q+1}.$$

*Asymptotic analysis.* Let $x>0$ be so that $g_q$ is increasing on $[x,\infty)$. Then for any $a\geqslant 0$, we have

$$\begin{aligned}|\gamma_q(x+a)-\gamma_q(x)| &\leqslant \lceil a\rceil\left|\frac{(\ln x)^q}{x}\right|;\\ \left|\gamma_q(x)+\frac{(\ln x)^{q+1}}{q+1}\right| &\leqslant \left|\frac{(\ln x)^q}{x}\right|.\end{aligned}$$

In particular, we have

$$\gamma_q(x+a)-\gamma_q(x)\;\to\;0 \quad\text{and}\quad \gamma_q(x)+\frac{(\ln x)^{q+1}}{q+1}\;\to\;0 \qquad \text{as } x\to\infty.$$

The latter convergence result is the analogue of Stirling's formula. It expresses the fact that the function $J^1[\Sigma g_q]$ vanishes at infinity. We also note that so do all its derivatives. For instance, we have

$$\gamma_q'(x)+\frac{(\ln x)^q}{x}\;\to\;0 \qquad \text{as } x\to\infty.$$

Also, for any $a \geqslant 0$, we have

$$\gamma_q(x+a) \sim -\frac{(\ln x)^{q+1}}{q+1} \qquad \text{as } x \to \infty.$$

Finally, for any $x > 0$ we have the inequalities

$$\left|\gamma_q\left(x+\tfrac{1}{2}\right) - g_q(x)\right| \leqslant |J^1[\Sigma g_q](x)| \leqslant |g_q(x)|,$$

which shows that the analogue of Burnside's formula

$$\gamma_q(x) + \tfrac{1}{q+1}(\ln(x-\tfrac{1}{2}))^{q+1} \to 0, \qquad \text{as } x \to \infty,$$

provides a better approximation of $\gamma_q(x)$ for large values of $x$ than the generalized Stirling formula. For $0 < x \leqslant 1$, we use the following approximations (see [68, p. 148])

$$\left|\gamma_0(x) - \frac{1}{x}\right| \leqslant \gamma \quad \text{and} \quad \left|\gamma_q(x) - \frac{(\ln x)^q}{x}\right| \leqslant \frac{(3+(-1)^q)(2q)!}{q^{q+1}(2\pi)^q}, \quad n \in \mathbb{N}^*.$$

*Eulerian and Weierstrassian forms.* For any $x > 0$, we have

$$\gamma_q(x) = \gamma_q + \frac{(\ln x)^q}{x} + \sum_{k=1}^{\infty}\left(\frac{(\ln(x+k))^q}{x+k} - \frac{(\ln k)^q}{k}\right)$$

and

$$\gamma_q(x) = \frac{(\ln x)^q}{x} + \sum_{k=1}^{\infty}\left(\frac{(\ln(x+k))^q}{x+k} - \frac{(\ln(k+1))^{q+1} - (\ln k)^{q+1}}{q+1}\right).$$

Both series can be differentiated and integrated term by term. Also, if $n \in \mathbb{N}^*$ is so that $g_q$ is increasing on $[n, \infty)$, then for any $x > 0$

$$\left|\gamma_q(x) - \gamma_q - \frac{(\ln x)^q}{x} - \sum_{k=1}^{n-1}\left(\frac{(\ln(x+k))^q}{x+k} - \frac{(\ln k)^q}{k}\right)\right| \leqslant \lceil x \rceil \left|\frac{(\ln n)^q}{n}\right|.$$

*Alternative series expression and Fontana-Mascheroni's series.* For any $x > 0$ satisfying the assumptions of Proposition 6.8, we obtain

$$\begin{aligned}\gamma_q(x) + \frac{(\ln x)^{q+1}}{q+1} &= \sum_{n=0}^{\infty} G_{n+1}\Delta_x^n \frac{(\ln x)^q}{x} \\ &= \sum_{n=0}^{\infty} |G_{n+1}| \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(\ln(x+k))^q}{x+k}.\end{aligned}$$

Setting $x = 1$ in this identity (provided that $x = 1$ satisfies the assumptions of Proposition 6.8), we obtain the Fontana-MascheroniâĂŹs series expression of $\gamma_q$

$$\gamma_q \;=\; \sum_{n=0}^{\infty}|G_{n+1}|\sum_{k=0}^{n}(-1)^k\tbinom{n}{k}\frac{(\ln(k+1))^q}{k+1}\,.$$

This latter expression was obtained by Blagouchine [17, p. 383].

*Antiderivative of* $\gamma_q(x)$*.* All eventually concave solutions $f\colon \mathbb{R}_+ \to \mathbb{R}$ to the equation

$$f(x+1)-f(x) \;=\; G_q(x),$$

where

$$G_q(x) \;=\; \int_1^x g_q(t)\,dt \;=\; -\frac{(\ln x)^{q+1}}{q+1} \;=\; \int_x^{x+1}\gamma_q(t)\,dt$$

are of the form

$$f(x) = c_q + \int_1^x \gamma_q(t)\,dt$$

for some $c_q \in \mathbb{R}$.

*Gauss' multiplication formula.* The following analogue of Gauss' multiplication formula was previously known (see also [16, p. 542]) but it can be derived straightforwardly from our results. For any $m \in \mathbb{N}^*$ and any $x > 0$, we have

$$\sum_{j=0}^{m-1}\gamma_q\left(\frac{x+j}{m}\right) \;=\; -\frac{m}{q+1}\left(\ln\frac{1}{m}\right)^{q+1} + m\sum_{j=0}^{q}\tbinom{q}{j}\left(\ln\frac{1}{m}\right)^{j}\gamma_{q-j}(x).$$

In particular,

$$\sum_{j=1}^{m}\gamma_q\left(\frac{j}{m}\right) \;=\; -\frac{m}{q+1}\left(\ln\frac{1}{m}\right)^{q+1} + m\sum_{j=0}^{q}\tbinom{q}{j}\left(\ln\frac{1}{m}\right)^{j}\gamma_{q-j}\,.$$

Also, Corollary 8.10 provides the following limits for $x > 0$

$$\lim_{m\to\infty}\sum_{j=0}^{q}\tbinom{q}{j}\left(\ln\frac{1}{m}\right)^{j}(\gamma_{q-j}(mx)-\gamma_{q-j}(m)) \;=\; -\frac{(\ln x)^{q+1}}{q+1}\,,$$

$$\lim_{m\to\infty}\left(-\frac{1}{q+1}\left(\ln\frac{1}{m}\right)^{q+1} + \sum_{j=0}^{q}\tbinom{q}{j}\left(\ln\frac{1}{m}\right)^{j}\gamma_{q-j}(mx)\right) \;=\; -\frac{(\ln x)^{q+1}}{q+1}\,.$$

For instance, setting $q = 1$ in these formulas yields

$$\lim_{m\to\infty}\gamma_1(mx)-\gamma_1(m)+(\ln m)(\psi(mx)-\psi(m)) \;=\; -\frac{1}{2}(\ln x)^2\,,$$

$$\lim_{m\to\infty}\gamma_1(mx)-\frac{1}{2}(\ln m)^2+\psi(mx)\ln m \;=\; -\frac{1}{2}(\ln x)^2\,.$$

Setting $m = 2$ in the multiplication formula, we obtain the following analogue of Legendre's duplication formula

$$\gamma_q\left(\frac{x}{2}\right) + \gamma_q\left(\frac{x+1}{2}\right) \;=\; -\frac{2}{q+1}\left(\ln\frac{1}{2}\right)^{q+1} + 2\sum_{j=0}^{q}\tbinom{q}{j}\left(\ln\frac{1}{2}\right)^{j}\gamma_{q-j}(x).$$

When $q = 0$ and $q = 1$, the multiplication formula reduces to the known formulas

$$\begin{aligned}\sum_{j=0}^{m-1}\psi\left(\frac{x+j}{m}\right) &= m(\psi(x) - \ln m)\,;\\ \sum_{j=0}^{m-1}\gamma_1\left(\frac{x+j}{m}\right) &= -\frac{m}{2}(\ln m)^2 + m(\ln m)\,\psi(x) + m\,\gamma_1(x).\end{aligned}$$

Moreover, Proposition 8.14 yields the following asymptotic expansion as $x \to \infty$. For any $m, q \in \mathbb{N}^*$ we have

$$\begin{aligned}\frac{1}{m}\sum_{j=0}^{m-1}\gamma_1\left(x+\frac{j}{m}\right) &= \frac{(\ln x)^2}{2} - (\ln x)\,\frac{1}{m}\sum_{j=0}^{m-1}\psi\left(x+\frac{j}{m}\right)\\ &\quad + \sum_{k=1}^{q}\frac{(-1)^{k-1}\,B_k\,H_{k-1}}{k(mx)^k} + O\left(\frac{1}{x^{q+1}}\right).\end{aligned}$$

Setting $m = 1$ in this latter formula, we obtain

$$\gamma_1(x) \;=\; \frac{(\ln x)^2}{2} - \psi(x)\ln x + \sum_{k=1}^{q}\frac{(-1)^{k-1}\,B_k\,H_{k-1}}{k\,x^k} + O\left(\frac{1}{x^{q+1}}\right).$$

Thus, we have

$$\gamma_1(x) \;=\; \frac{(\ln x)^2}{2} - \psi(x)\ln x - \frac{1}{12x^2} + \frac{11}{720x^4} + O\left(\frac{1}{x^6}\right).$$

*Wallis's product formula.* The analogue of Wallis's formula for the function $g_q(x)$ is

$$\sum_{k=1}^{\infty}(-1)^k\frac{(\ln k)^q}{k} \;=\; -\frac{(\ln 2)^{q+1}}{q+1} + \sum_{j=0}^{q-1}\tbinom{q}{j}\,(\ln 2)^{q-j}\gamma_j\,. \tag{81}$$

This formula was established by Briggs and Chowla [22, Eq. (8)]. For $q = 1$, it reduces to

$$\sum_{k=1}^{\infty}(-1)^k\frac{\ln k}{k} \;=\; -\frac{(\ln 2)^2}{2} + \gamma\ln 2\,.$$

For $q = 2$, we obtain

$$\sum_{k=1}^{\infty}(-1)^k\,\frac{(\ln k)^2}{k} \;=\; -\frac{(\ln 2)^3}{3}+\gamma(\ln 2)^2+2\gamma_1\ln 2\,.$$

These latter two formulas were also established by Hardy [40].

As an aside, let us establish conversion formulas between the sequences $q\mapsto\gamma_q$ and $q\mapsto\eta^{(q)}(1)$, where $\eta(s)$ is the Dirichlet eta function introduced in (79) and $\eta^{(q)}(1)$ stands for the limiting value of $\eta^{(q)}(s)$ as $s\to 1$. To ease the computations, let us instead consider the conversion formulas between the sequences $q\mapsto\gamma_q$ and $q\mapsto\lambda_q$, where

$$\lambda_q \;=\; \frac{1}{q+1}\,(\ln 2)^{q+1}+(-1)^{q+1}\,\eta^{(q)}(1)\,,\qquad q\in\mathbb{N}.$$

Using (81), we can readily derive the following equations

$$\lambda_q \;=\; \sum_{k=0}^{q-1}\tbinom{q}{k}\,(\ln 2)^{q-k}\,\gamma_k\,,\qquad q\in\mathbb{N}. \tag{82}$$

These equations actually consist of an infinite consistent triangular system. Solving this system provides the following conversion formula

$$\gamma_q \;=\; \sum_{k=0}^{q}\tbinom{q}{k}\,\frac{B_{q-k}}{k+1}\,(\ln 2)^{q-k-1}\,\lambda_{k+1}\,,\qquad q\in\mathbb{N}, \tag{83}$$

that is,

$$\gamma_q \;=\; -\frac{B_{q+1}}{q+1}\,(\ln 2)^{q+1}+\sum_{k=0}^{q}(-1)^k\,\tbinom{q}{k}\,\frac{B_{q-k}}{k+1}(\ln 2)^{q-k-1}\eta^{(k+1)}(1)\,,\quad q\in\mathbb{N}.$$

Indeed, plugging (83) in the right side of (82) we obtain for any $q\in\mathbb{N}$

$$\begin{aligned}\sum_{k=0}^{q-1}\tbinom{q}{k}\,(\ln 2)^{q-k}\,\gamma_k \;&=\; \sum_{k=0}^{q-1}\tbinom{q}{k}\,(\ln 2)^{q-k}\,\sum_{j=0}^{k}\tbinom{k}{j}\,\frac{B_{k-j}}{j+1}\,(\ln 2)^{k-j-1}\,\lambda_{j+1}\\ &=\; \sum_{j=0}^{q-1}\tbinom{q}{j}\,(\ln 2)^{q-j-1}\,\frac{\lambda_{j+1}}{j+1}\,\sum_{k=j}^{q-1}\tbinom{q-j}{k-j}\,B_{k-j}\,,\end{aligned}$$

where the inner sum reduces to $0^{q-j-1}$. The latter quantity then reduces to $\lambda_q$, as expected.

*Remark* 9.3. The conversion formulas (82) and (83) are not new. In essence, they were established by Liang and Todd [54, Eq. (3.6)] and Nan-Yue and Williams [68, Eqs. (1.9) and (7.1)].

*Webster's functional equation.* For any $m \in \mathbb{N}^*$, there is a unique eventually monotone solution $f\colon \mathbb{R}_+ \to \mathbb{R}$ to the equation $\sum_{j=0}^{m-1} f(x+\frac{j}{m}) = g_q(x)$, namely

$$f(x) \;=\; \gamma_q\left(x+\frac{1}{m}\right) - \gamma_q(x)\,.$$

*Rational arguments theorem.* Let us apply Proposition 8.19 to the function $g_q(x)$. For any $a, b \in \mathbb{N}^*$ with $a < b$ and any $j \in \{0, \ldots, b-1\}$ we have

$$S_j^b[g_q] \;=\; b\,(-1)^{q+1} \sum_{i=0}^{q} \tbinom{q}{i} (\ln b)^{q-i} D_s^i \operatorname{Li}_s(z)\big|_{(s,z)=(1,\omega_b^j)}$$

and hence

$$\gamma_q\left(\frac{a}{b}\right) - \gamma_q \;=\; (-1)^{q+1} \sum_{i=0}^{q} \tbinom{q}{i} (\ln b)^{q-i} \sum_{j=0}^{b-1} (1-\omega_b^{-aj}) D_s^i \operatorname{Li}_s(z)\big|_{(s,z)=(1,\omega_b^j)}\,.$$

We note that a more a practical formula was derived in the special case when $q = 1$ by Blagouchine [16] as a generalization of Gauss' digamma theorem.

## 9.8 Higher derivatives of the Hurwitz zeta function

Let $s \in \mathbb{R} \setminus \{1\}$ and $q \in \mathbb{N}$. Differentiating $q$ times both sides of (77) we obtain

$$\zeta^{(q)}(s, x+1) - \zeta^{(q)}(s, x) \;=\; (-1)^{q+1} x^{-s} (\ln x)^q\,, \qquad x > 0,$$

where $\zeta^{(q)}(s, x)$ stands for $D_s^q\, \zeta(s, x)$. This equation shows that the investigation of the higher derivatives of the Hurwitz zeta function can be carried out using our results. For an earlier investigation, see, e.g., [15, p. 36 *et seq.*].

*ID card.*

| $g_{s,q}(x)$ | Membership | $\deg g_{s,q}$ | $\Sigma g_{s,q}(x)$ |
|---|---|---|---|
| $-x^{-s}(-\ln x)^q$ | $\mathcal{C}^\infty \cap \widetilde{\mathcal{D}}_{\mathbb{N}}^{-1} \cap \mathcal{K}^\infty$, if $s > 1$,<br>$\mathcal{C}^\infty \cap \mathcal{D}^{\lfloor 1-s \rfloor} \cap \mathcal{K}^\infty$, if $s < 1$. | $-1$<br>$+\lfloor 1-s \rfloor_+$ | $\zeta^{(q)}(s, x)$<br>$-\zeta^{(q)}(s)$ |

We also observe that this investigation can be regarded as a simultaneous generalization of the studies of the Hurwitz zeta function and the generalized Stieltjes constants. For the latter, we observe that

$$(-1)^q \lim_{s \to 1} g_{s,q}(x) \;=\; -\frac{1}{x} (\ln x)^q.$$

Setting $s = 0$ in our results may also be very informative as it produces formulas involving the well-studied quantities $\zeta^{(q)}(0)$ and $\zeta^{(q)}(0, x) - \zeta^{(q)}(0)$ for any $q \in \mathbb{N}$.

*Characterization.* The function $\zeta^{(q)}(s, x)$ can be characterized as follows:

*All solutions* $f_{s,q}\colon \mathbb{R}_+ \to \mathbb{R}$ *to the equation* $f_{s,q}(x+1) - f_{s,q}(x) = g_{s,q}(x)$ *that lie in* $\mathcal{K}^{\lfloor 1-s \rfloor_+}$ *are of the form* $f_{s,q}(x) = c_{s,q} + \zeta^{(q)}(s,x)$, *where* $c_{s,q} \in \mathbb{R}$.

*Asymptotic constant, generalized Stirling's and Euler's constants, Raabe's formula.*

| $\overline{\sigma}[g_{s,q}]$ | $\sigma[g_{s,q}]$ | $\gamma[g_{s,q}]$ |
|---|---|---|
| $\infty$, if $s>1$, $-\zeta^{(q)}(s)$, if $s<1$. | $\frac{-q!}{(1-s)^{q+1}} - \zeta^{(q)}(s)$ | $\sigma[g_{s,q}] - \sum_{j=1}^{\lfloor 1-s \rfloor_+} G_j \Delta^{j-1} g_{s,q}(1)$ |

We have

$$
\begin{aligned}
\sigma[g_{s,q}] &= \lim_{n\to\infty}\left(\sum_{k=1}^{n-1} g_{s,q}(k) - \int_1^n g_{s,q}(t)\,dt + \sum_{j=1}^{\lfloor 1-s \rfloor_+} G_j \Delta^{j-1} g_{s,q}(n)\right)\\
&= \sum_{j=1}^{\lfloor 1-s \rfloor_+} G_j \Delta^{j-1} g_{s,q}(1)\\
&\quad - \sum_{k=1}^{\infty}\left(\int_k^{k+1} g_{s,q}(t)\,dt - \sum_{j=0}^{\lfloor 1-s \rfloor_+} G_j \Delta^{j} g_{s,q}(k)\right).
\end{aligned}
$$

Setting $s=0$ in the previous formulas, we obtain

$$
\begin{aligned}
(-1)^q(q! + \zeta^{(q)}(0)) &= \lim_{n\to\infty}\left(\sum_{k=1}^{n} (\ln k)^q - \int_1^n (\ln t)^q\,dt - \frac{1}{2}(\ln n)^q\right)\\
&= \sum_{k=1}^{\infty}\left(\frac{1}{2}(\ln k)^q - \int_k^{k+1} (\ln t)^q\,dt\right).
\end{aligned}
$$

On differentiating both sides of (80), we also obtain the following surprising identity

$$
(-1)^q(q! + \zeta^{(q)}(0)) = \sum_{n=0}^{\infty} \frac{\gamma_{n+q}}{n!}\,.
$$

We also have

$$
\int_1^x g_{s,q}(t)\,dt = \frac{q! - \Gamma(q+1,(s-1)\ln x)}{(1-s)^{q+1}}\,, \quad x>0,
$$

and hence the analogue of Raabe's formula is

$$
\begin{aligned}
\int_x^{x+1} \zeta^{(q)}(s,t)\,dt &= -\frac{\Gamma(q+1,(s-1)\ln x)}{(1-s)^{q+1}}\\
&= -q!\,\frac{x^{1-s}}{(1-s)^{q+1}} \sum_{j=0}^{q} \frac{((s-1)\ln x)^j}{j!}\,, \quad x>0.
\end{aligned}
$$

We also have for any $r \in \mathbb{N}$ and any $x > 0$

$$J^{r+1}[\Sigma g_{s,q}](x) \;=\; \zeta^{(q)}(s,x) - \int_x^{x+1} \zeta^{(q)}(s,t)\,dt + \sum_{j=1}^{r} G_j\,\Delta^{j-1} g_{s,q}(x).$$

*Restriction to the natural integers.* For any $n \in \mathbb{N}^*$ we have

$$\zeta^{(q)}(s,n) - \zeta^{(q)}(s) \;=\; \sum_{k=1}^{n-1} g_{s,q}(k).$$

Gregory's formula states that for any $n \in \mathbb{N}^*$ and any $r \in \mathbb{N}$ we have

$$\sum_{k=1}^{n-1} g_{s,q}(k) \;=\; \int_1^n g_{s,q}(t)\,dt - \sum_{j=1}^{r} G_j\left(\Delta^{j-1} g_s(n) - \Delta^{j-1} g_s(1)\right) - R^r_{s,q,n}\,,$$

with

$$|R^r_{s,q,n}| \;\leqslant\; \overline{G}_r\,|\Delta^r g_{s,q}(n) - \Delta^r g_{s,q}(1)|.$$

*Asymptotic analysis.* We have

$$\zeta^{(q)}(s,x+a) - \zeta^{(q)}(s,x) - \sum_{j=1}^{\lfloor 1-s\rfloor_+} \binom{a}{j}\,\Delta^{j-1} g_{s,q}(x) \;\to\; 0 \qquad \text{as } x \to \infty,$$

with equality if $a \in \{1, 2, \ldots, \lfloor 1-s\rfloor_+\}$. Also, we have the following analogue of Stirling's formula

$$\zeta^{(q)}(s,x) - \int_x^{x+1} \zeta^{(q)}(s,t)\,dt + \sum_{j=1}^{\lfloor 1-s\rfloor_+} G_j \Delta^{j-1} g_{s,q}(x) \;\to\; 0 \qquad \text{as } x \to \infty.$$

Setting $s = 0$ in this latter formula, we obtain

$$\zeta^{(q)}(0,x) + \Gamma(q+1, -\ln x) + \frac{1}{2}(-1)^{q+1}(\ln x)^q \;\to\; 0 \qquad \text{as } x \to \infty.$$

We also have

$$\zeta^{(q)}(s,x+a) \;\sim\; \int_x^{x+1} \zeta^{(q)}(s,t)\,dt \qquad \text{as } x \to \infty.$$

Finally, if $s > -1$, then we have the analogue of Burnside's formula

$$\zeta^{(q)}(s,x) - \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} \zeta^{(q)}(s,t)\,dt \;\to\; 0\,, \qquad \text{as } x \to \infty,$$

which provides a better approximation of $\zeta(s,x)$ than the analogue of Stirling's formula.

*Eulerian and Weierstrassian forms.* If $s > 1$, then for any $x > 0$, we simply have

$$\zeta^{(q)}(s,x) \;=\; -\sum_{k=0}^{\infty} g_{s,q}(x+k)$$

and this series can be integrated and differentiated term by term. If $s < 1$, then for any $x > 0$, the analogue of Gauss' limit is

$$\begin{aligned}\zeta^{(q)}(s,x)-\zeta^{(q)}(s) \;&=\; -\,g_{s,q}(x)\\ &\quad+\lim_{n\to\infty}\left(\sum_{k=1}^{n-1}(g_{s,q}(k)-g_{s,q}(x+k))+\sum_{j=1}^{\lfloor 1-s\rfloor}\tbinom{x}{j}\Delta^{j-1}g_{s,q}(n)\right).\end{aligned}$$

Also, the analogue of Euler's product form is

$$\begin{aligned}\zeta^{(q)}(s,x)-\zeta^{(q)}(s) \;&=\; -g_{s,q}(x)+\sum_{j=0}^{\lfloor -s\rfloor}\tbinom{x}{j+1}\Delta^{j}g_{s,q}(1)\\ &\quad+\sum_{k=1}^{\infty}\left(-g_{s,q}(x+k)+\sum_{j=0}^{\lfloor 1-s\rfloor}\tbinom{x}{j}\,\Delta^{j}g_{s,q}(k)\right)\end{aligned}$$

and the Weierstrassian form can be obtained similarly. Again, the series can be integrated and differentiated term by term. Note that the case where $(s,q) = (0,2)$ can be found in Ramanujan's second notebook [15, p. 26–27].

*Alternative series expression and Fontana-Mascheroni's series.* For any $x > 0$ satisfying the assumptions of Proposition 6.8, we obtain

$$\begin{aligned}\zeta^{(q)}(s,x) \;&=\; \int_x^{x+1}\zeta^{(q)}(s,t)\,dt-\sum_{n=0}^{\infty}G_{n+1}\Delta^{n}g_{s,q}(x)\\ &=\; \int_x^{x+1}\zeta^{(q)}(s,t)\,dt-\sum_{n=0}^{\infty}|G_{n+1}|\sum_{k=0}^{n}(-1)^k\tbinom{n}{k}\,g_{s,q}(x+k)\,.\end{aligned}$$

Setting $x = 1$ in this identity (provided that $x = 1$ satisfies the assumptions of Proposition 6.8) yields a series expression for $\zeta^{(q)}(s)$ that is the analogue of Fontana-Mascheroni series

$$\zeta^{(q)}(s) \;=\; \frac{-q!}{(1-s)^{q+1}}-\sum_{n=0}^{\infty}|G_{n+1}|\sum_{k=0}^{n}(-1)^k\tbinom{n}{k}\,g_{s,q}(k+1)\,,$$

which can also be obtained differentiating the analogue of Fontana-Mascheroni series for the Hurwitz zeta function. For instance, we have

$$\zeta''(0) \;=\; -\,2+\sum_{n=0}^{\infty}|G_{n+1}|\sum_{k=0}^{n}(-1)^k\tbinom{n}{k}\,(\ln(k+1))^2$$

and this value is also known to be (see, e.g., [15, p. 25])

$$\frac{1}{2}\gamma^2-\frac{\pi^2}{24}-\frac{1}{2}(\ln(2\pi))^2+\gamma_1\,.$$

*Gauss' multiplication formula.* Upon differentiating the analogue of Gauss' multiplication formula for the Hurwitz zeta function, we immediately obtain the following multiplication formula. For any $m\in\mathbb{N}^*$ and any $x>0$, we have

$$\sum_{j=0}^{m-1}\zeta^{(q)}\left(s,\frac{x+j}{m}\right) ~=~ m^s\sum_{j=0}^{q}\tbinom{q}{j}(\ln m)^{q-j}\,\zeta^{(j)}(s,x).$$

Also, Corollary 8.10 provides the following limit for any $x>0$ and any $s<1$

$$\lim_{m\to\infty}\sum_{j=0}^{q}\tbinom{q}{j}(\ln m)^{q-j}\,\frac{\zeta^{(j)}(s,mx)}{m^{1-s}} ~=~ -\frac{\Gamma(q+1,(s-1)\ln x)}{(1-s)^{q+1}}\,.$$

Also, for any $s\neq 1$, we have

$$\lim_{m\to\infty}\sum_{j=0}^{q}\tbinom{q}{j}(\ln m)^{q-j}\,\frac{\zeta^{(j)}(s,mx)-\zeta^{(j)}(s,m)}{m^{1-s}} ~=~ \frac{q!-\Gamma(q+1,(s-1)\ln x)}{(1-s)^{q+1}}\,.$$

*Wallis's product formula.* When $s<1$, the form of the analogue of Wallis's product formula strongly depends upon the value of $s$. If $s>1$, then we have

$$\begin{aligned}\eta^{(q)}(s) &= \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^s}(-\ln k)^q\\ &= \zeta^{(q)}(s)-2^{1-s}\sum_{j=0}^{q}\tbinom{q}{j}\left(\ln\frac{1}{2}\right)^{q-j}\zeta^{(j)}(s),\end{aligned}$$

where $s\mapsto\eta(s)$ is Dirichlet's eta function. Just as we did for the formulas (82) and (83), we can easily establish the following conversion formulas for $s>1$

$$\begin{aligned}\mu_q(s) &= \sum_{k=0}^{q-1}\tbinom{q}{k}\left(\ln\frac{1}{2}\right)^{q-k}\zeta^{(k)}(s)\,, \qquad q\in\mathbb{N}\,,\\ \zeta^{(q)}(s) &= \sum_{k=0}^{q}\tbinom{q}{k}\frac{B_{q-k}}{k+1}\left(\ln\frac{1}{2}\right)^{q-k-1}\mu_{k+1}(s)\,, \qquad q\in\mathbb{N}\,,\end{aligned}$$

where

$$\mu_q(s) ~=~ 2^{s-1}(\zeta^{(q)}(s)-\eta^{(q)}(s))-\zeta^{(q)}(s)\,, \qquad q\in\mathbb{N}.$$

*Webster's functional equation.* For any $m\in\mathbb{N}^*$, there is a unique solution $f_{s,q}\colon\mathbb{R}_+\to\mathbb{R}$ to the equation $\sum_{j=0}^{m-1}f_{s,q}(x+\frac{j}{m})=g_{s,q}(x)$ that lies in $\mathcal{K}^{\lfloor -s\rfloor_+}$, namely

$$f_{s,q}(x) ~=~ \zeta^{(q)}\left(s,x+\frac{1}{m}\right)-\zeta^{(q)}(s,x)\,.$$

## 9.9 The principal indefinite sum of the Hurwitz zeta function

For any $s \in \mathbb{R} \setminus \{1\}$, we define the function $\zeta_2(s,\cdot)\colon \mathbb{R}_+ \to \mathbb{R}$ by the equation

$$\zeta_2(s,x) \;=\; \Sigma_x \zeta(s,x).$$

Thus defined, this function can be studied through our results. Contrary to the previous examples, here we introduce a completely new function that has seemingly no closed form in terms of known elementary functions. Hence we give it a new symbol and a new name. To keep this investigation simple we restrict ourselves to the case when $s > 2$, for which the sequence $n \mapsto \zeta(s,n)$ is summable. We then introduce

$$\kappa(s) \;=\; \sum_{k=1}^{\infty} \zeta(s,k)$$

and we note that

$$\int_1^{\infty} \zeta(s,t)\,dt \;=\; \frac{\zeta(s-1)}{s-1}\,.$$

*ID card.*

| $g_s(x)$ | Membership | $\deg g_s$ | $\Sigma g_s(x)$ |
|---|---|---|---|
| $\zeta(s,x)$ | $\mathcal{C}^{\infty} \cap \widetilde{\mathcal{D}}_{\mathbb{N}}^{-1} \cap \mathcal{K}^{\infty}$ | $-1$ | $\zeta_2(s,x)$ |

*Characterization.* The function $\zeta(s,x)$ can be characterized as follows:

> *All eventually monotone solutions* $f_s\colon \mathbb{R}_+ \to \mathbb{R}$ *to the equation* $f_s(x+1) - f_s(x) = \zeta(s,x)$ *are of the form* $f_s(x) = c_s + \zeta_2(s,x)$, *where* $c_s \in \mathbb{R}$.

*Asymptotic constant, generalized Stirling's and Euler's constants, Raabe's formula.*

| $\overline{\sigma}[g_s]$ | $\sigma[g_s]$ | $\gamma[g_s]$ |
|---|---|---|
| $\infty$ | $\kappa(s) - \frac{\zeta(s-1)}{s-1}$ | $\gamma[g_s] = \sigma[g_s]$ |

We have the inequality $|\sigma[g_s]| \leqslant \zeta(s)$ and the following representations

$$\sigma[g_s] \;=\; \int_0^1 \zeta_2(s,t+1)\,dt \;=\; \int_1^{\infty} \big(\zeta(s,\lfloor t\rfloor) - \zeta(s,t)\big)\,dt\,.$$

Also, the analogue of Raabe's formula is

$$\int_x^{x+1} \zeta_2(s,t)\,dt \;=\; \kappa(s) - \frac{\zeta(s-1,x)}{s-1}\,, \qquad x > 0.$$

We also have for any $q \in \mathbb{N}$ and any $x > 0$

$$J^{q+1}[\Sigma g_s](x) \;=\; \zeta_2(s,x) - \kappa(s) + \frac{\zeta(s-1,x)}{s-1} + \sum_{j=1}^{q} G_j \, \Delta_x^{j-1} \zeta(s,x).$$

*Derivatives of* $\Sigma g_s(x)$ *at* $x = 1$. We have

$$(\Sigma g_s)^{(k)}(1) \;=\; (-1)^{k-1} k! \tbinom{s}{k} \kappa(s+k), \qquad k \in \mathbb{N}^*,$$

and

$$\sigma[g_s^{(k)}] \;=\; (-1)^{k-1}(k-1)! \tbinom{s}{k-1} \zeta(s+k-1) + (-1)^k k! \tbinom{s}{k} \kappa(s+k), \quad k \in \mathbb{N}^*.$$

The Taylor series expansion of $\zeta_2(s, x+1)$ about $x = 0$ is

$$\zeta_2(s, x+1) \;=\; -\sum_{k=1}^{\infty} \tbinom{s}{k} \kappa(s+k)(-x)^k \,.$$

*Asymptotic analysis.* For any $a \geqslant 0$ and any $x > 0$, we have

$$\begin{aligned} |\zeta_2(s, x+a) - \zeta_2(s,x)| \quad &\leqslant \quad \lceil a \rceil \, \zeta(s,x)\,; \\ \left| \zeta_2(s,x) - \kappa(s) + \frac{\zeta(s-1,x)}{s-1} \right| \quad &\leqslant \quad \zeta(s,x). \end{aligned}$$

In particular, we have $\zeta_2(s,x) \to \kappa(s)$ as $x \to \infty$.

We also have

$$\zeta_2(s,x) \;\sim\; \kappa(s) - \frac{\zeta(s-1, x-\frac{1}{2})}{s-1} \qquad \text{as } x \to \infty.$$

*Eulerian and Weierstrassian forms.* For any $x > 0$, we have

$$\zeta_2(s,x) \;=\; \kappa(s) - \sum_{k=0}^{\infty} \zeta(s, x+k).$$

and this series can be integrated and differentiated term by term. In particular,

$$\left| \frac{\zeta(s-1,x)}{s-1} - \sum_{k=0}^{\infty} \zeta(s,x+k) \right| \;\leqslant\; \zeta(s,x)$$

and

$$\sum_{k=0}^{\infty} \zeta(s, x+k) \;\to\; 0 \qquad \text{as } x \to \infty.$$

*Alternative series expression and Fontana-Mascheroni's series.* Proposition 6.8 gives the following series representation: for any $x > 0$ we have

$$\begin{aligned}\zeta_2(s,x) &= \kappa(s) - \frac{\zeta(s-1,x)}{s-1} - \sum_{n=0}^{\infty} G_{n+1}\,\Delta_x^n\,\zeta(s,x)\\ &= \kappa(s) - \frac{\zeta(s-1,x)}{s-1} - \sum_{n=0}^{\infty} |G_{n+1}| \sum_{k=0}^{n} (-1)^k \tbinom{n}{k} \zeta(s,x+k)\,.\end{aligned}$$

Setting $x = 1$ in this identity yields the analogue of Fontana-Mascheroni series:

$$\sum_{n=0}^{\infty} |G_{n+1}| \sum_{k=0}^{n} (-1)^k \tbinom{n}{k} \zeta(s,k+1) \;=\; \kappa(s) - \frac{\zeta(s-1)}{s-1}\,.$$

*Wallis's product formula.* We have

$$\begin{aligned}\sum_{k=1}^{\infty} (-1)^{k-1} \zeta(s,k) &= (2-2^{1-s})\zeta(s) + (1-2^{1-s})\kappa(s)\\ &\qquad - 2^{1-s} \sum_{k=0}^{\infty} \zeta\left(s, k+\frac{1}{2}\right).\end{aligned}$$

This formula is obtained by combining Proposition 6.4 with the duplication formula for the Hurwitz zeta function

$$2\,\zeta(s,2x) \;=\; 2^{1-s}\zeta(s,x) + 2^{1-s}\zeta\left(s, x+\frac{1}{2}\right).$$

*Webster's functional equation.* For any $m \in \mathbb{N}^*$, there is a unique eventually monotone solution $f\colon \mathbb{R}_+ \to \mathbb{R}$ to the equation $\sum_{j=0}^{m-1} f(x+\frac{j}{m}) = \zeta(s,x)$, namely

$$f(x) \;=\; \zeta_2\left(s, x+\frac{1}{m}\right) - \zeta_2(s,x)\,.$$

## 9.10 The Catalan number function

The Catalan number function is the restriction to $\mathbb{R}_+$ of the map $x \mapsto C_x$ defined on $(-\frac{1}{2},\infty)$ by $C_x = \frac{1}{x+1}\binom{2x}{x}$. This function satisfies the equation

$$C_{x+1} \;=\; \left(4 - \frac{6}{x+2}\right) C_x\,.$$

The additive version of this equation reads $\Delta f = g$, where the function $g$ is the logarithm of a rational function. We observe that such equations have been thoroughly investigated by Anastassiadis [6, p. 41] (see also [49]).

*ID card.*

| $g(x)$ | Membership | $\deg g$ | $\Sigma g(x)$ |
|---|---|---|---|
| $\ln\left(4-\frac{6}{x+2}\right)$ | $\mathcal{C}^\infty \cap \mathcal{D}^1 \cap \mathcal{K}^\infty$ | $0$ | $\ln C_x$ |

*Characterization.* The function $C_x$ can be characterized as follows:

> *All solutions* $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ *to the equation* $(x+2)f(x+1) = (4x+2)f(x)$ *for which* $\ln f$ *lies in* $\mathcal{K}^1$ *are of the form* $f(x) = c\, C_x$*, where* $c > 0$.

*Asymptotic constant, generalized Stirling's and Euler's constants, Raabe's formula.*

| $\exp(\overline{\sigma}[g])$ | $\sigma[g]$ | $\gamma[g]$ |
|---|---|---|
| $\frac{1}{2\sqrt{2\pi}}\, e^{3/2}$ | $\frac{1}{2}\left(3+\ln\frac{8}{27\pi}\right)$ | $\frac{1}{2}\left(3+\ln\frac{4}{27\pi}\right)$ |

We have the inequality $|\gamma[g]| \leqslant \frac{1}{2}\ln\frac{5}{4}$ and the following representations

$$\begin{aligned}\gamma[g] &= \int_1^\infty (t-\lfloor t\rfloor-\tfrac{1}{2})\tfrac{3}{(t+2)(2t+1)}\,dt\,;\\ \sigma[g] &= \int_0^1 \ln C_{t+1}\,dt.\end{aligned}$$

Also, Raabe's formula is

$$\int_x^{x+1} \ln C_t\,dt ~=~ \ln\left(\frac{e^{\frac{3}{2}}(4x+2)^{x+\frac{1}{2}}}{\sqrt{\pi}\,(x+2)^{x+2}}\right), \qquad x>0.$$

*Restriction to the natural integers.* For any $n \in \mathbb{N}^*$ we have $C_n = \frac{1}{n+1}\binom{2n}{n}$.

*Asymptotic analysis.* For any $a \geqslant 0$, we have

$$\frac{C_{x+a}}{C_x} ~\sim~ 4^a \quad\text{and}\quad C_x ~\sim~ \frac{4^x}{x^{3/2}\sqrt{\pi}} \qquad \text{as } x\to\infty.$$

Also, the analogue of Burnside's formula gives

$$\ln C_x - \ln\left(\frac{e^{\frac{3}{2}}(4x)^x}{\sqrt{\pi}\,(x+\frac{3}{2})^{x+\frac{3}{2}}}\right) ~\to~ 0 \qquad \text{as } x\to\infty.$$

*Eulerian and Weierstrassian forms.* For any $x>0$, we have

$$C_x ~=~ \frac{x+2}{4x+2}\,2^x \prod_{k=1}^\infty \frac{\left(2-\frac{3}{k+3}\right)^x}{\left(2-\frac{3}{k+2}\right)^{x-1}\left(2-\frac{3}{x+k+2}\right)}$$

and

$$C_x ~=~ \frac{x+2}{4x+2}\,e^{-\frac{x}{2}} \prod_{k=1}^\infty \frac{1+\frac{x}{k+2}}{1+\frac{2x}{2k+1}}\,e^{\frac{3x}{(k+2)(2k+1)}}.$$

# 10 Further examples

The scope of applications of our theory is very wide since it applies to any function lying in $\cup_{p\geqslant 0}(\mathcal{D}^p\cap\mathcal{K}^p)$. In Section 9, we have made a thorough study of some special functions. In the present section, we briefly discuss further examples that the reader may want to explore in more detail.

## 10.1 The multiple gamma functions

The multiple gamma functions introduced in Subsection 5.2 can also be studied through the sequence of functions $G_0, G_1, \ldots$, defined by (see [76, p. 56])

$$G_p(x) \;=\; \Gamma_p(x)^{(-1)^{p-1}}, \qquad p\in\mathbb{N}.$$

Equivalently, we have $G_0(x)=x$ and $\ln G_p=\Sigma\ln G_{p-1}$ for all $p\in\mathbb{N}^*$. Clearly, the function $\ln G_{p-1}$ lies in $\mathcal{C}^\infty\cap\mathcal{D}^p\cap\mathcal{K}^\infty$ and we have $\deg\ln G_p=p$. Also, the sequence can naturally be extended to $p=-1$ by setting $G_{-1}(x)=1+1/x$.

Just as for the gamma function and the Barnes G-function, we can derive the following asymptotic equivalence: for any $a\geqslant 0$,

$$\frac{G_p(x+a)}{G_p(x)} \;\sim\; \prod_{j=0}^{p-1} G_{p-j-1}(x)^{\binom{a}{j+1}} \qquad \text{as } x\to\infty,$$

with equality if $a\in\{1,2,\ldots,p\}$. We also have the following product representation

$$G_p(x) \;=\; \frac{1}{G_{p-1}(x)}\prod_{k=1}^{\infty}\frac{G_{p-1}(k)}{G_{p-1}(x+k)}\, G_{p-2}(k)^x G_{p-3}(k)^{\binom{x}{2}}\,\cdots\, G_{-1}(k)^{\binom{x}{p}}$$

and the recurrence formula

$$\ln G_p(x) \;=\; -\,(x-1)\,\sigma[D\ln G_{p-1}]+\int_1^x \Sigma D\ln G_{p-1}(t)\,dt.$$

For example, one can show that

$$\begin{aligned}\ln G_3(x) \;=\; &-\frac{1}{8}\,x(x-1)(2x-5)+\frac{1}{4}\,x(x-2)\ln(2\pi)+\tbinom{x-1}{2}\ln\Gamma(x)\\ &-\frac{1}{2}(2x-3)\,\psi_{-2}(x)+\psi_{-3}(x)-x\,\psi_{-3}(1).\end{aligned}$$

(This formula can also be checked by means of the characterization of $G_3$ as a 3-convex solution to the equation $\Delta f=\ln G_2$.) More generally, from the known expressions for $G_0$, $G_1$, $G_2$, and $G_3$, we can derive the general formula

$$\ln G_p(x) \;=\; h_p(x)+\sum_{j=0}^{p-1}(-1)^{p-1-j}\tbinom{x-1}{j}\left(\psi_{j-p}(x)-x\,\psi_{j-p}(1)\right),$$

where $h_0(x) = \ln x$, $h_1(x) = 0$, $h_2(x) = -\binom{x}{2}$, and

$$h_3(x) \;=\; -\frac{1}{8}\,x(x-1)(2x-5) - \frac{1}{2}x^2\,\psi_{-2}(1) + \frac{1}{2}\,\psi_{-2}(x).$$

## 10.2 The hyperfactorial function

The hyperfactorial function (or K-function) is the solution $K\colon \mathbb{R}_+ \to \mathbb{R}$ to the equation $K(x+1) = x^x K(x)$ defined by the identity $\ln K = \Sigma\Delta \ln K$. Consider the function $g = \Delta \ln K$, that is, $g(x) = x \ln x$. Using the identity

$$\Delta\psi_{-2}(x) \;=\; \int_0^x \ln t\,dt + \psi_{-2}(1),$$

we derive $g(x) = x + \Delta\psi_{-2}(x) - \psi_{-2}(1)$, and hence

$$\ln K(x) \;=\; \Sigma g(x) \;=\; \binom{x}{2} + \psi_{-2}(x) - x\,\psi_{-2}(1) \;=\; (x-1)\ln\Gamma(x) - \ln G(x).$$

The integer sequence $n \mapsto K(n)$ has entry A002109 in the OEIS database [75].

## 10.3 The Hurwitz Lerch transcendent

The Hurwitz Lerch transcendent $\Phi(z,s,a)$ is a generalization of the Hurwitz zeta function defined as an analytic continuation of the series $\sum_{k=0}^{\infty} z^k(a+k)^{-s}$ when $|z| < 1$ and $a \in \mathbb{C} \setminus \mathbb{N}$ (see, e.g., [76]). It satisfies the difference equation

$$\Phi(z,s,a+1) - z^{-1}\Phi(z,s,a) \;=\; -z^{-1}a^{-s}.$$

It follows that the modified function $\overline{\Phi}(z,s,a) = -z^a\,\Phi(z,s,a)$ satisfies the difference equation

$$\overline{\Phi}(z,s,a+1) - \overline{\Phi}(z,s,a) \;=\; z^a a^{-s}.$$

Thus, for some real values of $z$ and $s$, the restriction to $\mathbb{R}_+$ of the map $a \mapsto \overline{\Phi}(z,s,a)$ fits the assumptions of our theory. Its complete investigation through our results is left to the reader.

## 10.4 The regularized incomplete gamma function

Consider the 2-variable function $Q(x,s) = \Gamma(x,s)/\Gamma(x)$ on $\mathbb{R}^2_+$, where $\Gamma(x,s)$ is the upper incomplete gamma function. Thus defined, the function $Q(x,s)$ satisfies the difference equation

$$Q(x+1,s) - Q(x,s) \;=\; e^{-s}s^x/\Gamma(x+1).$$

For any $s > 0$, we define the function $g_s\colon \mathbb{R}_+ \to \mathbb{R}$ by $g_s(x) = e^{-s}s^x/\Gamma(x+1)$. This function lies in $\mathcal{C}^\infty \cap \widetilde{\mathcal{D}}^{-1}_{\mathbb{N}} \cap \mathcal{K}^\infty$ and has the property that $\Sigma g_s(x) = Q(x,s) - e^{-s}$.

We also note that the Eulerian form of $Q(x,s)$ is

$$
\begin{aligned}
Q(x,s) &= 1-\sum_{k=0}^{\infty} g_s(x+k) \;=\; 1-\frac{e^{-s}s^x}{\Gamma(x+1)}\sum_{k=0}^{\infty}\frac{\Gamma(x+1)}{\Gamma(x+k+1)}\,s^k \\
&= 1-\frac{e^{-s}s^x}{\Gamma(x+1)}\sum_{k=0}^{\infty} x^{\underline{-k}}\,s^k.
\end{aligned}
$$

## 10.5 The regularized incomplete beta functions

Consider the restriction to $(0,1]\times\mathbb{R}_+^2$ of the map

$$
(x,a,b)\;\mapsto\; I_x(a,b) \;=\; \frac{B(x;a,b)}{B(a,b)}\,,
$$

where $B(x;a,b)$ is the incomplete beta function. Thus defined, the function $I_x(a,b)$ satisfies the difference equation

$$
I_x(a+1,b)-I_x(a,b) \;=\; -\frac{x^a(1-x)^b}{a\,B(a,b)}\,.
$$

Defining the function $g_{b,x}\colon\mathbb{R}_+\to\mathbb{R}$ by $g_{b,x}(a)=-x^a(1-x)^b/(a\,B(a,b))$ for any fixed $x$ and $b$, we can investigate the difference equation above. We leave it as an exercise.

## 10.6 The error function

The function $g\colon\mathbb{R}_+\to\mathbb{R}$ defined by the equation

$$
g(x) \;=\; \tfrac{2}{\sqrt{\pi}}\,e^{-x^2}
$$

lies in $\mathcal{C}^\infty\cap\widetilde{\mathcal{D}}^{-1}_{\mathbb{N}}\cap\mathcal{K}^\infty$ and satisfies the equation $\mathbf{erf}(x)=\int_0^x g(t)\,dt$, where $\mathbf{erf}$ is the Gauss error function. For $x>0$ we then have

$$
\Sigma g(x) \;=\; \frac{2}{\sqrt{\pi}}\sum_{k=0}^{\infty}\left(e^{-(k+1)^2}-e^{-(k+x)^2}\right)
$$

The generalized Stirling formula yields the following convergence result

$$
\mathbf{erf}(x)+\frac{2}{\sqrt{\pi}}\sum_{k=0}^{\infty}e^{-(k+x)^2} \;\to\; 1 \qquad \text{as } x\to\infty.
$$

Also, the analogue of Legendre's duplication formula provides the surprising identity

$$
\sum_{k=0}^{\infty}\left(-e^{-(k+\frac{x}{2})^2}+e^{-(k+1)^2}-e^{-(k+\frac{x+1}{2})^2}+e^{-(k+\frac{1}{2})^2}-e^{-(\frac{k+1}{2})^2}+e^{-(\frac{k+x}{2})^2}\right) \;=\; 0.
$$

## 10.7 The exponential integral

The function $g\colon \mathbb{R}_+ \to \mathbb{R}$ defined by the equation $g(x) = e^{-x}/x$ lies in $\mathcal{C}^\infty \cap \widetilde{\mathcal{D}}^{-1}_{\mathbb{N}} \cap \mathcal{K}^\infty$ and satisfies the equation $E_1(x) = \int_x^\infty g(t)\,dt$, where $E_1$ is the exponential integral. For $x > 0$ we then have

$$\Sigma g(x) \;=\; \sum_{k=0}^{\infty}\left(\frac{e^{-(k+1)}}{k+1} - \frac{e^{-(k+x)}}{k+x}\right).$$

The generalized Stirling formula easily provides the following convergence result

$$E_1(x) - \sum_{k=0}^{\infty}\frac{e^{-(k+x)}}{k+x} \;\to\; 0 \qquad \text{as } x \to \infty.$$

Also, the analogue of Raabe's formula is

$$\int_x^{x+1} \Sigma g(t)\,dt \;=\; 1 - \ln(e-1) - E_1(x), \qquad x > 0.$$

## 10.8 The Bernoulli polynomials

Recall that, for any $n \in \mathbb{N}$, the $n$th degree Bernoulli polynomial $B_n(x)$ is defined by the equation

$$B_n(x) \;=\; \sum_{k=0}^{n}\tbinom{n}{k}\,B_{n-k}\,x^k\,,$$

where $B_k$ is the $k$th Bernoulli number. These polynomials satisfy the difference equation $B_n(x+1) - B_n(x) = n\,x^{n-1}$. Thus, the function $g_n\colon \mathbb{R}_+ \to \mathbb{R}$ defined by the equation $g_n(x) = n\,x^{n-1}$ lies in $\mathcal{C}^\infty \cap \mathcal{D}^n \cap \mathcal{K}^\infty$ and has the property that $\Sigma g_n(x) = B_n(x) - B_n(1)$. The form of the function $g_n$ also shows that

$$B_n(x) - B_n(1) \;=\; n\zeta(1-n) - n\zeta(1-n,x)\,, \qquad n \in \mathbb{N}.$$

Thus, the Bernoulli polynomials can be characterized as follows:

> *All solutions $f_n\colon \mathbb{R}_+ \to \mathbb{R}$ to the equation $f_n(x+1) - f_n(x) = n\,x^{n-1}$ that lie in $\mathcal{K}^n$ are of the form $f_n(x) = c_n + B_n(x)$, where $c_n \in \mathbb{R}$.*

We also easily retrieve the multiplication formula:

$$\sum_{j=0}^{m-1} B_n\left(\frac{x+j}{m}\right) \;=\; \frac{1}{m^{n-1}}\,B_n(x) \qquad x > 0.$$

### 10.9 The Bernoulli polynomials of the second kind

For any $n \in \mathbb{N}$, the $n$th degree Bernoulli polynomial of the second kind is defined by the equation

$$\psi_n(x) \;=\; \int_x^{x+1} \binom{t}{n}\, dt.$$

In particular, we have $\psi_n(0) = G_n$. Also, these polynomials satisfy the difference equation

$$\psi_{n+1}(x+1) - \psi_{n+1}(x) \;=\; \psi_n(x).$$

Thus, the function $g_n \colon \mathbb{R}_+ \to \mathbb{R}$ defined by the equation $g_n(x) = \psi_n(x)$ lies in $\mathcal{C}^\infty \cap \mathcal{D}^{n+1} \cap \mathcal{K}^\infty$ and has the property that $\Sigma g_n(x) = \psi_{n+1}(x) - \psi_{n+1}(1)$.

Thus, the Bernoulli polynomials of the second kind can be characterized as follows:

> *All solutions* $f_n \colon \mathbb{R}_+ \to \mathbb{R}$ *to the equation* $f_n(x+1) - f_n(x) = \psi_n(x)$ *that lie in* $\mathcal{K}^{n+1}$ *are of the form* $f_n(x) = c_n + \psi_{n+1}(x)$, *where* $c_n \in \mathbb{R}$.

# 11 Conclusion

Krull-Webster's theory has proved to be a very nice and useful contribution to the resolution of the difference equation $\Delta f = g$ on the real half-line $\mathbb{R}_+$. In this paper, we have provided a significant generalization of Krull-Webster's theory by considerably relaxing the asymptotic condition imposed on function $g$, and we have demonstrated through various examples how this generalization provides a unified framework to investigate the properties of many special functions. This framework has indeed enabled us to derive several general formulas that now constitute a powerful toolbox and even a genuine Swiss Army knife to investigate a large variety of functions.

The key point of this generalization was the discovery of expression (2) for the sequences $n \mapsto f_n^p[g](x)$, $p \in \mathbb{N}$. We also observe that our uniqueness and existence results strongly rely on Lemma 2.4 together with identities (13) and (18). These results actually constitute the common core and even the fundamental cornerstone of all the subsequent formulas that we derived in this paper. For instance, the generalized Stirling formula (40) has been obtained almost miraculously by merely integrating both sides of the inequality given in Lemma 2.4. Also, Gregory's summation formula (44) has been derived instantly by integrating both sides of identity (18), and we have shown how its remainder can be controlled using Lemma 2.4 again.

Our results clearly shed light on the way many of the classical special functions, such as the polygamma functions and the derivatives of the Hurwitz zeta function, can be systematically studied, sometimes by deriving identities and formulas almost mechanically.

Beyond this systematization aspect, our theory has enabled us to introduce a number of new important and useful objects. For instance, the map $\Sigma$ itself is a new concept that appears to be as fundamental as the basic antiderivative operation (cf. Definition 5.1). In this respect, it would be interesting to extend the map $\Sigma$ to a larger domain, e.g., a linear space of functions that would include not only the current admissible functions but also every function that has an exponential growth. Other concepts such as the Binet-like function and the asymptotic constant also appear to be new fundamental objects that merit further study.

In conclusion, we can clearly see that this area of investigation is very intriguing. We have just skimmed the surface, and there are a lot of questions that emerge naturally. We now list a few below.

- Find necessary and sufficient conditions on function $g$ to ensure both the uniqueness and existence of solutions lying in $\mathcal{K}^p$ to the equation $\Delta f = g$ (cf. Webster's question in Appendix B).

- Find general methods to determine analogues of Euler's reflection formula and Gauss' digamma theorem for any multiple $\log\Gamma$-type function.

- Find necessary and sufficient conditions on function $g$ for the function $\Sigma g$ to be real analytic.

- Show how our results can be used and interpreted when extending some multiple $\log\Gamma$-type functions to complex domains.

*Remark* 11.1. At some places in this paper (e.g., in Theorem 6.5), we have made the assumption that $g$ (resp. $g^{(r)}$ for some $r \in \mathbb{N}^*$) is continuous to ensure the existence of certain integrals. Although we can often relax this condition by simply requiring that $g$ (resp. $g^{(r)}$) is locally integrable, we have kept this continuity assumption for simplicity and consistency with similar results where higher order differentiability is assumed.

Recall also that any monotone function $f$ defined on a compact interval $[a, b]$ is integrable. Thus, for any function $g$ lying in $\cup_{p\geqslant 0}(\mathcal{D}^p \cap \mathcal{K}^p)$, the integral of $\Sigma g$ on $[x, x+1]$ exists for sufficiently large $x$. Nevertheless, most of our results that involve this latter integral also use the asymptotic constant $\sigma[g]$ and the integral of $g$ on the interval $[1, x]$. Thus, for the sake of simplicity, we have always ensured integrability on compact intervals by assuming continuity on the whole of $\mathbb{R}_+$.

# A On Krull-Webster's asymptotic condition

*Summary: We show that our uniqueness and existence results fully generalize a recent attempt by Rassias and Trif [72] to solve the particular case when* $p = 2$*.*

Recall that the asymptotic condition imposed by Krull and Webster on function $g$ is that, for each $x > 0$, $g(t+x) - g(t) \to 0$ as $t \to \infty$. Using our notation, this means that the function $g$ lies in $\mathcal{R}^1_{\mathbb{R}}$. Geometrically, this condition also means that the chord to the graph of $g$ on any fixed length interval has an asymptotic zero slope. Only fixed length intervals whose left endpoints are integers are to be considered if the condition reduces to requiring that $g \in \mathcal{R}^1_{\mathbb{N}}$. Our uniqueness and existence results show that this condition can actually be relaxed into $g \in \mathcal{D}^1_{\mathbb{N}}$, which means that the chord to the graph of $g$ on any interval of the form $[n, n+1]$, $n \in \mathbb{N}^*$, has an asymptotic zero slope. The function $g(x) = \ln x$ is a typical example that shows, just as does every function whose derivative vanishes at infinity, that those functions need not behave asymptotically like constant functions.

It remains, however, that Krull-Webster's asymptotic condition is rather restrictive. For instance, it is not satisfied for the functions $g(x) = \ln \Gamma(x)$ and $g(x) = x \ln x$. To overcome this restriction, Rassias and Trif [72] proposed a modification of Webster's results by considering solutions lying in $\mathcal{K}^2$ and replacing the asymptotic condition with a more appropriate one. Specifically, they considered any function $g\colon \mathbb{R}_+ \to \mathbb{R}$ for which there exists a number $a > 0$ such that

$$\lim_{t\to\infty} g(x+t) - g(t) - x \ln t \;=\; x \ln a, \qquad \text{for all } x > 0. \tag{84}$$

It turns out that both functions $g(x) = \ln \Gamma(x)$ and $g(x) = x \ln x$ satisfy this alternative condition.

Let us now show that our asymptotic condition that $g \in \mathcal{D}^2_{\mathbb{R}}$ generalizes not only Rassias and Trif's (84) but also many other similar conditions.

**Proposition A.1.** *Let* $\varphi\colon \mathbb{R}_+ \to \mathbb{R}$ *and suppose that* $g\colon \mathbb{R}_+ \to \mathbb{R}$ *has the property that, for each* $x > 0$, $g(x+t) - g(t) - x\,\varphi(t) \to 0$ *as* $t \to \infty$*. Then* $g$ *lies in* $\mathcal{R}^2_{\mathbb{R}} \subset \mathcal{D}^2_{\mathbb{R}}$*. In particular,* $\mathcal{R}^2_{\mathbb{R}}$ *contains all the functions that satisfy Rassias and Trif's condition.*

*Proof.* For any $t > 0$ and any $g\colon \mathbb{R}_+ \to \mathbb{R}$, define the function $\rho^{\varphi}_t[g]\colon [0,\infty) \to \mathbb{R}$ by

$$\rho^{\varphi}_t[g](x) \;=\; g(x+t) - g(t) - x\,\varphi(t).$$

Let also $\mathcal{R}^{\varphi}_{\mathbb{R}}$ be the set of functions $g\colon \mathbb{R}_+ \to \mathbb{R}$ with the property that, for each $x > 0$, $\rho^{\varphi}_t[g](x) \to 0$ as $t \to \infty$. Then we immediately see that

$$\rho^2_t[g](x) \;=\; \rho^{\varphi}_t[g](x) - x\rho^{\varphi}_t[g](1),$$

which shows that $\mathcal{R}^{\varphi}_{\mathbb{R}} \subseteq \mathcal{R}^2_{\mathbb{R}}$. Now, if $g$ satisfies Rassias and Trif's condition, then it lies in the set $\cup_{a>0}\mathcal{R}^{\varphi_a}_{\mathbb{R}}$, where $\varphi_a(x) = \ln(ax)$, and hence it also lies in $\mathcal{R}^2_{\mathbb{R}}$. ☐

Proposition A.1 can be generalized to $\mathcal{R}^p_{\mathbb{R}}$ for any value of $p \geqslant 2$ as follows.

**Proposition A.2.** *Let $p \geqslant 2$ be an integer, let $\varphi_1, \ldots, \varphi_{p-1} \colon \mathbb{R}_+ \to \mathbb{R}$, and suppose that $g \colon \mathbb{R}_+ \to \mathbb{R}$ has the property that, for each $x > 0$,*

$$g(x+t) - g(t) - \sum_{j=1}^{p-1} \tbinom{x}{j}\, \varphi_j(t) \;\to\; 0 \qquad \text{as } t \to \infty.$$

*Then $g$ lies in $\mathcal{R}^p_{\mathbb{R}} \subset \mathcal{D}^p_{\mathbb{R}}$.*

*Proof.* For any $t > 0$ and any $g \colon \mathbb{R}_+ \to \mathbb{R}$, define the function $\rho^{\varphi}_t[g] \colon [0, \infty) \to \mathbb{R}$ by

$$\rho^{\varphi}_t[g](x) \;=\; g(x+t) - g(t) - \sum_{j=1}^{p-1} \tbinom{x}{j}\, \varphi_j(t).$$

Define also the functions $\psi^{\varphi,1}_t[g], \ldots, \psi^{\varphi,p}_t[g] \colon [0, \infty) \to \mathbb{R}$ recursively by $\psi^{\varphi,1}_t[g] = \rho^{\varphi}_t[g]$ and

$$\psi^{\varphi,j+1}_t[g] \;=\; \psi^{\varphi,j}_t[g] - \tbinom{x}{j}\, \psi^{\varphi,j}_t[g](j), \qquad j = 1, \ldots, p-1.$$

Then, it is not difficult to see that

$$\psi^{\varphi,j}_t[g](x) \;=\; \rho^{\varphi}_t[g](x) - \sum_{i=1}^{j-1} \tbinom{x}{i}(\Delta^i g(t) - \varphi_i(t))$$

and hence $\psi^{\varphi,p}_t[g] = \rho^p_t[g]$. Thus, if the function $g \colon \mathbb{R}_+ \to \mathbb{R}$ has the property that, for each $x > 0$, $\rho^{\varphi}_t[g](x) \to 0$ as $t \to \infty$, then it lies in $\mathcal{R}^p_{\mathbb{R}}$. ☐

# B On a question raised by Webster

*Summary: We discuss conditions on function $g$ to ensure both the uniqueness (up to an additive constant) and existence of solutions to the equation $\Delta f = g$ that lie in $\mathcal{K}^p$.*

A natural question raised by Webster [80, p. 606], and that we now extend to any value of $p \in \mathbb{N}$, is the following: *Find necessary and sufficient conditions on function $g$ to ensure both the uniqueness (up to an additive constant) and existence of solutions lying in $\mathcal{K}^p_+$ (resp. $\mathcal{K}^p_-$) to the equation $\Delta f = g$.*

Lemma 2.2(b) shows that a necessary condition for this to occur is that $g \in \mathcal{K}^{p-1}_+$ (resp. $g \in \mathcal{K}^{p-1}_-$). Also, our uniqueness and existence results show that a sufficient condition is that $g \in \mathcal{D}^p \cap \mathcal{K}^p_-$ (resp. $g \in \mathcal{D}^p \cap \mathcal{K}^p_+$). It is tempting to believe that this latter condition is also necessary. The following two examples support this idea.

(a) Both functions $\ln\Gamma(x)$ and $\ln(1+\frac{1}{2}\sin(2\pi x))+\ln\Gamma(x)$ are solutions to the equation $\Delta f = g$ that lie in $\mathcal{K}^0_+$, where $g(x) = \ln x$ does not lie in $\mathcal{D}^0\cup\mathcal{K}^0_-$.

(b) Both functions $2^x$ and $2^x+\sin(2\pi x)$ are solutions to the equation $\Delta f = g$ that lie in $\mathcal{K}^p_+$ for any $p\in\mathbb{N}$, where $g(x) = 2^x$ does not lie in $\mathcal{D}^p\cup\mathcal{K}^p_-$.

Nevertheless, the following proposition shows that in general the condition above is not necessary.

**Proposition B.1.** *There exists a function* $f\in\mathcal{C}^0\cap\mathcal{K}^0$ *such that*

*(a)* $\Delta f$ *does not lie in* $\mathcal{D}^0\cup\mathcal{K}^0$*, and*

*(b) for any function* $\varphi\in\mathcal{K}^0$ *satisfying* $\Delta\varphi = \Delta f$ *we have that* $f-\varphi$ *is constant.*

*Proof.* Let $f\in\mathcal{K}^0_+$ be the function whose graph is the polygonal line through the points $(4n,4n)$ and $(4n+2,4n+4)$ for all $n\in\mathbb{N}$. Thus the sequence $n\mapsto\Delta f(n)$ is the 4-periodic sequence $2,0,0,2,2,0,0,2,\ldots$ and hence condition (a) holds. Now, let $\varphi\in\mathcal{K}^0$ be such that $\Delta\varphi = \Delta f$. Clearly, we must have $\varphi\in\mathcal{K}^0_+$. For the sake of a contradiction, suppose that the 1-periodic function $\omega = f-\varphi$ is not constant. That is, there exist $0<x<y\leqslant 1$ such that $\omega(x)\neq\omega(y)$. There are two exclusive cases to consider.

(a) Suppose that $\omega(x)<\omega(y)$. For large integer $n$, we then have

$$0 \;\leqslant\; \varphi(y+4n+2)-\varphi(x+4n+2) \;=\; \omega(x)-\omega(y) \;<\; 0.$$

(a) Suppose that $\omega(x)>\omega(y)$. For large integer $n$, we then have

$$0 \;\leqslant\; \varphi(x+4n+3)-\varphi(y+4n+2) \;=\; \omega(y)-\omega(x) \;<\; 0.$$

In both cases we reach a contradiction, and hence condition (b) holds. $\square$

We note that the function $f$ arising from Propositon B.1 is such that $g = \Delta f$ does not lie in $\mathcal{D}^0\cup\mathcal{K}^0$. The following proposition shows that if the equation $\Delta f = g$ has a unique solution (up to an additive constant) and if $g\in\mathcal{K}^p$ for some $p\in\mathbb{N}$, then necessarily $g\in\mathcal{D}^p\cap\mathcal{K}^p$ (see also Corollary 4.13).

**Proposition B.2.** *Let* $g\colon\mathbb{R}_+\to\mathbb{R}$ *and* $p\in\mathbb{N}$*, and suppose that the sequence* $n\mapsto\Delta^p g(n)$ *is eventually decreasing. Suppose also that there exists a unique (up to an additive constant) function* $f\in\mathcal{K}^p_+$ *satisfying the equation* $\Delta f = g$*. Then* $g$ *lies in* $\mathcal{D}^p_{\mathbb{N}}$*.*

*Proof.* For the sake of a contradiction, suppose that the assumptions are satisfied and that the sequence $n\mapsto\Delta^p g(n)$ does not approach zero. Since this sequence is eventually nonnegative (because we eventually have $\Delta^p g =$

$\Delta^{p+1}f \geqslant 0$), it must converge to a value $C > 0$. It follows that the function $\tilde{g}(x) = g(x) - C\binom{x}{p}$ lies in $\mathcal{D}^p \cap \mathcal{K}^p_-$ and hence $\Sigma\tilde{g}$ lies in $\mathcal{K}^p_+$. Now, for any $0 < \tau < C/(2\pi)^p$, the functions

$$\begin{aligned} f(x) &= \Sigma\tilde{g}(x) + C\tbinom{x}{p+1}, \\ \varphi(x) &= \Sigma\tilde{g}(x) + C\tbinom{x}{p+1} + \tau\sin(2\pi x), \end{aligned}$$

lie in $\mathcal{K}^p_+$; indeed, we have

$$D^{p+1}(C\tbinom{x}{p+1} + \tau\sin(2\pi x)) \ \geqslant\ C + \tau(2\pi)^p \ >\ 0.$$

Moreover, these functions are solutions to the equation $\Delta f = g$ and satisfy $\varphi(1) = f(1)$. This contradicts the uniqueness assumption. □

*Remark* B.3. We observe that if $f$ and $\varphi$ are solutions to $\Delta f = g$, then for any $x > 0$ and any $p \in \mathbb{N}^*$, we have $\Delta^p f(x) \geqslant 0$ if and only if $\Delta^p \varphi(x) \geqslant 0$. Indeed, suppose on the contrary that $\Delta^p f(x) \geqslant 0$ and $\Delta^p \varphi(x) < 0$ for some $x > 0$. Then

$$0 \ \leqslant\ \Delta^p f(x) \ =\ \Delta^{p-1} g(x) \ =\ \Delta^p \varphi(x) \ <\ 0,$$

a contradiction.

Thus, Webster's question still remains a very interesting open problem whose solution would certainly shed light on the theory developed here.

Regarding uniqueness issues only, the following two results due to John [42] are also worth mentioning. Generalizations of these results to higher convexity properties would be welcome.

**Proposition B.4** (see [42]). *Let $g\colon \mathbb{R}_+ \to \mathbb{R}$ have the property that*

$$\inf_{x \in \mathbb{R}_+} g(x) \ =\ 0.$$

*Then there is at most one (up to an additive constant) solution $f$ to the equation $\Delta f = g$ that is increasing.*

**Proposition B.5** (see [42]). *Let $g\colon \mathbb{R}_+ \to \mathbb{R}$ have the property that*

$$\liminf_{x \to \infty} \frac{g(x)}{x} \ =\ 0.$$

*Then there is at most one (up to an additive constant) solution $f$ to the equation $\Delta f = g$ that is convex.*

# C Asymptotic behaviors and bracketing

*Summary: We show that by considering higher and higher values of $p$ in Theorem 6.5 we obtain closer and closer bounds for the Binet-like function $J^{p+1}[\Sigma g]$.*

We have seen in Example 6.7 that the inequalities

$$\left(1+\frac{1}{x}\right)^{-\frac{1}{2}} \leqslant \frac{\Gamma(x)}{\sqrt{2\pi}\, e^{-x}\, x^{x-\frac{1}{2}}} \leqslant \left(1+\frac{1}{x}\right)^{\frac{1}{2}}$$

hold for any $x > 0$ and that tighter inequalities can also be obtained by using different values of the integer $p \geqslant 1$ in Theorem 6.5. In this appendix we show that and how this feature applies in general to multiple $\log\Gamma$-type functions.

Let $g \in \mathcal{C}^0 \cap \mathcal{D}^p \cap \mathcal{K}^p$, where $p = 1 + \deg p$. By Theorem 6.5, for any $x > 0$ such that $g$ is p-convex or p-concave on $[x, \infty)$ we have the inequalities

$$-\overline{G}_p\,|\Delta^p g(x)| \leqslant J^{p+1}[\Sigma g](x) \leqslant \overline{G}_p\,|\Delta^p g(x)|\,.$$

Let us now show how tighter inequalities can be obtained. For any $r \in \mathbb{N}$, define the functions $\alpha_r[\Sigma g]\colon \mathbb{R}_+ \to \mathbb{R}$ and $\beta_r[\Sigma g]\colon \mathbb{R}_+ \to \mathbb{R}$ by the equations

$$\begin{aligned}
\alpha_r[\Sigma g](x) &= -\overline{G}_{p+r}\left|\Delta^{p+r} g(x)\right| - \sum_{j=p+1}^{p+r} G_j \Delta^{j-1} g(x)\,;\\
\beta_r[\Sigma g](x) &= \overline{G}_{p+r}\left|\Delta^{p+r} g(x)\right| - \sum_{j=p+1}^{p+r} G_j \Delta^{j-1} g(x)\,.
\end{aligned}$$

We immediately see that the equality $\alpha_r[\Sigma g](x) = \beta_r[\Sigma g](x)$ holds if and only if $\Delta^{p+r} g(x) = 0$. Also, by Theorem 6.5, if $g \in \mathcal{K}^{p+r}$ and if $x > 0$ is so that $g$ is $(p+r)$-convex or $(p+r)$-concave on $[x, \infty)$, then the following inequalities hold:

$$\alpha_r[\Sigma g](x) \leqslant J^{p+1}[\Sigma g](x) \leqslant \beta_r[\Sigma g](x).$$

The following proposition shows that these inequalities get tighter and tighter as the value of $r$ increases.

**Proposition C.1.** *Let $g \in \mathcal{C}^0 \cap \mathcal{D}^p \cap \mathcal{K}^{p+r+1}$ for some $r \in \mathbb{N}$, where $p = 1 + \deg g$. Let $x > 0$ be so that $g|_{[x,\infty)}$ lies in $\mathcal{K}^{p+r}([x,\infty)) \cap \mathcal{K}^{p+r+1}([x,\infty))$. Then, we have*

$$\alpha_r[\Sigma g](x) \leqslant \alpha_{r+1}[\Sigma g](x) \leqslant \beta_{r+1}[\Sigma g](x) \leqslant \beta_r[\Sigma g](x).$$

*These inequalities are strict if $\Delta^{p+r} g(x+1) \neq 0$.*

*Proof.* We already know that the central inequality holds. Now, using Corollary 4.14, we can assume that $g$ is $(p+r)$-convex and $(p+r+1)$-concave on $[x, \infty)$; the other case can be dealt with similarly. By Lemma 2.3, it follows that $\Delta^{p+r} g \leqslant 0$ and $\Delta^{p+r+1} g \geqslant 0$ on $[x, \infty)$. Let us show that the first inequality holds; the third one can be established similarly.

We have two exclusive cases to consider.

- If $G_{p+r+1} < 0$, then

$$\begin{aligned}\Delta_r\,\alpha_r[\Sigma g](x) &= -\overline{G}_{p+r+1}\left(\Delta^{p+r+1}g(x)+\Delta^{p+r}g(x)\right)\\ &= -\overline{G}_{p+r+1}\Delta^{p+r}g(x+1).\end{aligned}$$

- If $G_{p+r+1} > 0$, then

$$\Delta_r\,\alpha_r[\Sigma g](x) \;=\; -\overline{G}_{p+r}\Delta^{p+r}g(x+1)+G_{p+r+1}\left(\Delta^{p+r+1}g(x)-\Delta^{p+r}g(x)\right).$$

In both cases, we can see that $\Delta_r\,\alpha_r[\Sigma g](x) \geqslant 0$. Moreover, we have $\Delta_r\,\alpha_r[\Sigma g](x) > 0$ if $\Delta^{p+r}g(x+1) \neq 0$. □

It is natural to wonder how the inequalities in Proposition C.1 behave as $r \to_{\mathbb{N}} \infty$. The following proposition, which is a reformulation of Proposition 6.8, answers this question and provides a series representation for $J^{p+1}[\Sigma g]$.

**Proposition C.2.** *Let* $g \in \mathcal{C}^0 \cap \mathcal{D}^p \cap \mathcal{K}^\infty$*, where* $p = 1 + \deg g$*, and let* $x > 0$*. Suppose that for every integer* $q \geqslant p$ *the function* $g$ *is* $q$*-convex or* $q$*-concave on* $[x, \infty)$*. Suppose also that the sequence* $q \mapsto \Delta^q g(x)$ *is bounded. Then the following assertions hold.*

*(a) The sequence* $q \mapsto \beta_q[\Sigma g](x) - \alpha_q[\Sigma g](x)$ *converges to zero.*

*(b) The sequence* $n \mapsto G_n\Delta^{n-1}g(x)$ *is summable.*

*(c) We have*

$$\Sigma g(x) \;=\; \sigma[g] + \int_1^x g(t)\,dt - \sum_{j=1}^{\infty} G_j\Delta^{j-1}g(x).$$

*Equivalently,*

$$J^{p+1}[\Sigma g](x) \;=\; -\sum_{j=p+1}^{\infty} G_j\Delta^{j-1}g(x).$$

# References


[1] V. S. Adamchik. The multiple gamma function and its application to computation of series. *Ramanujan J.*, 9(3):271–288, 2005.

[2] V. S. Adamchik. Contributions to the theory of the Barnes function. *Int. J. Math. Comput. Sci.*, 9(1):11–30, 2014.

[3] H. Alzer and J. Matkowski. A convexity property and a new characterization of Euler's gamma function. *Arch. Math. (Basel)*, 100(2):131–137, 2013

[4] J. Anastassiadis. Remarques sur quelques équations fonctionnelles. (French). *C. R. Acad. Sci. Paris*, 250:2663–2665, 1960.

[5] J. Anastassiadis. Sur les solutions de l'ÃĪquation fonctionnelle $f(x+1) = \phi(x)f(x)$. (French). *C. R. Acad. Sci. Paris*, 253:2446–2447, 1961.

[6] J. Anastassiadis. *Définition des fonctions eulériennes par des équations fonctionnelles*. (French). Mémor. Sci. Math., Fasc. 156 Gauthier-Villars & Cie, Éditeur-Imprimeur, Paris, 1964.

[7] T. M. Apostol. An elementary view of Euler's summation formula. *Amer Math Monthly*, 106(5):409-âĂŞ418, 1999.

[8] M. Ardjomande. Sur l'équation fonctionnelle $f(x+1) - f(x) = \delta(x)$. (French). *Enseign. Math. II*, 13:287–304, 1968.

[9] E. Artin. *Einführung in die Theorie der Gammafunktion*. Teubner, Leipzig, Germany, 1931.

[10] E. Artin. *The gamma function*. Dover Books on Mathematics. Dover Publications Inc., New York, 2015.

[11] E. W. Barnes. The genesis of the double gamma functions. *Proc. Lond. Math. Soc.*, 31:358–381, 1899.

[12] E. W. Barnes. The theory of the double gamma function. *Lond. Phil. Trans. (A)*, 196:265–387, 1901.

[13] E. W. Barnes. On the theory of the multiple gamma function. *Trans. Cambridge Philos. Soc.*, 19:374–425, 1904.

[14] I. S. Berezin and N. P. Shidkov. *Computing methods*. Vols. I, II. Pergamon Press, London, 1965.

[15] B. C. Berndt. Chapter 8 of Ramanujan's second notebook. *J. Reine Angew. Math.*, 338:1–55, 1983.

[16] I. V. Blagouchine. A theorem for the closed-form evaluation of the first generalized Stieltjes constant at rational arguments and some related summations. *J. Number Theory*, 148: 537–592, 2015.

[17] I. V. Blagouchine. Expansions of generalized Euler's constants into the series of polynomials in $\pi^{-2}$ and into the formal enveloping series with rational coefficients only. *J. Number Theory*, 158: 365–396, 2016.

[18] I. V. Blagouchine. A note on some recent results for the Bernoulli numbers of the second kind. *J. Integer Seq.*, 20(3), Art. 17.3.8, 2017.

[19] I. V. Blagouchine and M.-A. Coppo. A note on some constants related to the zeta-function and their relationship with the Gregory coefficients. Em Ramanujan J., 47(2):457–473, 2018.

[20] H. Bohr and J. Mollerup. *Laerebog i matematisk analyse.* (Danish). Vol. III. pp. 149–164. Copenhagen, 1922.

[21] N. Bourbaki. "Éléments de mathématique, Book IV, Chap. VII, La fonction gamma," Hermann et Cie, Paris, 1951.

[22] W. E. Briggs and S. Chowla. The power series coefficients of $\zeta(s)$. *Amer. Math. Monthly*, 62:323–325, 1955.

[23] T. Burić. Improvements of asymptotic approximation formulas for the factorial function. *Appl. Anal. Discrete Math.*, 13(3):895–904, 2019.

[24] W. Burnside. A rapidly convergent series for $\log N!$. *Messenger Math.*, 46:157–159, 1917.

[25] C.-P. Chen and J.-Y. Liu. Inequalities and asymptotic expansions for the gamma function. *J. Number Theory*, 149:313–326, 2015.

[26] C.-P. Chen and L. Lin. Inequalities and asymptotic expansions for the gamma function related to Mortici's formula. *J. Number Theory*, 162:578–588, 2016.

[27] H. Cohen and E. Friedman. RaabeâĂŹs formula for p-adic gamma and zeta functions. *Ann. Inst. Fourier (Grenoble)*, 58(1):363–376, 2008.

[28] A. Cuyt, V. B. Vigdis, B. Verdonk, H. Waadeland, and W. B. Jones. *Handbook of continued fractions for special functions. With contributions by F. Backeljauw and C. Bonan-Hamada.* Springer, New York, 2008.

[29] A. Dinghas. Zur Theorie der Gammafunktion. (German). *Math.-Phys. Semesterber.*, 6:245–252, 1958/1959.

[30] J. Dufresnoy and Ch. Pisot. Sur la relation fonctionnelle $f(x+1) - f(x) = \varphi(x)$. (French). *Bull. Soc. Math. Belg.*, 15: 259–270, 1963.

[31] L. Feng and W. Wang. A continued product approximation for the gamma function. *Integral Transforms Spec. Funct.*, 24(10):831–839, 2013.

[32] S. R. Finch. *Mathematical constants.* Encyclopedia of Mathematics and its Applications 94. Cambridge University Press, Cambridge, UK, 2003.

[33] R. W. Gosper. Decision procedure for indefinite hypergeometric summation. *Proc. Nat. Acad. Sci. U.S.A.*, 75(1):40–42, 1978.

[34] R. L. Graham, D. E. Knuth, and O. Patashnik. *Concrete Mathematics: A Foundation for Computer Science.* 2nd edition. Addison-Wesley Longman Publishing Co., Boston, MA, USA, 1994.

[35] D. Gronau. Normal solutions of difference equations, Euler's functions and spirals. *Aeq. Math.*, 68:230–247, 2004.

[36] D. Gronau. The spiral of Theodorus. *Amer. Math. Monthly*, 111(3):230–237, 2004.

[37] D. Gronau and J. Matkowski. Geometrical convexity and generalizations of the Bohr-Mollerup theorem on the gamma function. *Math. Pannon.*, 4(2):153–160, 1993.

[38] D. Gronau and J. Matkowski. Geometrically convex solutions of certain difference equations and generalized Bohr-Mollerup type theorems. *Results Math.*, 26(3-4):290–297, 1994.

[39] K. Guan. Geometrically convex solutions of a generalized gamma functional equation. *Aequationes Math.*, 89(4):1003–1013, 2015.

[40] G. H. Hardy. Note on Dr. Vacca's Series for $\gamma$. *Quart. J. Pure Appl. Math.*, 43:215–216, 1912.

[41] M. Himmel and J. Matkowski. Directional convexity and characterizations of beta and gamma functions. *J. Convex Anal.*, 25(3):927–938, 2018.

[42] F. John. Special solutions of certain difference equations. *Acta Math.*, 71:175–189, 1939.

[43] Ch. Jordan. *Calculus of finite differences.* 3rd Edition. Chelsea Publishing Company, New York, 1965.

[44] H.-H. Kairies. Über die logarithmische Ableitung der Gammafunktion. (German). *Math. Ann.*, 184:157–162, 1970.

[45] D. E. Knuth. *The Art of Computer Programming. Volume 1: Fundamental Algorithms.* 3rd edition. Addison-Wesley, Bonn, Germany, 1997.

[46] W. Krull. Bemerkungen zur Differenzengleichung $g(x+1)-g(x)=\varphi(x)$. (German). *Math. Nachr.*, 1:365–376, 1948.

[47] W. Krull. Bemerkungen zur Differenzengleichung $g(x+1)-g(x)=\varphi(x)$. II. (German). *Math. Nachr.*, 2:251–262, 1949.

[48] M. Kuczma. O równaniu funkcyjnym $g(x+1)-g(x)=\varphi(x)$. *Zeszyty Naukowe Uniw. Jagiell, Mat.-Fiz.-Chem.*, 4:27–38, 1958

[49] M. Kuczma. Remarques sur quelques théorèmes de J. Anastassiadis. (French). *Bull. Sci. Math.* (2), 84:98–102, 1960.

[50] M. Kuczma. Sur une équation aux différences finies et une caractérisation fonctionnelle des polynômes. (French). *Fund. Math.*, 55:77–86, 1964.

[51] M. Kuczma. Remark on a difference equation. *Prace Mat.*, 9:1–8, 1965.

[52] M. Kuczma. *Functional equations in a single variable*. Monografie Matematyczne, Tom 46. Państwowe Wydawnictwo Naukowe, Warsaw, 1968.

[53] M. Kuczma. *An introduction to the theory of functional equations and inequalities. Cauchy's equation and Jensen's inequality*. 2nd edition. Birkhäuser Verlag AG, Basel, Switzerland, 2009.

[54] J. J. Y. Liang and J. Todd. The Stieltjes constants. *J. Res. Nat. Bur. Standards Sect. B*, 76B:161–178, 1972.

[55] Z. Liu. A new version of the Stirling formula. *Tamsui Oxf. J. Math. Sci.*, 23(4):389–392, 2007.

[56] D. Lu, X. Liu, and T. Qu. Continued fraction approximations and inequalities for the gamma function by Burnside. *Ramanujan J.*, 42(2):491–500, 2017.

[57] J.-L. Marichal. On indefinite sums weighted by periodic sequences. *Results Math.*, 74(3), Paper No. 95, 15 pp., 2019.

[58] J. Matkowski. Some characterizations of the Euler gamma function. *Rocky Mountain J. Math.*, 45(4):1225-1231, 2015.

[59] A. E. Mayer. Konvexe Lösung der Funktionalgleichung $1/f(x+1) = xf(x)$. (German). *Acta Math.*, 70(1):57–62, 1939.

[60] M. Merkle and M. M. Ribeiro Merkle. Krull's theory for the double gamma function. *Appl. Math. Comput.*, 218(3):935–943, 2011.

[61] D. Merlini, R. Sprugnoli, M. C. Verri. The Cauchy numbers. *Discrete Mathematics*, 306:1906–1920, 2006.

[62] L. M. Milne-Thomson *The calculus of finite differences*. Macmillan and Co. LTD, London, 1951.

[63] C. Mortici. A class of integral approximations for the factorial function. *Comput. Math. Appl.*, 59(6):2053–2058, 2010.

[64] C. Mortici. On Gospers formula for the gamma function. *J. Math. Inequal.*, 5(4):611–613, 2011.

[65] C. Mortici. On the monotonicity and convexity of the remainder of the Stirling formula. *Appl. Math. Lett.*, 24(6):869–871, 2011.

[66] C. Mortici. New sharp bounds for gamma and digamma functions. *An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.)*, 57(1):57–60, 2011.

[67] M.-E. Muldoon. Some monotonicity properties and characterizations of the gamma function. *Aeq. Math.*, 18(1-2):54–63, 1978.

[68] Z. Nan-Yue and K. S. Williams. Some results on the generalized Stieltjes constants. *Analysis*, 14(2-3):147–162, 1994.

[69] N. E. Nörlund. *Vorlesungen über Differenzenrechnung.* Springer-Verlag, Berlin, 1924.

[70] T. Popoviciu. *Sur quelques propriétés des fonctions d'une ou de deux variables réelles.* Thèses de l'entre-deux-guerres, no. 146 (1933). 96 p. (http://www.numdam.org)

[71] A. Quarteroni, R. Sacco, and F. Saleri. *Numerical mathematics.* 2nd Edition. TAM 37. Springer, Berlin, Germany, 2010.

[72] T. M. Rassias and T. Trif. Log-convex solutions of the second order to the functional equation $f(x+1) = g(x)f(x)$. *J. Math. Anal. Appl.*, 331:1440–1451, 2007.

[73] A. W. Roberts and D. E. Varberg. *Convex functions.* Pure and Applied Mathematics, Vol. 57. Academic Press, New York-London, 1973.

[74] H.-W. Rohde. Bemerkung zur Funktionalgleichung $f(s+1) = \Phi(s)f(s)$. (German). *Rend. Circ. Mat. Palermo*, 14(2):43–60, 1965.

[75] N. J. A. Sloane (editor). The On-Line Encyclopedia of Integer Sequences. http://www.oeis.org

[76] H. M. Srivastava and J. Choi. *Zeta and* $q$*-Zeta functions and associated series and integrals.* Elsevier, Inc., Amsterdam, 2012.

[77] J. Stoer and R. Bulirsch. *Introduction to numerical analysis.* 3rd Edition. TAM 12. Springer-Verlag, New York, USA, 2010.

[78] H. P. Thielman. On the convex solution of a certain functional equation. *Bull. Amer. Math. Soc.*, 47:118–120, 1941.

[79] R. Webster. On the Bohr-Mollerup-Artin characterization of the gamma function. *Rev. Anal. Numér. Théor. Approx.*, 26(1-2):249–258, 1997.

[80] R. Webster. Log-convex solutions to the functional equation $f(x+1) = g(x)f(x)$: $\Gamma$-type functions. *J. Math. Anal. Appl.*, 209:605–623, 1997.

[81] J. G. Wendel. Note on the gamma function. *Amer. Math. Monthly*, 55:563–564, 1948.

[82] A. Xu, Y. Hu, and P. Tang. Asymptotic expansions for the gamma function. *J. Number Theory*, 169:134–143, 2016.

[83] Z.-H. Yang and J.-F. Tian. On Burnside type approximation for the gamma function. *Rev. R. Acad. Cienc. Exactas FÃŋs. Nat. Ser. A Mat. RACSAM*, 113(3):2665–2677, 2019.