Hidden automatic sequences

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Abstract

An automatic sequence is a letter-to-letter coding of a fixed point of a uniform morphism. More generally, we have morphic sequences, which are letter-to-letter codings of fixed points of arbitrary morphisms. There are many examples where an, a priori, morphic sequence with a *non-uniform* morphism happens to be an automatic sequence. An example is the Lysënok morphism $a \to aca$, $b \to d$, $c \to b$, $d \to c$, the fixed point of which is also a 2-automatic sequence. Such an identification is useful for the description of the dynamical systems generated by the fixed point. We give several ways to uncover such hidden automatic sequences, and present many examples. We focus in particular on morphisms associated with Grigorchuk(-like) groups.

1 Introduction

The substitution $a \to aca$, $b \to d$, $c \to b$, $d \to c$ was used by Lysënok [28] to provide a presentation by generators and (infinitely many) defining relations of the first Grigorchuk group. More recently Vorobets [34] proved several properties of the fixed point of this substitution. In an unpublished 2011 note the first and third authors proved among other things that the fixed point of this substitution is also the fixed point of the 2-substitution $a \to ac$, $b \to ad$, $c \to ab$, $d \to ac$, and so that this fixed point is 2-automatic [1]. This result was obtained again more recently in [20, 21], also see [5].

This phenomenon is rare, but was already encountered. One example is the proof by Berstel [11] that the Istrail squarefree sequence [26], defined as the unique fixed point of the morphism $\sigma_{\rm IS}$ given by

$$\sigma_{\rm IS}(0) = 12,\ \sigma_{\rm IS}(1) = 102\ , \sigma_{\rm IS}(2) = 0,$$

can also be obtained as the letter-to-letter image by the reduction modulo 3 of the fixed point beginning with 1 of the uniform morphism $0 \to 12$, $1 \to 13$, $2 \to 20$, $3 \to 21$.

This phenomenon is also interesting since substitutions of constant length d are "simpler" than general substitutions in particular, because they are related to d-ary expansions of the indexes of their terms. (Recall that letter-to-letter images of substitutions of constant length are called *automatic sequences*. For results about automatic sequences the reader can look at [32] and [2], and at the references therein.)

In view of what precedes, a natural question arises: *how to recognize that the fixed point of a non-uniform morphism is an automatic sequence?*

Of course not every iterative fixed point of a non-uniform morphism is q-automatic for some q, as the Fibonacci binary sequence (i.e., the iterative fixed point of $0 \to 01$, $1 \to 0$) shows, since the frequencies of its letters are not rational. However, it is true that any q-automatic sequence ($q \geq 2$) can be obtained as a non-uniformly morphic sequence, i.e., as the letter-to-letter image of an iterative fixed point of a non-uniform morphism [3, Theorem 5].

In Section 2 we will revisit a 1978 theorem of the second author to give a sufficient condition for a fixed point of a non-uniform morphism to be automatic. This is Theorem 1 below. Section 3 will show an interplay between this Theorem 1 and a result of [3] stating that any automatic sequence can also be obtained as the letter-to-letter image of the fixed point of a non-uniform morphism. In Section 4, using several sequences in the OEIS [31] as examples, we will show how to prove with this theorem (actually a particular case, the "Anagram Theorem") that these sequences, defined as fixed points of non-uniform morphisms, are automatic. We will recall the 2-automaticity of the fixed point of the Lysënok morphism in Section 5, and give several examples of sequences related to Grigorchuk groups and similar groups.

2 A general theorem revisited

Note that the vector of lengths of the Istrail morphism σ_{IS}: $(2, 3, 1)$ is a left eigenvector of the incidence matrix of the morphism. So Berstel's result also follows from [14, Section V, Theorem 1], as noted as an example in the same paper [14, Section IV, Example 8][1]. Since this theorem is stated in [14] in the context of dynamical systems, we will give an equivalent reformulation in Theorem 1 below. Before stating the theorem, we need a lemma on nonnegative matrices, which does not use any result à la Perron-Frobenius: see, e.g., the proof in [25, Corollary 8.1.30, p. 493].

Lemma 1 *Let M be a matrix whose entries are all nonnegative. If v is an eigenvector of M with positive coordinates associated with a real eigenvalue λ, then λ is equal to the spectral radius of M.*

We invite the reader to verify that the statement in this lemma is not true if v is only supposed to have nonnegative coordinates.

Theorem 1 ([14]) *Let σ be a morphism on $\{0, \ldots, r-1\}$ with length vector $L = (|\sigma(0)|, \ldots, |\sigma(r-1)|)$, for some integer $r > 1$. Suppose that σ is non-erasing (i.e., for all $i \in [0, r-1]$ one has $|\sigma(i)| \geq 1$). Let x be a fixed point of σ, and let M be the incidence matrix of σ. If L is a left eigenvector of M, then x is q-automatic, where q is the spectral radius of M.*

We give a sketch of the proof of this result, which will be useful in the sequel. Let $L_i = |\sigma(i)|$ be the length of $\sigma(i)$ for $i \in [0, r-1]$. The idea is to define a morphism τ on an alphabet of $L_0 + \cdots + L_{r-1}$ symbols $a(i,j)$, $0 \leq i < r$, $1 \leq j \leq L_i$ by setting

$$\tau(a(i,j)) = a(i^*, 1) \ldots a(i^*, L_{i^*}) \quad \text{if } \sigma(i)_j = i^*.$$

If σ is non-uniform, then τ is still non-uniform, but the uniqueness of the occurrences of the symbols $a_{i,j}$ permits to 'reshuffle' τ to a morphism τ' which is uniform, and the eigenvector criterium ensures that this can be done consistently. Rather than going into the details, we illustrate the argument with the Istrail morphism σ_{IS}. Here the alphabet is $\{a(0,1), a(0,2), a(1,1), a(1,2), a(1,3), a(2,1)\}$. We obtain

$$\begin{aligned}
&\tau_{\mathrm{IS}}(a(0,1)) = a(1,1)a(1,2)a(1,3),\ \tau_{\mathrm{IS}}(a(0,2)) = a(2,1)\\
&\tau_{\mathrm{IS}}(a(1,1)) = a(1,1)a(1,2)a(1,3),\ \tau_{\mathrm{IS}}(a(1,2)) = a(0,1)a(0,2),\ \tau_{\mathrm{IS}}(a(1,3)) = a(2,1)\\
&\tau_{\mathrm{IS}}(a(2,1)) = a(0,1)a(0,2).
\end{aligned}$$

Coding $a = a(0,1), b = a(0,2), c = a(1,1), d = a(1,2), e = a(1,3), f = a(2,1)$, the reshuffled τ'_{IS} is given by

$$\tau'_{\mathrm{IS}}(a) = cd,\ \tau'_{\mathrm{IS}}(b) = ef,\ \tau'_{\mathrm{IS}}(c) = cd,\ \tau'_{\mathrm{IS}}(d) = ea,\ \tau'_{\mathrm{IS}}(e) = bf,\ \tau'_{\mathrm{IS}}(f) = ab.$$

The letter-to-letter projection λ is given by $a \to 1, b \to 2, c \to 1, d \to 0, e \to 2, f \to 0$. This gives the Istrail sequence as a 2-automatic sequence by projection of a fixed point of the uniform morphism τ'_{IS} on a six letter

[1]The matrix there is the transpose of what is now considered to be the incidence matrix of a morphism.

alphabet. But, since $\tau'_{\rm IS}(a) = \tau'_{\rm IS}(c)$, and $\lambda(a) = \lambda(c)$, we can merge a and c. Finally, since $\lambda(b) = \lambda(e)$, and the first letters of $\tau'_{\rm IS}(b) = \tau'_{\rm IS}(e)$ are b and e, and the second letters are equal, also b and e can be merged. After a recoding, this gives Berstel's morphism above.

Let q be the constant length of the morphism τ'. We show in general why q is equal to the spectral radius of M. Let λ be the eigenvalue associated with the left eigenvector $(L_0, L_1, \ldots, L_{r-1})$, and let $r' := L_0 + \cdots + L_{r-1}$. Then

$$qr' = \sum_i \sum_j |\tau'(a(i,j))| = \sum_i \sum_j |\tau(a(i,j))| = \sum_i (LM)_i = \lambda \sum_i L_i = \lambda r'.$$

This implies that q has to be equal to λ. But, from Lemma 1 above, λ must be equal to the spectral radius of M. $\square$

The condition about the length vector given in Theorem 1 is not necessary. We will see in Section 5 an example of a sequence that is defined as the fixed point of a non-uniform morphism (the Lysënok morphism), that does not satisfy the length vector condition of Theorem 1, but that is 2-automatic.

For an alphabet of two letters we have the following obstruction for a morphism to satisfy the left eigenvector condition.

Proposition 1 *Let μ be a morphism on $\{0,1\}$. Then $\gcd(L_0, L_1) = 1$ implies that $L = (L_0, L_1)$ can not be a left eigenvector of the incidence matrix M of μ.*

Proof. Let $L = (L_0, L_1)$ be a left eigenvector > 0 of M. By Lemma 1, L is associated with $\rho(M)$. Let λ_2 be the other eigenvalue of M. By Cayley-Hamilton, $(L_0 - \lambda_2, L_1 - \lambda_2)$ is a left eigenvector of M. So $\lambda_2 = 0$. But then $\det(M) = 0$, and the columns are proportional, i.e., one is a multiple of the other. This implies that $\gcd(L_0, L_1) > 1$. $\square$

3 From k-automatic to non-uniformly morphic and back

The paper [3] gives an algorithm to represent any k-automatic sequence with associated morphism γ as a morphic sequence, where the morphism γ' is *non*-uniform[2]. We call this algorithm the CUP-algorithm, standing for Create Unique Pair. The question arises: if we are given this non-uniform representation, how do we find the uniform representation? The answer lies, once more, directly in the left eigenvector criterium.

We first give an example. We start with a famous 2-automatic sequence: the Thue-Morse sequence. For technical reasons we do not take the Thue-Morse morphism $0 \to 01, 1 \to 10$ as γ, but its cube. So let γ be the third power of the Thue-Morse morphism:

$$\gamma(0) = 01101001, \quad \gamma(1) = 10010110.$$

We define morphisms γ' on an extended alphabet $\{0, 1, b', c'\}$, where $b' = 0'$ will be projected on 0, and $c' = 1'$ will be projected on 1. Define the two non-empty words z and t as any concatenation which gives

$$zt = \gamma(01) = 0110100110010110,$$

for example $z = 0, t = 110100110010110$. Then define γ' on $\{0, 1, b', c'\}$ by

$$\gamma'(0) = 011b'c'001,\ \gamma'(1) = \gamma(1),\ \gamma'(b') = z,\ \gamma'(c') = t.$$

As in [3] it is easy to see that the infinite fixed point of γ' starting with 0 maps to the Thue-Morse sequence under the projection D given by $D(0) = 0, D(1) = 1, D(b') = 0, D(c') = 1$.

[2] See [15] for a simple version of this construction.

The incidence matrix of these morphisms is

$$M' := \begin{pmatrix} 3 & 4 & m_0 & 8-m_0 \\ 3 & 4 & m_1 & 8-m_1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

where m_0 is the number of 0's in z, and m_1 is the number of 1's in z. Let $L' = (8, 8, m_0 + m_1, 16 - m_0 - m_1)$ be the length vector of γ'. Then the following holds for any choice of z and t:

$$L'M' = 8L'.$$

This is exactly the left eigenvector criterium of Theorem 1. The general result is the following theorem.

Theorem 2 *Let x be a k-automatic sequence, and let γ' be the non-uniform morphism turning x into a (non-uniformly) morphic sequence in the CUP algorithm. Then the incidence matrix of γ' satisfies the left eigenvector criterium.*

Proof. Let γ be the uniform morphism of length k on the alphabet $\{0, 1, \dots, r-1\}$ such that x is a letter-to-letter projection of a fixed point y of γ. It is easy to see that, as in the proof in [3], we may suppose that $y = x$. Let $L = (k, k, \dots, k)$ be the length vector of γ, and let M be the incidence matrix of γ. Note that M satisfies the eigenvector criterium: $LM = kL$. Without loss of generality we assume that $b = 0$, and $c = 1$ are the two letters which give two extra letters $b' = r$ and $c' = r + 1$ in the CUP algorithm. Let m_0 be the number of 0's in z, and m_1 the number of 1's in z, of the CUP splitting $\gamma(01) = zt$. Then the length vector L' of γ' is equal to

$$L' = (k, k, \dots, k, m_0 + m_1, 2k - m_0 - m_1).$$

The first column of M' is equal to

$$(m_{00} - 1, m_{10} - 1, m_{20}, m_{30}, \dots, m_{r-1,0}, 1, 1)^\top.$$

The inner product of the length vector L' with this first column is equal to

$$k(m_{00} - 1) + k(m_{10} - 1) + km_{20} + \cdots + km_{r-1,0}m_0 + m_1 + 2k - m_0 - m_1 = k(m_0 + \dots m_{r-1,0}) = k^2.$$

Obviously the inner product of L' with the second till r^{th} column is also equal to k^2. The inner product of the length vector L' with the $(r+1)^{\text{th}}$ column is equal to

$$km_0 + km_1 = k(m_0 + m_1).$$

The inner product of the length vector L' with the $(r+2)^{\text{th}}$ column is equal to

$$k(k - m_0) + k(k - m_1) = k(2k - m_0 - m_1).$$

This finishes the checking of the left eigenvector criterium $L'M' = kL'$. $\square$

4 First examples of hidden automatic sequences

We start this section with the following *Anagram Theorem*, which is actually a particular case of Theorem 1. The interest of this simpler theorem is that it permits to prove that some fixed points of non-uniform morphisms are automatic in a purely "visual" (but rigorous) way.

Theorem 3 ["Anagram Theorem"] *Let $\mathcal{A}$ be a finite set. Let W be a set of anagrams on $\mathcal{A}$ (the words in W are also said to be abelian equivalent; they have the same Parikh vector). Let ψ be a morphism on $\mathcal{A}$ admitting an iterative fixed point, such that the image of each letter is a concatenation of words in W. Then the iterative fixed point of ψ is d-automatic, where d is the quotient of the length of $\psi(w)$ by the length of w, which is the same for all $w \in W$.*

Proof. Let $W = \{w_1, w_2, \ldots, w_k\}$. Let $m_a = N_a(w_1)$ be the number of times the letter $a \in \mathcal{A}$ occurs in w_1, or in any other word in the set W. Let n_a be the number of words from W used to build $\psi(a)$. Let m and d be defined by

$$m := \sum_{a \in \mathcal{A}} m_a, \quad d := \sum_{a \in \mathcal{A}} n_a m_a.$$

Then m is the length of the words in W. Let M be the incidence matrix of the morphism ψ. The coordinates of the column with index a of M are $n_a m_b$ where b runs through $\mathcal{A}$. The length vector L of the morphism ψ is the vector with entries $|\psi(b)| = mn_b$. It follows that the coordinate of the product of LM with index a is

$$\sum_{b \in \mathcal{A}} (mn_b)(n_a m_b) = mn_a \sum_{b \in \mathcal{A}} n_b m_b = mn_a d.$$

We see that the morphism ψ satisfies the left eigenvector criterium of Theorem 1 (with eigenvalue d), and so any iterative fixed point of ψ is d-automatic. $\square$

Example 1 Let ψ be the morphism on a three letter alphabet given by

$$\psi(a) = aabc, \ \psi(b) = bacaaabc, \ \psi(c) = bacabacabaca.$$

By taking the set $W = \{aabc, baca\}$, we see immediately from Theorem 3 that the fixed points of ψ are 7-automatic.

Example 2 The sequence A285249 from [31] is called the 0-limiting word of the morphism f which maps $0 \to 10$, $1 \to 0101$ on $\{0,1\}^*$, i.e., A285249 is the fixed point of f^2 starting with 0, where f^2 is given by $f^2(0) = 010110$, $f^2(1) = 100101100101$. The images of 0 and of 1 by f^2 can be respectively written www' and $w'www'ww$, with $w = 01$ and $w' = 10$. Again, Theorem 3 gives that the fixed points of f^2 are 9-automatic, which is equivalent to being 3-automatic.

More examples like sequence A285249 are collected in the following corollary to Theorem 3.

Corollary 1 *The following automaticity properties for sequences in the OEIS hold.*

- *The four sequences* A284878, A284905, A285305, *and* A284912 *are generated by morphisms f, where $f(0)$ and $f(1)$ can be written as concatenations of one, respectively two of the two words $w = 01$ and $w' = 10$. So Theorem 3 immediately implies that they are all* 3*-automatic.*
- *The sequences* A285252, A285255 *and* A285258*, are fixed points of squares of such morphisms, and so they are* 9*-automatic (hence* 3*-automatic).*
- *Finally the fact that* A284878*, is* 3*-automatic easily implies that* A284881 *is* 3*-automatic.*

Remark 1 Other sequences in the OEIS that do not satisfy the hypotheses of Theorem 3 can be proved automatic because they satisfy the hypotheses of Theorem 1: for example the sequences A285159 and A285162 (replace the morphism given in the OEIS by its square to obtain these two sequences as fixed points of morphisms), A285345, A284775 and A284935 are 3-automatic.

5 Hidden automatic sequences and self-similar groups

The substitution τ defined by $\tau(a) = aca$, $\tau(b) = d$, $\tau(c) = b$, $\tau(d) = c$ was used by Lysënok to provide a presentation by generators and (infinitely many) defining relations of the first Grigorchuk group. Note that this substitution does not satisfy the "left eigenvector criterium". The proof given in [1] consisted of the introduction of the morphism ψ defined by

$$\psi(a) := ac, \quad \psi(b) := ad, \quad \psi(c) := ab, \quad \psi(d) := ac$$

and of the remark that $\tau \circ \psi = \psi \circ \psi$, which easily implies that τ and ψ have the same fixed point beginning with a. A similar proof was given in [20].

Another proof (essentially the one in [21] and [5]) introduces a non-overlapping—2-block morphism (i.e., a morphism that, starting from a sequence $u_0, u_1, u_2, u_3 \ldots$, yields a sequence on the new "letters" u_0u_1, u_2u_3, ...), namely the substitution (coding $ab = 1, ac = 2, ad = 3$)

$$1 \to 23, \ 2 \to 21, \ 3 \to 22,$$

from which we see immediately that the Lysënok fixed point is also generated by a substitution of constant length 2.

We may ask whether this second approach works in other "similar" situations, i.e., for morphisms related to Grigorchuk or "Grigorchuk-like groups". Before we address this question, it is worthwhile to give a general result on automatic sequences in terms of "non-overlapping–k-block morphisms".

Theorem 4 *Let $q \geq 2$ and let $\mathbf{u} = (u(n))_{n \geq 0}$ be a sequence with values in $\mathcal{A}$. Then, $\mathbf{u}$ is q-automatic if and only if there exist a positive integer r and a q-uniform morphism μ on $\mathcal{A}^{q^r}$ such that the sequence of q^r-blocks obtained by grouping in $\mathbf{u}$ the terms q^r at a time (namely the sequence $(u(q^r n), u(q^r n + 1), \ldots u(q^r n + q^r - 1))_{n \geq 0}$) is a fixed point of μ.*

Proof. This is essentially Theorem 1 in [13]. $\square$

Theorem 4 is indeed illustrated by the Lysënok fixed point, and by the following example (which, contrary to the Lysënok morphism, is primitive).

Corollary 2 *Let σ be the morphism defined by*

$$\sigma : \quad a \to acaba, \ b \to bac, \ c \to cab.$$

Then the iterative fixed fixed point of σ beginning with a is 4-automatic (hence 2-automatic).

Proof. There are only the 2-blocks ac, ab occurring at even positions in the fixed point $x := acabacab \ldots$ of σ. In fact σ induces the following morphism $\sigma^{[2]}$ on non-overlapping—2-blocks:

$$\sigma^{[2]} : \quad ab \to ac\,ab\,ab\,ac, \ ac \to ac\,ab\,ac\,ab.$$

The fact that $\sigma^{[2]}$ has constant length 4 implies that x is a 4-automatic sequence, hence a 2-automatic sequence. $\square$

Another general result will prove useful.

Proposition 2 *If the incidence matrix of a primitive non-uniform morphism has an irrational dominant eigenvalue, then an iterative fixed point of this morphism cannot be automatic.*

Proof. Since the morphism is primitive, the frequency of each letter exists, and the vector of frequencies is the unique normalized eigenvector of the matrix for the dominant eigenvalue. If the sequence were automatic, all the frequencies of letters would be rational, which gives a contradiction with the irrationality of the eigenvalue and the fact that the entries of the matrix are integers. $\square$

We deduce the following corollary.

Corollary 3 *We can give the nature (i.e., whether they are automatic or not automatic) of the following fixed points of morphisms related to Girgorchuk-like groups.*

- The fixed point of the morphism $a \to aba$, $b \to d$, $c \to b$, $d \to c$ (see, e.g., [8, Proposition 5.6]) is 2-automatic (with the same proof as for the fixed point of the Lysënok morphism).

- The fixed point of the morphism $a \to aca$, $b \to d$, $c \to aba$, $d \to c$ (see [9, Theorem 4.1]) is not automatic. (Namely the matrix of this morphism is primitive and its characteristic polynomial, which is equal to $x^4 - 2x^3 - 2x^2 - x + 2$, clearly has no rational root; the result follows from Proposition 2 above.)

- The fixed point of the morphism $x \to xzy$, $y \to xx$, $z \to yy$ (see [7, Proof of Proposition 4.7]) is not automatic. (Again this is an application of Proposition 2 above, since the characteristic polynomial of the –primitive– incidence matrix is equal to $x^3 - x^2 - 2x - 4$ which has no rational root.)

- The fixed point beginning with 2^* of the morphism $1 \to 2$, $1^* \to 2^*$, $2 \to 1^*2^*$, $2^* \to 21$ (see [30]) is not automatic. Namely, putting $2^*2 := A$ and $11^* := B$ it can be written $ABABAABAABA$... which is a fixed point of the morphism $A \to ABABA$, $B \to ABA$, which easily seen to be Sturmian from the criterion [33, Proposition 1.2] since $AB \to ABABAABA = ABA(BA)ABA$ while $BA \to ABAABABA = ABA(AB)ABA$. Actually a more precise result holds: this morphism is conjugate to f^3 where f is the Fibonacci morphism $A \to AB$, $B \to A$ (see the comments of the second author for the sequences A334413 and A006340 in [31], where the alphabet $\{1, 0\}$ corresponds to our $\{A, B\}$ here).

- The fixed point of the morphism $a \to aba$, $b \to c$, $c \to b$ (see, e.g., [29, p. 40]) can also be generated by the morphism on the non-overlapping–2-blocks $0 = ab$ and $1 = ac$ defined by $0 \to 01$, $1 \to 00$, i.e., the "period-doubling" morphism, and so this fixed point is 2-automatic.

- The morphism $a \to b$, $b \to c$, $c \to aba$ (see, [22, Theorem 3.1], also see [29, p. 40]) has the property that its cube has a fixed point. This fixed point is not automatic since the frequencies of letters exist and are irrational.

- The morphism $a \to c$, $b \to aba$, $c \to b$ (see, e.g., [29, p. 46]) has the property that its cube has three fixed points. None of them is automatic. (Namely, the characteristic polynomial of the –primitive– incidence matrix is equal to $x^3 - 7x^2 + 12x - 8$, which has no rational root.)

We end this section with a theorem which will apply to two morphisms related to other Grigorchuk-like groups (see our Corollary 5 below).

Theorem 5 *Let $\mathbf{x} = (x_n)_{n \geq 0}$ be a sequence on some alphabet $\mathcal{A}$. Let $\mathcal{A}'$ be a proper subset of $\mathcal{A}$. Suppose that there exists a sequence $\mathbf{y} = (y_n)_{n \geq 0}$ on $\mathcal{A}'$ with the property that each of its prefixes is a factor of $\mathbf{x}$. Let $d \geq 2$. If no sequence in the closed orbit of $\mathbf{y}$ under the shift is d-automatic, then $\mathbf{x}$ is not d-automatic.*

Proof. Define an order on $\mathcal{A}$ such that each element of $\mathcal{A} \setminus \mathcal{A}'$ is larger than each element of $\mathcal{A}$. The set of sequences on $\mathcal{A}$ is equipped with the lexicographical order induced by the order on $\mathcal{A}$. Let $\mathbf{z} = (z_n)_{n \geq 0}$ be the lexicographically least sequence in the orbit closure of $\mathbf{x}$. Since the sequence $\mathbf{y}$ has its values in $\mathcal{A}'$ and since each prefix of $\mathbf{y}$ is a factor of $\mathbf{x}$, the orbit closure of $\mathbf{y}$ under the shift is included in the orbit closure of $\mathbf{x}$. Now, since the elements of $\mathcal{A} \setminus \mathcal{A}'$ are larger than the elements of $\mathcal{A}'$, the least element of the orbit closure of $\mathbf{y}$ is equal to the least element of the orbit closure of $\mathbf{x}$, i.e., is equal to $\mathbf{z}$. Now, if $\mathbf{x}$ were d-automatic for some $d \geq 2$, then $\mathbf{z}$ would be d-automatic [4, Theorem 6], which contradicts the hypothesis on the orbit of $\mathbf{y}$. $\square$

Corollary 4 *Let Let $\mathbf{x} = (x_n)_{n \geq 0}$ be a sequence on some alphabet $\mathcal{A}$. Let $\mathcal{A}'$ be a proper subset of $\mathcal{A}$. Suppose that there exists a sequence $\mathbf{y} = (y_n)_{n \geq 1}$ on $\mathcal{A}'$ with the property that each of its prefixes is a factor of $\mathbf{x}$. Suppose that $\mathbf{y}$ is Sturmian, or that $\mathbf{y}$ is uniformly recurrent*[3] *and that its complexity is not $\mathcal{O}(n)$, then $\mathbf{x}$ is not d-automatic for any $d \geq 2$.*

[3]Uniformly recurrent sequences are also called *minimal*.

Proof. If $\mathbf{y}$ is Sturmian, all sequences in its orbit closure are Sturmian (they have complexity $n+1$), hence cannot be d-automatic. If $\mathbf{y}$ is uniformly recurrent, all the sequences in its orbit closure have the same complexity –which is not $\mathcal{O}(n)$– and thus none of them can be d-automatic. $\square$

We are ready for our last corollary about the two fixed points of morphisms respectively given in [10, Theorem 2.9] and [6, Theorem 4.5].

Corollary 5 *The fixed point beginning with a of the morphism $a \to aca$, $b \to bc$, $c \to b$ is not automatic. The fixed point beginning with a of the morphism $a \to aca$, $c \to cd$, $d \to c$ is not automatic.*

Proof. Note that the fixed point of the morphism $b \to bc$, $c \to b$ (respectively $c \to cd$, $d \to c$) is a Sturmian sequence and apply Corollary 4 above.

6 Final remarks

For more on the Grigorchuk group or similar groups, the reader can also consult, e.g., [24, 19, 18, 27, 23]. Note that *automata groups* appear to be close to morphic or automatic sequences, while *automatic groups* (see, e.g., [17]) seem to be rather away from these sequences. Note that substitutions can also be used, in a different context, to characterize *families* of groups: for example it is proved in [12] that *a finite group is an extension of a nilpotent group by a 2-group if and only if it satisfies a Thue-Morse identity for all elements x, y*, where the nth Thue-Morse identity between x and y is defined by $\varphi^n_{x,y}(x) = \varphi^n_{x,y}(y)$ for every $n \geq 0$, and the Thue-Morse substitution $\varphi_{x,y}$ is defined by $\varphi_{x,y}(x) := xy$ and $\varphi_{x,y}(y) := yx$.

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References

[1] J.-P. Allouche, M. Queffélec, A note on substitutions and the first Grigorchuk group, Unpublished draft (2011).

[2] J.-P. Allouche, J. Shallit, *Automatic Sequences. Theory, Applications, Generalizations*, Cambridge University Press, 2003.

[3] J.-P. Allouche, J. Shallit, Automatic sequences are also non-uniformly morphic, Preprint, 2019, available at `https://arxiv.org/abs/1910.08546`. To appear in *Discrete Mathematics and Applications*, A. M. Raigorodskii, M. T. Rassias (Eds.), Springer, 2020.

[4] J.-P. Allouche, N. Rampersad, J. Shallit, Periodicity, repetitions, and orbits of an automatic sequence, *Theoret. Comput. Sci.* **410** (2009), 2795–2803.

[5] M. Baake, J. Roberts, R. Yassawi, Reversing and extended symmetries of shift spaces, *Discrete Contin. Dyn. Syst.* **38** (2018), 835–866.

[6] L. Bartholdi, Endomorphic presentations of branch groups, *J. Algebra* **268** (2003), 419–443.

[7] L. Bartholdi, A. Eschler, Growth of permutational extensions, *Invent. Math.* **189** (2012), 431–455.

[8] L. Bartholdi, R. I. Grigorchuk, On parabolic subgroups and Hecke algebras of some fractal groups, *Serdica Math. J.* **28** (2002), 47–90.

[9] L. Bartholdi, O. Siegenthaler, The twisted twin of the Grigorchuk group, *Internat. J. Algebra Comput.* **20** (2010), 465–488.

[10] M. G. Benli, Profinite completion of Grigorchuk's group is not finitely presented, *Internat. J. Algebra Comput.* **22** (2012), 1250045.

[11] J. Berstel, Sur la construction de mots sans carré, *Sém. Théorie Nombres Bordeaux* **8** (1978-1979), 1–16.

[12] M. Boffa, F. Point, Identités de Thue-Morse dans les groupes, *C. R. Acad. Sci. Paris, Sér. 1* **312** (1991), 667–670.

[13] A. Cobham, Uniform tag sequences, *Math. Systems Theory* **6** (1972), 164–192.

[14] F. M. Dekking, The spectrum of dynamical systems arising from substitutions of constant length, *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **41** (1977/78), 221–239.

[15] F. M. Dekking, On the structure of Thue–Morse subwords, with an application to dynamical systems, *Theoret. Comput. Sci.* **550** (2014), 107–112.

[16] F. Durand, Cobham's theorem for substitutions, *J. Eur. Math. Soc.* **13** (2011), 1799–1814.

[17] D. B. A. Epstein, J. W. Cannon, D. F. Holt, Derek, S. V. F. Levy, M. S. Paterson, W. Thurston, *Word Processing in Groups*, Jones and Bartlett Publishers, Boston, MA, 1992.

[18] R. Grigorchuk, Solved and unsolved problems around one group in *Infinite Groups: Geometric, Combinatorial, and Dynamical Aspects*, Prog. Math. **248**, Birkhäuser, Basel, 2005, 117–218.

[19] R. Grigorchuk, A. Zuk, On a torsion-free weakly branch group defined by a three state automaton, *Internat. J. Algebra Comput.* **12** (2002), 1–24.

[20] R. Grigorchuk, D. Lenz, T. Nagnibeda, Schreier graphs of Grigorchuk's group and a subshift associated to a nonprimitive substitution, in *Groups, Graphs and Random Walks*, London Math. Soc. Lecture Note Ser. **436**, Cambridge Univ. Press, Cambridge, 2017, 250–299.

[21] R. Grigorchuk, D. Lenz, T. Nagnibeda, Spectra of Schreier graphs of Grigorchuk's group and Schroedinger operators with aperiodic order, *Math. Ann.* **370** (2018), 1607–1637.

[22] R. Grigorchuk, D. Savchuk, Z. Šunić, The spectral problem, substitutions and iterated monodromy, in *Probability and Mathematical Physics*, CRM Proc. Lecture Notes **42**, Amer. Math. Soc., Providence, RI, 2007, 225–248.

[23] R. Grigorchuk, Y. Leonov, V. Nekrashevych, V. Sushchansky, Self-similar groups, automatic sequences, and unitriangular representations, *Bull. Math. Sci.* **6** (2016), 231–285.

[24] P. de la Harpe, *Topics in geometric group theory*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 2000.

[25] R. A. Horn, C. R. Johnson, *Matrix Analysis*, Cambridge University Press, 2013.

[26] S. Istrail, On irreductible [sic] languages and nonrational numbers, *Bull. Math. Soc. Sci. Math. R. S. Roumanie* **21 (69)** (1977), 301–308.

[27] G. A. Jones, Maps related to Grigorchuk's group, *Eur. J. Comb.* **32**, (2011), 478–494.

[28] I. G. Lysënok, A system of defining relations for a Grigorchuk group, *Mat. Zametki* **38** (1985), 503–516, [*Math. Notes* **38** (1985), 784–792].

[29] Y. Muntian, Automata groups, PhD Thesis, 2009, available at `https://oaktrust.library.tamu.edu/bitstream/handle/1969.1/ETD-TAMU-2009-05-751/1_Muntyan.pdf`.

[30] V. Nekrashevych, Palindromic subshifts and simple periodic groups of intermediate growth, *Ann. of Math.* **187** (2018), 667–719.

[31] *On-Line Encyclopedia of Integer Sequences*, founded by N. J. A. Sloane, electronically available at `http://oeis.org`.

[32] M. Queffélec, *Substitution dynamical systems—Spectral analysis*, Second edition, Lect. Notes in Math. **1294**, Springer-Verlag, Berlin, 2010.

[33] B. Tan, Z.-Y. Wen, Invertible substitutions and Sturmian sequences, *European J. Combin.* **24** (2003), 983–1002.

[34] Y. Vorobets, On a substitution subshift related to the Grigorchuk group, in *Differential equations and topology. II, Collected papers. In commemoration of the centenary of the birth of Academician Lev Semenovich Pontryagin*, (MAIK Nauka/Interperiodica, Moscow, 2010), *Tr. Mat. Inst. Steklova* **271**, 319–334, [*Proc. Steklov Inst. Math.* **271**, (2010) 306–32].