

# DENOMINATORS OF COEFFICIENTS OF THE BAKER–CAMPBELL–HAUSDORFF SERIES

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## Abstract

For the computation of terms of the Baker–Campbell–Hausdorff series  $H = \log(e^A e^B)$  some a priori knowledge about the denominators of the coefficients of the series can be beneficial. In this paper an explicit formula for the computation of common denominators for the rational coefficients of the homogeneous components of the series is derived. Explicit computations up to degree 30 show that the common denominators obtained by this formula are as small as possible, which suggests that the formula is in a sense optimal. The sequence of integers defined by the formula seems to be interesting also from a number-theoretic point of view. There is, e.g., a connection with the denominators of the Bernoulli numbers and the Bernoulli polynomials.

## 1. Introduction

We consider the Baker–Campbell–Hausdorff (BCH) series which is formally defined as the element

$$H = \log(e^A e^B) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} (e^A e^B - 1)^k = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \left( \sum_{p+q>0} \frac{1}{p!q!} A^p B^q \right)^k$$

in the ring  $\mathbb{Q}\langle\langle A, B \rangle\rangle$  of formal power series in the non-commuting variables  $A$  and  $B$  with rational coefficients. A classical result known as the Baker–Campbell–Hausdorff theorem states that  $H$  is a *Lie series*, i.e., a sum  $H = \sum_{n=1}^{\infty} H_n$  of homogeneous components  $H_n$  which can be written as linear combinations of  $A$  and  $B$  and (possibly nested) commutator terms in  $A$  and  $B$ . For an accessible proof of the BCH theorem see, e.g., [1] or [3].

The algorithmic determination of the homogeneous components  $H_n$  turns out to be quite nontrivial, especially if the  $H_n$  are to be represented as linear combinations of linearly independent commutators. The determination of the coefficients  $h_w \in \mathbb{Q}$  in the representations

$$H_n = \sum_{w \in \{A, B\}^n} h_w w, \quad n = 1, 2, \dots$$

can be an important first step for this task. Here,  $\{\mathbf{A}, \mathbf{B}\}^n$  denotes the finite set of all words  $w$  of length (degree)  $|w| = n$  over the alphabet  $\{\mathbf{A}, \mathbf{B}\}$ . It would be beneficial if some a priori information about the denominators of the coefficients  $h_w = \text{coeff}(w, H)$  would be available, which would allow to compute a common denominator valid for all  $h_w \in \{\mathbf{A}, \mathbf{B}\}^n$ , because it would then be possible to compute these coefficients using pure integer arithmetic rather than less efficient rational arithmetic (cf. [4]). In this context, K. Goldberg stated in the penultimate paragraph of [2]:

[T]he chief difficulty is that computation with rationals is unavoidable until some idea of the factorization of the denominators of the coefficients is known. However for the small degrees,  $n \leq 10$ , all the denominators for the same degree  $n$  divide the denominator of  $(B_{n-1} + B_{n-2})/n!$  and this may be the general case.

Here  $B_n$  denote the Bernoulli numbers. Unfortunately, already for degree  $n = 11$  the denominator of the given formula is not a valid common denominator for all coefficients corresponding to this degree.<sup>1</sup> Indeed, for the word  $w = \mathbf{A}^8\mathbf{B}^3 \in \{\mathbf{A}, \mathbf{B}\}^{11}$  we have<sup>2</sup>  $h_w = 1/1247400$  and  $(B_{10} + B_9)/11! = 1/526901760$  but  $526901760/1247400 = 2212/5 \notin \mathbb{Z}$ . However, a valid such common denominator is given by the following theorem, which is the main result of the present paper.

**Theorem 1.** For  $n \geq 1$  define

$$d_n = \prod_{p \text{ prime, } p < n} p^{\max\{t: p^t \leq s_p(n)\}}, \quad (1)$$

where  $s_p(n) = \alpha_0 + \alpha_1 + \dots + \alpha_r$  is the sum of the digits in the  $p$ -adic expansion  $n = \alpha_0 + \alpha_1 p + \dots + \alpha_r p^r$ . Then  $n! d_n$  is a valid common denominator for all coefficients of words of length  $n$  in the Baker–Campbell–Hausdorff series  $H = \log(e^{\mathbf{A}} e^{\mathbf{B}})$ , or, equivalently,<sup>3</sup>

$$\text{denom}(\text{coeff}(w, H)) \mid n! d_n, \quad w \in \{\mathbf{A}, \mathbf{B}\}^n.$$

**Remark 1.** The theorem can be extended to the case of  $K \geq 2$  exponentials,

$$\text{denom}(\text{coeff}(w, \log(e^{\mathbf{A}_1} \dots e^{\mathbf{A}_K}))) \mid n! d_n, \quad w \in \{\mathbf{A}_1, \dots, \mathbf{A}_K\}^n.$$

Our proof of Theorem 1 in Section 3 will cover this more general case as well.

An explicit computation [4] yields<sup>4</sup>

$$\text{lcm}\{\text{denom}(\text{coeff}(w, H)) : w \in \{\mathbf{A}, \mathbf{B}\}^n\} = n! d_n, \quad n = 1, 2, \dots, 30,$$

<sup>1</sup>Also for  $n \leq 3$  the given formula is not entirely correct and has to be properly interpreted.

<sup>2</sup>This value can for example be looked up in the table given in [7].

<sup>3</sup>Formally, we define  $\text{denom}(r)$  for  $r \in \mathbb{Q}$  as the smallest positive integer  $d$  such that  $r \cdot d \in \mathbb{Z}$ . In particular,  $\text{denom}(0) = 1$ .

<sup>4</sup>Here  $\text{lcm } \mathcal{M}$  denotes the *least common multiple* of the elements of the finite set  $\mathcal{M} \subset \mathbb{Z}$ .

which shows that at least for  $n \leq 30$  the common denominators  $n!d_n$  are as small as possible. The first few values of  $d_n$  are

$$d_n = 1, 1, 2, 1, 6, 2, 6, 3, 10, 2, 6, 2, 210, 30, 12, 3, 30, 10, 210, 42, 330, 30, 60, 30, 546, \dots$$

A search in the Online Encyclopedia of Integer Sequences [8] does not (yet) result in a match for  $\{d_n\}$ , but remarkably there is a near match, namely the sequence A195441,

$$\tilde{d}_n = 1, 1, 2, 1, 6, 2, 6, 3, 10, 2, 6, 2, 210, 30, 6, 3, 30, 10, 210, 42, 330, 30, 30, 30, 546, \dots,$$

which is investigated in [5].<sup>5</sup> For  $n \leq 25$  we have  $d_n \neq \tilde{d}_n$  only for  $n = 15$  and  $n = 23$ . As it turns out,  $\tilde{d}_n$  is the square-free kernel of  $d_n$ ,

$$\tilde{d}_n = \prod_{p|d_n} p = \prod_{p < n: s_p(n) \geq p} p,$$

and there is a connection with the Bernoulli numbers and the Bernoulli polynomials,

$$\tilde{d}_n = \text{denom}(B_n(x) - B_n),$$

see [5].

The rest of the paper is organized as follows. First, still in this introduction, we prove two corollaries to Theorem 1 which for special degrees  $n$  provide information about the numerators of the BCH coefficients  $h_w$ . Then we give an example in which the results of Theorem 1 and Corollary 1 are verified by explicit computations. Our proof of Theorem 1 is naturally divided into a combinatorial and a number-theoretical part. The combinatorial part is given in Section 2 and leads to a preliminary result in Proposition 1. Based on this preliminary result we finally prove Theorem 1 in the number-theoretical part in Section 3 .

**Corollary 1.** *Let  $p \geq 2$  prime, and let  $w \in \{\mathbf{A}, \mathbf{B}\}^p \setminus \{\mathbf{A}^p, \mathbf{B}^p\}$  be a word of length  $p$  different from  $\mathbf{A}^p$  and  $\mathbf{B}^p$ . If the coefficient of  $w$  in the Baker–Campbell–Hausdorff series  $H = \log(e^{\mathbf{A}}e^{\mathbf{B}})$  is written with denominator  $p!d_p$ ,*

$$h_w = \text{coeff}(w, H) = \frac{a_w}{p!d_p},$$

*then the numerator  $a_w \in \mathbb{Z}$  satisfies*

$$a_w \equiv -d_p \pmod{p}.$$

*Proof.* In [9, Section IV.A] it is shown that for  $w \in \{\mathbf{A}, \mathbf{B}\}^p \setminus \{\mathbf{A}^p, \mathbf{B}^p\}$  the coefficient  $h_w$  can be written as

$$h_w = \frac{1}{p} + C_w,$$

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<sup>5</sup>In [5] the sequence  $\{\tilde{d}_n\}$  is denoted  $\{q_n\}$  and starts with index  $n = 0$  such that  $\tilde{d}_n = q_{n-1}$ ,  $n = 1, 2, \dots$

where  $C_w$  is a rational number whose denominator is not divisible by  $p$ . It follows

$$a_w = \left(\frac{1}{p} + C_w\right) p! d_p \equiv (p-1)! d_p \equiv -d_p \pmod{p},$$

where in the last step we used Wilson's theorem.  $\square$

**Corollary 2.** *Let  $n = p + 1$  with  $p \geq 3$  an odd prime. Define the exceptional set*

$$\mathcal{Z}_{p+1} = \{\mathbf{AB}^p, \mathbf{B}^p\mathbf{A}, \mathbf{A}^p\mathbf{B}, \mathbf{B}^p\mathbf{A}\} \cup \{w = w_1 \cdots w_{p+1} \in \{\mathbf{A}, \mathbf{B}\}^{p+1} : w_1 = w_{p+1}\}.$$

*Let  $w \in \{\mathbf{A}, \mathbf{B}\}^{p+1} \setminus \mathcal{Z}_{p+1}$ . If the coefficient of  $w$  in the Baker–Campbell–Hausdorff series  $H = \log(e^{\mathbf{A}}e^{\mathbf{B}})$  is written with denominator  $(p+1)! d_{p+1}$ ,*

$$h_w = \text{coeff}(w, H) = \frac{a_w}{(p+1)! d_{p+1}},$$

*then the numerator  $a_w \in \mathbb{Z}$  satisfies*

$$a_w \equiv \frac{p-1}{2} d_{p+1} \pmod{p}.$$

*If, on the other hand,  $w \in \mathcal{Z}_{p+1}$ , then  $\text{coeff}(w, H) = 0$ .*

*Proof.* In [9, Section IV.B] it is shown that  $h_w = 0$  for  $w \in \mathcal{Z}_{p+1}$ , and that for  $w \in \{\mathbf{A}, \mathbf{B}\}^{p+1} \setminus \mathcal{Z}_{p+1}$  the coefficient  $h_w$  can be written as

$$h_w = \frac{1}{2p} + C_w,$$

where  $C_w$  is a rational number whose denominator is not divisible by  $p$ . It follows

$$a_w = \left(\frac{1}{2p} + C_w\right) (p+1)! d_{p+1} \equiv (p-1)! \frac{p+1}{2} d_{p+1} \equiv \frac{p-1}{2} d_{p+1} \pmod{p}.$$

$\square$

**Example 1.** We consider the case  $n = 11$ . The coefficients  $h_w$  corresponding to the  $2^{11} - 2 = 2046$  words  $w \in \{\mathbf{A}, \mathbf{B}\}^{11} \setminus \{\mathbf{A}^{11}, \mathbf{B}^{11}\}$  only take values from a set of 30 elements. These 30 possible values of the coefficients are displayed in Table 1 and can be looked up in [7].<sup>6</sup> Also displayed are the prime factorizations of the denominators of the coefficients. The smallest common denominator for all these coefficients is given by the least common multiple of the denominators, which using

<sup>6</sup>The fact that so many coefficients have the same value is not a coincidence but a consequence of certain symmetries satisfied by the coefficients, see [2].

$h_w$	denom( $h_w$ )	$a_w$	$h_w$	denom( $h_w$ )	$a_w$
1/47900160	$2^9 \cdot 3^5 \cdot 5 \cdot 7 \cdot 11$	5	1/739200	$2^7 \cdot 3 \cdot 5^2 \cdot 7 \cdot 11$	324
-1/4790016	$2^8 \cdot 3^5 \cdot 7 \cdot 11$	-50	-13/554400	$2^5 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11$	-5616
1/1064448	$2^9 \cdot 3^3 \cdot 7 \cdot 11$	225	17/4435200	$2^8 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11$	918
1/1247400	$2^3 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11$	192	1/88704	$2^7 \cdot 3^2 \cdot 7 \cdot 11$	2700
-1/399168	$2^6 \cdot 3^4 \cdot 7 \cdot 11$	-600	-17/5322240	$2^9 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11$	-765
-13/6652800	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11$	-468	1/332640	$2^5 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11$	720
-1/277200	$2^4 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11$	-864	1/3991680	$2^7 \cdot 3^4 \cdot 5 \cdot 7 \cdot 11$	60
-1/712800	$2^5 \cdot 3^4 \cdot 5^2 \cdot 11$	-336	13/665280	$2^6 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11$	4680
1/228096	$2^8 \cdot 3^4 \cdot 11$	1050	13/7983360	$2^8 \cdot 3^4 \cdot 5 \cdot 7 \cdot 11$	390
7/2851200	$2^7 \cdot 3^4 \cdot 5^2 \cdot 11$	588	-1/124740	$2^2 \cdot 3^4 \cdot 5 \cdot 7 \cdot 11$	-1920
1/158400	$2^6 \cdot 3^2 \cdot 5^2 \cdot 11$	1512	-1/33264	$2^4 \cdot 3^3 \cdot 7 \cdot 11$	-7200
1/1900800	$2^8 \cdot 3^3 \cdot 5^2 \cdot 11$	126	-1/10395	$3^3 \cdot 5 \cdot 7 \cdot 11$	-23040
-1/190080	$2^7 \cdot 3^3 \cdot 5 \cdot 11$	-1260	-1/73920	$2^6 \cdot 3 \cdot 5 \cdot 7 \cdot 11$	-3240
-1/887040	$2^8 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	-270	1/27720	$2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	8640
-17/1663200	$2^5 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11$	-2448	-1/2772	$2^2 \cdot 3^2 \cdot 7 \cdot 11$	-86400

Table 1: Possible values and factorizations of their denominators for the coefficients  $h_w = \text{coeff}(w, \log(e^A e^B))$  corresponding to words  $w$  of length  $n = 11$ . The  $a_w$  are the numerators of these coefficients if written with denominator  $11!d_{11} = 239500800$ .

the factorizations is readily determined to be  $2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11 = 239500800$ . The computations

$$\begin{aligned}
11 &= 1 \cdot 2^3 + 1 \cdot 2^1 + 1 \cdot 2^0, & s_2(11) &= 3, & \max\{t : 2^t \leq s_2(11)\} &= 1, \\
11 &= 1 \cdot 3^2 + 2 \cdot 3^0, & s_3(11) &= 3, & \max\{t : 3^t \leq s_3(11)\} &= 1, \\
11 &= 2 \cdot 5^1 + 1 \cdot 5^0, & s_5(11) &= 3, & \max\{t : 5^t \leq s_5(11)\} &= 0, \\
11 &= 1 \cdot 7^1 + 4 \cdot 5^0, & s_7(11) &= 5, & \max\{t : 7^t \leq s_7(11)\} &= 0
\end{aligned}$$

result in  $d_{11} = 2 \cdot 3 = 6$  for the value defined by (1). Together with  $11! = 39916800$  this gives  $11!d_{11} = 239500800$  which is indeed the smallest possible common denominator. Furthermore, Table 1 shows the numerators  $a_w$  of the coefficients  $h_w = a_w/(11!d_{11})$  written with denominator  $11!d_{11} = 239500800$ . Since  $n = 11$  is prime, we expect that  $a_w \equiv -d_{11} = -6 \equiv 5 \pmod{11}$  holds by Corollary 1. It is readily verified that this is indeed the case, e.g., by computing the alternating digit sums of the numerators  $a_w$ , as in the well-known divisibility rule for  $n = 11$ .

## 2. A preliminary result

If some information about the denominators of the coefficients of the sub-expressions  $X_1, \dots, X_K \in \mathbb{Q}\langle\langle \mathcal{A} \rangle\rangle$  is available, one can expect that from it something can be learned about the denominators of the coefficients of the compound expressions  $X_1 +$

$\dots + X_K$  and  $X_1 \cdots X_K$ . The following technical lemma makes this idea concrete. We will apply this lemma to obtain a preliminary result about the denominators of the coefficients of the BCH series  $H = \log(e^A e^B)$  in Proposition 1, which will be the starting point for the proof of Theorem 1 in Section 3.

**Lemma 1.** *Let  $n \geq 0$ . Let  $X_i \in \mathbb{Q}\langle\langle A \rangle\rangle$  and let  $\delta_j(X_i) \in \mathbb{Z}_{>0}$  such that*

$$\text{denom}(\text{coeff}(v, X_i)) \mid \delta_j(X_i), \quad v \in \mathcal{A}^j, \quad j = 0, \dots, n, \quad i = 1, \dots, K.$$

Then for  $w \in \mathcal{A}^n$ , we have

(i)

$$\text{denom}(\text{coeff}(w, \frac{a}{b} X_i)) \mid \frac{b \delta_n(X_i)}{\gcd(b \delta_n(X_i), a)}, \quad a \in \mathbb{Z}, \quad b \in \mathbb{Z}_{>0},$$

(ii)

$$\text{denom}(\text{coeff}(w, X_1 + \dots + X_K)) \mid \text{lcm}\{\delta_n(X_1), \dots, \delta_n(X_K)\},$$

(iii)

$$\text{denom}(\text{coeff}(w, X_1 \cdots X_K)) \mid \text{lcm}\{\delta_{j_1}(X_1) \cdots \delta_{j_K}(X_K) : j_i \geq 0, j_1 + \dots + j_K = n\}.$$

(iv) *If the  $X_i$  have no constant terms,  $\text{coeff}(1, X_i) = 0$ ,  $i = 1, \dots, K$ , then the last divisibility relation can be tightened to*

$$\text{denom}(\text{coeff}(w, X_1 \cdots X_K)) \mid \text{lcm}\{\delta_{j_1}(X_1) \cdots \delta_{j_K}(X_K) : j_i \geq 1, j_1 + \dots + j_K = n\}.$$

*Proof.* (i) follows from

$$\text{denom}\left(\frac{a}{b} \frac{c}{d}\right) \mid \frac{bd}{\gcd(bd, a)}, \quad a, c \in \mathbb{Z}, \quad b, d \in \mathbb{Z}_{>0}.$$

(ii) follows from

$$\text{denom}(r_1 + \dots + r_K) \mid \text{lcm}\{\text{denom}(r_1), \dots, \text{denom}(r_K)\}, \quad r_1, \dots, r_K \in \mathbb{Q}. \quad (2)$$

Ad (iii). By distributing the subwords  $v^{(1)}, \dots, v^{(K)}$  of all partitions  $w = v^{(1)} \cdots v^{(K)}$  of  $w$  into  $K$  subwords of length  $|v^{(i)}| \geq 0$  among the factors  $X_1, \dots, X_K$  and summing over all such partitions we obtain

$$\text{coeff}(w, X_1 \cdots X_K) = \sum_{v^{(1)} \cdots v^{(K)} = w} \text{coeff}(v^{(1)}, X_1) \cdots \text{coeff}(v^{(K)}, X_K).$$

Each partition  $w = v^{(1)} \cdots v^{(K)}$  into  $K$  subwords uniquely corresponds to a partition  $n = j_1 + \dots + j_K$  of  $n = |w|$  into  $K$  summands  $j_1, \dots, j_K \geq 0$ , where the correspondence is given by  $(v^{(1)}, \dots, v^{(K)}) \mapsto (j_1, \dots, j_K) = (|v^{(1)}|, \dots, |v^{(K)}|)$ . Using (2) and

$$\text{denom}(r_1 \cdots r_K) \mid \text{denom}(r_1) \cdots \text{denom}(r_K), \quad r_1, \dots, r_K \in \mathbb{Q}$$

it follows

$$\text{denom}(\text{coeff}(w, X_1 \cdots X_K)) \mid \text{lcm}\{\delta_{j_1}(X_1) \cdots \delta_{j_K}(X_K) : j_i \geq 0, j_1 + \cdots + j_K = n\}.$$

The proof of (iv) is the same as the one of (iii) except that now only partitions  $w = v^{(1)} \cdots v^{(K)}$  into subwords of length  $|v^{(1)}| \geq 1$  have to be considered. Such partitions now correspond to partitions  $n = j_1 + \cdots + j_K$  into summands  $j_1, \dots, j_K \geq 1$ .  $\square$

**Proposition 1.** *Define*

$$D_n = \text{lcm}\{k j_1! \cdots j_k! : j_i \geq 1, j_1 + \cdots + j_k = n, k = 1, \dots, n\}, \quad n = 1, 2, \dots \quad (3)$$

*Then*

$$\text{denom}(\text{coeff}(w, \log(e^{\mathbf{A}_1} \cdots e^{\mathbf{A}_K}))) \mid D_n, \quad w \in \mathcal{A}^n, \mathcal{A} = \{\mathbf{A}_1, \dots, \mathbf{A}_K\}.$$

*Proof.* Because  $\text{coeff}(w, e^{\mathbf{A}_i}) \in \{0, 1/n!\}$  for  $w \in \mathcal{A}^n$  it is clear that

$$\text{denom}(\text{coeff}(w, e^{\mathbf{A}_i})) \mid n!, \quad w \in \mathcal{A}^n, i = 1, \dots, K, n = 0, 1, 2, \dots,$$

which using Lemma 1 (iii) implies

$$\text{denom}(\text{coeff}(w, e^{\mathbf{A}_1} \cdots e^{\mathbf{A}_K})) \mid \text{lcm}\{j_1! \cdots j_K! : j_i \geq 0, j_1 + \cdots + j_K = n\} = n!.$$

We set  $Y = e^{\mathbf{A}_1} \cdots e^{\mathbf{A}_K} - 1$  so that  $\text{coeff}(1, Y) = 0$  and

$$\text{denom}(\text{coeff}(w, Y)) \mid n!, \quad w \in \mathcal{A}^n, n = 1, 2, \dots,$$

which using Lemma 1 (i), (iv) implies

$$\text{denom} \left( \text{coeff} \left( w, \frac{(-1)^{k+1}}{k} Y^k \right) \right) \mid \text{lcm}\{k j_1! \cdots j_k! : j_i \geq 1, j_1 + \cdots + j_k = n\}$$

for  $k = 1, \dots, n$  and  $w \in \mathcal{A}^n$ . Because  $\text{coeff}(w, Y^k) = 0$  for  $k > |w|$  we have

$$\text{coeff}(w, \log(1 + Y)) = \text{coeff} \left( w, \sum_{k=1}^n \frac{(-1)^{k+1}}{k} Y^k \right), \quad |w| \leq n,$$

from which

$$\text{denom}(\text{coeff}(w, \log(1 + Y))) \mid \text{lcm}\{k j_1! \cdots j_k! : j_i \geq 1, j_1 + \cdots + j_k = n, k = 1, \dots, n\}$$

follows for  $w \in \mathcal{A}^n, n = 1, 2, \dots$  by an application of Lemma 1 (ii).  $\square$

### 3. Proof of Theorem 1

For the proof of Theorem 1 we will show that

$$n! d_n = D_n, \quad (4)$$

where  $d_n$  is defined by (1) and  $D_n$  is defined by (3). Then, Theorem 1 will be an immediate consequence of Proposition 1.

For a prime  $p \geq 2$  the  $p$ -adic valuation  $v_p(n)$  of  $n$  is defined as the exponent of the highest power of  $p$  that divides  $n$ . The function  $v_p$  satisfies

$$v_p(n \cdot m) = v_p(n) + v_p(m),$$

which implies that (4) is equivalent to

$$v_p(n!) + v_p(d_n) = v_p(D_n) \quad \text{for all primes } p \geq 2,$$

where  $v_p(d_n) = \max\{t : p^t \leq s_p(n)\}$ . Here and in the following  $s_p(n) = \alpha_0 + \dots + \alpha_r$  is the sum of digits in the  $p$ -adic expansion  $n = \alpha_0 + \alpha_1 p + \dots + \alpha_r p^r$ . To compute  $v_p(D_n)$  we need some further properties of the function  $v_p$ .

For nonempty finite subsets  $\mathcal{M} \subset \mathbb{Z}_{\geq 0}$  we have

$$v_p(\text{lcm } \mathcal{M}) = \max_{m \in \mathcal{M}} v_p(m)$$

and, by convention,  $\text{lcm}(\emptyset) = 1$  such that  $v_p(\text{lcm}(\emptyset)) = 0$ .

For the computation of  $v_p$  for factorials we have *Legendre's formula*

$$v_p(n!) = \frac{n - s_p(n)}{p - 1},$$

see, e.g., [6].

Now, let  $j_1, \dots, j_k \geq 1$  with  $j_1 + \dots + j_k = n$ . Then

$$\begin{aligned} v_p(k j_1! \cdots j_k!) &= v_p(k) + v_p(j_1!) + \dots + v_p(j_k!) \\ &= v_p(k) + \frac{1}{p-1} (n - s_p(j_1) - \dots - s_p(j_k)) \\ &= v_p(n!) + v_p(k) - \frac{1}{1-p} (s_p(j_1) + \dots + s_p(j_k) - s_p(n)). \end{aligned}$$

It follows

$$\begin{aligned} v_p(D_n) &= \max_{k=1, \dots, n} \max_{j_i \geq 1, j_1 + \dots + j_k = n} v_p(k j_1! \cdots j_k!) \\ &= v_p(n!) + \max_{k=1, \dots, n} ((v_p(k) - h_p(n, k))), \end{aligned} \quad (5)$$



where

$$h_p(n, k) = \frac{1}{p-1} \min_{j_i \geq 1, j_1 + \dots + j_k = n} (s_p(j_1) + \dots + s_p(j_k) - s_p(n)).$$

To complete the computation of  $v_p(D_n)$  we need some properties of the function  $h_p(n, k)$  which follow from the following two lemmas.

**Lemma 2.** *If  $k \leq s_p(n)$ , then*

$$\min_{j_i \geq 1, j_1 + \dots + j_k = n} (s_p(j_1) + \dots + s_p(j_k) - s_p(n)) = 0.$$

*Proof.* With the multinomial coefficient  $\binom{n}{j_1, \dots, j_k} = \frac{n!}{j_1! \dots j_k!}$  we have

$$\frac{1}{p-1} (s_p(j_1) + \dots + s_p(j_k) - s_p(n)) = v_p \left( \binom{n}{j_1, \dots, j_k} \right) \geq 0, \quad (6)$$

and thus

$$s_p(j_1) + \dots + s_p(j_k) - s_p(n) \geq 0$$

for all  $j_1, \dots, j_k \geq 1$  with  $j_1 + \dots + j_k = n$ .

Using the assumption  $k \leq s_p(n)$  we now construct an assignment of the variables  $j_1, \dots, j_k \geq 1$  for which  $j_1 + \dots + j_k = n$  and  $s_p(j_1) + \dots + s_p(j_k) = s_p(n)$  hold. The existence of such an assignment suffices to prove the lemma.

Corresponding to the  $p$ -adic expansion

$$n = \alpha_0 + \alpha_1 p + \dots + \alpha_r p^r$$

let  $x$  ( $0 \leq x \leq r$ ) be uniquely defined by the inequalities

$$\alpha_0 + \dots + \alpha_{x-1} \leq k-1 < \alpha_0 + \dots + \alpha_x,$$

and let  $y$  ( $0 \leq y < \alpha_x$ ) be defined by the equation

$$k-1 = \alpha_0 + \dots + \alpha_{x-1} + y.$$

Note that here for the existence of  $x$  the requirement  $k-1 < s_p(n) = \alpha_0 + \dots + \alpha_r$  is necessary. Define  $j_1, \dots, j_{k-1}$  by

$$\begin{aligned} j_i &= 1, & i &= 1, \dots, \alpha_0, \\ j_i &= p, & i &= \alpha_0 + 1, \dots, \alpha_0 + \alpha_1, \\ j_i &= p^2, & i &= \alpha_0 + \alpha_1 + 1, \dots, \alpha_0 + \alpha_1 + \alpha_2, \\ & \vdots & & \vdots \\ j_i &= p^{x-1}, & i &= \alpha_0 + \dots + \alpha_{x-2} + 1, \dots, \alpha_0 + \dots + \alpha_{x-2} + \alpha_{x-1}, \\ j_i &= p^x, & i &= \alpha_0 + \dots + \alpha_{x-1} + 1, \dots, \alpha_0 + \dots + \alpha_{x-1} + y = k-1, \end{aligned}$$

and  $j_k$  by

$$j_k = (\alpha_x - y)p^x + \alpha_{x+1}p^{x+1} + \dots + \alpha_r p^r.$$

Then  $s_p(j_i) = 1$ ,  $i = 1, \dots, k-1$  and  $s_p(j_k) = (\alpha_x - y) + \alpha_{x-1} + \dots + \alpha_r$ , and thus

$$\begin{aligned} s_p(j_1) + \dots + s_p(j_{k-1}) + s_p(j_k) &= k-1 + \alpha_x - y + \alpha_{x+1} + \dots + \alpha_r \\ &= \alpha_0 + \dots + \alpha_{x-1} + y + \alpha_x - y + \alpha_{x+1} + \dots + \alpha_r \\ &= \alpha_0 + \dots + \alpha_r = s_p(n). \end{aligned}$$

Similarly, it is easy to check that  $j_1 + \dots + j_k = n$ , and it is clear that  $j_i, \dots, j_k \geq 1$  (for  $j_k$  this follows from  $y < \alpha_x$ ).

□

**Lemma 3.** For  $n \geq 1$  let  $l = \max\{t : p^t \leq s_p(n)\}$  such that  $p^l \leq s_p(n) < p^{l+1}$ , and let  $k = p^{l+m}x > s_p(n)$  with  $m \geq 1$  and  $x \geq 1$ . Then

$$\frac{1}{p-1}(s_p(j_1) + \dots + s_p(j_k) - s_p(n)) \geq m \quad (7)$$

for all  $j_1, \dots, j_k \geq 1$  with  $j_1 + \dots + j_k = n$ .

*Proof.* We have

$$\begin{aligned} \frac{1}{p-1}(s_p(j_1) + \dots + s_p(j_k) - s_p(n)) &> \frac{1}{p-1}(p^{l+m}x - p^{l+1}) \\ &\geq \frac{p^{l+1}}{p-1}(p^{m-1} - 1) \\ &= p^{l+1}(p^{m-2} + p^{m-3} + \dots + 1) \\ &\geq 2(m-1) \geq m-1. \end{aligned}$$

(Here, if  $m = 1$ , the sum  $p^{m-2} + p^{m-3} + \dots + 1$  in the next-to-last row is understood to be  $= 0$ .) Since  $(s_p(j_1) + \dots + s_p(j_k) - s_p(n))/(p-1)$  is an integer according to (6), this implies (7). □

We are now in the position to complete the computation (5) of  $v_p(D_n)$ . Let  $l = v_p(d_n) = \max\{t : p^t \leq s_p(n)\}$  such that  $p^l \leq s_p(n) < p^{l+1}$ . For  $k \in \{1, \dots, n\}$  we have the following 3 mutually exclusive possibilities:

- (i) If  $k \leq s_p(n)$ , then  $v_p(k) = l$  and  $h_p(n, k) = 0$  by Lemma 2; thus  $v_p(k) - h_p(n, k) = l$ .
- (ii) If  $k = p^{l+m}x > s_p(n)$ ,  $m \geq 1$ ,  $x \geq 1$ ,  $p \nmid x$ , then  $v_p(k) = l + m$  and  $h_p(n, k) \geq m$  by Lemma 3; thus  $v_p(k) - h_p(n, k) \leq l$ .
- (iii) If  $k = p^t x > s_p(n)$ ,  $t \leq l$ ,  $x \geq 1$ ,  $p \nmid x$ , then  $v_p(k) = t \leq l$  and  $h_p(n, k) \geq 0$ ; thus  $v_p(k) - h_p(n, k) \leq l$ .

Altogether this implies

$$v_p(D_n) = v_p(n!) + \max_{k=1, \dots, n} ((v_p(k) - h_p(n, k))) = v_p(n!) + l = v_p(n!) + v_p(d_n)$$

for all primes  $p \geq 2$ , which as already mentioned is equivalent to (4), and thus completes the proof of Theorem 1.

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