Counting Path Configurations in Parallel Diffusion

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Abstract Parallel Diffusion is a variant of Chip-Firing introduced in 2018 by Duffy et al. In Parallel Diffusion, chips move from places of high concentration to places of low concentration through a discrete-time process. At each time step, every vertex sends a chip to each of its poorer neighbours, allowing for some vertices to perhaps fall into debt (represented by negative stack sizes). In their recent paper, Long and Narayanan proved a conjecture from the original paper by Duffy et al. that every Parallel Diffusion process eventually, after some pre-period, exhibits periodic behaviour. With this result, we are now able to count the number of these periods that exist up to a definition of isomorphism. We determine a recurrence relation for calculating this number for a path of any length. If T_n is the number of configurations with period length 2 that can exist on P_n up to isomorphism and n is an integer greater than 4, we conclude that $T_n = 3T_{n-1} + 2T_{n-2} + T_{n-3} - T_{n-4}$.

Keywords Graph Theory · Discrete-Time Processes · Chip-Firing · Diffusion

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1 Introduction

Introduced by Duffy et al. [2], Parallel Diffusion is a process defined on a simple finite graph, G, in which at every time step, chips are diffused throughout the graph following specific rules. Each vertex is assigned a *stack size* which is an integral number that represents the number of chips a vertex has. An assignment of stack sizes to the vertices of a graph G is referred to as a *configuration*, denoted $C = \{(v, |v|^C) : v \in V(G)\}$, where $|v|^C$ is the stack size of v in C. We omit the superscript when the configuration is clear. At each time step, the chips are redistributed via the following rules: If a vertex is adjacent to a vertex with fewer chips, it takes a chip from its stack and adds it to the stack of the poorer vertex. This creates a new configuration (see Figure 1). We call this action the *firing* of a vertex. At each time step of the diffusion process, every vertex fires simultaneously. Note that when a vertex with no poorer neighbours fires, it does not send any chips.

Long and Narayanan showed this process to be periodic [3], with every configuration eventually (after some number of steps) leading either to a single configuration in which every stack size is equal (period length of 1) or a pair of configurations which yield each other (period length of 2). As a byproduct of Long and Narayanan's work, we can now count the configurations that can exist on a given graph. Clearly, the number of configurations that can exist on a given graph. Clearly, the number of configurations that can exist on a given graph is infinite because there are infinitely many integers. But what if we only were interested in counting those configurations that yield each other, described by Long and Narayanan? In this paper, we use Long and Narayanan's result that every period is length 1 or 2 to count the number of different configurations that can exist on a path up to a definition of isomorphism. Our method will involve viewing the transfer of chips as a mixed graph and excluding every mixed graph on P_n that cannot possibly represent the flow of chips within the period. From there, we count the number of period configurations that exist on P_n for each of the mixed graphs that were not excluded.

A vertex v is said to be richer than another vertex u in configuration C if $|v|^C > |u|^C$. In this instance, u is said to be poorer than v in C. If $|v|^C < 0$, we say v is in debt in C.

We are interested in counting the number of configurations, T_n , on P_n , $n \ge 1$ (up to a definition of isomorphism). We will show, in Theorem 6, that T_n can be calculated for all $n \ge 5$ by the recurrence relation $T_n = 3T_{n-1} + 2T_{n-2} + T_{n-3} - T_{n-4}$. This relation has an asymptotic growth rate of roughly 3.6096 (Corollary 6).

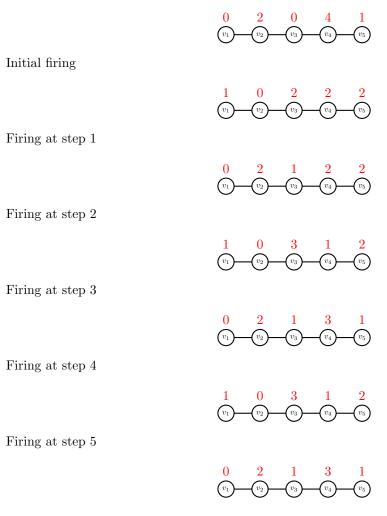


Fig. 1 Several steps in a Parallel Diffusion game on P_5

Unlike some other chip-firing processes like the original Chip-Firing game [1] and Brushing [5], in Parallel Diffusion it is possible for a stack size to initially be positive but to become negative as time goes on. For example if some vertex v with a stack size of $n, n \in \mathbb{N}$, is adjacent to n + 1vertices, each of which having a stack size of 0, then after firing, v would have at stack size of -1. However in [2], it was shown that Parallel Diffusion is such that an addition of some constant k, $k \in \mathbb{Z}$, to each stack size will have no effect on determining when and if a chip will move from one vertex to another. So if one wanted to view diffusion as a process in which stack sizes are never negative, one would only need to add a sufficient constant $k, k \in \mathbb{Z}$, to each stack size. Some results pertaining to locating an appropriate k value for any given graph can be found in "Uniform Bounds for Non-Negativity of the Diffusion Game" by Carlotti and Herman, arXiv:1805.05932v1.

We begin with some necessary terminology.

Let G be a finite simple undirected graph with vertex set V(G) and edge set E(G). Let $A \subseteq E(G)$. A graph orientation of a graph G is a mixed graph obtained from G by choosing an orientation $(x \to y)$ or $y \to x$ for each edge xy in A. We refer to the edges that are in $E(G) \setminus A$ as flat. We refer to the assignment of either $x \to y$, $y \to x$, or flat to an edge xy as xy's edge orientation. Let R be a graph orientation of a graph G. A suborientation R' of R is a graph orientation of some subgraph G' of G such that every edge xy in G' is assigned the same edge orientation as in R.

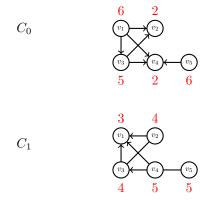


Fig. 2 Configuration C_0 fires, yielding C_1 . Directed edges depict the flow of chips from richer vertices to poorer vertices.

In Parallel Diffusion, the assigned value of a vertex, v, at step t, is referred to as its stack size at time t. If the initial configuration is C, then the stack size at time t is denoted $|v|_t^C$. This implies that $|v|^C = |v|_0^C$. We omit the superscript when the configuration is clear. Given a graph G and an initial configuration C_0 , then $C_t = \{(v, |v|_t^C) : v \in V(G)\}$. The configuration sequence $Seq(C_0) = \{C_0, C_1, C_2, \ldots\}$ is the sequence of configurations that arises as the time increases. The configuration sequence clearly depends on both the initial configuration and the graph G. However, it will always be clear to which graph we are referring, so we omit any reference to G in our notation, $Seq(C_0)$. The *initial firing* is the firing of the vertices in C_0 , yielding C_1 . Likewise, the *firing at step* m is the firing of the vertices of C_m , yielding C_{m+1} .

Given two configurations, C and D, of a graph G, in which the vertices are labelled, C and D are equal if $|v|^C = |v|^D$ for all $v \in V(G)$. Let $Seq(C_0) = \{C_0, C_1, C_2, ...\}$ be the configuration sequence on a graph G with initial configuration C_0 . The positive integer p is a period length if $C_t = C_{t+p}$ for all $t \geq N$ for some N. In this case, N is the preperiod length. For such a value, N, if $k \geq N$, then we say that the configuration, C_k , is inside the period. For the purposes of this paper, all references to period length will refer to the minimum period length p in a given configuration sequence. Also, all references to preperiod length will refer to the least preperiod length that yields that minimum period length p in a given configuration sequence. In Figure 1, the period length is 2 and the preperiod length is 3.

In their paper [3], Long and Narayanan proved the following theorem which was presented as a conjecture by Duffy et al. [2].

Theorem 1 [3] Every configuration sequence, regardless of initial configuration, has period 1 or 2.

In the proof of Theorem 1, Long and Narayanan show that once inside the period, if a chip fires from u to v at step t, then a chip must fire from v to u at step t + 1. We will be using this result, so we set it aside as the following corollary.

Corollary 1 [3] In Parallel Diffusion, let C_t be the configuration at time t and suppose C_t is inside the period. If a vertex u is richer than an adjacent vertex v at step t, then v is richer than u at step t+1. Note that this implies that if an edge is flat inside the period, then upon firing, it must remain flat. Both the previous corollary and the following observation will prove crucial in counting period configurations on paths.

Observation 1 A step in Parallel Diffusion induces a graph orientation.

Proof Let G be a graph and C_t a configuration on G. For all pairs of adjacent vertices u, v in G at step t, either u gives a chip to v, v gives a chip to u, the stack sizes of u and v are equal in C_t . Let uv be an edge. Assign directions as follows:

- If u gives a chip to v at time t, assign uv the edge orientation $u \to v$.
- If v gives a chip to u at time t, assign uv the edge orientation $v \to u$.
- If the stack sizes of u and v are equal at time t, do not direct the edge uv.

Thus, a graph orientation on G results.

We say that this graph orientation is *induced* by C_t , the configuration of G at time t. We see an example of a graph orientation induced by a configuration in Parallel Diffusion in Figure 3.

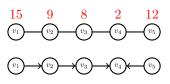


Fig. 3 Configuration on P_5 and its induced graph orientation.

Let $L(G) = \{Seq(C) : C \text{ is a configuration on } G\}$. Since the set of integers is infinite, on any graph G, L(G) is an infinite set.

Let $\overline{Seq(C_0)}$ be the singleton or ordered pair of configurations contained within the period of a configuration sequence $Seq(C_0)$. If $Seq(C_0)$ has period 2, define the first element of the ordered pair $\overline{Seq(C_0)}$ to be the one which occurs first in the configuration sequence. A configuration D on a graph G is a period configuration if D is in $\overline{Seq(C)}$ for some configuration C. A configuration D on a graph G is a period configuration if D is in $\overline{Seq(C)}$ for some configuration C and $\overline{Seq(C)}$ has 2 elements. A configuration D on a graph G is a fixed configuration if D is in $\overline{Seq(C)}$ for some configuration that is induced by a period configuration. A period orientation is a graph orientation that is induced by a period configuration. A p₂-orientation is a graph orientation that is induced by a p₂-configuration. A fixed orientation is a graph orientation that is induced by a fixed configuration.

Let C be a configuration on a graph G. Let C + k be the configuration created by adding an integer k to every stack size in the configuration C. $Seq(C), Seq(D) \in L(G)$ are *isomorphic* if $\overline{Seq(C+k)} = \overline{Seq(D)}$ for some integer k. Let $\overline{L(G)} = \{\overline{Seq(C)} : C \text{ a configuration on } G\}$. Let $\overline{L'(G)}$ be the largest subset of $\overline{L}(G)$ such that no two elements are isomorphic. In this paper, we will determine the cardinality of $\overline{L'(P_n)}$ for all $n \geq 1$.

We see an example of isomorphic configuration sequences in Figure 5. We see an example of $\overline{L}(G)$ and $\overline{L}'(G)$ in Figure 4.

\longrightarrow

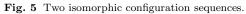
 $\overline{L}(P_2) = \{\{(0,0)\}, \{(0,1), (1,0)\}, \{(1,0), (0,1)\}, \{(1,2), (2,1)\}, \{(2,1), (1,2)\}, \{(3,2), (2,3)\}, \dots \}$

$$\overline{L}'(P_2) = \{\{(0,0)\}, \{(0,1), (1,0)\}, \{(1,0), (0,1)\}\}$$

Fig. 4 $\overline{L}(P_2)$ and $\overline{L}'(P_2)$

Configuration sequence $Seq(C_0)$

	C_0	$\begin{array}{cccccccccccccccccccccccccccccccccccc$
Firing at step 0		
Fining at stop 1	C_1	$1 0 1 0 1$ $v_1 \rightarrow v_2 \leftarrow v_3 \rightarrow v_4 \leftarrow v_5$
Firing at step 1		
	C_2	$\begin{array}{cccccccccccccccccccccccccccccccccccc$
Firing at step 2		
	C_3	$1 0 1 0 1 \\ v_1 \longrightarrow v_2 \longleftarrow v_3 \longrightarrow v_4 \longleftarrow v_5$
Configuration sequence $Seq(C'_0)$		
		1 1 0 1 1
	C_0'	$ \begin{array}{cccc} -1 & 1 & -2 & 1 & -1 \\ \hline v_1 \longrightarrow v_2 \longleftarrow v_3 \longrightarrow v_4 \longleftarrow v_5 \end{array} $
Firing at step 0		
	C_1'	$\begin{array}{cccccccccccccccccccccccccccccccccccc$
Firing at step 1		
	C'_2	$\begin{array}{cccccccccccccccccccccccccccccccccccc$
Firing at step 2		
	C'_3	$\begin{array}{cccccccccccccccccccccccccccccccccccc$
Fig. 5 Two isomorphic configuration sequences		



Lemma 1 (Lemma 3.1.16 from "On Variants of Diffusion", T. Mullen, PhD Thesis) Let G be a graph. Up to isomorphism, the only fixed configuration on G is the one in which every vertex has 0 chips.

Lemma 2 (Lemma 3.1.1 from "On Variants of Diffusion", T. Mullen, PhD Thesis) Let C and D be configurations on a graph G. Let k be an integer. Suppose that for all $v \in V(G)$, $|v|^C = |v|^D + k$. Then for all t, $|v|_t^C = |v|_t^D + k$.

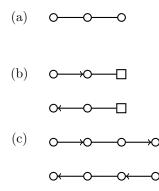
2 Paths

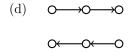
We now approach the problem of counting all of the p_2 -orientations on a path. We will draw our paths along a horizontal axis and label the vertices from *right to left* with the rightmost vertex labelled v_1 . We will fix v_1 at zero chips. By Theorem 1, we know every configuration is eventually periodic with either period 1 or 2. Since we are fixing v_1 at zero chips, by Lemma 1, the only possible period 1 configuration on any path is the one in which every vertex has zero chips. So, we will restrict our view to only counting p_2 -configurations. We will make frequent use of Corollary 1 in justifying the flow of chips inside the period.

Since our paths are drawn along a horizontal axis, the terms "left" and "right" have an obvious meaning. On a path drawn along a horizontal axis, a *left* edge is a directed edge in which the head is to the left of the tail and a *right* edge is a directed edge in which the head is to the right of the tail. When referring to edges contained within a path defined on a horizontal axis, two edges *agree* if they are either both right edges or both left edges. Two edges *disagree* if one is right and the other is left. An *alternating arrow orientation* is a path orientation in which every pair of adjacent edges disagree. Note that this means an alternating arrow orientation cannot contain any flat edges.

We will proceed by dividing P_n , $n \ge 3$, into all of its possible p_2 -orientations and then determining, for each p_2 -orientation, R, how many different nonisomorphic p_2 -configurations induce R. We begin by characterizing when a path orientation is a p_2 -orientation.

Theorem 2 A path orientation is a p_2 -orientation if and only if none of the following mixed graphs exist as a suborientation (let a square represent a leaf and a circle represent a vertex that may or may not be a leaf).





can only exist within the respective subgraphs

We express the proof with a series of lemmas. In particular, we prove the necessary condition with Lemmas 3-6, and we prove the sufficient statement with Lemma 7.

Lemma 3 Case (a) in Theorem 2: No p_2 -configuration on a path induces a mixed graph with two adjacent flat edges.

Proof Suppose, by contradiction, that it were possible to have two flat edges adjacent to each other in a graph orientation induced by a p_2 -configuration. Call these edges e_k and e_{k+1} , and call the vertices v_k , v_{k+1} , and v_{k+2} . So, the subpath in question is $X = v_{k+2}e_{k+1}v_{k+1}e_kv_k$. Since the period is two, we know that some edge in the graph must not be flat by Lemma 1. Without loss of generality, suppose that the edge immediately to the right of X, e_{k-1} , is oriented left or right. This would result in v_{k+1} maintaining its number of chips in the initial firing, but as for the firing at step 1, v_{k+1} is now adjacent to a vertex that either increased or decreased its stack size in the initial firing. Thus, $|v_{k+1}|_2 \neq |v_{k+1}|_0$. Therefore, the orientation is not induced by a p_2 -configuration. This is a contradiction.

Lemma 4 Case (b) in Theorem 2: No p_2 -configuration on a path induces a graph orientation which contains a flat edge incident with a leaf.

Proof Suppose, by contradiction, that there exists a flat edge, e_1 , incident with a leaf, v_1 , in a graph orientation, R, induced by a p_2 -configuration, C. We know by Lemma 1 that there exists at least one more edge, e_2 , in this graph since we are supposing that C has period 2. In step 1, v_1 and its neighbour, v_2 , have equal stack sizes. By Lemma 3, we know that e_2 , the edge adjacent to e_1 , is not flat. So, v_2 has either gained or lost a chip in the initial firing, while $|v_1|$ has gone unchanged. So, in the firing at step 1, v_1 will gain or lose a chip, indicating that $|v_1|_2 \neq |v_1|_0$. This is a contradiction.

Lemma 5 Case (c) in Theorem 2: Every flat edge in a graph orientation induced by a p_2 -configuration on a path is adjacent to two edges: one right and one left.

Proof Suppose, by contradiction, that there exists some flat edge that is not adjacent to both a left edge and a right edge in some graph orientation, R, induced by a p_2 -configuration, C. Call this flat edge e_k and its endpoints v_k and v_{k+1} . In the initial firing, no chips will move across e_k since it is flat. In the firing at step 1, the same must be true since we supposed that C is a p_2 -configuration.

Case 1: e_k is incident with a leaf. By Lemma 4, we know that no flat can be incident with a leaf. This is a contradiction.

Case 2: At least one edge adjacent to e_k is flat. By Lemma 3, this is impossible. This is a contradiction.

Case 3: e_k is adjacent to two left edges. Then $|v_k|_1 = |v_k|_0 + 1$ and $|v_{k+1}|_1 = |v_k|_0 - 1$. So, the stack sizes of v_k and v_{k+1} are not equal at step t + 1. This is a contradiction.

Case 4: e_k is adjacent to two right edges. Then $|v_k|_1 = |v_k|_0 - 1$ and $|v_{k+1}|_1 = |v_k|_0 + 1$. So, the stack sizes of v_k and v_{k+1} are not equal at step t + 1. This is a contradiction.

Thus, every flat edge in a p_2 -configuration on a path is incident with two edges: one right and one left.

Lemma 6 Case (d) in Theorem 2: Let H be a suborientation of a p_2 -orientation, R, on a path, P_n . If H is a directed path consisting of three vertices and two right edges, then H must be incident with two left edges in P_n . Conversely, if H is a directed path consisting of three vertices and two left edges, then H must be incident with two right edges in P_n .

Proof Suppose that $H = v_{k+1}e_kv_ke_{k-1}v_{k-1}$ contains three vertices and, without loss of generality, two right edges. In the initial firing, v_k gives and receives a single chip, maintaining its stack size. Initially, we have $|v_{k+1}|_0 > |v_k|_0 > |v_{k-1}|_0$. In the configuration at step 1, these inequalities must be reversed since we are already inside the period, by Corollary 1. Since $|v_k|_1 = |v_k|_0$, we know that v_{k+1} must lose at least 2 chips in the initial firing and v_{k-1} must gain at least 2 chips in the initial firing. However, this is only possible if both edges in $P_n - H$ that are incident with H are left edges.

Lemma 7 Any path orientation with no suborientations of the forms outlined in Theorem 2, is a p_2 -orientation.

Proof We must now show that every orientation that does not contain any of the suborientations from Theorem 2 is a p_2 -orientation. Our method will involve taking an arbitrary orientation R that does not contain any of the suborientations listed in Theorem 2, and proving that there exists an assignment of stack sizes that both induces R and exists within a period of length 2. There are 3 orientations that an edge may have: flat, left, and right. We will assume that moving from right to left, every vertex v_i has been assigned an initial stack size to create the configuration, C, using the following rule:

$$|v_i|_0 = \begin{cases} |v_{i-1}|_0 + 1 & \text{if edge } v_{i-1}v_i & \text{is directed right.} \\ |v_{i-1}|_0 - 1 & \text{if edge } v_{i-1}v_i & \text{is directed left.} \\ |v_{i-1}|_0 & \text{if edge } v_{i-1}v_i & \text{is flat.} \end{cases}$$

and v_1 has been assigned 0 chips.

We now inspect an edge $e_j = v_j v_{j+1}$ in R with the goal of determining if its incident vertices will restore their initial stack size after two firings.

Case 1: The edge $e_j = v_j v_{j+1}$ is flat.

We know that neither v_j nor v_{j+1} is a leaf by Lemma 4. In C, v_j and v_{j+1} have both been initially assigned to have the same number of chips. However, in order for C to be a p_2 -configuration, we must also have that $|v_j|_1 = |v_{j+1}|_1$, by Corollary 1. In order to determine this, we must know the stack sizes of v_j and v_{j+1} at step 1. This will depend on the initial orientation of edges $e_{j-1} = v_{j-1}v_j$ and $e_{j+1} = v_{j+1}v_{j+2}$. We know that since e_j is flat, no adjacent edge can be flat by Lemma 3. Also, e_{j-1} and e_{j+1} cannot be both right or both left by Lemma 5. So, e_{j-1} and e_{j+1} must disagree. Without loss of generality, suppose e_{j-1} is directed right and e_{j+1} is directed left. So, our rule dictates that $|v_{j-1}|_0 + 1 = |v_j|_0 = |v_{j+1}|_0 = |v_{j+2}|_0 + 1$. Thus, with both vertices receiving a total of one chip at step 0, we have that $|v_j|_1 = |v_{j+1}|_1$. So, $|v_j|_2 = |v_j|_0$ and $|v_{j+1}|_2 = |v_j|_0$.

Case 2: The edge $e_j = v_j v_{j+1}$ is directed.

Suppose, without loss of generality, that e_j is directed right. In C, $|v_{j+1}|_0 = |v_j|_0 + 1$. In order for C to be a p_2 -configuration, we must have that $|v_{j+1}|_1 < |v_j|_1$. In order to determine this, we must know the stack sizes of v_j and v_{j+1} at step 1. This will depend on the initial orientation of edges $e_{j-1} = v_{j-1}v_j$ and $e_{j+1} = v_{j+1}v_{j+2}$. Note that either e_{j-1} or e_{j+1} may not exist depending on if either v_j or v_{j+1} is a leaf. However, the absence of either of these edges has the same effect on the stack size of the incident vertices as a flat edge would. We consider the possible orientations of e_{j-1} and e_{j+1} .

(i) Both e_{j-1} and e_{j+1} are flat.

So, $|v_{j+1}|_1 = |v_{j+1}|_0 - 1$ and $|v_j|_1 = |v_j|_0 + 1$. Thus, $|v_{j+1}|_1 = |v_j|_0 < |v_j|_0 + 1 = |v_j|_1$.

- (ii) e_{j-1} is flat and e_{j+1} is directed left. So, $|v_{j+1}|_1 = |v_{j+1}|_0 - 2$ and $|v_j|_1 = |v_j|_0 + 1$. Thus, $|v_{j+1}|_1 = |v_j|_0 - 1 < |v_j|_0 + 1 = |v_j|_1$.
- (iii) e_{j-1} is flat and e_{j+1} is directed right. This suborientation cannot exist within the period by Lemma 6.
- (iv) e_{j-1} is directed right and e_{j+1} is flat. This suborientation cannot exist within the period by Lemma 6.
- (v) e_{j-1} is directed right and e_{j+1} is directed left.

So,
$$|v_{j+1}|_1 = |v_{j+1}|_0 - 2$$
 and $|v_j|_1 = |v_j|_0$. Thus, $|v_{j+1}|_1 = |v_j|_0 - 1 < |v_j|_0 = |v_j|_1$.

- (vi) Both e_{j-1} and e_{j+1} are directed right. This suborientation cannot exist within the period by Lemma 6.
- (vii) e_{j-1} is directed left and e_{j+1} is flat. So, $|v_{j+1}|_1 = |v_{j+1}|_0 - 1$ and $|v_j|_1 = |v_j|_0 + 2$. Thus, $|v_{j+1}|_1 = |v_j|_0 < |v_j|_0 + 2 = |v_j|_1$.
- (viii) Both e_{j-1} and e_{j+1} are directed left. So, $|v_{j+1}|_1 = |v_{j+1}|_0 - 2$ and $|v_j|_1 = |v_j|_0 + 2$. Thus, $|v_{j+1}|_1 = |v_j|_0 - 1 < |v_j|_0 + 2 = |v_j|_1$.
- (ix) e_{j-1} is directed left and e_{j+1} is directed right. So, $|v_{j+1}|_1 = |v_{j+1}|_0$ and $|v_j|_1 = |v_j|_0 + 2$. Thus, $|v_{j+1}|_1 = |v_j|_0 + 1 < |v_j|_0 + 2 = |v_j|_1$.

So, for all possible graph orientations R, either R is a p_2 -orientation or R contains a suborientation listed in Theorem 2.

Let R_n be the number of p_2 -orientations on P_n . Quick calculations show that $R_1 = 0$, $R_2 = 2$, $R_3 = 2$, and $R_4 = 4$.

Theorem 3 The number of p_2 -orientations, R_n , on a path P_n , $n \ge 5$, is given by the recurrence relation $R_n = R_{n-1} + 2R_{n-2} - R_{n-4}$ with initial values $R_1 = 0$, $R_2 = 2$, $R_3 = 2$, and $R_4 = 4$.

Proof Let R be a p_2 -orientation on $P_n = v_1 e_1 v_2 e_2 v_3 \dots v_{n-1} e_{n-1} v_n$. There are three mutually exclusive and exhaustive cases: e_{n-2} is flat, e_{n-2} agrees with e_{n-3} , or e_{n-2} is neither flat nor agreeing with e_{n-3} . We will add together the total number of p_2 -configurations of each form to reach $R_n = R_{n-1} + 2R_{n-2} - R_{n-4}$.

Case 1: e_{n-2} is flat. Let R' be the induced suborientation of R on

 $P_{n-2} = v_1 e_1 v_2 \dots v_{n-3} e_{n-3} v_{n-2}$. We now check that R' is a p_2 -orientation by using our criteria from Lemmas 3 - 6.

Lemma 3 states the non-existence of adjacent flat edges. Since R is a p_2 -orientation, it does not contain adjacent flat edges. Therefore R', being an induced suborientation of R, also does not contain adjacent flat edges.

Lemma 4 states the non-existence of flat edges incident with a leaf. Since R is a p_2 -orientation, it does not contain a flat edge incident with a leaf. The vertex v_{n-2} is a leaf in R' but not in R. However, we know, by Lemma 5, that e_{n-3} and e_{n-1} disagree. This implies that e_{n-3} , the only edge incident with v_{n-2} in R', is not flat. Therefore, R' does not contain a flat edge incident with a leaf.

Lemma 5 states that flat edges must be adjacent to disagreeing edges. Since R is a p_2 -orientation, each flat edge in R is adjacent to disagreeing edges. Since e_{n-3} is not flat, every flat edge in R' is adjacent to the same set of edges in both R and R'. Therefore, every flat edge in R' is adjacent to disagreeing edges.

Lemma 6 states that any edge, e, adjacent to an edge f, with which it agrees must also be adjacent to an edge, d, with which it disagrees. We know by Lemma 6, that since R is a p_2 -orientation and e_{n-2} is flat, e_{n-3} does not agree with e_{n-4} . So, every pair of adjacent agreeing edges in R' is adjacent to the same set of edges in both R and R'. Therefore, in R', every edge, e, adjacent to an edge, f, with which it agrees is also adjacent to an edge, d, with which it disagrees.

By Theorem 2, we can conclude that R' is a p_2 -orientation. Also, R is uniquely determined by R'. That is, given R' and that e_{n-2} is flat, we know that R must have an e_{n-1} that disagrees with e_{n-3} . Therefore, there exist exactly R_{n-2} different p_2 -orientations of this form on P_n .

Case 2: e_{n-2} agrees with e_{n-3} .

Let R' be the induced suborientation of R on $P_{n-2} = v_1 e_1 v_2 \dots v_{n-3} e_{n-3} v_{n-2}$. We must check that R' is a p_2 -orientation by using our criteria from Lemmas 3 - 6.

Lemma 3 states the non-existence of adjacent flat edges. Since R is a p_2 -orientation, it does not contain adjacent flat edges. Therefore R', being an induced suborientation of R, also does not contain adjacent flat edges.

Lemma 4 states the non-existence of flat edges incident with a leaf. Since R is a p_2 -orientation, it does not contain a flat edge incident with a leaf. The vertex v_{n-2} is a leaf in R' but not in R. However, we know that e_{n-3} and e_{n-2} agree. This implies that e_{n-3} , the only edge incident with v_{n-2} in R', is not flat. Therefore, R' does not contain a flat edge incident with a leaf.

Lemma 5 states that flat edges must be adjacent to disagreeing edges. Since R is a p_2 -orientation, each flat edge in R is adjacent to disagreeing edges. Since e_{n-3} is not flat, every flat edge in R' is adjacent to the same set of edges in both R and R'. Therefore, every flat edge in R' is adjacent to disagreeing edges.

Lemma 6 states that any edge, e, adjacent to an edge, f, with which it agrees must also be adjacent to an edge, d, with which it disagrees. We know by Lemma 6, that since R is a p_2 -orientation and e_{n-2} agrees with e_{n-3} , then e_{n-3} disagrees with e_{n-4} . So, every pair of adjacent agreeing edges in R' is adjacent to the same set of edges in both R and R'. Therefore, in R', every edge, e, adjacent to an edge, f, with which it agrees is also adjacent to an edge, d, with which it disagrees.

By Theorem 2, we can conclude that R' is a p_2 -orientation. Also, R is uniquely determined by R'. That is, given R' and that e_{n-2} agrees with e_{n-3} , we know that R must have an e_{n-1} that disagrees with e_{n-2} . So, the number of p_2 -orientations of P_n in which e_{n-2} agrees with e_{n-3} is equal to the number of p_2 -orientations of R' in which e_{n-3} disagrees with e_{n-4} . We can determine this

value recursively. From Case 1, we can see that the number of p_2 -orientations of R' in which e_{n-3} disagrees with e_{n-4} is equal to $R_{n-2} - R_{n-4}$.

Case 3: e_{n-2} is neither flat nor agreeing with e_{n-3} .

Let R' be the induced suborientation of R on $P_{n-1} = v_1 e_1 v_2 \dots v_{n-3} e_{n-3} v_{n-2} e_{n-2} v_{n-1}$. We must check that R' is a p_2 -orientation by using our criteria from Lemmas 3 - 6.

Lemma 3 states the non-existence of adjacent flat edges. Since R is a p_2 -orientation, it does not contain adjacent flat edges. Therefore R', being an induced suborientation of R, also does not contain adjacent flat edges.

Lemma 4 states the non-existence of flat edges incident with a leaf. Since R is a p_2 -orientation, it does not contain a flat edge incident with a leaf. The vertex v_{n-1} is a leaf in R' but not in R. However, we know that e_{n-2} , the only edge incident with v_{n-1} in R', is not flat. Therefore, R' does not contain a flat edge incident with a leaf.

Lemma 5 states that flat edges must be adjacent to disagreeing edges. Since R is a p_2 -orientation, each flat edge in R is adjacent to disagreeing edges. Since e_{n-2} is not flat, every flat edge in R' is adjacent to the same set of edges in both R and R'. Therefore, every flat edge in R' is adjacent to disagreeing edges.

Lemma 6 states that any edge, e, adjacent to an edge, f, with which it agrees must also be adjacent to an edge, d, with which it disagrees. Since e_{n-2} does not agree with e_{n-3} , every pair of adjacent agreeing edges in R' is adjacent to the same set of edges in both R and R'. Therefore, in R', every edge, e, adjacent to an edge, f, with which it agrees is also adjacent to an edge, d, with which it disagrees.

By Theorem 2, we can conclude that R' is a p_2 -orientation. Also, R is uniquely determined by R'. That is, given R', we know that R must have an e_{n-1} which disagrees with e_{n-2} . Therefore, there exist exactly R_{n-1} different p_2 -orientations of this form on P_n .

Adding together the values from our three cases, we get that $R_n = R_{n-1} + 2R_{n-2} - R_{n-4}$.

In the OEIS [4], this sequence: 0, 2, 2, 4, 8, 14, 28, 52, 100, 190, 362... generated by the recurrence in Theorem 3, is A052535. The generating sequence is $\frac{(1-x^2)}{(1-x-2x^2+x^4)}$. The asymptotic solution for the k^{th} term of this recurrence is approximately $(0.3017)(1.9052)^k$. This is shown in "On Variants of Diffusion", T. Mullen, PhD Thesis.

$3 p_2$ -Configurations on Paths

We have already calculated the number of p_2 -orientations that exist on a path. Now, we will calculate, given a p_2 -orientation, the number of p_2 -configurations that exist. For each vertex, we determine the number of possible stack sizes that that vertex can have. We call this number the *multiplier* of that vertex. We now look at an example:



Fig. 6 P_{10} under orientation R

Example 31 We assign P_{10} to have configuration R pictured in Figure 6. Moving from right to left, we determine the number of possible stack sizes each vertex can take on:

- 1. We fix v_1 at 0 chips by convention. So, v_1 has a multiplier of 1.
- 2. We have that v_2 must have a negative stack size (so as to receive from v_1 in the initial firing) that is large enough to already be in the period. We know that the stack size of v_1 will decrease by 1 in the first step and the stack size of v_2 will increase by 2 in the first step. This means that $|v_2|_0 < |v_1|_0 = 0$ and $|v_2|_0 + 2 > |v_1|_0 - 1 = -1$. So, $0 > |v_2|_0 > -3$. Thus, the two possible values that $|v_2|_0$ can take on are -1 and -2. So, v_2 has a multiplier of 2.
- 3. Given a value for $|v_2|_0$, we calculate the number of possible initial stack sizes that v_3 can take on. We know $|v_3|_0 > |v_2|_0$ and $|v_3|_0 - 2 < |v_2|_0 + 2$. So, we have that $|v_2|_0 < |v_3|_0 < |v_2|_0 + 4$. Thus, the three possible initial stack sizes for v_3 are $|v_2|_0 + 1$, $|v_2|_0 + 2$, and $|v_2|_0 + 3$. So, v_3 has a multiplier of 3.
- 4. Given a value for $|v_3|_0$, we calculate the number of possible initial stack sizes that v_4 can take on. We know $|v_4|_0 < |v_3|_0$ and $|v_4|_0 + 2 > |v_3|_0 - 2$. So, we have that $|v_3|_0 > |v_4|_0 > |v_3|_0 - 4$. Thus, the three possible initial stack sizes for v_4 are $|v_3|_0 - 1$, $|v_3|_0 - 2$, and $|v_3|_0 - 3$. So, v_4 has a multiplier of 3.
- 5. Given a value for $|v_4|_0$, we calculate the number of possible initial stack sizes that v_5 can take on. We know $|v_5|_0 > |v_4|_0$ and $|v_5|_0 + 1 - 1 < |v_4|_0 + 2$. So, we have that $|v_4|_0 < |v_5|_0 < |v_4|_0 + 2$. Thus, the only possible initial stack size for v_5 is $|v_4|_0 + 1$. So, v_5 has a multiplier of 1.
- 6. Given a value for $|v_5|_0$, we calculate the number of possible initial stack sizes that v_6 can take on. We know $|v_6|_0 > |v_5|_0$ and $|v_6|_0 - 2 < |v_5|_0 + 1 - 1$. So, we have that $|v_5|_0 + 2 > |v_6|_0 > |v_5|_0$. Thus, the only possible initial stack size for v_6 is $|v_5|_0 + 1$. So, v_6 has a multiplier of 1.
- 7. Given a value for $|v_6|_0$, we calculate the number of possible initial stack sizes that v_7 can take on. We know that $|v_7|_0 < |v_6|_0$ and $|v_7|_0 + 1 > |v_6|_0 - 2$. So, we have that $|v_6|_0 - 3 < |v_7|_0 < |v_6|_0$. Thus, the two possible initial stack sizes for v_7 are $|v_6|_0 - 2$ and $|v_6|_0 - 1$. So, v_7 has a multiplier of 2.
- 8. Given a value for $|v_7|_0$, we calculate the number of possible initial stack sizes that v_8 can take on. We know $|v_8|_0 = |v_7|_0$. Thus, the only possible initial stack size for v_8 is $|v_7|_0$. So, v_8 has a multiplier of 1.
- 9. Given a value for $|v_8|_0$, we calculate the number of possible initial stack sizes that v_9 can take on. We know that $|v_9|_0 > |v_8|_0$ and $|v_9| - 2 < |v_8| + 1$. So, we have that $|v_8|_0 < |v_9|_0 < |v_8| + 3$. Thus, the two possible initial stack sizes for v_9 are $|v_8|_0 + 1$ and $|v_8|_0 + 2$. So, v_9 has a multiplier of 2.
- 10. Given a value for $|v_9|_0$, we calculate the number of possible initial stack sizes that v_{10} can take on. We know that $|v_{10}|_0 < |v_9|_0$ and $|v_{10}|_0 + 1 > v_9 - 2$. So, we have that $|v_9|_0 - 3 < |v_{10}|_0 < |v_9|_0$. Thus, the two possible initial stack sizes for v_{10} are $|v_9|_0 - 2$ and $|v_9|_0 - 1$. Thus, the multiplier for v_{10} is 2. So, v_10 has a multiplier of 2.

Multiplying all of these possibilities together we get $1 \times 2 \times 3 \times 3 \times 1 \times 1 \times 2 \times 1 \times 2 \times 2 = 144$ period configurations on this period orientation.

We now formally define the multiplier of a vertex.

Definition 1 Given a graph orientation R on a path P_n , the **multiplier** assigned to a vertex v represents the number of possible initial stack sizes v could have in a p_2 -orientation, supposing that an initial stack size has already been chosen for every vertex to the right of v.

Given an assignment of stack sizes to the vertices $v_1, v_2, \dots v_{i-1}$, the multiplier of v_i in R is the number of different stack sizes v_i can have in a p_2 -configuration which induces R.

Since we are dealing with paths and we are assuming that we are already inside the period of a configuration with period 2, these calculations can be conducted locally, as is evidenced by Example 31. That is, the multiplier of a vertex v_k depends only on the orientation of the edges incident to v_k and those incident to v_{k-1} .

Our goal now is to determine the multiplier for any vertex v_k in any p_2 -configuration on a path.

In order to determine the multiplier of a given vertex v_k in a path P_n , we would like to be able to assume that v_k and v_{k-1} are each incident with two edges. We will begin with a smaller theorem that deals with calculating the multiplier for vertices in which this assumption fails.

Theorem 4 (Little Multiplier Theorem) Let $P_n = v_1 e_1 v_2 e_2 \dots e_{n-1} v_n$ be a path on $n \ge 3$ vertices and let R be a p_2 -orientation on P_n . Then

- $-v_1$ has a multiplier of 1.
- If e_2 is flat, then the multiplier of v_2 is 1.
- If e_2 is directed, then the multiplier of v_2 is 2.
- If e_{n-2} is flat, then the multiplier of v_n is 1.
- If e_{n-2} is directed, then the multiplier of v_n is 2.

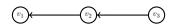
Proof The multiplier for v_1 is always 1 because, by convention, we set v_1 at 0 chips. By Lemma 4, we know that e_1 cannot be flat. By Lemma 6, we know that e_2 does not agree with the e_1 . So, when calculating the multiplier for v_2 , there are two cases. Either e_2 disagrees with e_1 , or e_2 is flat. That is, we can exclude the following suborientations

$$v_1$$
 v_2 v_3









We will suppose first that e_2 is flat. There are two possibilities.

(i)
$$e_1$$
 is directed right.
 v_1 v_2 v_3

The net effect of the initial firing on v_2 is a decrease of one chip, and the net effect of the initial firing on v_1 is an increase of one chip. This means that $|v_2|_0 > |v_1|_0$ and $|v_2|_0 - 1 < |v_1|_0 + 1$.

So, $|v_1|_0 < |v_2|_0 < |v_1|_0 + 2$. Therefore, $|v_1|_0 + 1$ is the only possible initial stack sizes for v_2 . Since there is only one possible initial stack size, we say that v_2 has a multiplier of 1.

(ii) e_1 is directed left. v_1 v_2 v_3

> The net effect of the initial firing on v_2 is an increase of one chip, and the net effect of the initial firing on v_1 is a decrease of one chip. This means that $|v_2|_0 < |v_1|_0$ and $|v_2|_0 + 1 > |v_1|_0 - 1$. So, $|v_1|_0 > |v_2|_0 > |v_1|_0 - 2$. Therefore, $|v_1|_0 - 1$ is the only possible initial stack sizes for v_2 . Since there is only one possible initial stack size, we say that v_2 has a multiplier of 1.

We now suppose instead that e_2 disagrees with e_1 . There are two possibilities.

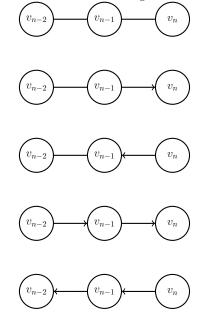
(i)
$$e_2$$
 is directed right.
 $v_1 \rightarrow v_2 \leftarrow v_3$

The net effect of the initial firing on v_2 is an increase of two chips, and the net effect of the initial firing on v_1 is a decrease of one chip. This means that $|v_2|_0 < |v_1|_0$ and $|v_2|_0 + 2 > |v_1|_0 - 1$. So, $|v_1|_0 > |v_2|_0 > |v_1|_0 - 3$. Therefore, $|v_1|_0 - 1$ and $|v_1|_0 - 2$ are the only possible initial stack sizes for v_2 . Since there are only two possible initial stack sizes, we say that v_2 has a multiplier of 2.

(ii) e_2 is directed left. $v_1 \longrightarrow v_2 \longrightarrow v_3$

> The net effect of the initial firing on v_2 is a decrease of two chips, and the net effect of the initial firing on v_1 is an increase of one chip. This means that $|v_2|_0 > |v_1|_0$ and $|v_2|_0 - 2 < |v_1|_0 + 1$. So, $|v_1|_0 < |v_2|_0 < |v_1|_0 + 3$. Therefore, $|v_1|_0 + 1$ and $|v_1|_0 + 2$ are the only possible initial stack sizes for v_2 . Since there are only two possible initial stack sizes, we say that v_2 has a multiplier of 2.

We now turn our attention to v_n . By Lemma 4, we know that the edge e_{n-1} is not flat. By Lemma 6, we know that the edge e_{n-2} does not agree with the edge e_{n-1} . So, when calculating the multiplier for v_n , there are two cases. Either e_{n-2} disagrees with e_{n-1} or e_{n-2} is flat. That is, we can exclude the following suborientations



We will suppose first that e_{n-2} is flat. There are two possibilities.

(i) e_{n-1} is directed right. v_{n-2} v_{n-1} v_n

The net effect of the initial firing on v_n is a decrease of one chip, and the net effect of the initial firing on v_{n-1} is an increase of one chip. This means that $|v_n|_0 > |v_{n-1}|_0$ and $|v_n|_0 - 1 < |v_{n-1}|_0 + 1$. So, $|v_{n-1}|_0 < |v_n|_0 < |v_{n-1}|_0 + 2$. Therefore, $|v_{n-1}|_0 + 1$ is the only possible initial stack size for v_n . Since there is only one possible initial stack size, we say that v_n has a multiplier of 1.

(ii) e_{n-1} is directed left.

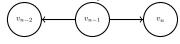
The net effect of the initial firing on v_n is an increase of one chip, and the net effect of the initial firing on v_{n-1} is a decrease of one chip. This means that $|v_n|_0 < |v_{n-1}|_0$ and $|v_n|_0 + 1 > |v_{n-1}|_0 - 1$. So, $|v_{n-1}|_0 > |v_n|_0 > |v_{n-1}|_0 - 2$. Therefore, $|v_{n-1}|_0 - 1$ is the only possible initial stack size for v_n . Since there is only one possible initial stack size, we say that v_n has a multiplier of 1.

We now suppose instead that e_n disagrees with e_{n-1} . There are two possibilities.

1. e_n is directed right. v_{n-2} v_{n-1} v_{n-1}

The net effect of the initial firing on v_n is a decrease of one chip, and the net effect of the initial firing on v_{n-1} is an increase of two chips. This means that $|v_n|_0 > |v_{n-1}|_0$ and $|v_n|_0 - 1 < |v_{n-1}|_0 + 2$. So, $|v_{n-1}|_0 < |v_n|_0 < |v_{n-1}|_0 + 3$. Therefore, $|v_{n-1}|_0 + 1$ and $|v_1|_0 + 2$ are the only possible initial stack sizes for v_n . Since there are only two possible initial stack sizes, we say that v_n has a multiplier of 2.

2. e_n is directed left.



The net effect of the initial firing on v_n is an increase of one chip, and the net effect of the initial firing on v_{n-1} is a decrease of two chips. This means that $|v_n|_0 < |v_{n-1}|_0$ and $|v_n|_0 + 1 > |v_{n-1}|_0 - 2$. So, $|v_{n-1}|_0 > |v_n|_0 > |v_1|_0 - 3$. Therefore, $|v_{n-1}|_0 - 1$ and $|v_{n-1}|_0 - 2$ are the only possible initial stack sizes for v_n . Since there are only two possible initial stack sizes, we say that v_n has a multiplier of 2.

We will now look at the multipliers of the other vertices.

Theorem 5 (The Multiplier Theorem) Let R be a p_2 -orientation on a path

 $P_n = v_1 e_1 v_2 e_2 \dots e_{n-1} v_n$ with $n \ge 4$. If a vertex, v_k , and its neighbour, v_{k-1} , each have exactly two neighbours, then the multiplier of v_k is 1, 2, or 3 depending on the suborientation within which it exists, as outlined in Table 1.

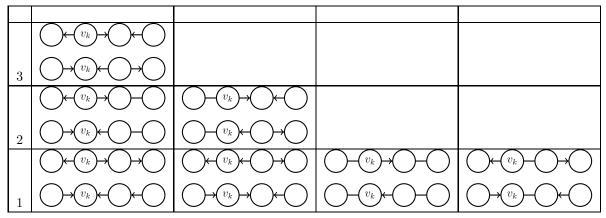
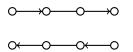


Table 1 Multipliers (listed in the leftmost column) of v_k based on neighbourhood

Proof We will begin by proving that no suborientation omitted from Table 1 can be contained within a p_2 -orientation.

Every edge has 3 possible orientations. Therefore, there exist $3^3 = 27$ graph orientations of P_4 . However, we know several of these orientations cannot exist as suborientations within a p_2 -orientation by Theorem 2. We now list these orientations which cannot exist within a p_2 -orientation. The following 5 suborientations cannot exist within a p_2 -orientation by Lemma 3.

The following 2 suborientations cannot exist within a p_2 -orientation by Lemma 5.



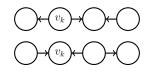
The following 6 suborientations cannot exist within a p_2 -orientation by Lemma 6.

 $0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0$ $0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0$ $0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0$ $0 \longleftarrow 0 \longleftarrow 0 \longleftarrow 0 \longleftarrow 0$ $0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0$ $0 \longrightarrow 0 \longrightarrow 0$

Fig. 7 List of suborientations which cannot exist within a p_2 -orientation

What remains are the 27 - 13 = 14 suborientations listed in Table 1. We will break these 14 suborientations into 7 pairs of suborientations and show their multipliers using a case analysis. Each orientation will be paired with the orientation created by reversing the direction of every directed edge contained within. We will see that these pairs always have the same multiplier and can be proven using similar arguments. Note that by Corollary 1, every period that contains one of these orientations must also contain the one with which it is paired.

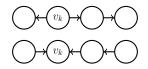
Case 1: Alternating arrow suborientation.



First assume that v_k is losing two chips in the initial firing. The net effect of the initial firing on v_k is a decrease of two chips, and the net effect of the initial firing on v_{k-1} is an increase of two chips. This means that $|v_k|_0 > |v_{k-1}|_0$ and $|v_k|_0 - 2 < |v_{k-1}|_0 + 2$. So, $|v_{k-1}|_0 < |v_k|_0 < |v_{k-1}|_0 + 4$. Therefore, $|v_{k-1}|_0 + 1$, $|v_{k-1}|_0 + 2$, and $|v_{k-1}|_0 + 3$ are the only possible initial stack sizes for v_k . Since there are only three possible initial stack sizes, v_k has a multiplier of 3.

Now assume that instead, v_k is gaining two chips in the initial firing. The net effect of the initial firing on v_k is an increase of two chips, and the net effect of the initial firing on v_{k-1} is a decrease of two chips. This means that $|v_k|_0 < |v_{k-1}|_0$ and $|v_k|_0 + 2 > |v_{k-1}|_0 - 2$. So, $|v_{k-1}|_0 > |v_k|_0 > |v_{k-1}|_0 - 4$. Therefore, $|v_{k-1}|_0 - 1$, $|v_{k-1}|_0 - 2$, and $|v_{k-1}|_0 - 3$ are the only possible initial stack sizes for v_k . Since there are only three possible initial stack sizes, v_k has a multiplier of 3.

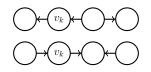
Case 2:



First assume that v_k is losing two chips in the initial firing. The net effect of the initial firing on v_k is a decrease of two chips, and the net effect of the initial firing on v_{k-1} is no change in the number of chips. This means that $|v_k|_0 > |v_{k-1}|_0$ and $|v_k|_0 - 2 < |v_{k-1}|_0$. So, $|v_{k-1}|_0 < |v_k|_0 < |v_{k-1}|_0 + 2$. Therefore, $|v_{k-1}|_0 + 1$ is the only possible initial stack size for v_k . Since there is only one possible initial stack size, v_k has a multiplier of 1.

Now assume that instead, v_k is gaining two chips in the initial firing. The net effect of the initial firing on v_k is an increase of two chips, and the net effect of the initial firing on v_{k-1} is no change in the number of chips. This means that $|v_k|_0 < |v_{k-1}|_0$ and $|v_k|_0 + 2 > |v_{k-1}|_0$. So, $|v_{k-1}|_0 > |v_k|_0 > |v_{k-1}|_0 - 2$. Therefore, $|v_{k-1}|_0 - 1$ is the only possible initial stack size for v_k . Since there is only one possible initial stack size, v_k has a multiplier of 1.

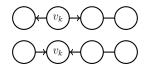
Case 3:



First assume that v_{k-1} is losing two chips in the initial firing. The net effect of the initial firing on v_k is no change in the number of chips, and the net effect of the initial firing on v_{k-1} is a decrease of two chips. This means that $|v_k|_0 < |v_{k-1}|_0$ and $|v_k|_0 > |v_{k-1}|_0 - 2$. So, $|v_{k-1}|_0 > |v_k|_0 > |v_{k-1}|_0 - 2$. Therefore, $|v_{k-1}|_0 - 1$ is the only possible initial stack size for v_k . Since there is only one possible initial stack size, v_k has a multiplier of 1.

Now assume that instead, v_{k-1} is gaining two chips in the initial firing. The net effect of the initial firing on v_k is no change in the number of chips, and the net effect of the initial firing on v_{k-1} is an increase of two chips. This means that $|v_k|_0 > |v_{k-1}|_0$ and $|v_k|_0 < |v_{k-1}|_0 + 2$. So, $|v_{k-1}|_0 < |v_k|_0 < |v_{k-1}|_0 + 2$. Therefore, $|v_{k-1}|_0 + 1$ is the only possible initial stack size for v_k . Since there is only one possible initial stack size, v_k has a multiplier of 1.

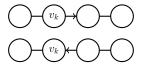
Case 4:



First assume that v_k is losing two chips in the initial firing. The net effect of the initial firing on v_k is a decrease of two chips, and the net effect of the initial firing on v_{k-1} is an increase of one chip. This means that $|v_k|_0 > |v_{k-1}|_0$ and $|v_k|_0 - 2 < |v_{k-1}|_0 + 1$. So, $|v_{k-1}|_0 < |v_k|_0 < |v_{k-1}|_0 + 3$. Therefore, $|v_{k-1}|_0 + 1$ and $|v_{k-1} + 2$ are the only possible initial stack sizes for v_k . Since there are only two possible initial stack sizes, v_k has a multiplier of 2.

Now assume that instead, v_k is gaining two chips in the initial firing. The net effect of the initial firing on v_k is an increase of two chips, and the net effect of the initial firing on v_{k-1} is a decrease of one chip. This means that $|v_k|_0 < |v_{k-1}|_0$ and $|v_k|_0 + 2 > |v_{k-1}|_0 - 1$. So, $|v_{k-1}|_0 > |v_k|_0 > |v_{k-1}|_0 - 3$. Therefore, $|v_{k-1}|_0 - 1$ and $|v_{k-1}|_0 - 2$ are the only possible initial stack sizes for v_k . Since there are only two possible initial stack sizes, v_k has a multiplier of 2.

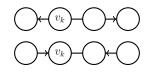
Case 5:



First assume that v_k is losing one chip in the initial firing. The net effect of the initial firing on v_k is a decrease of one chip, and the net effect of the initial firing on v_{k-1} is an increase of one chip. This means that $|v_k|_0 > |v_{k-1}|_0$ and $|v_k|_0 - 1 < |v_{k-1}|_0 + 1$. So, $|v_{k-1}|_0 < |v_k|_0 < |v_{k-1}|_0 + 2$. Therefore, $|v_{k-1}|_0 + 1$ is the only possible initial stack size for v_k . Since there is only one possible initial stack size, v_k has a multiplier of 1.

Now assume that instead, v_k is gaining one chip in the initial firing. The net effect of the initial firing on v_k is an increase of one chip, and the net effect of the initial firing on v_{k-1} is a decrease of one chip. This means that $|v_k|_0 < |v_{k-1}|_0$ and $|v_k|_0 + 1 > |v_{k-1}|_0 - 1$. So, $|v_{k-1}|_0 > |v_k|_0 > |v_{k-1}|_0 - 2$. Therefore, $|v_{k-1}|_0 - 1$ is the only possible initial stack size for v_k . Since there is only one possible initial stack size, v_k has a multiplier of 1.

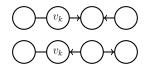
Case 6:



First assume that v_k is losing one chip in the initial firing. The net effect of the initial firing on v_k is a decrease of one chip, and the net effect of the initial firing on v_{k-1} is a decrease of one chip. This means that $|v_k|_0 = |v_{k-1}|_0$ and $|v_k|_0 - 1 = |v_{k-1}|_0 - 1$. Therefore, $|v_{k-1}|_0$ is the only possible initial stack size for v_k . Since there is only one possible initial stack size, v_k has a multiplier of 1.

Now assume that instead, v_k is gaining one chip in the initial firing. The net effect of the initial firing on v_k is an increase of one chip, and the net effect of the initial firing on v_{k-1} is an increase of one chip. This means that $|v_k|_0 = |v_{k-1}|_0$ and $|v_k|_0 + 1 = |v_{k-1}|_0 + 1$. Therefore, $|v_{k-1}|_0$ is the only possible initial stack size for v_k . Since there is only one possible initial stack size, v_k has a multiplier of 1.

Case 7:



First assume that v_k is losing one chip in the initial firing. The net effect of the initial firing on v_k is a decrease of one chip, and the net effect of the initial firing on v_{k-1} is an increase of two chips. This means that $|v_k|_0 > |v_{k-1}|_0$ and $|v_k|_0 - 1 < |v_{k-1}|_0 + 2$. So, $|v_{k-1}|_0 < |v_k|_0 < |v_{k-1}|_0 + 3$. Therefore, $|v_{k-1}|_0 + 1$ and $|v_{k-1}|_0 + 2$ are the only possible initial stack sizes for v_k . Since there are only one two possible initial stack sizes, v_k has a multiplier of 2.

Now assume that instead, v_k is gaining one chip in the initial firing. The net effect of the initial firing on v_k is an increase of one chip, and the net effect of the initial firing on v_{k-1} is a decrease of two chips. This means that $|v_k|_0 < |v_{k-1}|_0$ and $|v_k|_0 + 1 > |v_{k-1}|_0 - 2$. So, $|v_{k-1}|_0 > |v_k|_0 > |v_{k-1}|_0 - 3$. Therefore, $|v_{k-1}|_0 - 1$ and $|v_{k-1}|_0 - 2$ are the only possible initial stack sizes for v_k . Since there are only one two possible initial stack sizes, v_k has a multiplier of 2.

We now state a number of corollaries that come from the results regarding the multipliers of specific vertices found in Theorem 5. In particular, these corollaries will allow us to break the problem of counting all p_2 -configurations on P_n into three cases: p_2 -configurations that exist on alternating arrow orientations on n vertices, p_2 -configurations in which, moving from right to left, a flat edge appears before the first pair of adjacent agreeing edges, and p_2 -configurations in which, moving from right to left, the first pair of adjacent agreeing edges appears before the first flat. We will then add up these three totals to determine the number of p_2 -configurations that exist on P_n .

Corollary 2 The number of period configurations that exist on alternating arrow orientations on P_n $(n \ge 3)$ is $8 \times 3^{n-3}$.

Proof In an alternating arrow orientation, every edge e_i disagrees with the previous edge e_{i-1} . So, an alternating arrow orientation is unique based on the orientation of $e_1 = v_1v_2$. Therefore, there exist two alternating arrow orientations on a given path P_n , n > 1.

From Theorems 4 and 5, we get that the multiplier for v_1 is 1, the multiplier for both v_2 and v_n is 2, and every other multiplier is 3.

Thus, the number of period configurations on a particular alternating arrow orientation on P_n is $1 \times 2 \times 2 \times 3^{n-3}$. Multiplying by two different alternating arrow orientations depending on the orientation of the first edge, we get that the number of period configurations that exist on alternating arrow orientations on P_n , $n \ge 3$, is $1 \times 2 \times 2 \times 3^{n-3} \times 2 = 8 \times 3^{n-3}$.

Define a sequence A_n to represent the number of period configurations on an alternating path on *n* vertices. $A_n = 0, 2, 8, 24, 72, 216, 648, ..., A_k, 3A_k, 3 \times 3A_k,$

Corollary 3 For all $n \geq 3$, $3A_n = A_{n+1}$.

Claim Let R be a p_2 -orientation on P_n , $n \ge 2$. Let R_1, R_2, \ldots, R_k , be the suborientations of R on the k disjoint paths created by removing k - 1 flat edges from P_n . Then, for $1 \le i \le k$, R_i is a p_2 -orientation on its respective path.

Proof Let R be a period orientation on P_n , $n \ge 2$, with at least one flat edge. Let R_1 and R_2 be the suborientations of R on the disjoint paths created by removing a single flat edge from P_n . We will run through our checklist from Theorem 2 to determine whether or not these suborientations are themselves period orientations of their respective subgraphs.

- (a) Since there is no pair of adjacent flat edges in R, there cannot be a flat pair of adjacent edges in either R_1 or R_2 .
- (b) In both R_1 and R_2 , there is an edge incident with a leaf that is not incident with a leaf in R. Call these edges e_a and e_b . However, in R, we know that every edge incident with a flat edge must be directed (not flat). Thus, neither e_a nor e_b is flat.
- (c) Since every flat edge in R is incident with both a right and left edge, this is also true of R_1 and R_2 .
- (d) The removal of flat edges can have no effect on this rule for adjacent agreeing edges.

So, we can conclude that R_1 and R_2 , and thus, any number of disjoint orientations created by removing flats from a period orientation, are themselves, period orientations.

Corollary 4 Let R be a period orientation of P_n . Let R_1, R_2, \ldots, R_k , be the suborientations of R on the k disjoint paths created by removing k - 1 flat edges from P_n . The number of period configurations that exist on R is equal to the product of the number of period configurations that exist on the suborientations R_1, R_2, \ldots, R_k .

Proof Let R be a period orientation on P_n with at least one flat edge. Let R_1 and R_2 be the suborientations of R on the disjoint paths created by removing a single flat edge from P_n .

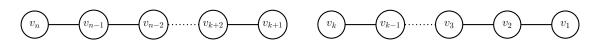


Fig. 8 P_n with edge $v_k v_{k+1}$ removed.

Suppose we removed just one flat edge, $v_k v_{k+1} = e_k$. The only vertices that could have an altered multiplier are those which are endpoints of e_k or e_{k+1} . The vertices in question are v_k , v_{k+1} , and

 v_{k+2} . However, since what is measured in calculating the multiplier is the net effect of the firing, being incident to a flat edge is equivalent to not being incident to an edge at all. In particular, note that v_{k+1} , appearing to the left of the flat edge e_k , has only one possible initial stack size, that being $|v_k|_0$. This is equivalent to v_{k+1} having only one possible initial stack size, by convention, when viewed as the right leaf in R_2 . So, it follows that any number of flat edge removals will still maintain this result.

Next, we present a corollary of the multiplier theorem (Theorem 5) which will be useful in determining the number of p_2 -configurations that exist which induce orientations with adjacent agreeing arrows. It will be shown that, given a p_2 -orientation of P_n which contains some suborientation $v_{k+2}e_{k+1}v_{k+1}e_kv_k$ such that e_{k+1} agrees with e_k , the orientation of P_{n-2} created by contracting the edges e_{k+1} and e_k and reversing the direction of all directed edges e_i , i > k + 1, is induced by the same number of p_2 -configurations. We see an example of two such graph orientations in Figure 9.

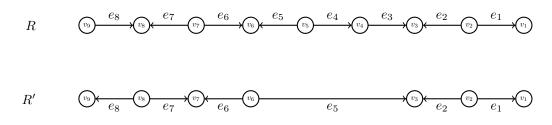


Fig. 9 Graph orientations R and R', created by contracting two adjacent agreeing edges and reversing the direction of all subsequent directed edges

Note how the edges e_3 and e_4 have been contracted, removing v_4 , and v_5 , and every directed edge occurring to the left of the contraction has reversed direction.

Corollary 5 Suppose there exist adjacent agreeing edges $e_k = v_k v_{k+1}$ and

 $e_{k+1} = v_{k+1}v_{k+2}$ in a period orientation, R, on a path, P_n , $n \ge 4$. Let R' be the graph orientation created by contracting e_k and e_{k+1} , reversing direction of every directed edge e_i , i > k + 1, and maintaining every other edge orientation from R. The number of p_2 -configurations on R is equal to the number of p_2 -configurations on R'.

Proof By Theorem 5, given a suborientation $v_{k+2}e_{k+1}v_{k+1}e_kv_k$ of a p_2 -orientation on a path P_n in which e_k and e_{k+1} agree, the multipliers of v_{k+2} and v_{k+1} are both equal to one. By contracting e_k and e_{k+1} , we are removing v_{k+2} and v_{k+1} from the orientation. By removing these vertices, assuming every other multiplier has been maintained, the number of p_2 -configurations that exist inducing the resulting orientation is the same as the number of p_2 -configurations that exist inducing the original orientation. Due to the reversing direction of every subsequent directed edge, v_k remains within the same P_4 suborientation from Theorem 5 and thus, maintains the same multiplier. Finally, every vertex appearing to the left of this contraction has had any incident directed edges reverse direction. However, by Theorem 5, such a flipping of directed edges does not change a vertex's multiplier. Therefore, the product of multipliers must only be divided by 1×1 (the product of the multipliers of the removed vertices) to accommodate the edge contraction, and thus, the number of p_2 -configurations does not change. So, we get that the number of configurations on R is equal to the number of configurations on R' and this process can be repeated until all pairs of adjacent agreeing edges have been removed.

Let T_n be the number of p_2 -configurations that exist on P_n .

Theorem 6 For all paths P_n , $n \ge 4$,

$$T_{n+4} = 3T_{n+3} + 2T_{n+2} + T_{n+1} - T_n$$

with $T_1 = 0$, $T_2 = 2$, $T_3 = 8$, and $T_4 = 26$.

We will prove this theorem with the help of a number of claims.

In order to count the number of p_2 -configurations on P_n , we will divide the set of all p_2 orientations into 3 cases. We have already solved for the number of p_2 -configurations that exist
on alternating arrow orientations on n vertices, A_n . Our other two cases will be the case in which,
moving from right to left, a flat edge appears before the first pair of adjacent agreeing edges, and
the case in which, moving from right to left, the first pair of adjacent agreeing edges appears before
the first flat. We will then add up these three totals to determine the number of p_2 -configurations
that exist on P_n .

Claim The number of p_2 -configurations on P_n , $n \ge 4$, in which, moving from right to left, a flat appears before the first pair of adjacent agreeing edges is

$$\sum_{k=2}^{n-2} \frac{1}{2} A_k \times T_{n-k}$$

Proof Suppose that, moving from right to left, the graph orientation, R, is alternating until the first flat appears. That is, the first flat appears before the first pair of adjacent agreeing edges appear. By Corollary 4, the number of p_2 -configurations that exist on a graph with some flat edge e_k is equal to the product of the numbers of p_2 -configurations that exist on the two suborientations created by removing e_k . Let $e_k = v_{k+1}v_k$ be the flat edge with the least index. We know that the suborientation $R_1 = v_k e_{k-1}v_{k-1}e_{k-2} \dots e_1v_1$ is an alternating arrow orientation by supposition. We know less about the suborientation $R_2 = v_n e_{n-1}v_{n-1}e_{n-2} \dots e_{k+1}v_{k+1}$. By Lemma 5, we know that e_{k+1} disagrees with e_{k-1} . The number of configurations of P_{n-k} which induce a graph orientation in which the orientation of the edge with the least index is given (without loss of generality, suppose it is right) is equal to $\frac{1}{2}F_{n-k}$ since half of the possibilities are excluded because the direction of the first edge is already known. So by Corollary 4, the number of p_2 -configurations which induce R is $\frac{1}{2}F_{n-k} \times A_k$. Summing this value over all possible edges that could represent the first flat edge, we get

$$\sum_{k=2}^{n-2} \frac{1}{2} A_k \times T_{n-k}$$

Definition 2 The k^{th} stage of a path is the total number of p_2 -configurations that exist on that path in which either a pair of adjacent agreeing edges or a flat appears within the first k + 1 edges.

For example, on P_8 , the 1st stage is the number of period configurations that exist in which the second edge is flat. The 2nd stage is the number of period configurations that exist in which either the second or third edge is flat, or the third edge agrees with the second edge. And the 3rd stage is

the number of period configurations that exist in which either the second, third, or fourth edge is flat, or either the third or fourth edge agrees with its previous edge. We denote the k^{th} stage of P_n by s_n^k .

Claim If n > 2, then $s_n^k = T_n$ for all $k \ge n-1$ and $s_n^k = T_n - A_n$ if k = n-2 or k = n-3.

Proof Case 1: k = n - 3

From the definition of stage, we are counting the number of period configurations that exist on P_n in which either a pair of adjacent agreeing edges or a flat appears within the first n-2 edges. The edge e_{n-1} cannot be flat or agree with e_{n-2} by Theorem 2. So, s_n^{n-3} counts every p_2 -configuration except for those that induce an alternating arrow orientation. Thus, $s_n^{n-3} = T_n - A_n$.

Case 2: k = n - 2

We are counting every configuration from Case 1, but also including the possibility of e_{n-1} being flat and the possibility of e_{n-1} agreeing with e_{n-2} . However, by Theorem 2, there are no p_2 -orientations in which either of these situations arise. So, $s_n^{n-2} = s_n^{n-3} = T_n - A_n$.

Case 3: $k \ge n - 1$

We are counting every configuration from Case 2, but also including the possibility that we fail to find a flat edge or pair of adjacent agreeing edges within the n-1 edges. So, every p_2 -configuration must be counted. So, $s_n^k = T_n$ for all $k \ge n-1$.

Claim The number of p_2 -configurations on P_n , $n \ge 5$, in which, moving from right to left, a pair of adjacent agreeing edges appear before a flat is

$$\sum_{k=3}^{n-3} T_{n-2} - s_{n-2}^{k-2}$$

Proof We say $n \ge 5$ since, by Theorem 2, no pair of adjacent agreeing edges can exist in a p_2 configuration on a path with fewer than 5 vertices. We are assuming that, moving from right to left,
the graph orientation is entirely alternating until the first pair of adjacent agreeing edges appears.
That is, the first pair of adjacent agreeing edges appears before the first flat appears. When adjacent
agreeing edges appear, the number of p_2 -configurations is equal to the number of p_2 -configurations
on the path with two fewer vertices in which the agreeing edges are removed and subsequent directed
edges are reversed as outlined in Corollary 5. So every graph orientation of this form on P_n can be
viewed as a similar graph orientation on P_{n-2} without changing the multipliers of any vertices. This
allows for a recurrence, helping us to evaluate F_n using F_{n-2} . However, we have supposed that up
to some edge, e_k , the graph orientation is alternating. So, we must subtract the proper stage of the
path on n-2 vertices. This will remove the possibility of agreeing edges and flat edges appearing
to the right of e_k . Taking this sum over all possible edges that could represent, moving from right
to left, the first edge that agrees with its immediate predecessor, we get

$$\sum_{k=3}^{n-3} T_{n-2} - s_{n-2}^{k-2}$$

We now calculate T_n based on T_{n-1} , T_{n-2} , T_{n-3} , and T_{n-4} . Given a p_2 -orientation on P_n , there are 4 mutually exclusive cases: e_{n-2} is flat, e_{n-3} is flat, e_{n-2} and e_{n-3} agree, e_{n-2} and e_{n-3} disagree. We know that these are the only possibilities by Theorem 2.

For each of these four cases, we will determine the number of p_2 -orientations that exist in that case. We then add up these four totals to calculate T_n .

Case 1: e_{n-2} is flat.

This calculation is equivalent to the first flat edge being e_2 . We know that this is $A_2 \times \frac{1}{2}T_{n-2} = T_{n-2}$. This is the k = 2 summand from Claim 3.

Case 2: e_{n-3} is flat.

This calculation is equivalent to the first flat edge being e_3 . $A_3 \times \frac{1}{2}T_{n-3} = 4T_{n-3}$. This is the k = 3 summand from Claim 3.

Case 3: e_{n-2} and e_{n-3} agree.

We use our rule from Corollary 5 for compacting agreeing arrows. What we get is every solution on P_{n-2} that begins with two disagreeing arrows. This is equivalent to just subtracting the possibility that the first edge is flat. When e_2 is flat, we get $A_2 \times T_{n-4} = T_{n-4}$. So, we get $T_{n-2} - T_{n-4}$.

Case 4: e_{n-2} and e_{n-3} disagree.

This can be viewed as adding a new leftmost vertex to P_{n-1} . This vertex adds a multiplier of 3 (being amongst an alternating arrow suborientation) unless e_{n-3} is flat. However, since we know e_{n-3} to not be flat, we can exclude it from our calculation. If e_{n-3} is flat in P_{n-1} , then there are $A_2 \times \frac{1}{2}T_{n-3} = T_{n-3} p_2$ -configurations. So, we get $3(T_{n-1} - T_{n-3})$.

The total sum is thus, $T_n = T_{n-2} + 4T_{n-3} + T_{n-2} - T_{n-4} + 3(T_{n-1} - T_{n-3}) = 3T_{n-1} + 2T_{n-2} + T_{n-3} - T_{n-4}$.

In order to find the explicit formula, we must perform some algebra:

$$T_n = 3T_{n-1} + 2T_{n-2} + T_{n-3} - T_{n-4}$$

$$T_n - 3T_{n-1} - 2T_{n-2} - T_{n-3} + T_{n-4} = 0$$

$$x^n - 3x^{n-1} - 2x^{n-2} - x^{n-3} + x^{n-4} = 0$$

$$x^{n-4}(x^4 - 3x^3 - 2x^2 - x + 1) = 0$$

$$x^4 - 3x^3 - 2x^2 - x + 1 = 0$$

The roots of this equation are $\alpha_1 \approx 3.6096$, $\alpha_2 \approx 0.4290$, $\alpha_3 \approx -0.5193 - 0.6133i$, and $\alpha_4 \approx -0.5193 + 0.6133i$.

The solution for the k^{th} value of this recurrence is

$$\sum_{i=1}^{4} -\frac{\left(-2\alpha_{i}^{-2}-6\alpha_{i}^{-1}+2\right)(\alpha_{i})^{k}}{\left(4\,\alpha_{i}^{-3}-3\,\alpha_{i}^{-2}-4\,\alpha_{i}^{-1}-3\right)\alpha_{i}^{-1}}$$

This can be rewritten as $T_k = c_1(\alpha_1)^k + c_2(\alpha_2)^k + c_3(\alpha_3)^k + c_4(\alpha_4)^k$. The dominating term, out of these four roots, is the one which has the greatest modulus. These values are roughly 3.6096, 0.4290, 0.8036, and 0.8036. Thus, in the equation $T_k = c_1(\alpha_1)^k + c_2(\alpha_2)^k + c_3(\alpha_3)^k + c_4(\alpha_4)^k$, the dominant term is $c_1(\alpha_1)^k \approx (0.1564)(3.6096)^k$.

Corollary 6 T_k has an asymptotic value of 0.1564×3.6096^k .

Suppose now that a graph G_k is composed of some graph G_0 connected to a path P_k , $k \ge 4$, with a bridge (an edge which, upon removal, would disconnect the graph). Due to the fact that multiplier calculations are localized for each vertex in a path, we conjecture that if a new vertex vwere added to the end of this path then the new vertex will be such a distance away from G_0 that this recurrence relation will hold. That is, if we know the number of p_2 - configurations that exist on the four graphs G_i , i = 0, 1, 2, 3, then this same recurrence relation can calculate the number of p_2 -configurations that exist on G_4 . In this way, we conjecture that our recurrence relation solution extends to any graph connected to a path of length at least 4.

Conjecture 1 Let G_k be a graph composed of some graph G_0 connected to a path P_k , $k \ge 3$, with a bridge. Then the number of p_2 -configurations on G_k , $F(G_k)$, can be determined using the recurrence

$$F(G_k) = 3F(G_{k-1}) + 2F(G_{k-2}) + F(G_{k-3}) - F(G_{k-4}).$$

4 Conclusion

This result for paths sits alongside similar results for complete graphs from "On Variants of Diffusion", T. Mullen, PhD Thesis. When compared to the methods used on complete graphs with regards to polyominoes, the methods from this paper are rather crude. It is the opinion of the authors that results on other graph classes will not be so simple as those found here and in "On Variants of Diffusion". We have plucked the low-hanging fruit and we believe that further results on the number of non-isomorphic configurations on specific graph classes will require computer data and may not result in (comparatively) nice third- and fourth-order recurrence relations.

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