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SUMS OF FOUR SQUARES WITH CERTAIN RESTRICTIONS

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Abstract. Let $a, b \in \mathbb{N} = \{0, 1, 2, \ldots\}$ and $\lambda \in \{2, 3\}$. We show that $4^a(4b+1)$ can be written as $x^2+y^2+z^2+w^2$ with $x, y, z, w \in \mathbb{N}$ such that $x+2y+\lambda z$ is a positive square. We also pose some open conjectures; for example, we conjecture that any positive odd integer can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that $x + 2y + 3z$ is a positive power of two.

1. Introduction

Lagrange's four-square theorem established in 1770 states that each $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$. In 2017 the author [6] proved that this can be refined in various ways.

As in [6], we call a polynomial $P(x, y, z, w) \in \mathbb{Z}[x, y, z, w]$ *suitable* if any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that $P(x, y, z, w)$ is a square. The author [6] showed that the linear polynomial $x, 2x, x - y, 2(x - y)$ are suitable, and conjectured that

$$x + 2y,\ x + 3y,\ x + 24y,\ 2x - y,\ 4x - 3y,\ 6x - 2y$$

are also suitable. Based on the idea of [6], Y.-C. Sun and the author [5] proved that $x + 2y$ is suitable, moreover any $m \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that $x + 2y$ is a positive square. Recently, Y.-F. Sue and H.-L. Wu [4] proved that $x + 3y$ is also suitable via the arithmetic theory of ternary quadratic forms, and A. Machiavelo et al. [3, 2] proved the author's 1-3-5 conjecture which states that $x + 3y + 5z$ is suitable by using Hamilton quaternions.

Conjecture 4.15(ii) of the author's paper [7] states that for any $m \in \mathbb{Z}^+$ we can write $m^2 = x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that $x + 2y + 3z \in \{4^a : a \in \mathbb{N}\}$. This is implied by the author's following new conjecture formulated on Oct. 10, 2020.

Conjecture 1.1 (1-2-3 Conjecture). (i) (Weak version) *Any positive odd integer m can be written as $x^2 + y^2 + z^2 + w^2$ $(x, y, z, w \in \mathbb{N})$ such that $x + 2y + 3z \in \{2^a : a \in \mathbb{Z}^+\}$.*

(ii) (Strong version) *Any integer $m > 4627$ with $m \not\equiv 0, 2 \pmod 8$ can be written as $x^2+y^2+z^2+w^2$ $(x, y, z, w \in \mathbb{N})$ with $x+2y+3z \in \{4^a : a \in \mathbb{Z}^+\}$.*

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Remark 1.1. By [6, Theorem 1.2(v)], any positive integer can be written as $x^2+y^2+z^2+4^a$ with $a,x,y,z\in\mathbb{N}$. We have verified the 1-2-3 Conjecture for $m\le 5\times 10^6$. See [8, A338096 and A338103] for some data concerning the 1-2-3 Conjecture.

Motivated by the 1-2-3 Conjecture, we establish the following result.

Theorem 1.1. *Let $m\in\mathbb{Z}^+$ and let $\lambda\in\{2,3\}$. If $m=4^a(4b+1)$ for some $a,b\in\mathbb{N}$, then we can write m as $x^2+y^2+z^2+w^2$ $(x,y,z,w\in\mathbb{N})$ such that $x+2y+\lambda z$ is a positive square.*

Remark 1.2. By [7, Theorem 1.4], any $m\in\mathbb{Z}^+$ can be written $x^2+y^2+z^2+w^2$ with $x,y,z,w\in\mathbb{Z}$ and $x+2y+2z\in\{4^a:\ a\in\mathbb{N}\}$. By [5, Theorem 1.7(iv)], any $n\in\mathbb{N}$ can be written as $x^2+y^2+z^2+w^2$ $(x,y,z,w\in\mathbb{Z})$ with $x+2y+3z$ a square. Note that x,y,z here are just integers while x,y,z in Theorem 1.1 are nonnegative integers.

As any positive square has the form $4^a(8b+1)$ with $a,b\in\mathbb{N}$, Theorem 1.1 has the following consequence.

Corollary 1.1. *Let $\lambda\in\{2,3\}$. Then any positive square can be written as $x^2+y^2+z^2+w^2$ $(x,y,z,w\in\mathbb{N})$ such that $x+2y+\lambda z$ is a positive square.*

Remark 1.3. Actually, our computation via a computer suggests that those $m\in\mathbb{Z}^+$ which cannot be written as $x^2+y^2+z^2+w^2$ $(x,y,z,w\in\mathbb{N})$ with $x+2(y+z)$ a positive square are

$$7\times 2^{4a},\ 3\times 2^{4a+3},\ 15\times 2^{4a+3},\ 55\times 2^{4a},\ 255\times 2^{4a}$$

with $a\in\mathbb{N}$, and that those $m\in\mathbb{Z}^+$ which cannot be written as $x^2+y^2+z^2+w^2$ $(x,y,z,w\in\mathbb{N})$ with $x+2y+3z$ a positive square are

$$3\times 2^{4a+2},\ 9\times 2^{4a+3},\ 19\times 2^{4a+2},\ 23\times 2^{4a+2}$$

with $a\in\mathbb{N}$.

In contrast with the 1-2-3 Conjecture, we have the following result.

Theorem 1.2. *Any integer $m>1$ with $m\ne 10$ can be written as $x^2+y^2+z^2+w^2$ with $x,y,z,w\in\mathbb{Z}$ and $x+2y+3z\in\{4^a:\ a\in\mathbb{Z}^+\}$.*

Remark 1.4. In [7, Conjecture 4.5(ii)], the author conjectured that any $m\in\mathbb{Z}^+$ can be written as $x^2+y^2+z^2+w^2$ with $x,y,z,w\in\mathbb{N}$ and $|x+2y-3z|\in\{4^a:\ a\in\mathbb{N}\}$.

We are going to provide three lemmas in the next section. We will prove Theorems 1.1-1.2 in Section 3 and pose some conjectures in Section 4.

2. Three lemmas

For any real numbers $a_1,\ldots,a_n,x_1,\ldots,x_n$, we have the Cauchy-Schwarz inequality (cf. [1, p. 178])

$$(a_1x_1+\ldots+a_nx_n)^2\le(a_1^2+\ldots+a_n^2)(x_1^2+\ldots x_n^2).$$

We will make use of the inequality in our proof of the following lemma.

Lemma 2.1. *Let a, b, c, d, m be nonnegative real numbers with $a^2+b^2+c^2+d^2 \neq 0$. Suppose that x, y, z, w are real numbers satisfying*

$$\begin{cases} x^2+y^2+z^2+w^2=m, \\ ax+by+cz+dw=s, \end{cases} \tag{2.1}$$

where

$$s \geq \sqrt{m(a^2+b^2+c^2+d^2-\min(\{a^2,b^2,c^2,d^2\}\setminus\{0\})}. \tag{2.2}$$

Then all the numbers ax, by, cz, dw are nonnegative.

Proof. Let

$$t = ay-bx+cw-dz,\ \ u = az-bw-cx+dy,\ \ v = aw+bz-cy-dx.$$

By Euler's four-square identity, we have

$$(a^2+b^2+c^2+d^2)(x^2+y^2+z^2+w^2) = s^2+t^2+u^2+v^2. \tag{2.3}$$

Solving the system of equations

$$\begin{cases} ax+by+cz+dw=s, \\ ay-bx+cw-dz=t, \\ az-bw-cx+dy=u, \\ aw+bz-cy-dx=v, \end{cases} \tag{2.4}$$

as in [5], we find that

$$\begin{cases} x=\frac{as-bt-cu-dv}{a^2+b^2+c^2+d^2}, \\ y=\frac{bs+at+du-cv}{a^2+b^2+c^2+d^2}, \\ z=\frac{cs-dt+au+bv}{a^2+b^2+c^2+d^2}, \\ w=\frac{ds+ct-bu+av}{a^2+b^2+c^2+d^2}. \end{cases} \tag{2.5}$$

Suppose that $a>0$. Then

$$s^2 \geq m(a^2+b^2+c^2+d^2-a^2) = (b^2+c^2+d^2)m$$

and hence

$$(a^2+b^2+c^2+d^2)s^2 \geq (b^2+c^2+d^2)(a^2+b^2+c^2+d^2)m = (b^2+c^2+d^2)(s^2+t^2+u^2+v^2).$$

Thus $a^2s^2 \geq (b^2+c^2+d^2)(t^2+u^2+v^2)$. By the Cauchy-Schwarz inequality,

$$(bt+cu+dv)^2 \leq (b^2+c^2+d^2)(t^2+u^2+v^2).$$

Therefore $as \geq |bt+cu+dv|$ and hence $x>0$ in view of (2.5).

Similarly, $y \geq 0$ if $b>0$, and $z \geq 0$ if $d>0$. This concludes the proof. □

The Gauss-Legendre theorem (cf. [1, p. 23]) states that $n \in \mathbb{N}$ can be written as the sum of three squares if and only if n does not belong to the set

$$E = \{4^s(8t+7):\ s,t \in \mathbb{N}\}. \tag{2.6}$$

Lemma 2.2. *Let $m, n \in \mathbb{Z}^+$ with $3 \nmid n$ and $9m-n^4 \in \mathbb{N}\setminus E$. Then m can be written as $x^2+y^2+z^2+w^2$ with $x, y, z, w \in \mathbb{Z}$ and $x+2y+2z=n^2$.*

Proof. By the Gauss-Legendre theorem, $9m - n^4 = a^2 + b^2 + c^2$ for some $a, b, c \in \mathbb{Z}$. As $3 \nmid n$, we have $3 \mid abc$. Without loss of generality, we suppose $c = 3w$ with $w \in \mathbb{Z}$. Since $a^2 + b^2 \equiv -n^4 \equiv 2 \pmod 3$, we have $3 \nmid ab$. Without loss of generality, we may assume that $a \equiv 2 \pmod 3$ and $b \equiv -2 \pmod 3$ (otherwise we may change the signs of a and b suitably). Clearly, $a = 3u + 2n^2$ and $b = 3v - 2n^2$ for some $u, v \in \mathbb{Z}$. Observe that

$$12n^2(u - v) + 8n^4 \equiv (3u + 2n^2)^2 + (3v - 2n^2)^2 = a^2 + b^2 \equiv -n^4 \pmod 9$$

and hence $u \equiv v \pmod 3$ since $3 \nmid n$. Set

$$y = -\frac{2u + v}{3} \quad \text{and} \quad z = \frac{u + 2v}{3}.$$

Then

$$\begin{aligned} 9m - n^4 =& a^2 + b^2 + c^2 = (3u + 2n^2)^2 + (3v - 2n^2)^2 + 9w^2 \\ =& (3(-2y - z) + 2n^2)^2 + (3(2z + y) - 2n^2)^2 + 9w^2 \\ =& 9(2y + z)^2 + 9(2z + y)^2 - 36n^2(y + z) + 8n^4 + 9w^2 \end{aligned}$$

and hence

$$m = n^4 + w^2 + (2y + z)^2 + (2z + y)^2 - 4n^2(y + z) = x^2 + y^2 + z^2 + w^2,$$

where $x = n^2 - 2(y + z)$. Note that $x + 2y + 2z = n^2$ as desired. □

Lemma 2.3. *Let $m, n \in \mathbb{Z}^+$ with $14m - n^4 \in \mathbb{N} \setminus E$. Then there are $x, y, z, w \in \mathbb{Z}$ such that $x^2 + y^2 + z^2 + w^2 = m$ and $x + 2y + 3z = n^2$.*

Proof. The norm of the Hamilton quaternion $\zeta = 1 + 2i + 3j + 0k$ is $N(\zeta) = 1^2 + 2^2 + 3^2 + 0^2 = 14$. Applying [3, Theorem 2], we immediately get the desired result. □

3. Proof of Theorems 1.1-1.2

Proof of Theorem 1.1. If there are $x, y, z, w \in \mathbb{N}$ such that $m = x^2 + y^2 + z^2 + w^2$ with $x + 2y + \lambda z$ a positive square, then $16m = (4x)^2 + (4y)^2 + (4z)^2 + (4w)^2$, and $4x + 2(4y) + \lambda(4z) = 4(x + 2y + \lambda z)$ is also a positive square. So, it suffices to handle the case $16 \nmid m$.

Below we suppose $m = 4^a(4b + 1)$ with $a \in \{0, 1\}$ and $b \in \mathbb{N}$.

Case 1. $\lambda = 2$.

If $m \le 40125453$, then we can use a computer to verify that m can be written as $x^2 + y^2 + z^2 + w^2$ ($x, y, z, w \in \mathbb{N}$) with $x + 2y + 2z$ a positive square. Now assume that $m = 4^a(4b + 1) \ge 40125454$. Then

$$m \ge \left(\frac{4}{9^{1/4} - 8^{1/4}}\right)^4 \approx 40125453.161$$

and hence the interval $I = [\sqrt[4]{8m}, \sqrt[4]{9m}]$ has length at least four. It follows that there is an integer $n \in I$ with $n \equiv \pm 2 \pmod 6$. Note that

$$9m - n^4 \equiv 9m = 4^a(4(9b + 2) + 1) \pmod{16}.$$

So we have $9m - n^4 \in \mathbb{N} \setminus E$. By Lemma 2.2, there are $x, y, z, w \in \mathbb{Z}$ such that $x^2 + y^2 + z^2 + w^2 = m$ and $x + 2y + 2z = n^2 > 0$. As

$$n^2 \geq \sqrt{m(1^2 + 2^2 + 2^2 + 0^2 - 1^2)} = \sqrt{8m},$$

by Lemma 2.1 we have $x, y, z \in \mathbb{N}$ as desired.

Case 2. $\lambda = 3$.

If $m \leq 10065600$, then we can use a computer to verify that m^2 can be written as $x^2 + y^2 + z^2 + w^2$ $(x, y, z, w \in \mathbb{N})$ with $x + 2y + 3z$ a positive square. Now assume that $m = 4^a(4b+1) \geq 10065601$. Then

$$m \geq \left(\frac{2}{14^{1/4} - 13^{1/4}}\right)^4 \approx 10065600.518$$

and hence the interval $J = [\sqrt[4]{13m}, \sqrt[4]{14m}]$ has length at least two. It follows that J contains an even integer n. If $a = 0$ then

$$14m - n^4 \equiv 14(4b+1) \equiv 6 \pmod{8};$$

if $a = 1$ then

$$14m - n^4 \equiv 56(4b+1) \equiv 8 \pmod{16}.$$

So we have $14m - n^4 \in \mathbb{N} \setminus E$. By Lemma 2.2, there are $x, y, z, w \in \mathbb{Z}$ such that $x^2 + y^2 + z^2 + w^2 = m$ and $x + 2y + 3z = n^2 > 0$. As

$$n^2 \geq \sqrt{m(1^2 + 2^2 + 3^2 + 0^2 - 1^2)} = \sqrt{13m},$$

by Lemma 2.1 we have $x, y, z \in \mathbb{N}$ as desired. □

Proof of Theorem 1.2. In view of Lemma 2.3, it suffices to find $a \in \mathbb{Z}^+$ with $14m - 2^{4a} \in \mathbb{N} \setminus E$. If $14m - 2^{4a} \in \mathbb{N} \setminus E$ then $14(16m) - 2^{4(a+1)} = 4^2(14m - 2^{4a}) \in \mathbb{N} \setminus E$. Note also that

$$16 = 4^2 + 0^2 + 0^2 + 0^2 \text{ and } 160 = 4^2 + 0^2 + 0^2 + 12^2 \text{ with } 4 + 2 \times 0 + 3 \times 0 = 4^1.$$

So we only need to handle the case $m \not\equiv 0 \pmod{16}$. If $m \in \{2, \ldots, 18\} \setminus \{10, 16\}$, then we can verify the desired result directly.

Below we assume that $m > 18$ and $16 \nmid m$. Note that $14m \geq 14 \times 19 = 266 > 2^8$.

If $2 \nmid m$, then $14m - 2^4 \in \mathbb{N} \setminus E$ since $14m \equiv 2 \pmod{4}$.

In the case $m \equiv 2 \pmod{4}$, we have $14m = 4q$ for some odd integer $q > 64$. If $q \not\equiv 7 \pmod{8}$, then $14m - 2^8 = 4(q - 64) \in \mathbb{N} \setminus E$. If $q \equiv 7 \pmod{8}$, then $14m - 2^4 = 4(q - 4) \in \mathbb{N} \setminus E$.

If $m \equiv 4 \pmod{8}$, then $14m \equiv 8 \pmod{16}$ and hence $14m - 2^4 \in \mathbb{N} \setminus E$.

In the case $m \equiv 8 \pmod{16}$, we have $14m = 16q$ for some odd integer $q > 16$. If $q \not\equiv 7 \pmod{8}$, then $14m - 2^8 = 16(q - 16) \in \mathbb{N} \setminus E$. If $q \equiv 7 \pmod{8}$, then $14m - 2^4 = 16(q - 1) \in \mathbb{N} \setminus E$.

This completes the proof of Theorem 1.2. □

4. Some conjectures

The following two conjectures are similar to the 1-2-3 Conjecture.

Conjecture 4.1 (2020-10-10)**.** *Any odd integer $m > 1$ can be written as $x^2 + y^2 + z^2 + w^2$ $(x, y, z, w \in \mathbb{N})$ with $x + y \in \{2^a : a \in \mathbb{Z}^+\}$. Moreover, the only positive integers $m \not\equiv 0, 6 \pmod 8$ which cannot be written as $x^2 + y^2 + z^2 + w^2$ $(x, y, z, w \in \mathbb{N})$ with $x + y \in \{4^a : a \in \mathbb{Z}^+\}$ are*

$$1, 2, 3, 4, 5, 7, 31, 43, 67, 79, 85, 87, 103, 115,$$
$$475, 643, 1015, 1399, 1495, 1723, 1819, 1939, 1987.$$

Remark 4.1. We have verified this for m up to 3×10^7. See [8, A338094 and A338121] for related data. By [7, Theorem 1.1(ii)], any positive integer can be written as $x^2+y^2+z^2+w^2$ $(x, y, z, w \in \mathbb{N})$ with $x-y \in \{2^a : a \in \mathbb{N}\}$.

Conjecture 4.2 (2020-10-10)**.** *Any positive odd integer m can be written as $x^2 + y^2 + z^2 + w^2$ $(x, y, z, w \in \mathbb{N})$ with $x + y + 2z \in \{2^a : a \in \mathbb{Z}^+\}$. Moreover, any integer $m > 10840$ with $m \not\equiv 0, 2 \pmod 8$ can be written as $x^2 + y^2 + z^2 + w^2$ $(x, y, z, w \in \mathbb{N})$ with $x + y + 2z \in \{4^a : a \in \mathbb{Z}^+\}$.*

Remark 4.2. We have verified this for m up to 5×10^6. See [8, A338095 and A338119] for related data. By [7, Theorem 1.4(i)], any $m \in \mathbb{Z}^+$ can be written as $x^2+y^2+z^2+w^2$ with $x, y, z, w \in \mathbb{Z}$ and $x+y+2z \in \{4^a : a \in \mathbb{N}\}$.

Conjecture 4.3 (2020-10-12)**.** *Any $m \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + w^2$ $(x, y, z, w \in \mathbb{N})$ with $2x^2 + 4y^2 - 7xy \in \{2^a : a \in \mathbb{N}\}$. Moreover, any positive integer $m \equiv 1, 2 \pmod 4$ can be written as $x^2 + y^2 + z^2 + w^2$ $(x, y, z, w \in \mathbb{N})$ with $2x^2 + 4y^2 - 7xy \in \{4^a : a \in \mathbb{Z}^+\}$.*

Remark 4.3. We have verified this for $m \le 10^8$. See [8, A337082] for related data.

Conjecture 4.4 (2020-10-12)**.** *Any $m \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + w^2$ $(x, y, z, w \in \mathbb{N})$ with $x^2 + 26y^2 - 11xy \in \{2^a : a \in \mathbb{N}\}$. Moreover, any positive integer $m \equiv 1, 2 \pmod 4$ can be written as $x^2 + y^2 + z^2 + w^2$ $(x, y, z, w \in \mathbb{N})$ with $x^2 + 26y^2 - 11xy \in \{4^a : a \in \mathbb{N}\}$.*

Remark 4.4. We have verified this for $m \le 10^8$. See [8, A338139] for related data.

Conjecture 4.5 (2018-02-21)**.** *Let $\lambda \in \{2, 3, 4\}$. Any positive square can be written as $x^2 + y^2 + z^2 + w^2$ $(x, y, z, w \in \mathbb{N})$ with $x + 2y + 2z + \lambda w$ a square.*

Conjecture 4.6 (2018-02-22)**.** *For each $\lambda \in \{1, 2, 3\}$, any positive square can be written as $x^2 + y^2 + z^2 + w^2$ $(x, y, z, w \in \mathbb{N})$ with $x + 2y + 4z + \lambda w \in \{2^a : a \in \mathbb{N}\}$. Also, any positive square can be written as $x^2 + y^2 + z^2 + w^2$ $(x, y, z, w \in \mathbb{N})$ with $x + 3y + 3z + 4w$ a power of two.*

Similar to [7, Conjecture 4.16], we have the following conjectures.

Conjecture 4.7 (2018-03-01)**.** *Let $\delta \in \{0, 1\}$ and $m \in \mathbb{N}$ with $m > \delta$. Then m^2 can be written as $x^2 + y^2 + z^2 + w^2$ $(x, y, z, w \in \mathbb{N})$ such that $\{2^{2a+\delta} : a \in \mathbb{N}\}$ contains $x + 3y$, and also x or y.*

Remark 4.5. Note that 81503^2 cannot be written as $x^2+y^2+z^2+w^2$ $(x,y,z,w\in\mathbb{N})$ with $\{x,x+3y\}\subseteq\{4^a:\ a\in\mathbb{N}\}$. However,

$$81503^2=16372^2+4^2+52372^2+60265^2$$

with $4=4^1$ and $16372+3\times 4=4^7$.

Conjecture 4.8 (2018-03-04)**.** *Let $\lambda\in\{2,8\}$ and let $\delta\in\{0,1\}$. Then any positive square can be written as $x^2+y^2+z^2+w^2$ $(x,y,z,w\in\mathbb{N})$ such that x or y is a power of 2, and $x+\lambda y\in\{2^{2a+r}:\ a\in\mathbb{N}\}$.*

Conjecture 4.9 (2018-03-04)**.** *Let $\delta\in\{0,1\}$ and $m\in\mathbb{N}$ with $m>\delta$. Then m^2 can be written as $x^2+y^2+z^2+w^2$ $(x,y,z,w\in\mathbb{N})$ with $\{2^{2a+\delta}:\ a\in\mathbb{N}\}$ contains $x+15y$, and also x or $2y$.*

Conjecture 4.10 (2018-03-04)**.** *Let $\delta\in\{0,1\}$ and $m\in\mathbb{N}$ with $m>\delta$. Then m^2 can be written as $x^2+y^2+z^2+w^2$ $(x,y,z,w\in\mathbb{N})$ with $16x-15y\in\{2^{2a+\delta}:\ a\in\mathbb{N}\}$.*

Conjecture 4.11 (2018-03-05)**.** *Any positive square can be written as $x^2+y^2+z^2+w^2$ $(x,y,z,w\in\mathbb{N})$ with $x+63y\in\{2^{2a+1}:\ a\in\mathbb{N}\}$ such that $2x$ or y is a power of 4.*

For $P(x,y,z,w)\in\mathbb{Z}[x,y,z,w]$, we define its exceptional set $E(P)$ as the set of all those $n\in\mathbb{N}$ for which there are no $x,y,z,w\in\mathbb{N}$ with $n=x^2+y^2+z^2+w^2$ such that $P(x,y,z,w$ is a square.

Conjecture 4.12 (2020-10-09)**.** *Any $m\in\mathbb{N}$ not divisible by 8 can be written as $x^2+y^2+z^2+w^2$ $(x,y,z,w\in\mathbb{N})$ with $x+3y+4z$ a square. Moreover,*

$$E(x+3y+4z)=\{2^{4a+3}q:\ a\in\mathbb{N},\ q\in\{1,3,5,43\}\}.$$

Remark 4.6. We have verified the former assertion for $m\le 6\times 10^6$. See [8, A335624] for related data.

Conjecture 4.13 (2020-10-09)**.** *Any $m\in\mathbb{N}$ not divisible by 8 can be written as $x^2+y^2+z^2+w^2$ $(x,y,z,w\in\mathbb{N})$ with $3x+10y+36z$ a positive square. Moreover,*

$$E(3x+10y+36z)=\{2^{4a+3}q:\ a\in\mathbb{N},\ q\in\{1,3,5,61\}\}.$$

Remark 4.7. We have verified the former assertion for $m\le 5\times 10^6$. See [8, A338019] for related data.

Conjecture 4.14 (2020-10-08)**.** (i) *We have*

$$E(x-2y)=\{43\times 2^{4a}:\ a\in\mathbb{N}\},\ E(4x-y)=\{7\times 2^{4a}:\ a\in\mathbb{N}\},$$
$$E(3x-2y)=E(5x-y)=E(7x-3y)=E(32x-15y)=\{3\times 2^{4a+3}:\ a\in\mathbb{N}\},$$
$$E(x+4y)=\{2^{4a+2}q:\ a\in\mathbb{N},\ q\in\{3,23\}\},$$
$$E(2x+7y)=\{35\times 2^{4a}:\ a\in\mathbb{N}\},\ E(8x+9y)=\{47\times 2^{4a}:\ a\in\mathbb{N}\}.$$

(ii) *We have*

$$\begin{aligned} E(x+2y+4z) &= \{3\times 2^{4a}:\ a\in\mathbb{N}\},\\ E(x+2y+6z) &= \{15\times 2^{4a}:\ a\in\mathbb{N}\},\\ E(2x+3y+4z) &= \{3\times 2^{4a+1}:\ a\in\mathbb{N}\},\\ E(2x+4y+5z) &= \{3\times 2^{4a+2}:\ a\in\mathbb{N}\},\\ E(4x+5y+8z) &= \{23\times 2^{4a}:\ a\in\mathbb{N}\},\\ E(2x+6y+14z) &= \{2^{4a+2}q:\ a\in\mathbb{N},\ q\in\{7,31\}\}. \end{aligned}$$

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