The feasible regions for consecutive patterns of pattern-avoiding permutations

Jacopo Borga*[1] and Raul Penaguiao†[1]

[1]Institut für Mathematik, Universität Zürich

Abstract

We study the feasible region for consecutive patterns of pattern-avoiding permutations. More precisely, given a family $\mathcal{C}$ of permutations avoiding a fixed set of patterns, we study the limit of proportions of consecutive patterns on large permutations of $\mathcal{C}$. These limits form a region, which we call the *pattern-avoiding feasible region for* $\mathcal{C}$. We show that, when $\mathcal{C}$ is the family of τ-avoiding permutations, with either τ of size three or τ a monotone pattern, the pattern-avoiding feasible region for $\mathcal{C}$ is a polytope. We also determine its dimension using a new tool for the monotone pattern case whereby we are able to compute the dimension of the image of a polytope after a projection.

We further show some general results for the pattern-avoiding feasible region for any family $\mathcal{C}$ of permutations avoiding a fixed set of patterns, and we conjecture a general formula for its dimension.

Along the way, we discuss connections of this work with the problem of packing patterns in pattern-avoiding permutations and to the study of local limits for pattern-avoiding permutations.

1 Introduction

1.1 The pattern-avoiding feasible regions

The study of limits of pattern-avoiding permutations is a very active field in combinatorics and discrete probability theory. There are two main ways of investigating these limits:

- The most classical one is to look at the limits of various statistics for pattern-avoiding permutations. For instance, the limit distributions of the longest increasing subsequences in uniform pattern-avoiding permutations have been considered in [DHW03, MY19]. Another example is the general problem of studying the limiting distribution of the number of occurrences of a fixed pattern π in a uniform random permutation avoiding a fixed set of patterns when the size tends to infinity (see for instance Janson's papers [Jan17, Jan18a, Jan18b], where the author studied this problem in the model of uniform permutations avoiding a fixed family of patterns of size three). Many other statistics have been considered, for instance in [BKL+18] the authors studied the distribution of ascents, descents, peaks, valleys, double ascents, and double descents over pattern-avoiding permutations.

Key words and phrases. Feasible region, pattern-avoiding permutations, cycle polytopes, overlap graphs, consecutive patterns.

2010 *Mathematics subject classification.* 52B11, 05A05, 60C05.

*jacopo.borga@math.uzh.ch

†raul.penaguiao@math.uzh.ch

- The second way is to look at the geometric limit of large pattern-avoiding permutations. Two main notions of convergence for permutations have been defined: a global notion of convergence (called permuton convergence, [HKM+13]) and a local notion of convergence[1] (called Benjamini–Schramm convergence, [Bor20b]). For an intuitive explanation of them we refer the reader to [BP19, Section 1.1], where additional references can be found. We just mention here that permuton convergence is equivalent to the convergence of all *pattern density* statistics (see [BBF+19, Theorem 2.5]); and Benjamini–Schramm convergence is equivalent to the convergence of all *consecutive pattern density* statistics (see [Bor20b, Theorem 2.19]). The latter are the subject of this paper.

In this paper we study the feasible region for consecutive patterns of pattern-avoiding permutations. This is strongly connected with both ways of studying limits of these families of permutations refered above. We start by defining this region, and then we comment on these connections.

Let $\mathcal{S}_k$ denote the set of permutations of size k and $\mathcal{S}$ the set of all permutations. Many basic concepts and notation on permutations will be provided in Section 1.5. In this introduction, we use the classical terminology and we briefly introduce essential notation along the way, like $\widetilde{\text{c-occ}}(\pi,\sigma)$ which denotes the proportion of consecutive occurrences of a pattern π in a permutation σ. Given a set of patterns $B \subset \mathcal{S}$, we denote by $\text{Av}_n(B)$ the set of B-avoiding permutations of size n, and by $\text{Av}(B) \coloneqq \bigcup_{n\in\mathbb{Z}_{\geq 1}} \text{Av}_n(B)$ the set of B-avoiding permutations of arbitrary finite size. We consider the *pattern-avoiding feasible region for consecutive patterns for* $\text{Av}(B)$, defined by

$$P_k^{\text{Av}(B)} \coloneqq \{\vec{v} \in [0,1]^{\mathcal{S}_k} \big| \exists (\sigma^m)_{m\in\mathbb{Z}_{\geq 1}} \in \text{Av}(B)^{\mathbb{Z}_{\geq 1}} \text{ such that } \\ |\sigma^m| \to \infty \text{ and } \widetilde{\text{c-occ}}(\pi,\sigma^m) \to \vec{v}_\pi, \forall \pi \in \mathcal{S}_k\}.$$

For different choices of the family $\text{Av}(B)$, we refer to these regions as *pattern-avoiding feasible regions*. In words, the region $P_k^{\text{Av}(B)}$ is formed by the set of $k!$-dimensional vectors $\vec{v}$ for which there exists a sequence of permutations in $\text{Av}(B)$ whose size tends to infinity and whose proportion of consecutive patterns of size k tends to $\vec{v}$.

We can now comment on the connection between these regions and the two ways of studying limits of pattern-avoiding permutations mentioned above. For the first one, i.e. the study of various statistics for pattern-avoiding permutations, the statistic that we consider here is the number of consecutive occurrences of a pattern. For the second one, i.e. the study of geometric limits, the relation is with Benjamini–Schramm limits. In particular, having a precise description of the regions $P_k^{\text{Av}(B)}$ for all $k \in \mathbb{Z}_{\geq 1}$ determines all the Benjamini–Schramm limits that can be obtained through sequences of permutations in $\text{Av}(B)$.

An orthogonal motivation for investigating the pattern avoiding feasible regions is the problem of *packing patterns* in pattern avoiding permutations. The classical question of *packing patterns* in permutations is to describe the maximum number of occurrences of a pattern π in any permutation of $\mathcal{S}_n$ (see for instance [AAH+02, Bar04, Pri97]). More recently, a similar question in the context of pattern-avoiding permutations has been addressed by Pudwell [Pud20]. It consists in describing the maximum number of occurrences of a pattern π in any pattern-avoiding permutation. Describing the pattern avoiding feasible region $P_k^{\text{Av}(B)}$ is a fundamental step for solving the question of finding the asymptotic maximum number of consecutive occurrences of a pattern $\pi \in \mathcal{S}_k$ in large permutations of $\text{Av}(B)$ (indeed the latter problem can be translated into a linear optimization problem in the feasible region $P_k^{\text{Av}(B)}$).

[1] A third and new notion of convergence was recently introduced in [Bev20]; it interpolates between the two main notions mentioned in the paper.

Additional motivations for studying the regions $P_k^{\mathrm{Av}(B)}$ are the novelties of the results in this paper compared with a previous work [BP19] where we studied the (non-restricted) *feasible region* for consecutive patterns (for a more precise discussion on this point see Sections 1.2 and 1.3). The latter is defined for all $k \in \mathbb{Z}_{\geq 1}$ as

$$P_k := \left\{ \vec{v} \in [0,1]^{\mathcal{S}_k} \middle| \exists (\sigma^m)_{m \in \mathbb{Z}_{\geq 1}} \in \mathcal{S}^{\mathbb{Z}_{\geq 1}} \text{ s.t. } |\sigma^m| \to \infty \text{ and } \widetilde{\text{c-occ}}(\pi, \sigma^m) \to \vec{v}_\pi, \forall \pi \in \mathcal{S}_k \right\} . \quad (1)$$

We refer the reader to [BP19, Section 1.1] for motivations to investigate this region and to [BP19, Section 1.2] for a summary of the related literature.

1.2 Previous results on the standard feasible region for consecutive patterns

Before presenting our results on the pattern-avoiding feasible regions, we recall two key definitions from [BP19] and review some results presented in that paper.

Definition 1.1. The *overlap graph* $\mathcal{O}v(k)$ is a directed multigraph with labeled edges, where the vertices are elements of $\mathcal{S}_{k-1}$ and for every $\pi \in \mathcal{S}_k$ there is an edge labeled by π from the pattern induced by the first $k-1$ indices of π to the pattern induced by the last $k-1$ indices of π.

For an example for $k = 3$ see the left-hand side of Fig. 1. Given a permutation π, we denote the pattern induced by the first $k-1$ indices by $\mathrm{beg}_{k-1}(\pi)$ and the pattern induced by the last $k-1$ indices by $\mathrm{end}_{k-1}(\pi)$.

Definition 1.2. Let $G = (V, E)$ be a directed multigraph. For each non-empty cycle $\mathcal{C}$ in G, define $\vec{e}_{\mathcal{C}} \in \mathbb{R}^E$ such that

$$(\vec{e}_{\mathcal{C}})_e := \frac{\#\text{ of occurrences of } e \text{ in } \mathcal{C}}{|\mathcal{C}|}, \quad \text{for all} \quad e \in E.$$

We define the *cycle polytope* of G to be the polytope $P(G) := \mathrm{conv}\{\vec{e}_{\mathcal{C}} | \mathcal{C} \text{ is a simple cycle of } G\}$.

We recall some results from [BP19].

Proposition 1.3 (Proposition 1.7 in [BP19])**.** *The cycle polytope of a strongly connected directed multigraph $G = (V, E)$ has dimension $|E| - |V|$.*

Theorem 1.4 (Theorem 1.6. in [BP19])**.** *P_k is the cycle polytope of the overlap graph $\mathcal{O}v(k)$. Its dimension is $k! - (k-1)!$ and its vertices are given by the simple cycles of $\mathcal{O}v(k)$.*

An instance of the result above is depicted in Fig. 1.

We also recall for later purposes the following construction related to the overlap graph $\mathcal{O}v(k)$. Given a permutation $\sigma \in \mathcal{S}_m$, for some $m \geq k$, we can associate with it a walk $W_k(\sigma) = (e_1, \dots, e_{m-k+1})$ in $\mathcal{O}v(k)$ of size $m-k+1$, where e_i is the edge of $\mathcal{O}v(k)$ labeled by the pattern of σ induced by the indices from i to $i+k-1$. The map W_k is not injective, but in [BP19] we proved the following.

Lemma 1.5 (Lemma 3.8 in [BP19])**.** *Fix $k \in \mathbb{Z}_{\geq 1}$ and $m \geq k$. The map W_k, from the set $\mathcal{S}_m$ of permutations of size m to the set of walks in $\mathcal{O}v(k)$ of size $m-k+1$, is surjective.*

This lemma was a key step in the proof of Theorem 1.4.

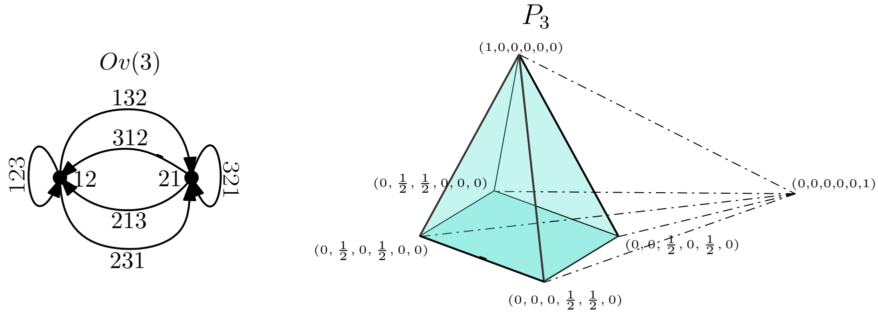


Figure 1: The overlap graph $\mathcal{O}v(3)$ and the four-dimensional polytope P_3. The coordinates of the vertices correspond to the patterns $(123, 231, 312, 213, 132, 321)$ respectively. Note that the top vertex (resp. the right-most vertex) corresponds to the loop indexed by 123 (resp. 321); the other four vertices correspond to the four cycles of length two. We highlight in light-blue one of the six three-dimensional faces of P_3. This face is a pyramid with a square base. The polytope itself is a four-dimensional pyramid, whose base is the highlighted face. From Theorem 1.4 we have that P_3 is the cycle polytope of $\mathcal{O}v(3)$.

1.3 Main results on the pattern-avoiding feasible regions

We start with a natural generalization of Definition 1.1 to pattern-avoiding permutations.

Definition 1.6. Fix a set of patterns $B \subset \mathcal{S}$ and $k \in \mathbb{Z}_{\geq 1}$. The *overlap graph* $\mathcal{O}v^{\mathrm{Av}(B)}(k)$ is a directed multigraph with labeled edges, where the vertices are elements of $\mathrm{Av}_{k-1}(B)$ and for every $\pi \in \mathrm{Av}_k(B)$ there is an edge labeled by π from the pattern induced by the first $k-1$ indices of π to the pattern induced by the last $k-1$ indices of π.

Informally, $\mathcal{O}v^{\mathrm{Av}(B)}(k)$ arises simply as the restriction of $\mathcal{O}v(k)$ to all the edges and vertices in $\mathrm{Av}(B)$. We have the following result, which is proved in Section 2.

Theorem 1.7. *Fix $k \in \mathbb{Z}_{\geq 1}$. For all sets of patterns $B \subset \mathcal{S}$, the feasible region $P_k^{\mathrm{Av}(B)}$ is a closed set and satisfies $P_k^{\mathrm{Av}(B)} \subseteq P(\mathcal{O}v^{\mathrm{Av}(B)}(k)) \subseteq P_k$.*

Moreover, if $\mathrm{Av}(B)$ is closed either for the direct or skew sum then the feasible region $P_k^{\mathrm{Av}(B)}$ is convex and $\dim(P_k^{\mathrm{Av}(B)}) \leq |\mathrm{Av}_k(B)| - |\mathrm{Av}_{k-1}(B)|$.

In particular, $P_k^{\mathrm{Av}(\tau)}$ is always convex for any pattern $\tau \in \mathcal{S}$, as $\mathrm{Av}(\tau)$ is either closed for the $\oplus$ operation (whenever τ is $\oplus$-indecomposable) or closed for the $\ominus$ operation (whenever τ is $\ominus$-indecomposable).

For some sets of patterns B, the region $P_k^{\mathrm{Av}(B)}$ is not even convex. For instance, if $B = \{132, 213, 231, 312\}$, then $\mathrm{Av}(B)$ is the set of monotone permutations. Therefore, the resulting pattern-avoiding feasible region is formed by two distinct points, hence it is not convex. This shows that the last hypothesis in Theorem 1.7 is not superfluous.

We will show later in Fact 1.10 that sometimes $P_k^{\mathrm{Av}(B)} \neq P(\mathcal{O}v^{\mathrm{Av}(B)}(k))$ (see also the right-hand side of Fig. 2) but we believe that the bound on the dimension of the feasible regions given above is tight whenever $|B| = 1$.

Conjecture 1.8. *Fix $k \in \mathbb{Z}_{\geq 1}$. For all patterns $\tau \in \mathcal{S}$, we have that*

$$\dim(P_k^{\mathrm{Av}(\tau)}) = |\mathrm{Av}_k(\tau)| - |\mathrm{Av}_{k-1}(\tau)|. \tag{2}$$

It is natural to wonder what happens for $|B| \geq 2$. In the case $B = \{132, 213, 231, 312\}$ described above, the feasible region is not a polytope. However, using the general notion of dimension in real spaces due to Hausdorff, we can still talk about the dimension of this reagion, and we have that $\dim P_k^{\mathrm{Av}(B)} = 0$, which seems to agree with the prediction of our conjecture for $k \geq 3$. However, when we consider $B = \mathcal{S}_{k+1}$, then $P_k^{\mathrm{Av}(B)} = \emptyset$, which by convention has dimension -1, whereas we have $|\operatorname{Av}_k(B)| - |\operatorname{Av}_{k-1}(B)| = k! - (k-1)!$.

The main goal of this paper is to prove that Conjecture 1.8 is true when $|\tau| = 3$ or τ is a monotone pattern, i.e. $\tau = n \cdots 1$ or $\tau = 1 \cdots n$, for $n \in \mathbb{Z}_{\geq 2}$. More precisely, we will completely describe the feasible regions $P_k^{\mathrm{Av}(\tau)}$ for such patterns τ. By symmetry, we only need to study the cases $\tau = 312$ and $\tau = n \cdots 1$ for $n \in \mathbb{Z}_{\geq 2}$. Indeed, every other permutation arises as compositions of the reverse map (symmetry of the diagram w.r.t. the vertical axis) and the complementation map (symmetry of the diagram w.r.t. the horizontal axis) of the permutations $\tau = 312$ or $\tau = n \cdots 1$ for $n \in \mathbb{Z}_{\geq 2}$. Beware that the inverse map (symmetry of the diagram w.r.t. the principal diagonal) cannot be used since it does not preserve consecutive pattern occurrences.

1.3.1 312-avoiding permutations

When $\tau = 312$ we have the following result.

Theorem 1.9. *Fix $k \in \mathbb{Z}_{\geq 1}$. The feasible region $P_k^{\mathrm{Av}(312)}$ is the cycle polytope of the overlap graph $\mathcal{O}v^{\mathrm{Av}(312)}(k)$. Its dimension is $C_k - C_{k-1}$, where C_k is the k-th Catalan number, and its vertices are given by the simple cycles of $\mathcal{O}v^{\mathrm{Av}(312)}(k)$.*

An instance of the result above is depicted on the left-hand side of Fig. 2.

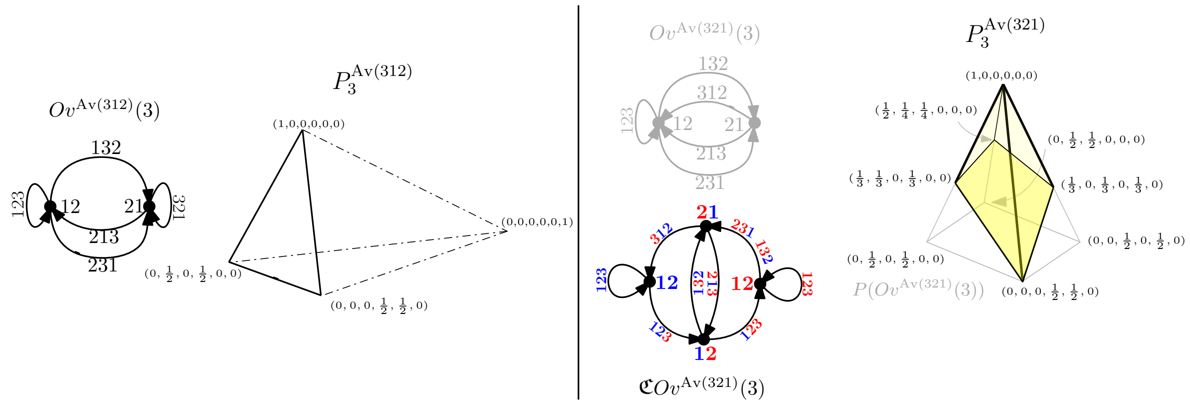


Figure 2: We suggest to compare this picture with the one in Fig. 1. In particular, we use the same conventions for the coordinates of the vertices of the polytopes. **Left:** The overlap graph $\mathcal{O}v^{\mathrm{Av}(312)}(3)$ and the three-dimensional polytope $P_3^{\mathrm{Av}(312)}$. Note that $P_3^{\mathrm{Av}(312)} \subseteq P_3$. From Theorem 1.9 we have that $P_3^{\mathrm{Av}(312)}$ is the cycle polytope of $\mathcal{O}v^{\mathrm{Av}(312)}(3)$. **Right:** In grey the overlap graph $\mathcal{O}v^{\mathrm{Av}(321)}(3)$ and the corresponding three-dimensional cycle polytope $P(\mathcal{O}v^{\mathrm{Av}(321)}(3))$, that is strictly larger than $P_3^{\mathrm{Av}(321)}$. The latter feasible region is highlighted in yellow. From Theorem 1.11 we have that $P_k^{\mathrm{Av}(321)}$ is the projection (defined precisely in Theorem 4.8) of the cycle polytope of the coloured overlap graph $\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(321)}(3)$ (see Definition 4.6 for a precise description). This graph is plotted on the bottom-left side. Note that $P_3^{\mathrm{Av}(312)} \subseteq P_3$.

1.3.2 Monotone-avoiding permutations

In this section we fix $\searrow_n = n \cdots 1$ for $n \in \mathbb{Z}_{\geq 2}$, the decreasing pattern of size n, and an integer $k \in \mathbb{Z}_{\geq 1}$. We start with the following result (compare it with the right-hand side of Fig. 2), which shows that the study of the monotone case deviates significantly from the one in Theorem 1.9.

Fact 1.10. The cycle polytope $P(\mathcal{O}v^{\mathrm{Av}(321)}(3))$ is different from the feasible region $P_k^{\mathrm{Av}(321)}$.

Proof. Consider the vector $\vec{v} = (0, 1/2, 1/2, 0, 0, 0)$, where the coordinates of the vector correspond to the patterns $(123, 231, 312, 213, 132, 321)$. We are going to show that $\vec{v} \in P(\mathcal{O}v^{\mathrm{Av}(321)}(3))$ but $\vec{v} \notin P_k^{\mathrm{Av}(321)}$.

Since the patterns $(231, 312)$ form a simple cycle in $\mathcal{O}v^{\mathrm{Av}(321)}(3)$ we have that $\vec{v} \in P(\mathcal{O}v^{\mathrm{Av}(321)}(3))$ by definition.

Now assume for sake of contradiction that $\vec{v} \in P_k^{\mathrm{Av}(321)}$. There exists a sequence $(\sigma^m)_{m \in \mathbb{Z}_{\geq 1}} \in \mathrm{Av}(321)^{\mathbb{Z}_{\geq 1}}$ such that $|\sigma^m| \to \infty$ and $\widetilde{\text{c-occ}}(\pi, \sigma^m) \to \frac{\mathbb{1}_{\{231,312\}}(\pi)}{2}$ for all $\pi \in \mathcal{S}_3$. Consider an interval $I = \{i, i+1, i+2\}$ such that $\mathrm{pat}_I(\sigma^m) = 312$ and $i + 3 \leq |\sigma^m|$. Note that since $\sigma^m \in \mathrm{Av}(321)$ then $\mathrm{pat}_{\{i+1,i+2,i+3\}}(\sigma^m) \neq 231$, otherwise we would have $\mathrm{pat}_{\{i,i+1,i+3\}}(\sigma^m) = 321$. Note also that it is not possible to have $\mathrm{pat}_{\{i+1,i+2,i+3\}}(\sigma^m) \neq 312$ since $\sigma^m(i+1) < \sigma^m(i+2)$. Therefore if $\mathrm{pat}_I(\sigma^m) = 312$ and $i + 3 \leq |\sigma^m|$, then $\mathrm{pat}_{\{i+1,i+2,i+3\}}(\sigma^m) \in \{123, 213, 132, 321\}$. This is a contradiction with the fact that

$$\widetilde{\text{c-occ}}(312, \sigma^m) \to 1/2, \quad \text{and} \quad \widetilde{\text{c-occ}}(\pi, \sigma^m) \to 0, \quad \text{for all } \pi \in \{123, 213, 132, 321\}. \qquad \square$$

As a consequence, the feasible region $P_k^{\mathrm{Av}(\searrow_n)}$ cannot be directly described as the cycle polytope of the overlap graph $\mathcal{O}v^{\mathrm{Av}(\searrow_n)}(k)$. In Section 4 we will see that by considering a coloured version of the graph $\mathcal{O}v^{\mathrm{Av}(\searrow_n)}(k)$, denoted $\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\searrow_n)}(k)$, we can overcome this problem (see in particular Definition 4.6).

The main results for the monotone patterns case are the following two.

Theorem 1.11. *Fix $\searrow_n = n \cdots 1$ for $n \in \mathbb{Z}_{\geq 2}$. There exists a projection map Π, explicitly described in Eq. (5), such that the pattern-avoiding feasible region $P_k^{\mathrm{Av}(\searrow_n)}$ is the Π-projection of the cycle polytope of the coloured overlap graph $\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\searrow_n)}(k)$. That is,*

$$P_k^{\mathrm{Av}(\searrow_n)} = \Pi(P(\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\searrow_n)}(k)))\,.$$

An instance of the result above is depicted on the right-hand side of Fig. 2.

Theorem 1.12. *The dimension of $P_k^{\mathrm{Av}(\searrow_n)}$ is $|\mathrm{Av}_k(\searrow_n)| - |\mathrm{Av}_{k-1}(\searrow_n)|$.*

Remark 1.13. Note that Proposition 1.3 gives the dimension of the polytope $P(\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\searrow_n)}(k))$, but one needs to carefully keep track of what happens in the projection in order to determine the dimension of $P_k^{\mathrm{Av}(\searrow_n)} = \Pi(P(\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\searrow_n)}(k)))$. A technique for that is developed in Section 4, where we explicitly compute what dimensions are lost after applying the projection to the original polytope.

For more information on the numbers $A(n,k) = |\mathrm{Av}_k(\searrow_n)|$ we refer to [Slo96, A214015]. We just recall that a closed formula for these numbers is not available. However, thanks to the Robinson-Schensted correspondence, $A(n,k)$ is equal to $\sum_\lambda f_\lambda^2$, where the sum runs over all partitions λ of k with at most $n-1$ parts and f_λ is the number of standard Young tableaux with shape λ.

Note that $A(3,k) = C_k$ is the k-th Catalan number. Therefore, Theorems 1.9 and 1.12 imply that $\dim(P_k^{\mathrm{Av}(\rho)})$ does not depend on the particular permutation $\rho \in \mathcal{S}_3$.

1.4 Future projects and open questions

We present here some ideas for future projects and some open questions.

- Theorem 1.9 and Theorem 1.11 give a description of the feasible regions $P_k^{\mathrm{Av}(\tau)}$ for all patterns τ of size three. Can we describe the feasible regions $P_k^{\mathrm{Av}(B)}$ for all subsets $B \subseteq \mathcal{S}_3$? It is easy to see that $P_k^{\mathrm{Av}(B)} \subseteq \bigcap_{\tau \in B} P_k^{\mathrm{Av}(\tau)}$, but the other inclusion is not trivial and does not hold in general. We believe that it would be interesting to investigate this question.

- It seems to be the case that the feasible region $P_k^{\mathrm{Av}(B)}$ can be precisely described for other specific sets of patterns B different from the ones already considered in this paper. In particular, we believe that a good choice would be a set of (possibly generalized) patterns B for which the corresponding family $\mathrm{Av}(B)$ have been enumerated through generating trees. Indeed, the first author of this article has recently shown in [Bor20a] that generating trees behave well in the analysis of consecutive patterns of permutations in these families. We believe that generating trees would be particularly helpful to prove some analogues of Lemma 4.14 - that is the key lemma in the proof of Theorem 1.11 - for other families of permutations.
- The main open question of this article is Conjecture 1.8.

1.5 Notation

Permutations and patterns. We recall that we denoted by $\mathcal{S}_n$ the set of permutations of size n, and by $\mathcal{S}$ the set of all permutations.

If $x_1, \dots, x_n$ is a sequence of distinct numbers, let $\mathrm{std}(x_1, \dots, x_n)$ be the unique permutation π in $\mathcal{S}_n$ whose elements are in the same relative order as $x_1, \dots, x_n$, i.e. $\pi(i) < \pi(j)$ if and only if $x_i < x_j$. Given a permutation $\sigma \in \mathcal{S}_n$ and a subset of indices $I \subseteq [n]$, let $\mathrm{pat}_I(\sigma)$ be the permutation induced by $(\sigma(i))_{i\in I}$, namely, $\mathrm{pat}_I(\sigma) \coloneqq \mathrm{std}\left((\sigma(i))_{i\in I}\right)$. For example, if $\sigma = 24637185$ and $I = \{2, 4, 7\}$, then $\mathrm{pat}_{\{2,4,7\}}(24637185) = \mathrm{std}(438) = 213$. In two particular cases, we use the following more compact notation: for $k \le |\sigma|$, $\mathrm{beg}_k(\sigma) \coloneqq \mathrm{pat}_{\{1,2,\dots,k\}}(\sigma)$ and $\mathrm{end}_k(\sigma) \coloneqq \mathrm{pat}_{\{|\sigma|-k+1,|\sigma|-k+2,\dots,|\sigma|\}}(\sigma)$.

Given two permutations, $\sigma \in \mathcal{S}_n$ for some $n \in \mathbb{Z}_{\ge 1}$ and $\pi \in \mathcal{S}_k$ for some $k \le n$, and a set of indices $I = \{i_1 < \dots < i_k\}$, we say that $\sigma(i_1)\dots\sigma(i_k)$ is an *occurrence* of π in σ if $\mathrm{pat}_I(\sigma) = \pi$ (we will also say that π is a *pattern* of σ). If the indices $i_1, \dots, i_k$ form an interval, then we say that $\sigma(i_1)\dots\sigma(i_k)$ is a *consecutive occurrence* of π in σ (we will also say that π is a *consecutive pattern* of σ). We denote intervals of integers as $[n, m] = \{n, n+1, \dots, m\}$ for $n, m \in \mathbb{Z}_{\ge 1}$ with $n \le m$.

Example 1.14. The permutation $\sigma = 1532467$ contains an occurrence of 1423 (but no such consecutive occurrences) and a consecutive occurrence of 321. Indeed $\mathrm{pat}_{\{1,2,3,5\}}(\sigma) = 1423$ but no interval of indices of σ induces the permutation 1423. Moreover, $\mathrm{pat}_{[2,4]}(\sigma) = 321$.

We denote by $\mathrm{occ}(\pi, \sigma)$ the number of occurrences of a pattern π in σ, more precisely

$$\mathrm{occ}(\pi, \sigma) \coloneqq \left|\left\{I \subseteq [n] \middle| \mathrm{pat}_I(\sigma) = \pi\right\}\right|.$$

We denote by $\text{c-occ}(\pi, \sigma)$ the number of consecutive occurrences of a pattern π in σ, more precisely

$$\text{c-occ}(\pi, \sigma) \coloneqq \left|\left\{I \subseteq [n] \middle| \, I \text{ is an interval, } \mathrm{pat}_I(\sigma) = \pi\right\}\right|.$$

Moreover, we denote by $\widetilde{\mathrm{occ}}(\pi, \sigma)$ (resp. by $\widetilde{\text{c-occ}}(\pi, \sigma)$) the proportion of occurrences (resp. consecutive occurrences) of a pattern $\pi \in \mathcal{S}_k$ in $\sigma \in \mathcal{S}_n$, that is,

$$\widetilde{\mathrm{occ}}(\pi, \sigma) \coloneqq \frac{\mathrm{occ}(\pi, \sigma)}{\binom{n}{k}} \in [0, 1], \qquad \widetilde{\text{c-occ}}(\pi, \sigma) \coloneqq \frac{\text{c-occ}(\pi, \sigma)}{n} \in [0, 1].$$

Remark 1.15. The natural choice for the denominator of the expression in the right-hand side of the equation above should be $n-k+1$ and not n, but we make this choice for later convenience. Moreover, for every fixed k, there is no difference in the asymptotics when n tends to infinity.

For a fixed $k \in \mathbb{Z}_{\ge 1}$ and a permutation $\sigma \in \mathcal{S}$, we let $\widetilde{\mathrm{occ}}_k(\sigma), \widetilde{\text{c-occ}}_k(\sigma) \in [0, 1]^{\mathcal{S}_k}$ be the vectors

$$\widetilde{\mathrm{occ}}_k(\sigma) \coloneqq \left(\widetilde{\mathrm{occ}}(\pi, \sigma)\right)_{\pi\in\mathcal{S}_k}, \qquad \widetilde{\text{c-occ}}_k(\sigma) \coloneqq \left(\widetilde{\text{c-occ}}(\pi, \sigma)\right)_{\pi\in\mathcal{S}_k}.$$

We say that σ *avoids* π if σ does not contain any occurrence of π. We point out that the definition of π-avoiding permutations refers to occurrences and not to consecutive occurrences. Given a set of patterns $B \subset \mathcal{S}$, we say that σ *avoids* B if σ avoids π, for all $\pi \in B$. We denote by $\mathrm{Av}_n(B)$ the set of B-avoiding permutations of size n and by $\mathrm{Av}(B) := \bigcup_{n \in \mathbb{Z}_{\geq 1}} \mathrm{Av}_n(B)$ the set of B-avoiding permutations of arbitrary finite size. The set $\mathrm{Av}(B)$ is often called a *permutation class*.

We also introduce two classical operations on permutations. We denote with $\oplus$ the *direct sum* of two permutations, i.e. for $\tau \in \mathcal{S}_m$ and $\sigma \in \mathcal{S}_n$,

$$\tau \oplus \sigma = \tau(1) \dots \tau(m)(\sigma(1)+m) \dots (\sigma(n)+m)\,,$$

and we denote with $\oplus_\ell \sigma$ the direct sum of ℓ copies of σ (we remark that the operation $\oplus$ is associative). A similar definition holds for the *skew sum* $\ominus$,

$$\tau \ominus \sigma = (\tau(1)+n) \dots (\tau(m)+n)\sigma(1) \dots \sigma(n)\,. \tag{3}$$

We say that a permutation is $\oplus$-indecomposable (resp. $\ominus$-indecomposable) if it cannot be written as the direct sum (resp. skew-sum) of two non-empty permutations.

Directed graphs.

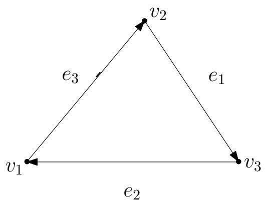
All graphs, their subgraphs and their subtrees are considered to be directed multigraphs in this paper (and we often refer to them as directed graphs or simply as graphs). In a directed multigraph $G = (V(G), E(G))$, the set of edges $E(G)$ is a multiset, allowing for loops and parallel edges. An edge $e \in E(G)$ is an oriented pair of vertices, (v, u), often denoted by $v \to u$. We write $\mathrm{s}(e)$ for the starting vertex v and $\mathrm{a}(e)$ for the arrival vertex u. We often consider directed graphs G with labelled edges, and write $\mathrm{lb}(e)$ for the label of the edge $e \in E(G)$. In a graph with labelled edges we refer to edges by using their labels. Given an edge $e \in E(G)$, we denote by $C_G(e)$ (for "set of *continuations* of e") the set of edges $e' \in E(G)$ such that $\mathrm{s}(e') = \mathrm{a}(e)$.

A *walk* of size k on a directed graph G is a sequence of k edges $(e_1, \dots, e_k) \in E(G)^k$ such that for all $i \in [k-1]$, $\mathrm{a}(e_i) = \mathrm{s}(e_{i+1})$. A walk is a *cycle* if $\mathrm{s}(e_1) = \mathrm{a}(e_k)$. A walk is a *path* if all the edges are distinct, as well as its vertices, with a possible exception that $\mathrm{s}(e_1) = \mathrm{a}(e_k)$ may happen. A cycle that is a path is called a *simple cycle*. Given two walks $w = (e_1, \dots, e_k)$ and $w' = (e'_1, \dots, e'_{k'})$ such that $\mathrm{a}(e_k) = \mathrm{s}(e'_1)$, we write $w \bullet w'$ for their concatenation, i.e. $w \bullet w' = (e_1, \dots, e_k, e'_1, \dots, e'_{k'})$. For a walk w, we denote by $|w|$ the number of *edges* in w.

Given a walk $w = (e_1, \dots, e_k)$ and an edge e, we denote by $n_e(w)$ the number of times the edge e is traversed in w, i.e. $n_e(w) := |\{i \leq k | e_i = e\}|$.

The *incidence matrix* of a directed graph G is the matrix $L(G)$ with rows indexed by $V(G)$, and columns indexed by $E(G)$, such that for any edge $e = v \to u$ with $v \neq u$, the corresponding column in $L(G)$ has $(L(G))_{v,e} = 1$, $(L(G)))_{u,e} = -1$ and is zero everywhere else. Moreover, if $e = v \to v$ is a loop, the corresponding column in $L(G)$ has zero everywhere.

For instance, we show in Fig. 3 a graph G with its incidence matrix $L(G)$.

$G =$ 

$$, \qquad L(G) = \begin{array}{c} \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} .$$

Figure 3: A graph G with its incidence matrix $L(G)$.

2 Topological properties of the pattern-avoiding feasible regions and an upper-bound on their dimensions

This section is devoted to the proof of Theorem 1.7.

Proposition 2.1. *Fix $k \in \mathbb{Z}_{\geq 1}$. For any set of patterns $B \subseteq \mathcal{S}$, the feasible region $P_k^{\mathrm{Av}(B)}$ is a closed set.*

This is a classical consequence of the fact that $P_k^{\mathrm{Av}(B)}$ is a set of limit points. For completeness, we include a simple proof of the statement. Recall that we defined $\widetilde{\text{c-occ}}_k(\sigma) := \big(\widetilde{\text{c-occ}}(\pi,\sigma)\big)_{\pi\in\mathcal{S}_k}$.

Proof. It suffices to show that, for any sequence $(\vec{v}_s)_{s\in\mathbb{Z}_{\geq 1}}$ in $P_k^{\mathrm{Av}(B)}$ such that $\vec{v}_s \to \vec{v}$ for some $\vec{v} \in [0,1]^{\mathcal{S}_k}$, we have that $\vec{v} \in P_k^{\mathrm{Av}(B)}$. For all $s \in \mathbb{Z}_{\geq 1}$, consider a sequence of permutations $(\sigma_s^m)_{m\in\mathbb{Z}_{\geq 1}} \in \mathrm{Av}(B)^{\mathbb{Z}_{\geq 1}}$ such that $|\sigma_s^m| \xrightarrow{m\to\infty} \infty$ and $\widetilde{\text{c-occ}}_k(\sigma_s^m) \xrightarrow{m\to\infty} \vec{v}_s$, and some index $m(s)$ of the sequence $(\sigma_s^m)_{m\in\mathbb{Z}_{\geq 1}}$ such that for all $m \geq m(s)$,

$$|\sigma_s^m| \geq s \quad \text{and} \quad ||\widetilde{\text{c-occ}}_k(\sigma_s^m) - \vec{v}_s||_2 \leq \tfrac{1}{s}\,.$$

Without loss of generality, assume that $m(s)$ is increasing. For every $\ell \in \mathbb{Z}_{\geq 1}$, define $\sigma^\ell := \sigma_\ell^{m(\ell)}$. It is easy to show that

$$|\sigma^\ell| \xrightarrow{\ell\to\infty} \infty \quad \text{and} \quad \widetilde{\text{c-occ}}_k(\sigma^\ell) \xrightarrow{\ell\to\infty} \vec{v}\,,$$

where we use the fact that $\vec{v}_s \to \vec{v}$. Furthermore, by assumption we have that $\sigma^\ell \in \mathrm{Av}(B)$. Therefore $\vec{v} \in P_k^{\mathrm{Av}(B)}$. □

The following result is an analogue of [BP19, Proposition 3.2], where it was proved that the feasible region P_k is convex.

Proposition 2.2. *Fix $k \in \mathbb{Z}_{\geq 1}$. Consider a set of patterns $B \subset \mathcal{S}$ such that the class $\mathrm{Av}(B)$ is closed for one of the two operations $\oplus, \ominus$. Then, the feasible region $P_k^{\mathrm{Av}(B)}$ is convex.*

In particular, $P_k^{\mathrm{Av}(\tau)}$ is convex for any pattern $\tau \in \mathcal{S}$.

Proof. We will present a proof for the case where $\mathrm{Av}(B)$ is closed for the $\oplus$ operation, however the arguments hold equally for the $\ominus$ operation.

Since $P_k^{\mathrm{Av}(B)}$ is a closed set (by Proposition 2.1) it is enough to consider rational convex combinations of points in $P_k^{\mathrm{Av}(B)}$, i.e. it is enough to establish that for all $\vec{v}_1, \vec{v}_2 \in P_k^{\mathrm{Av}(B)}$ and all $s, t \in \mathbb{Z}_{\geq 1}$, we have that

$$\frac{s}{s+t}\vec{v}_1 + \frac{t}{s+t}\vec{v}_2 \in P_k^{\mathrm{Av}(B)}.$$

Fix $\vec{v}_1, \vec{v}_2 \in P_k^{\mathrm{Av}(B)}$ and $s, t \in \mathbb{Z}_{\geq 1}$. Since $\vec{v}_1, \vec{v}_2 \in P_k^{\mathrm{Av}(B)}$, there exist two sequences $(\sigma_1^\ell)_{\ell\in\mathbb{Z}_{\geq 1}}$, $(\sigma_2^\ell)_{\ell\in\mathbb{Z}_{\geq 1}}$ such that $|\sigma_i^\ell| \xrightarrow{\ell\to\infty} \infty$, $\sigma_i^\ell \in \mathrm{Av}(B)$ and $\widetilde{\text{c-occ}}_k(\sigma_i^\ell) \xrightarrow{\ell\to\infty} \vec{v}_i$, for $i = 1, 2$.

Define $t_\ell := t \cdot |\sigma_1^\ell|$ and $s_\ell := s \cdot |\sigma_2^\ell|$.

We set $\tau^\ell := \big(\oplus_{s_\ell}\, \sigma_1^\ell\big) \oplus \big(\oplus_{t_\ell}\, \sigma_2^\ell\big)$. For a graphical interpretation of this construction we refer to Fig. 4. We note that for every $\pi \in \mathcal{S}_k$, we have

$$\text{c-occ}(\pi, \tau^\ell) = s_\ell \cdot \text{c-occ}(\pi, \sigma_1^\ell) + t_\ell \cdot \text{c-occ}(\pi, \sigma_2^\ell) + Er,$$

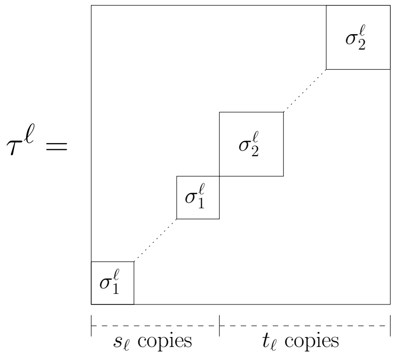


Figure 4: Schema for the definition of the permutation τ^ℓ.

where $Er \leq (s_\ell + t_\ell - 1) \cdot |\pi|$. This error term comes from the number of intervals of size $|\pi|$ that intersect the boundary of some copies of σ_1^ℓ or σ_2^ℓ. Hence

$$\widetilde{\text{c-occ}}(\pi, \tau^\ell) = \frac{s_\ell \cdot |\sigma_1^\ell| \cdot \widetilde{\text{c-occ}}(\pi, \sigma_1^\ell) + t_\ell \cdot |\sigma_2^\ell| \cdot \widetilde{\text{c-occ}}(\pi, \sigma_2^\ell) + Er}{s_\ell \cdot |\sigma_1^\ell| + t_\ell \cdot |\sigma_2^\ell|}$$
$$= \frac{s}{s+t}\widetilde{\text{c-occ}}(\pi, \sigma_1^\ell) + \frac{t}{s+t}\widetilde{\text{c-occ}}(\pi, \sigma_2^\ell) + O\left(|\pi| \left(\tfrac{1}{|\sigma_1^\ell|} + \tfrac{1}{|\sigma_2^\ell|}\right)\right).$$

As ℓ tends to infinity, we have

$$\widetilde{\text{c-occ}}_k(\tau^\ell) \to \frac{s}{s+t}\vec{v}_1 + \frac{t}{s+t}\vec{v}_2,$$

since $|\sigma_i^\ell| \xrightarrow{\ell \to \infty} \infty$ and $\widetilde{\text{c-occ}}_k(\sigma_i^\ell) \xrightarrow{m \to \infty} \vec{v}_i$, for $i = 1, 2$. Noting also that $|\tau^\ell| \to \infty$, we can conclude that $\frac{s}{s+t}\vec{v}_1 + \frac{t}{s+t}\vec{v}_2 \in P_k^{\text{Av}(B)}$. This ends the proof of the first part of the statement.

The second part of the statement follows from the fact that $\text{Av}(\tau)$ is either closed for the $\oplus$ operation (whenever τ is $\oplus$-indecomposable) or closed for the $\ominus$ operation (whenever τ is $\ominus$-indecomposable). $\square$

Proposition 2.3. *Fix $k \in \mathbb{Z}_{\geq 1}$. For any set of patterns $B \subset \mathcal{S}$, we have that*

$$P_k^{\text{Av}(B)} \subseteq P(\mathcal{O}v^{\text{Av}(B)}(k)) \subseteq P_k.$$

Recall that the map W_k associating a walk in $\mathcal{O}v(k)$ with every permutation was defined at the end of Section 1.2.

Proof. We start by proving the first inclusion. Consider any point $\vec{v} \in P_k^{\text{Av}(B)}$, and a corresponding sequence $\left(\sigma^\ell\right)_{\ell \geq 0} \in \text{Av}(B)^{\mathbb{Z}_{\geq 0}}$ such that $\widetilde{\text{c-occ}}_k(\sigma^\ell) \to \vec{v}$. Because $\sigma^\ell \in \text{Av}(B)$, we know that for each ℓ, $W_k(\sigma^\ell)$ is a walk in $\mathcal{O}v^{\text{Av}(B)}(k)$. Using the same method as in the proof of $P_k \subseteq P(\mathcal{O}v(k))$ in [BP19, Theorem 3.12], we can deduce that $\widetilde{\text{c-occ}}_k(\sigma^\ell)$ converges to a point in $P(\mathcal{O}v^{\text{Av}(B)}(k))$, and so $\vec{v} \in P(\mathcal{O}v^{\text{Av}(B)}(k))$. Because $\vec{v}$ is generic, it follows that $P_k^{\text{Av}(B)} \subseteq P(\mathcal{O}v^{\text{Av}(B)}(k))$.

The second inclusion follows from the definition of $\mathcal{O}v^{\text{Av}(B)}(k)$ and Theorem 1.4. $\square$

Proposition 2.4. *Fix $k \in \mathbb{Z}_{\geq 1}$ and a set of patterns $B \subset \mathcal{S}$ such that the class $\text{Av}(B)$ is closed for one of the two operations $\oplus, \ominus$. Then the graph $\mathcal{O}v^{\text{Av}(B)}(k)$ is strongly connected and* $\dim(P(\mathcal{O}v^{\text{Av}(B)}(k))) = |\text{Av}_k(B)| - |\text{Av}_{k-1}(B)|$.

In particular, this holds for $P_k^{\text{Av}(\tau)}$ for any pattern $\tau \in \mathcal{S}$.

Proof. Consider v_1, v_2 two vertices of $\mathcal{O}v^{\mathrm{Av}(B)}(k)$, and assume that $\mathrm{Av}(B)$ is closed for $\oplus$, for simplicity. Then $\mathrm{lb}(v_1)\oplus\mathrm{lb}(v_2)$ is a permutation in $\mathrm{Av}(B)$, so $W_k(\mathrm{lb}(v_1)\oplus\mathrm{lb}(v_2))$ is a walk in the graph $\mathcal{O}v^{\mathrm{Av}(B)}(k)$ that connects v_1 to v_2. We conclude that $\mathcal{O}v^{\mathrm{Av}(B)}(k)$ is strongly connected.

It follows from Proposition 1.3 that $\dim(P(\mathcal{O}v^{\mathrm{Av}(B)}(k))) = |\mathrm{Av}_k(B)| - |\mathrm{Av}_{k-1}(B)|$. □

Note that Propositions 2.1, 2.2, 2.3, 2.4 imply Theorem 1.7.

3 The feasible region for 312-avoiding permutations

This section is devoted to the proof of Theorem 1.9. The key step in this proof is to show an analogue of Lemma 1.5 for 312-avoiding permutations. More precisely, we have the following.

Lemma 3.1. *Fix $k \in \mathbb{Z}_{\geq 1}$ and $m \geq k$. The map W_k, from the set $\mathrm{Av}_m(312)$ of permutations of size m to the set of walks in $\mathcal{O}v^{\mathrm{Av}(312)}(k)$ of size $m-k+1$, is surjective.*

To prove the lemma above we have to introduce the following.

Definition 3.2. Given a permutation $\sigma \in \mathcal{S}_n$ and an integer $m \in [n+1]$, we denote by σ^{*m} the permutation obtained from σ by *appending a new final value* equal to m and shifting by $+1$ all the other values larger than or equal to m. Equivalently,

$$\sigma^{*m} := \mathrm{std}(\sigma(1), \dots, \sigma(n), m - 1/2).$$

The proof of Lemma 3.1 is based on the following result. Recall the definition of the set $C_G(e)$ of continuations of an edge e in a graph G, i.e. the set of edges $e' \in E(G)$ such that $\mathrm{s}(e') = \mathrm{a}(e)$.

Lemma 3.3. *Let σ be a permutation in $\mathrm{Av}(312)$ such that $\mathrm{end}_k(\sigma) = \pi$ for some $\pi \in \mathrm{Av}_k(312)$. Let $\pi' \in \mathrm{Av}_k(312)$ such that $\pi' \in C_{\mathcal{O}v^{\mathrm{Av}(312)}(k)}(\pi)$. Then there exists $m \in [|\sigma|+1]$ such that $\sigma^{*m} \in \mathrm{Av}(312)$ and $\mathrm{end}_k(\sigma^{*m}) = \pi'$.*

We first explain how Lemma 3.1 follows from Lemma 3.3 and then we prove the latter.

Proof of Lemma 3.1. In order to prove the claimed surjectivity, given a walk $w = (e_1, \dots, e_s)$ in $\mathcal{O}v^{\mathrm{Av}(312)}(k)$, we have to exhibit a permutation $\sigma \in \mathrm{Av}(312)$ of size $s+k-1$ such that $W_k(\sigma) = w$. We do that by constructing a sequence of s permutations $(\sigma_i)_{i\leq s} \in (\mathrm{Av}(312))^s$ with size $|\sigma_i| = i+k-1$, in such a way that σ is equal to σ_s. Moreover, we will have that $\mathrm{beg}_{|\sigma_{i+1}|-1}(\sigma_{i+1}) = \sigma_i$.

The first permutation is defined as $\sigma_1 = \mathrm{lb}(e_1)$. To construct σ_{i+1} from σ_i, note that from Lemma 3.3 there exists $\ell \in [|\sigma_i|+1]$ such that $\mathrm{end}_k(\sigma_i^{*\ell})$ is equal to the pattern $\mathrm{lb}(e_{i+1})$ and $\sigma_i^{*\ell}$ avoids the pattern 312. Then we define $\sigma_{i+1} := \sigma_i^{*\ell}$, determining the sequence $(\sigma_i)_{i\leq s} \in (\mathrm{Av}(312))^s$. Finally, setting $\sigma := \sigma_s$ we have by construction that $W_k(\sigma) = w$ and that $\sigma \in \mathrm{Av}_{s+k-1}(312)$. □

Proof of Lemma 3.3. We have to distinguish two cases.

Case 1: $\pi'(k) \in \{1, k\}$. We define $h := \mathbb{1}_{\{\pi'(k)=1\}} + (|\sigma|+1)\mathbb{1}_{\{\pi'(k)=k\}}$. In this case one can see that $\sigma^{*h} \in \mathrm{Av}(312)$ – the new final value h cannot create an occurrence of 312 in σ^{*h} – and that $\mathrm{end}_k(\sigma^{*h}) = \pi'$.

Case 2: $\pi'(k) \in [2, k-1]$. Consider the point just above $(k, \pi'(k))$ in the diagram of π' and the corresponding point in the last $k-1$ points of σ (for an example see the two red points in Fig. 5). Let i be the index in the diagram of σ of the latter point. We claim that $\sigma^{*\sigma(i)} \in \mathrm{Av}(312)$ and $\mathrm{end}_k(\sigma^{*\sigma(i)}) = \pi'$. The latter is immediate. It just remains to show that $\sigma' := \sigma^{*\sigma(i)} \in \mathrm{Av}(312)$.

Assume by contradiction that σ' contains an occurrence of 312. Since by assumption $\sigma \in \mathrm{Av}(312)$ then the value 2 of the occurrence 312 must correspond to the final value $\sigma'(|\sigma'|) = \sigma(i)$

of σ'. Moreover, since $\pi' \in \mathrm{Av}(312)$, the 312-occurrence cannot occur in the last k elements of σ', that is the 312-occurrence must occur at the values $\sigma'(j), \sigma'(r), \sigma'(|\sigma'|)$ for some indices $j \leq |\sigma'|-k$ and $j < r < |\sigma'|$. Because $\sigma'(j), \sigma'(r), \sigma'(|\sigma'|)$ is an occurence of 312, $\sigma'(j) > \sigma'(|\sigma'|)$. Moreover, since $\sigma'(i) = \sigma'(|\sigma'|) + 1$ by construction, it follows that $\sigma'(j) > \sigma'(i)$. Note that $r \neq i$ since $\sigma'(i) = \sigma'(|\sigma'|) + 1$ and $\sigma'(r) < \sigma'(|\sigma'|)$. Therefore, we have two cases:

- If $r < i$ then $\sigma'(j), \sigma'(r), \sigma'(i)$ is also an occurrence of 312. A contradiction to the fact that $\sigma \in \mathrm{Av}(312)$.
- If $r > i$ then $\sigma'(i), \sigma'(r), \sigma'(|\sigma'|)$ is also an occurrence of 312. A contradiction to the fact that $\pi' \in \mathrm{Av}(312)$.

This concludes the proof. □

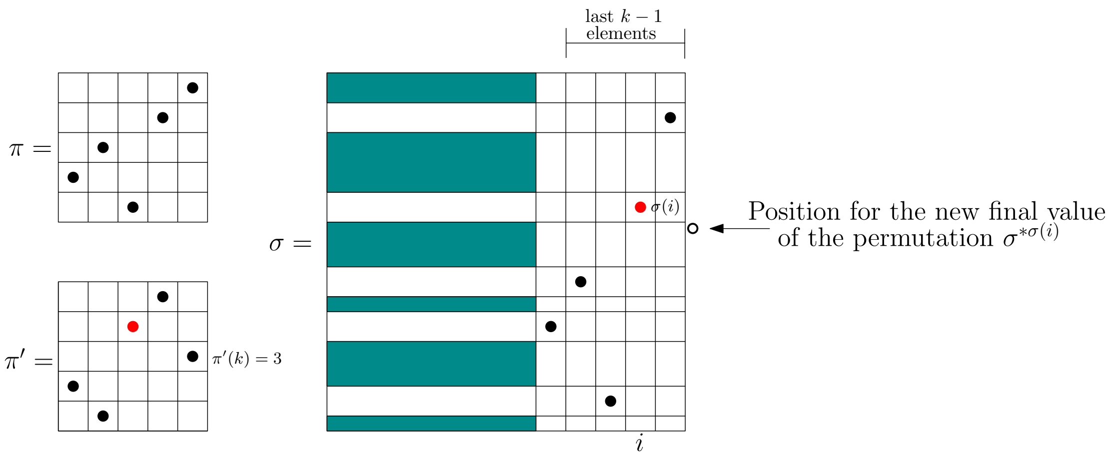


Figure 5: A schema for the proof of Lemma 3.3.

Building on Proposition 2.2 and Lemma 3.1 we can now prove Theorem 1.9.

Proof of Theorem 1.9. The fact that $P_k^{\mathrm{Av}(312)} = P(\mathcal{O}v^{\mathrm{Av}(312)}(k))$ follows using exactly the same proof of [BP19, Theorem 3.12] replacing Lemma 3.8 and Proposition 3.2 of [BP19] by Lemma 3.1 and Proposition 2.2 of this paper (note that in the proof of [BP19, Theorem 3.12] we also use the fact that the feasible region is closed and this is still true for $P_k^{\mathrm{Av}(312)}$, thanks to Proposition 2.1).

The fact that the dimension of $P_k^{\mathrm{Av}(312)}$ is $C_k - C_{k-1}$ follows from Proposition 2.4 and the well-known fact that the number of permutations of size k avoiding the pattern 312 is equal to the k-th Catalan number. Finally the fact that the vertices of $P_k^{\mathrm{Av}(312)}$ are given by the simple cycles of $\mathcal{O}v^{\mathrm{Av}(312)}(k)$ is a consequence of [BP19, Proposition 2.2]. □

4 The feasible region for monotone-avoiding permutations

Fix $\searrow_n = n \cdots 1$, the decreasing pattern of size $n \in \mathbb{Z}_{\geq 1}$. In this section we study $P_k^{\mathrm{Av}(\searrow_n)}$ and we show that it is related to the cycle polytope of the coloured overlap graph $\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\searrow_n)}(k)$, presented in Definition 4.6 – this is Theorem 1.11, more precisely stated in Theorem 4.8. We also compute the dimension of $P_k^{\mathrm{Av}(\searrow_n)}$ – this is Theorem 1.12.

4.1 Definitions and combinatorial constructions

We start by introducing colourings of permutations.

Definition 4.1 (Colourings and RITMO colourings)**.** Fix an integer $m \in \mathbb{Z}_{\geq 1}$. For a permutation σ, an m-colouring of σ is a map $\mathfrak{c} : [|\sigma|] \to [m]$, which is to be interpreted as a map from the set of *indices* of σ to $[m]$. An m-colouring $\mathfrak{c}$ is said to be *rainbow* when $\mathrm{im}(\mathfrak{c}) = [m]$. For any permutation σ, we define its *right-top monotone colouring* (simply RITMO colouring henceforth), which we denote as $\mathbb{C}(\sigma)$. This colouring is constructed iteratively, starting with the highest value of the permutation which receives the colour 1 and going down while assigning the lowest possible colour that prevents the occurrence of a monochromatic 21.

If a permutation is coloured with its RITMO colouring, the left-to-right maxima are coloured by 1, removing these left-to-right maxima, the left-to-right maxima of the resulting set of points are coloured by 2, and so on. We suggest to the reader to keep in mind both points of view (the one given in the definition and the one described now) on RITMO colourings.

Example 4.2. One can see in Fig. 6 an example of the RITMO colourings for permutations 312, 1427536 and 124376985. In all our examples, we paint in red the values coloured by one, in blue the ones coloured by two, and in green the ones coloured by three.

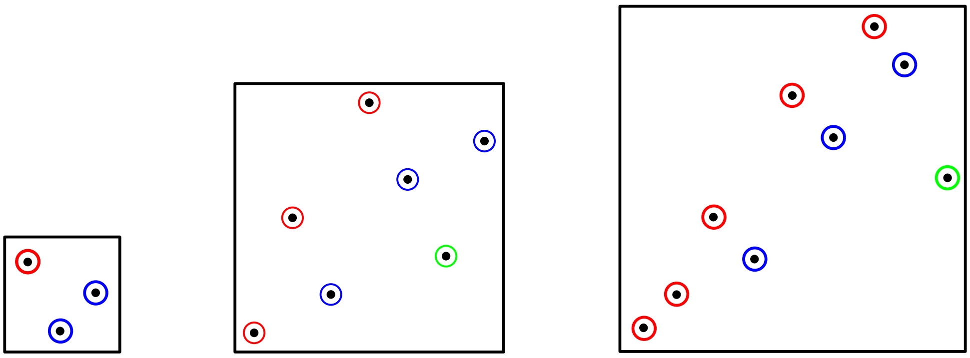


Figure 6: The RITMO colourings of 312, 1427536 and 124376985.

For the pair $(\sigma, \mathbb{C}(\sigma))$ we simply write $\mathbb{S}(\sigma)$. If σ avoids the permutation $\searrow_n$, it is known that its RITMO colouring is an $(n-1)$-colouring (the origins of this result are hard to trace, but it goes back at least to [Gre74] where it is already noted as something that is not hard to prove; see also [Bón12, Chapter 4.3]).

We furthermore allow for taking *restrictions of colourings*. Given a permutation σ of size k, a colouring $\mathfrak{c}$ of σ and a subset $I = \{i_1, \dots, i_j\} \subseteq [k]$, we consider the restriction $\mathrm{pat}_I(\sigma, \mathfrak{c})$ to be the pair $(\mathrm{pat}_I(\sigma), \mathfrak{c}')$, where $\mathfrak{c}'(\ell) = \mathfrak{c}(i_\ell)$ for all $\ell \in [j]$.

Observe that it may be the case that $\mathrm{pat}_I(\mathbb{S}(\sigma))$ and $\mathbb{S}(\mathrm{pat}_I(\sigma))$ are distinct inherited colourings of the permutation $\mathrm{pat}_I(\sigma)$. For instance, if $\sigma = 2134$ and $I = \{2, 3, 4\}$ then $\mathrm{pat}_I(\mathbb{S}(\sigma)) = \mathrm{pat}_{\{2,3,4\}}(2134) = 123$ but $\mathbb{S}(\mathrm{pat}_I(\sigma)) = \mathbb{S}(123) = 123$. This is unlike the relation between $\mathbb{S}$ and beg, as one can see in Observation 4.5. The following definition is fundamental in our results.

Definition 4.3. We say that an m-colouring $\mathfrak{c}$ of a permutation $\pi \in \mathrm{Av}(\searrow_n)$ of size k is *inherited* if there is some permutation $\sigma \in \mathrm{Av}(\searrow_n)$ of size $\ell \geq k$ such that $\mathrm{end}_k(\mathbb{S}(\sigma)) = (\pi, \mathfrak{c})$.

To sum up, we have introduced three notions of colourings, each more restricted than the previous one. In particular, any inherited colouring is a colouring, and the RITMO colouring is an inherited colouring.

Let $\mathcal{C}_m(\pi)$ be the set of all inherited m-colourings of a permutation $\pi \in \mathrm{Av}(\searrow_n)$. We also set $\mathcal{C}_m(k) = \{(\pi, \mathfrak{c}) | \pi \in \mathrm{Av}_k(\searrow_n),\ \mathfrak{c}$ is an inherited m-colouring of $\pi\}$, that is the set of all inherited m-colourings of permutations of size k.

Example 4.4. Let $n = 3$. In Table 1 we present all the inherited 2-colourings of permutations of size three. Thus,

$$\mathcal{C}_2(3) = \{123, 123, 123, 123, 132, 132, 213, 231, 312\} .$$

123	$123 = \mathbb{S}(123)$, $123 = \text{end}_3(2134)$, $123 = \text{end}_3(3124)$, $123 = \text{end}_3(4123)$
132	$132 = \mathbb{S}(132)$, $132 = \text{end}_3(3142)$
213	$213 = \mathbb{S}(213)$
231	$231 = \mathbb{S}(231)$
312	$312 = \mathbb{S}(312)$

Table 1: The permutations of size three, and their corresponding inherited 2-colourings. Note that all permutations of size four in this table are coloured according to their RITMO colouring. Observe also that the coloured permutation 213 is not inherited.

The following simple result is a key step for the next definition.

Observation 4.5. For all permutations $\sigma \in \text{Av}(\searrow_n)$ and all $j \leq |\sigma|$, we have that

$$\text{beg}_j(\mathbb{S}(\sigma)) = \mathbb{S}(\text{beg}_j(\sigma)) .$$

Definition 4.6. The *coloured overlap graph* $\mathfrak{C}\mathcal{O}v^{Av(\searrow_n)}(k)$ is defined with the vertex set

$$V := \mathcal{C}_{n-1}(k-1) = \{(\pi, \mathfrak{c}) | \pi \in \text{Av}_{k-1}(\searrow_n),\ \mathfrak{c} \text{ is an inherited } (n-1)\text{-colouring of } \pi\},$$

and the edge set

$$E := \mathcal{C}_{n-1}(k) = \{(\pi, \mathfrak{c}) | \pi \in \text{Av}_k(\searrow_n),\ \mathfrak{c} \text{ is an inherited } (n-1)\text{-colouring of } \pi\} ,$$

where the edge $(\pi, \mathfrak{c})$ connects $v_1 \to v_2$ with $v_1 = \text{beg}_{k-1}(\pi, \mathfrak{c})$ and $v_2 = \text{end}_{k-1}(\pi, \mathfrak{c})$.

In Fig. 7 we present the coloured overlap graph corresponding to $k = 3$ and $n = 3$.

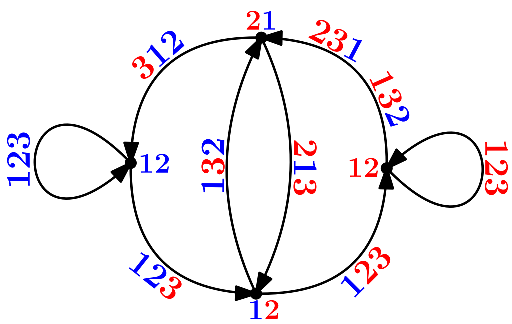


Figure 7: The coloured overlap graph for $k = 3$ and $n = 3$, that also appears in the right-hand side of Fig. 2. Note that in order to obtain a clearer picture we do not draw multiple edges, but we use multiple labels (for example the edge $12 \to 21$ is labeled with the permutations 231 and 132 and should be thought of as two distinct edges labeled with 231 and 132 respectively).

Lemma 4.7. *The coloured overlap graph is well-defined, i.e. that for any edge* $(\pi, \mathfrak{c}) \in \mathcal{C}_{n-1}(k)$, *then both* $\text{beg}_{k-1}(\pi, \mathfrak{c}) \in \mathcal{C}_{n-1}(k-1)$ *and* $\text{end}_{k-1}(\pi, \mathfrak{c}) \in \mathcal{C}_{n-1}(k-1)$.

Proof. We can equivalently show that given an inherited $(n-1)$-colouring $(\pi, \mathfrak{c})$ of size k, both $\text{beg}_{k-1}(\pi, \mathfrak{c})$ and $\text{end}_{k-1}(\pi, \mathfrak{c})$ are inherited $(n-1)$-colourings of size $k-1$.

By definition of inherited colouring, there exists $\sigma \in \mathrm{Av}(\searrow_n)$ such that $\mathrm{end}_k(\mathbb{S}(\sigma)) = (\pi, \mathfrak{c})$. Then, naturally, we have that $\mathrm{end}_{k-1}(\mathbb{S}(\sigma)) = \mathrm{end}_{k-1}(\pi, \mathfrak{c})$, and therefore $\mathrm{end}_{k-1}(\pi, \mathfrak{c}) \in \mathcal{C}_{n-1}(k-1)$. On the other hand, from Observation 4.5 we have that

$$\mathrm{beg}_{k-1}(\pi, \mathfrak{c}) = \mathrm{beg}_{k-1}(\mathrm{end}_k(\mathbb{S}(\sigma))) = \mathrm{end}_{k-1}(\mathrm{beg}_{|\sigma|-1}(\mathbb{S}(\sigma))) \overset{4.5}{=} \mathrm{end}_{k-1}(\mathbb{S}(\mathrm{beg}_{|\sigma|-1}(\sigma)))\,, \quad (4)$$

and so $\mathrm{beg}_{k-1}(\pi, \mathfrak{c}) \in \mathcal{C}_{n-1}(k-1)$. $\square$

We can now give a more precise formulation of Theorem 1.11. We denote by δ the Kronecker delta function.

Theorem 4.8. *Let Π be the projection map*

$$\Pi : \mathbb{R}^{\mathcal{C}_{n-1}(k)} \to \mathbb{R}^{\mathrm{Av}_k(\searrow_n)} \quad (5)$$

that sends the basis elements $(\delta_{(\pi,\mathfrak{c})}(x))_{x \in \mathcal{C}_{n-1}(k)}$ to $(\delta_\pi(x))_{x \in \mathrm{Av}_k(\searrow_n)}$, i.e. the map that "forgets" colourings.

In this way, the pattern-avoiding feasible region $P_k^{\mathrm{Av}(\searrow_n)}$ is the Π-projection of the cycle polytope of the overlap graph $\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\searrow_n)}(k)$. That is,

$$P_k^{\mathrm{Av}(\searrow_n)} = \Pi(P(\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\searrow_n)}(k)))\,.$$

4.2 The feasible region is the projection of the cycle polytope of the coloured overlap graph

To prove Theorem 4.8, we start by recalling that $P_k^{\mathrm{Av}(\searrow_n)}$ is a convex set, as established in Proposition 2.2. Thus we only need to describe its extremal points. The proof that these are given by the simple cycles of $\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\searrow_n)}(k)$ is split into two parts. We first establish that, for any vertex $\vec{v} \in P(\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\searrow_n)}(k))$, its projection $\Pi(\vec{v})$ is in the feasible region. To this end, we construct a walk map $\mathfrak{C}W_k^{\mathrm{Av}(\searrow_n)}$ (see Definition 4.9 below) that transforms a permutation $\sigma \in \mathrm{Av}(\searrow_n)$ into a walk on the graph $\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\searrow_n)}(k)$. Secondly, we see via a factorization theorem that any point in the feasible region results from a sequence of walks in $\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\searrow_n)}(k)$ that can be asymptotically decomposed into simple cycles; so the feasible region must be in the convex hull of the vectors given by simple cycles.

Definition 4.9 (The coloured walk function)**.** Let σ be a permutation in $\mathrm{Av}_m(\searrow_n)$. The walk $\mathfrak{C}W_k^{\mathrm{Av}(\searrow_n)}(\sigma)$ is the walk of size $m-k+1$ on $\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\searrow_n)}(k)$ given by:

$$\Big(\mathrm{pat}_{\{1,\dots,k\}}(\mathbb{S}(\sigma)), \dots, \mathrm{pat}_{\{m-k+1,\dots,m\}}(\mathbb{S}(\sigma))\Big)\,,$$

where we recall that $\mathbb{S}(\sigma) = (\sigma, \mathbb{C}(\sigma))$ and $\mathbb{C}(\sigma)$ is the RITMO colouring of σ, presented in Definition 4.1.

Remark 4.10. Given a permutation σ that avoids $\searrow_n$, each of the restrictions

$$\mathrm{pat}_{\{\ell-k+1,\dots,\ell\}}(\mathbb{S}(\sigma)), \quad \text{for all} \quad \ell \in [k, m],$$

is an inherited $(n-1)$-colouring. The fact that these are $(n-1)$-colourings follows because σ avoids $\searrow_n$, and the fact that these are inherited colourings follows from Observation 4.5 after computations similar to Eq. (4).

Example 4.11. We present the walk $\mathfrak{C}W_k^{\mathrm{Av}(\searrow_n)}(\sigma)$ corresponding to the permutation $\sigma = 1243756$, for $k=3$ and $n=3$. The RITMO colouring of σ is 1243756, and the corresponding walk is

$$(123, 132, 213, 132, 312)\,.$$

We can see in Fig. 8 this walk highlighted on the coloured overlap graph $\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(321)}(3)$.

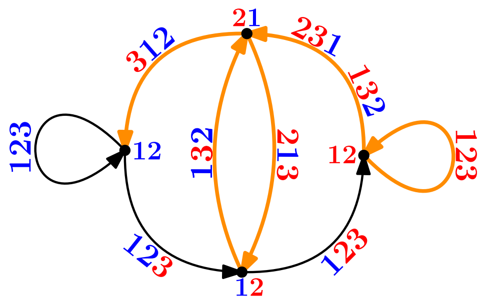


Figure 8: The walk $\mathfrak{C}W_3^{\mathrm{Av}(321)}(1243756)$ in the coloured overlap graph $\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(321)}(3)$ is highlighted in orange.

The following preliminary lemma is fundamental for the proof of Theorem 4.8.

Lemma 4.12. *There exists a constant $C = C(k,n)$ such that, for any walk $w = (e_1, \dots, e_j)$ in $\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\searrow_n)}(k)$ there exists a walk w' in $\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\searrow_n)}(k)$ of length $|w'| \leq C$ and a permutation σ of size $j + k - 1 + |w'|$ that satisfies $\mathfrak{C}W_k^{\mathrm{Av}(\searrow_n)}(\sigma) = w' \bullet w$.*

Remark 4.13. Note that, heuristically speaking, Lemma 4.12 states that the map $\mathfrak{C}W_k^{\mathrm{Av}(\searrow_n)}$ is "almost" surjective.

In the same spirit of the proof of Lemma 3.1, in order to prove Lemma 4.12 we need the following result (whose proof is postponed to Section 4.4). Recall the definition of the set $C_G(e)$ of continuations of an edge e in a graph G, i.e. the set of edges $e' \in E(G)$ such that $\mathrm{s}(e') = \mathrm{a}(e)$.

Lemma 4.14. *Let σ be a permutation in $\mathrm{Av}(\searrow_n)$ such that $\mathrm{end}_k(\mathbb{S}(\sigma)) = (\pi, \mathfrak{c})$ for some $(\pi, \mathfrak{c}) \in E(\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\searrow_n)}(k))$. Assume that $\mathbb{C}(\sigma)$ is a rainbow colouring. Let also $(\pi', \mathfrak{c}') \in C_{\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\searrow_n)}(k)}(\pi, \mathfrak{c})$. Then there exists $\iota \in [|\sigma| + 1]$ such that $\mathrm{end}_k(\mathbb{S}(\sigma^{*\iota})) = (\pi', \mathfrak{c}')$.*

Proof of Lemma 4.12. We start by defining the desired constant $C = C(k,n)$. Recall that the edges of the coloured overlap graph $\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\searrow_n)}(k)$ are inherited colourings of permutations. Therefore, for each edge $e = (\pi, \mathfrak{c}) \in E(\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\searrow_n)}(k))$ we can choose σ_e, one among the smallest $\searrow_n$-avoiding permutations such that $(\pi, \mathfrak{c}) = \mathrm{end}_k(\mathbb{S}(\sigma_e))$. Define $C(k,n) := \max_{e \in E(\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\searrow_n)}(k))} |\sigma_e| + n - k - 1$. We claim that this is the desired constant.

We will prove a stronger version of the lemma, by constructing a permutation $\sigma \in \mathrm{Av}(\searrow_n)$ such that $\mathbb{C}(\sigma)$ is a rainbow $(n-1)$-colouring and $\mathfrak{C}W_k^{\mathrm{Av}(\searrow_n)}(\sigma) = w' \bullet w$ for some walk w' bounded as above. This will be proven by induction on the length of the walk $j = |w|$.

We first consider the case $j = 1$. In this case, the walk $w = (e_1)$ has a unique edge, and we can select $\sigma = (n-1) \cdots 1 \oplus \sigma_{e_1}$. In this way, it is clear that $\mathbb{C}(\sigma)$ is a rainbow $(n-1)$-colouring, because σ has a monotone decreasing subsequence of size $n-1$, while it is clearly $\searrow_n$-avoiding. Furthermore, because $\mathrm{end}_k(\mathbb{S}(\sigma)) = \mathrm{end}_k(\mathbb{S}(\sigma_{e_1})) = e_1$, we have that $\mathfrak{C}W_k^{\mathrm{Av}(\searrow_n)}(\sigma) = w' \bullet e_1$ for some path w' such that $|w' \bullet e_1| = |w'| + 1 = |\sigma| - k + 1 = |\sigma_{e_1}| + n - k$. Therefore we have that $|w'| = |\sigma_{e_1}| + n - 1 - k \leq C$, concluding the base case.

We now consider the case $j \geq 2$. Take a walk $w = (e_1, \dots, e_j)$ in $\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\searrow_n)}(k)$, and consider (by induction hypothesis) the permutation σ such that $\mathfrak{C}W_k^{\mathrm{Av}(\searrow_n)}(\sigma) = w' \bullet (e_1, \dots, e_{j-1})$ for some walk w' of size at most C and such that $\mathbb{C}(\sigma)$ is a rainbow $(n-1)$-colouring.

From Lemma 4.14, we can find a value $\iota \in [|\sigma| + 1]$ such that $\mathrm{end}_k(\mathbb{S}(\sigma^{*\iota})) = e_j$. If so, the colouring $\mathbb{C}(\sigma^{*\iota})$ is clearly a rainbow $(n-1)$-colouring (hence, $\sigma^{*\iota} \in \mathrm{Av}(\searrow_n)$). Furthermore, we have that $\mathfrak{C}W_k^{\mathrm{Av}(\searrow_n)}(\sigma^{*\iota}) = w' \bullet (e_1, \dots, e_{j-1}, e_j)$, concluding the induction step, as $|w'| \leq C$ by hypothesis. □

We can now prove the main result of this section.

Proof of Theorem 4.8. Let $\sigma \in \operatorname{Av}(\searrow_n)$. Let us first establish a formula for $\widetilde{\text{c-occ}}_k(\sigma)$ with respect to the walk $\mathfrak{C}W_k^{\operatorname{Av}(\searrow_n)}(\sigma)$. Given a permutation ρ with a colouring $\mathfrak{c}$ we set $\operatorname{per}(\rho, \mathfrak{c}) = \rho$. Given a walk w in $\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\searrow_n)}(k)$ and a permutation π, define $[\pi : w]$ as the number of edges e in w such that $\operatorname{per}(e) = \pi$. Thus, it easily follows that

$$\widetilde{\text{c-occ}}_k(\sigma) = \frac{1}{|\sigma|} \sum_{\pi \in \operatorname{Av}_k(\searrow_n)} [\pi : \mathfrak{C}W_k^{\operatorname{Av}(\searrow_n)}(\sigma)] \vec{e}_\pi \,. \tag{6}$$

On the other hand, from [BP19, Proposition 2.2], the vertices of the cycle polytope $P(\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\searrow_n)}(k))$ are given by the simple cycles of the graph $\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\searrow_n)}(k)$. Specifically, the vertices are given by the vectors $\vec{e}_\mathcal{C} \in \mathbb{R}^{\mathcal{C}_{n-1}(k)}$, for each simple cycle $\mathcal{C}$ of $\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\searrow_n)}(k)$, as follows:

$$(\vec{e}_\mathcal{C})_{(\pi,\mathfrak{c})} = \frac{\mathbb{1}[(\pi, \mathfrak{c}) \in \mathcal{C}]}{|\mathcal{C}|} \,,$$

for each inherited coloured permutation $(\pi, \mathfrak{c})$. In this way, we have that

$$\Pi(\vec{e}_\mathcal{C}) = \frac{1}{|\mathcal{C}|} \sum_{\pi \in \operatorname{Av}_k(\searrow_n)} [\pi : \mathcal{C}] \vec{e}_\pi \,. \tag{7}$$

Now let us start by proving the inclusion $\Pi(P(\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\searrow_n)}(k))) \subseteq P_k^{\operatorname{Av}(\searrow_n)}$. Take a vertex of the polytope $P(\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\searrow_n)}(k))$, that is a vector $\vec{e}_\mathcal{C}$ for some simple cycle $\mathcal{C}$ of $\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\searrow_n)}(k)$. Because $\mathcal{C}$ is a cycle, we can define the walks $\mathcal{C}^{\bullet \ell}$ obtained by concatenating ℓ times the cycle $\mathcal{C}$. From Lemma 4.12, there exists a walk w'_ℓ with $|w'_\ell| \leq C(k, n)$ and a $\searrow_n$-avoiding permutation σ^ℓ of size $|w'_\ell| + \ell|\mathcal{C}| + k - 1$, such that $\mathfrak{C}W_k^{\operatorname{Av}(\searrow_n)}(\sigma^\ell) = w'_\ell \bullet \mathcal{C}^{\bullet \ell}$. The next step is to prove that

$$\widetilde{\text{c-occ}}_k(\sigma^\ell) \xrightarrow{\ell \to \infty} \Pi(\vec{e}_\mathcal{C}).$$

We have that

$$\begin{aligned}
\widetilde{\text{c-occ}}_k\left(\sigma^\ell\right) &\overset{(6)}{=} \frac{1}{|\sigma^\ell|} \sum_{\pi \in \operatorname{Av}_k(\searrow_n)} [\pi : \mathfrak{C}W_k^{\operatorname{Av}(\searrow_n)}(\sigma^\ell)] \vec{e}_\pi \\
&= \frac{\ell}{|\sigma^\ell|} \left(\sum_{\pi \in \operatorname{Av}_k(\searrow_n)} [\pi : \mathcal{C}] \vec{e}_\pi \right) + \frac{1}{|\sigma^\ell|} \sum_{\pi \in \operatorname{Av}_k(\searrow_n)} [\pi : w'_\ell] \vec{e}_\pi \\
&\overset{(7)}{=} \frac{\ell|\mathcal{C}|}{|\sigma^\ell|} \Pi(\vec{e}_\mathcal{C}) + \frac{1}{|\sigma^\ell|} \vec{z}_\ell \\
&= \left(1 - \frac{k - 1 + |w'_\ell|}{|\sigma^\ell|}\right) \Pi(\vec{e}_\mathcal{C}) + \frac{1}{|\sigma^\ell|} \vec{z}_\ell \,,
\end{aligned}$$

where $\vec{z}_\ell = \sum_{\pi \in \operatorname{Av}_k(\searrow_n)} [\pi : w'_\ell] \vec{e}_\pi$. However, because $|w'_\ell| \leq C(k, n)$, we have that

$$\frac{k - 1 + |w'_\ell|}{|\sigma^\ell|} \xrightarrow{\ell \to \infty} 0 \,,$$

$$\frac{1}{|\sigma^\ell|} ||\vec{z}_\ell||_1 = \frac{1}{|\sigma^\ell|} \sum_{\pi \in \operatorname{Av}_k(\searrow_n)} [\pi : w'_\ell] = \frac{|w'_\ell|}{|\sigma^\ell|} \xrightarrow{\ell \to \infty} 0 \,.$$

Therefore $\widetilde{\text{c-occ}}_k(\sigma^\ell) \to \Pi(\vec{e}_\mathcal{C})$. This, together with Proposition 2.2, shows the desired inclusion.

For the other inclusion, consider a vector $\vec{v} \in P_k^{\operatorname{Av}(\searrow_n)}$, so that there is a sequence of $\searrow_n$-avoiding permutations σ^ℓ such that $\widetilde{\text{c-occ}}_k(\sigma^\ell) \xrightarrow{\ell \to \infty} \vec{v}$ and that $|\sigma^\ell| \xrightarrow{\ell \to \infty} \infty$. Fix $\varepsilon > 0$, and let M be an integer such that $\ell \geq M$ implies $||\widetilde{\text{c-occ}}_k(\sigma^\ell) - \vec{v}||_2 < \frac{1}{2}\varepsilon$ and $|\sigma^\ell| > \frac{6k!}{\varepsilon}$. The set of edges of the walk $\mathfrak{C}W_k^{\mathrm{Av}(\searrow_n)}(\sigma^\ell)$ can be split into $\mathcal{C}_1^{(\ell)} \uplus \cdots \uplus \mathcal{C}_j^{(\ell)} \uplus \mathcal{T}^{(\ell)}$, where each $\mathcal{C}_i^{(\ell)}$ is a simple cycle of $\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\searrow_n)}(k)$ and $\mathcal{T}^{(\ell)}$ is a path that does not repeat vertices, so $|\mathcal{T}^{(\ell)}| < V(\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\searrow_n)}(k)) \leq (k-1)!$ (for a precise explanation of this fact see [BP19, Lemma 3.13]). Thus, we get

$$\begin{aligned}
\widetilde{\text{c-occ}}_k(\sigma^\ell) &\overset{(6)}{=} \frac{1}{|\sigma^\ell|} \sum_{\pi \in \mathrm{Av}_k(\searrow_n)} [\pi : \mathfrak{C}W_k^{\mathrm{Av}(\searrow_n)}(\sigma^\ell)] \vec{e}_\pi \\
&= \frac{1}{|\sigma^\ell|} \sum_{i=1}^{j} \sum_{\pi \in \mathrm{Av}_k(\searrow_n)} [\pi : \mathcal{C}_i^{(\ell)}] \vec{e}_\pi + \frac{1}{|\sigma^\ell|} \sum_{\pi \in \mathrm{Av}_k(\searrow_n)} [\pi : \mathcal{T}^{(\ell)}] \vec{e}_\pi \\
&\overset{(7)}{=} \frac{|\sigma^\ell| - |\mathcal{T}^{(\ell)}| - k + 1}{|\sigma^\ell|} \sum_{i=1}^{j} \frac{|\mathcal{C}_i^{(\ell)}|}{|\sigma^\ell| - |\mathcal{T}^{(\ell)}| - k + 1} \Pi(\vec{e}_{\mathcal{C}_i^{(\ell)}}) + \frac{1}{|\sigma^\ell|} \sum_{\pi \in \mathrm{Av}_k(\searrow_n)} [\pi : \mathcal{T}^{(\ell)}] \vec{e}_\pi \, .
\end{aligned}$$

Now we set $\vec{x} := \sum_{i=1}^{j} \frac{|\mathcal{C}_i^{(\ell)}|}{|\sigma^\ell| - |\mathcal{T}^{(\ell)}| - k + 1} \Pi(\vec{e}_{\mathcal{C}_i^{(\ell)}})$ and $\vec{y} := \frac{1}{|\sigma^\ell|} \sum_{\pi \in \mathrm{Av}_k(\searrow_n)} [\pi : \mathcal{T}^{(\ell)}] \vec{e}_\pi$. Note that $\vec{x} \in \Pi(P(\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\searrow_n)}(k)))$, indeed it is a convex combination (since $\sum_{i=1}^{j} |\mathcal{C}_i^{(\ell)}| = |\sigma^\ell| - |\mathcal{T}^{(\ell)}| - k + 1$) of vectors corresponding to simple cycles. We simply get that

$$\widetilde{\text{c-occ}}_k(\sigma^\ell) = \frac{|\sigma^\ell| - |\mathcal{T}^{(\ell)}| - k + 1}{|\sigma^\ell|} \vec{x} + \vec{y} \, .$$

Thus,

$$\operatorname{dist}\left(\widetilde{\text{c-occ}}_k(\sigma^\ell), \Pi\left(P(\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\searrow_n)}(k))\right)\right) \leq ||\widetilde{\text{c-occ}}_k(\sigma^\ell) - \vec{x}||_2 \leq \frac{|\mathcal{T}^{(\ell)}| + k - 1}{|\sigma^\ell|} ||\vec{x}||_2 + ||\vec{y}||_2 \, . \tag{8}$$

Observe that $||\vec{y}||_2 \leq \frac{1}{|\sigma^\ell|} \sum_{\pi \in \mathrm{Av}_k(\searrow_n)} [\pi : \mathcal{T}^{(\ell)}] = \frac{|\mathcal{T}^{(\ell)}|}{|\sigma^\ell|} \leq \frac{(k-1)!}{|\sigma^\ell|}$. Also, because the coordinates of $\vec{x}$ are non-negative and sum to one, we have that $||\vec{x}||_2 \leq 1$ and so that $\frac{|T^{(\ell)}| + k - 1}{|\sigma^\ell|} ||\vec{x}||_2 \leq \frac{(k-1)! + k - 1}{|\sigma^\ell|}$. Then, we can simplify Eq. (8) to

$$\operatorname{dist}(\widetilde{\text{c-occ}}_k(\sigma^\ell), \Pi(P(\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\searrow_n)}(k)))) \leq \frac{(k-1)! + k - 1 + (k-1)!}{|\sigma^\ell|} \leq \frac{3k!}{|\sigma^\ell|} \, ,$$

so that for $\ell \geq M$ we have that $\operatorname{dist}(\widetilde{\text{c-occ}}_k(\sigma^\ell), \Pi(P(\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\searrow_n)}(k)))) < \frac{1}{2}\varepsilon$. As a consequence, for $\ell \geq M$,

$$\operatorname{dist}(\vec{v}, \Pi(P(\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\searrow_n)}(k)))) \leq ||\vec{v} - \widetilde{\text{c-occ}}_k(\sigma^\ell)||_2 + \operatorname{dist}(\widetilde{\text{c-occ}}_k(\sigma^\ell), \Pi(P(\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\searrow_n)}(k)))) < \varepsilon \, .$$

Noting that $\Pi(P(\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\searrow_n)}(k)))$ is a closed set, since ε is generic, we obtain that $\vec{v}$ is in the polytope $\Pi(P(\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\searrow_n)}(k)))$, concluding the proof of the theorem. □

It just remains to prove Lemma 4.14. This is the goal of the next two sections.

4.3 Preliminary results: basic properties of RITMO colourings and their relations with active sites

We begin by stating (without proof) some basic properties of the RITMO colouring. We suggest to compare the following lemma with Fig. 9. We remark that Properties 2 and 3 in the following lemma arise as particular cases of a general result explained after the statement.

Lemma 4.15. *Let σ be a permutation, and consider $\mathbb{C}(\sigma)$ its RITMO colouring.*

1. *If $i < j \in [|\sigma|]$ such that $\sigma(i) > \sigma(j)$, then $\mathbb{C}(\sigma)(i) < \mathbb{C}(\sigma)(j)$.*

2. *If $i < j \in [|\sigma|]$ such that $\sigma(i) < \sigma(j)$ and $\mathbb{C}(\sigma)(i) < \mathbb{C}(\sigma)(j)$, then there exists k such that $i < k < j$, $\sigma(k) > \sigma(j)$ and $\mathbb{C}(\sigma)(k) = \mathbb{C}(\sigma)(i)$.*

3. *If $i < j \in [|\sigma|]$ such that $\sigma(i) < \sigma(j)$ and $\mathbb{C}(\sigma)(i) < \mathbb{C}(\sigma)(j)$, then there exists h such that $i < h < j$, $\sigma(h) > \sigma(j)$ and $\mathbb{C}(\sigma)(h) = \mathbb{C}(\sigma)(j) - 1$.*

As mentioned before, we explain that Properties 2 and 3 are particular cases of the same general result: consider $i < j \in [|\sigma|]$ with $\sigma(i) < \sigma(j)$. Let $c = \mathbb{C}(\sigma)(i)$ and $d = \mathbb{C}(\sigma)(j)$ and assume that $c < d$. Then there are indices $i < k_c < k_{c+1} < \cdots < k_{d-1} < j$ such that $\mathbb{C}(\sigma)(k_s) = s$ for all $s \in [c, d-1]$, and $\sigma(j) < \sigma(k_{d-1}) < \cdots < \sigma(k_c)$. We opt to single out Properties 2 and 3 because these will be enough for our applications.

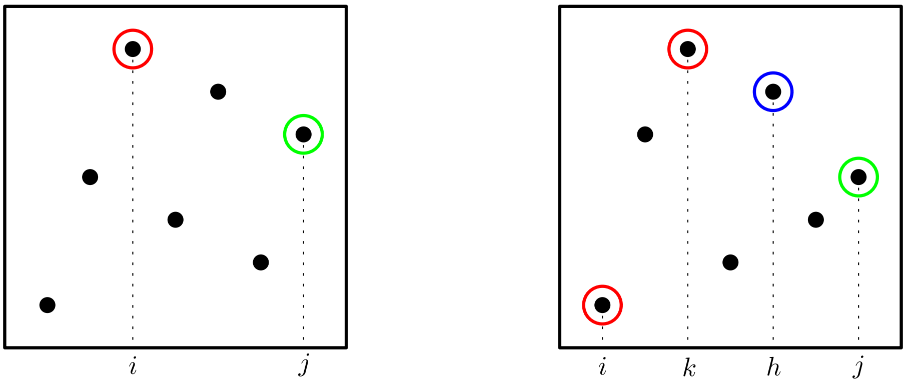


Figure 9: A schema for Lemma 4.15. The left-hand side illustrates Property 1 and the right-hand side illustrates Properties 2 and 3.

We now introduce a key definition.

Definition 4.16. Given a coloured permutation $(\pi, \mathfrak{c})$, and a pair (y, f) with $y \in [|\pi| + 1]$, $f \geq 1$, we define the coloured permutation $(\pi, \mathfrak{c})^{*(y,f)}$ to be the permutation π^{*y} together with the colouring $\mathfrak{c}^{*f}$. The latter is defined as a colouring $\mathfrak{c}^{*f} : [|\pi|+1] \to \mathbb{Z}_{\geq 1}$ such that $\mathfrak{c}^{*f}(i) = \mathfrak{c}(i)$ for all $i \in [|\pi|]$ and $\mathfrak{c}^{*f}(|\pi| + 1) = f$.

Let $(\pi, \mathfrak{c})$ be an inherited $(n-1)$-coloured permutation. An *active site* is a pair (y, f) with $y \in [|\pi| + 1]$ and $f \in [n-1]$, such that $(\pi, \mathfrak{c})^{*(y,f)}$ is an inherited $(n-1)$-coloured permutation.

We present the following analogue of Lemma 4.15.

Lemma 4.17. *Let (y, f) be an active site of an inherited coloured permutation $(\pi, \mathfrak{c})$, and consider some index $i \in [|\pi|]$. Then*

1. *if $\mathfrak{c}(i) \geq f$, then $y > \pi(i)$;*

2. *if $\pi(i) < y$ and $\mathfrak{c}(i) < f$, then there exists $k > i$ such that $\pi(k) \geq y$ and $\mathfrak{c}(k) = \mathfrak{c}(i)$.*

3. *if $\pi(i) < y$ and $\mathfrak{c}(i) < f$, then there exists $h > i$ such that $\pi(h) \geq y$ and $\mathfrak{c}(h) = f - 1$.*

Proof. Let σ be a permutation such that $\mathrm{end}_{|\pi|+1}(\mathbb{S}(\sigma)) = (\pi, \mathfrak{c})^{*(y,f)}$, which exists because (y, f) is an active site of $(\pi, \mathfrak{c})$. The lemma is an immediate consequence of Lemma 4.15, applied to the RITMO colouring $\mathbb{C}(\sigma)$, and for $j = |\sigma|$ (so that $\mathbb{C}(\sigma)(j) = f$ and $\sigma(j) = y$). $\square$

We now observe a correspondence between edges of $\mathfrak{CO}v^{\mathcal{A}v(\searrow_n)}(k)$ and active sites of some coloured permutations.

Observation 4.18. Fix an inherited coloured permutations $(\pi_1, \mathfrak{c}_1)$ of size $k-1$. Then there exists a bijection between the set of edge $e \in \mathfrak{CO}v^{\mathcal{A}v(\searrow_n)}(k)$ with $\mathrm{s}(e) = (\pi_1, \mathfrak{c}_1)$ and the set of active sites (y, f) of $(\pi_1, \mathfrak{c}_1)$. Specifically, this correspondence between edges and active sites is given by the following maps, which can be easily seen to be inverses of each other:

$$e = (\pi, \mathfrak{c}) \mapsto (\pi(k), \mathfrak{c}(k)), \quad (y, f) \mapsto (\pi_1, \mathfrak{c}_1)^{*(y,f)} .$$

Fix now an inherited coloured permutation $(\pi, \mathfrak{c})$. By definition, there exists some σ_0 that satisfies $\mathrm{end}_{|\pi|}(\mathbb{S}(\sigma_0)) = (\pi, \mathfrak{c})$. The goal of the next section is to show that, with some mild restrictions on the chosen permutation σ_0, if (y, f) is an active site of $(\pi, \mathfrak{c})$ then there exists an index $i \in [|\sigma_0| + 1]$ such that

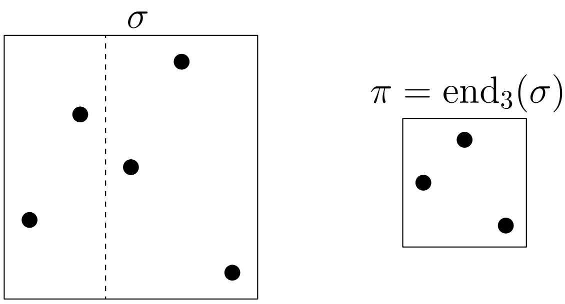
$$\mathrm{end}_{|\pi|+1}(\mathbb{S}(\sigma_0^{*i})) = (\pi, \mathfrak{c})^{*(y,f)} .$$

We already know that there exists a permutation σ_1 such that $\mathrm{end}_{|\pi|+1}(\mathbb{S}(\sigma_1)) = (\pi, \mathfrak{c})^{*(y,f)}$; here we are interested in finding out if σ_1 can arise as an extension of σ_0.

We introduce two definitions and give some of their simple properties.

Definition 4.19. Let π and σ be two permutations such that $\pi = \mathrm{end}_{k-1}(\sigma)$. For a point at height $\ell \in [|\pi|]$ in the diagram of π, we define $\widetilde{\ell}$ to be the height of the corresponding point in the diagram of σ. Algebraically we have that $\widetilde{\ell} = \sigma(|\sigma| - |\pi| + \pi^{-1}(\ell))$. We use the convention that $\widetilde{|\pi| + 1} = |\sigma| + 1$ and $\widetilde{0} = 0$.

See Fig. 10 for an example. We have the following simple result.



ℓ	0	1	2	3	4
$\tilde{\ell}$	0	1	3	5	6

Figure 10: A schema illustrating Definition 4.19. On the left-hand side the permutation $\sigma = 24351$, in the middle the pattern $\pi = 231$ induced by the last three indices of σ, and on the right-hand side the quantities $\tilde{\ell}$.

Lemma 4.20. *Let σ, π be permutations such that $\pi = \mathrm{end}_{k-1}(\sigma)$. Let $y \in [|\pi| + 1]$ and $\iota \in [|\sigma| + 1]$. Then we have that*

$$\mathrm{end}_k(\sigma^{*\iota}) = \pi^{*y} \iff \widetilde{y-1} < \iota \leq \widetilde{y} .$$

Definition 4.21. Fix a permutation σ and a colour $f \in \{1, 2, \dots\}$. If there exists a maximal index p of σ such that $\mathbb{C}(\sigma)(p) = f$ we set $z_\sigma(f) := \sigma(p) + 1$. Otherwise, if such a p does not exist, then $z_\sigma(f) := 1$. We use the convention that $z_\sigma(0) = |\sigma| + 2$.

See Fig. 11 for an example. We have the following simple result.

Lemma 4.22. *Let σ be a permutation, $f \in \mathbb{Z}_{\geq 1}$ a colour and $\iota \in [|\sigma| + 1]$. Then we have that*

$$\mathbb{C}(\sigma^{*\iota})(|\sigma| + 1) = f \iff z_\sigma(f) \leq \iota < z_\sigma(f-1) .$$

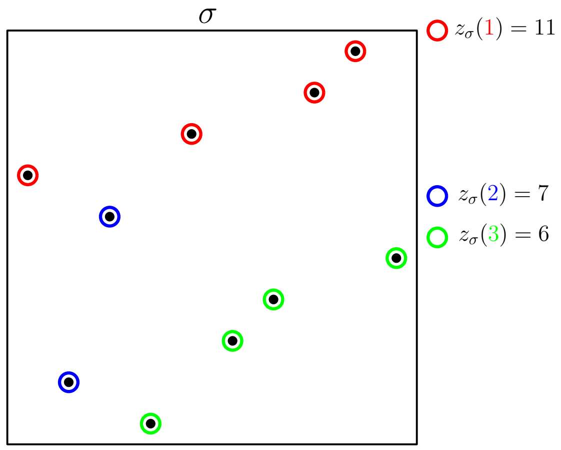


Figure 11: A permutation $\sigma \in \text{Av}(4321)$ coloured with its RITMO colouring. The quantities $z_\sigma(1), z_\sigma(2), z_\sigma(3)$, defined in Definition 4.21, are highlighted on the right of the diagram of the permutation σ.

4.4 The proof of the main lemma

We can now prove Lemma 4.14. We will do this as follows: in order to construct a larger permutation from σ we will find a suitable index ι so that $\sigma^{*\iota}$ has the desired pattern at the end. According to Lemma 4.20, fixing the permutation at the end of $\sigma^{*\iota}$ fixes ι to an interval, and according to Lemma 4.22fixing the colour of the last entry fixes ι to another interval. The bulk of the proof wil be dedicated to show that these intervals have non-trivial intersection.

Proof of Lemma 4.14. Observe that $\text{beg}_{k-1}(\pi', \mathfrak{c}') = \text{end}_{k-1}(\pi, \mathfrak{c}) = \text{end}_{k-1}(\mathbb{S}(\sigma))$. Let $(\rho, \mathfrak{d}) = \text{end}_{k-1}(\mathbb{S}(\sigma))$ be this common coloured permutation. For an entry of height $\ell \in [|\rho|]$ in the diagram of ρ, we recall that $\widetilde{\ell} \in [|\sigma|]$ denotes the height of the corresponding entry in the diagram of σ, as in Definition 4.19. Let (y, f) be the active site of $(\rho, \mathfrak{d})$ corresponding to the edge $(\pi', \mathfrak{c}')$, so that $f \in [n-1]$ and $y \in [|\rho| + 1]$ (see Observation 4.18).

From Lemma 4.20, we have that $\text{end}_k(\sigma^{*\iota}) = \pi'$ if and only if

$$\widetilde{y-1} < \iota \leq \widetilde{y}\,. \tag{9}$$

From Lemma 4.22, we have that $\mathbb{C}(\sigma^{*\iota})(|\sigma| + 1) = f$ if and only if

$$z_\sigma(f) \leq \iota < z_\sigma(f-1)\,. \tag{10}$$

This gives us two intervals that are, by Definition 4.19 and Definition 4.21, non-empty. Our goal is to show that these intervals have a non-trivial intersection, concluding that the desired index ι exists.

Claim. $z_\sigma(f) \leq \widetilde{y}$.

Assume by sake of contradiction that $z_\sigma(f) > \widetilde{y}$. If $y = |\rho| + 1$, then $\widetilde{y} = |\sigma| + 1$ by convention. This gives a contradiction because $f \geq 1$ and so $z_\sigma(f) \leq |\sigma| + 1$. Thus $y < |\rho| + 1$. Let $p \in [|\sigma|]$ be the maximal index such that $\mathbb{C}(\sigma)(p) = f$. We know that such a p exists, because $\mathbb{C}(\sigma)$ is a rainbow $(n-1)$-colouring. By maximality of p, it follows that $\sigma(p) + 1 = z_\sigma(f)$ (see Definition 4.21). We now split the proof into two cases: when p is included in the last $|\rho|$ indices of σ and when it is not.

- **Assume that** $p > |\sigma| - |\rho|$. Let $q = p - (|\sigma| - |\rho|) > 0$. Because $\text{end}_{k-1}(\mathbb{S}(\sigma)) = (\rho, \mathfrak{d})$, we have that $f = \mathbb{C}(\sigma)(p) = \mathfrak{d}(q)$. Since we know that $\sigma(p) + 1 = z_\sigma(f) > \widetilde{y}$, we have that $\rho(q) + 1 > y$. This contradicts Property 1 of Lemma 4.17, as the active site (y, f) satisfies both $\mathfrak{d}(q) \geq f$ and $\rho(q) \geq y$.

- **Assume that** $p \leq |\sigma| - |\rho|$. Then $\sigma(p) \neq \widetilde{y}$, so from $\sigma(p)+1 = z_\sigma(f) > \widetilde{y}$ we have that $\sigma(p) > \widetilde{y}$. Using Property 1 of Lemma 4.15 with $i = p$ and $j = \sigma^{-1}(\widetilde{y})$, we have that $f = \mathbb{C}(\sigma)(p) < \mathbb{C}(\sigma)(\sigma^{-1}(\widetilde{y}))$. So $\mathfrak{d}(\rho^{-1}(y)) = \mathbb{C}(\sigma)(\sigma^{-1}(\widetilde{y})) > f$. But this contradicts again Property 1 of Lemma 4.17 for $i = \rho^{-1}(y)$, as the active site (y, f) satisfies both $\mathfrak{d}(\rho^{-1}(y)) > f$ and $y \leq \rho(\rho^{-1}(y))$.

Therefore, in both cases we have a contradiction.

Claim. $z_\sigma(f-1) > \widetilde{y-1} + 1$.

Assume by contradiction that $z_\sigma(f-1) \leq \widetilde{y-1} + 1$. If $f = 1$, then recall that we use the convention that $z_\sigma(0) = |\sigma| + 2$, so we have $\widetilde{y-1} \geq |\sigma| + 1$. But $y \leq |\rho| + 1$ so $\widetilde{y-1} \leq |\sigma|$, a contradiction. Thus $f > 1$. Let p be the maximal index in $[|\sigma|]$ such that $\mathbb{C}(\sigma)(p) = f - 1$. We know that such a p exists, because $\mathbb{C}(\sigma)$ is a rainbow $(n-1)$-colouring. By construction, $\sigma(p) + 1 = z_\sigma(f-1) \leq \widetilde{y-1} + 1$ (see Definition 4.21), so $\sigma(p) \leq \widetilde{y-1}$. As above, we now split the proof into two cases: when p is included in the last $|\rho|$ indices of σ and when it is not.

- **Assume that** $p > |\sigma| - |\rho|$. Let $q = p - (|\sigma| - |\rho|) > 0$. Because $\mathrm{end}_{k-1}(\mathbb{S}(\sigma)) = (\rho, \mathfrak{d})$, we have that $f - 1 = \mathbb{C}(\sigma)(p) = \mathfrak{d}(q)$. Since we know that $\sigma(p) \leq \widetilde{y-1}$, we have that $\rho(q) \leq y - 1$. Thus, by Property 2 of Lemma 4.17, there exists some $k > q$ such that $\mathfrak{d}(k) = \mathfrak{d}(q) = f - 1$. The existence of such k contradicts the maximality of p, as we get that $k + (|\sigma| - |\rho|) > p$ has $\mathbb{C}(\sigma)(k + (|\sigma| - |\rho|)) = \mathfrak{d}(k) = f - 1$.

- **Assume that** $p \leq |\sigma| - |\rho|$. Let $r = \sigma^{-1}(\widetilde{y-1})$. Then $r > |\sigma| - |\rho| \geq p$ and so $p \neq r$. It follows that $\sigma(p) = z_\sigma(f-1) - 1 < \widetilde{y-1}$.

  We now claim that $\mathbb{C}(\sigma)(r) < f - 1$. Indeed, if $\mathbb{C}(\sigma)(r) = f - 1$, because $p < r$ we have immediately a contradiction with the maximality of p. Moreover, if $\mathbb{C}(\sigma)(r) > f - 1$, Property 2 of Lemma 4.15 guarantees that there is some $k > p$ such that $\sigma(k) > \sigma(r)$ and $\mathbb{C}(\sigma)(k) = f - 1$. Again, we have a contradiction with the maximality of p.

  Now let $q = r - (|\sigma| - |\rho|)$, and observe that $\mathfrak{d}(q) = \mathbb{C}(\sigma)(r) < f - 1$. On the other hand, because $r = \sigma^{-1}(\widetilde{y-1})$, we have $\rho(q) = y - 1$. Because (y, f) is an active site of $(\rho, \mathfrak{d})$, Property 3 of Lemma 4.17 guarantees that there is some index $k > q$ of ρ such that $\mathfrak{d}(k) = f - 1$. But this contradicts again the maximality of p, as we would have that $\mathbb{C}(\sigma)(k + |\sigma| - |\rho|) = f - 1$ while $k + |\sigma| - |\rho| > q + |\sigma| - |\rho| = r > p$.

Therefore, in both cases we have a contradiction.

Using the two claims above, we can conclude that the intervals in Eqs. (9) and (10) have a non-trivial intersection, and therefore the envisaged index ι we were looking for exists. Consequently, we can construct the desired permutation $\sigma^{*\iota}$. □

4.5 Dimension of the feasible region

The computation of the dimension of $P_k^{\mathrm{Av}(\searrow_n)}$ is based on the description given in Theorem 4.8. This allows us to compute a lower bound by carefully studying the kernel of the map Π.

Proof of Theorem 1.12. From Theorem 1.7 we have that $\dim(P_k^{\mathrm{Av}(\searrow_n)}) \leq |\mathrm{Av}_k(\searrow_n)| - |\mathrm{Av}_{k-1}(\searrow_n)|$. Therefore, we just have to establish that $\dim(P_k^{\mathrm{Av}(\searrow_n)}) \geq |\mathrm{Av}_k(\searrow_n)| - |\mathrm{Av}_{k-1}(\searrow_n)|$.

First, recall the definition of the projection $\Pi : \mathbb{R}^{\mathcal{C}_{n-1}(k)} \to \mathbb{R}^{\mathrm{Av}_k(\searrow_n)}$ in Eq. (5). Let $S = \mathrm{span}\{P(\mathfrak{CO}v^{\mathcal{A}v(\searrow_n)}(k))\}$. From the rank nullity theorem applied to the restriction $\Pi|_S$ we have that

$$\dim S = \dim \mathrm{im}\, \Pi|_S + \dim \ker \Pi|_S\,. \tag{11}$$

Note that the graph $\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\searrow_n)}(k)$ is strongly connected, as we can construct a path between any two vertices with $\mathfrak{C}W_k^{\mathrm{Av}(\searrow_n)}$. Therefore, from Proposition 1.3 and the fact that $\vec{0}$ is not in the affine span of $P(\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\searrow_n)}(k))$, we have that

$$\dim S = 1 + |E(\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\searrow_n)}(k))| - |V(\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\searrow_n)}(k))| = 1 + |\mathcal{C}_{n-1}(k)| - |\mathcal{C}_{n-1}(k-1)| \,. \quad (12)$$

In addition, from Theorem 4.8 we have $\operatorname{im}\Pi|_S = \operatorname{span}\{\Pi(P(\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\searrow_n)}(k)))\} = \operatorname{span}\{P_k^{\mathrm{Av}(\searrow_n)}\}$, and so

$$\dim \operatorname{im}\Pi|_S = \dim \operatorname{span}\{P_k^{\mathrm{Av}(\searrow_n)}\} = 1 + \dim P_k^{\mathrm{Av}(\searrow_n)} \,, \quad (13)$$

because $\vec{0}$ is not in the affine span of $P_k^{\mathrm{Av}(\searrow_n)}$. Eqs. (11) to (13) together give us that

$$1 + |\mathcal{C}_{n-1}(k)| - |\mathcal{C}_{n-1}(k-1)| = 1 + \dim P_k^{\mathrm{Av}(\searrow_n)} + \dim\ker\Pi|_S \,.$$

We now claim that $\dim\ker\Pi|_S \leq |\mathcal{C}_{n-1}(k)| - |\operatorname{Av}_k(\searrow_n)| - |\mathcal{C}_{n-1}(k-1)| + |\operatorname{Av}_{k-1}(\searrow_n)|$. This is enough to conclude, as we get that

$$\dim P_k^{\mathrm{Av}(\searrow_n)} = |\mathcal{C}_{n-1}(k)| - |\mathcal{C}_{n-1}(k-1)| - \dim\ker\Pi|_S \geq |\operatorname{Av}_k(\searrow_n)| - |\operatorname{Av}_{k-1}(\searrow_n)| \,.$$

To compute $\dim\ker\Pi|_S$, notice that $\ker\Pi|_S$ is a vector space given by two types of equations: the ones defining $\ker\Pi$ and the ones defining S. The description of $\ker\Pi$ as a set of equations is rather straightforward. On the other hand, the description of the vector space S can be computed from [BP19, Proposition 2.6], where it was shown that for any strongly connected graph G, the polytope $P(G)$ is described by the equations

$$P(G) = \left\{ \vec{v} \in \mathbb{R}_{\geq 0}^{E(G)} \middle| \sum_{\mathrm{a}(e)=v} \vec{v}_e = \sum_{\mathrm{s}(e)=v} \vec{v}_e \text{ for } v \in V(G), \sum_{e \in E(G)} \vec{v}_e = 1 \right\} .$$

Thus, for $G = \mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\searrow_n)}(k)$, we get that

$$S = \left\{ \vec{v} \in \mathbb{R}^{E(\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\searrow_n)}(k))} \middle| \sum_{\mathrm{a}(e)=v} \vec{v}_e = \sum_{\mathrm{s}(e)=v} \vec{v}_e \text{ for } v \in V(\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\searrow_n)}(k)) \right\} .$$

The vector space $\ker\Pi|_S$ arises then as the kernel of an $|\operatorname{Av}_k(\searrow_n)| + |\mathcal{C}_{n-1}(k-1)|$ by $|\mathcal{C}_{n-1}(k)|$ matrix A.

We now describe this matrix A. It can be split as $A = \begin{bmatrix} A_{\ker} \\ A_S \end{bmatrix}$, where $A_{\ker}$ is an $|\operatorname{Av}_k(\searrow_n)|$ by $|\mathcal{C}_{n-1}(k)|$ matrix defined for a permutation $\pi \in \operatorname{Av}_k(\searrow_n)$ and a coloured permutation $(\pi', \mathfrak{c}) \in \mathcal{C}_{n-1}(k)$ as $(A_{\ker})_{\pi,(\pi',\mathfrak{c})} = \mathbb{1}_{\{\pi=\pi'\}}$, and A_S is the $|\mathcal{C}_{n-1}(k-1)|$ by $|\mathcal{C}_{n-1}(k)|$ incidence matrix of $\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\searrow_n)}(k)$. In Example 4.23 one can see an example of this matrix for $k=3$, $n=3$ and see p. 27 for $k=3$, $n=4$. We have that $\dim\ker\Pi|_S = |\mathcal{C}_{n-1}(k)| - \operatorname{rk} A$, so our goal is to establish that

$$\operatorname{rk} A \geq |\mathcal{C}_{n-1}(k-1)| + |\operatorname{Av}_k(\searrow_n)| - |\operatorname{Av}_{k-1}(\searrow_n)| \,.$$

This will be done by finding a suitable non-singular minor of A with size $|\mathcal{C}_{n-1}(k-1)| + |\operatorname{Av}_k(\searrow_n)| - |\operatorname{Av}_{k-1}(\searrow_n)|$.

Construction of the minor. We are going to select a subsets $\mathfrak{C}\mathcal{E}$ of columns and a subset $\mathcal{V}$ of rows of the matrix A, both of cardinality $|\mathcal{C}_{n-1}(k-1)| + |\operatorname{Av}_k(\searrow_n)| - |\operatorname{Av}_{k-1}(\searrow_n)|$.

We start by determining the set $\mathfrak{C}\mathcal{E}$. For each vertex v of $\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\searrow_n)}(k)$, consider the active site $(k, 1)$, which is always an active site, and the corresponding edge e (according to Observation 4.18) which we write from now on as $e = \operatorname{comp}(v)$. We call this the *completion* *process* of v. Notice that in this case we have $\mathrm{beg}(e) = v$. As a result, we can define the set of edges $\mathfrak{C}\mathcal{E}^k(k)$ obtained by the completion process of all $v \in \mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\searrow_n)}(k)$ – note that the notation $\mathfrak{C}\mathcal{E}^k(k)$ indicates that the permutations in $\mathfrak{C}\mathcal{E}^k(k)$ are coloured, end with the value k and have size k. Because for each vertex $v \in \mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\searrow_n)}(k)$ there is exactly one edge $e \in \mathfrak{C}\mathcal{E}^k(k)$ such that $\mathrm{beg}(e) = v$ and these edges are distinct for different choices of v, we have that $|\mathfrak{C}\mathcal{E}^k(k)| = |V(\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\searrow_n)}(k))| = |\mathcal{C}_{n-1}(k-1)|$.

Let $\mathcal{N}\mathcal{E}^k(k)$ be the set of permutations $\sigma \in \mathrm{Av}_k(\searrow_n)$ that satisfy $\sigma(k) \neq k$ – note that the notation $\mathcal{N}\mathcal{E}^k(k)$ remarks that the permutations in $\mathcal{N}\mathcal{E}^k(k)$ are not coloured (indeed there is no $\mathfrak{C}$), do not end with the value k and have size k. Let $\mathfrak{C}\mathcal{N}\mathcal{E}^k(k)$ be a set of edges of $\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\searrow_n)}(k)$ such that for each permutation $\sigma \in \mathcal{N}\mathcal{E}^k(k)$, there is exactly one edge $e \in \mathfrak{C}\mathcal{N}\mathcal{E}^k(k)$ such that $e = (\sigma, \mathfrak{c})$ for some colouring $\mathfrak{c}$ and these edges are distinct for different choices of σ. It is clear that $|\mathfrak{C}\mathcal{N}\mathcal{E}^k(k)| = |\mathcal{N}\mathcal{E}^k(k)|$, which is the number of permutations in $\mathrm{Av}_k(\searrow_n)$ that do not end with k. Because the permutations in $\mathrm{Av}_k(\searrow_n)$ that end with k are of the form $\pi \oplus 1$, for $\pi \in \mathrm{Av}_{k-1}(\searrow_n)$, we have that $|\mathfrak{C}\mathcal{N}\mathcal{E}^k(k)| = |\,\mathrm{Av}_k(\searrow_n)| - |\,\mathrm{Av}_{k-1}(\searrow_n)|$. It is also clear that the sets $\mathfrak{C}\mathcal{E}^k(k)$ and $\mathfrak{C}\mathcal{N}\mathcal{E}^k(k)$ are disjoint. Define $\mathfrak{C}\mathcal{E} := \mathfrak{C}\mathcal{E}^k(k) \uplus \mathfrak{C}\mathcal{N}\mathcal{E}^k(k)$, so that $|\mathfrak{C}\mathcal{E}| = |\mathcal{C}_{n-1}(k-1)| + |\,\mathrm{Av}_k(\searrow_n)| - |\,\mathrm{Av}_{k-1}(\searrow_n)|$.

Consider $\gamma = 1 \cdots k$ to be the increasing permutation of size k. We prove (for later use) that $\mathfrak{C}\mathcal{E}^k(k)$ has a unique cycle, which is the loop $\mathbb{S}(\gamma) = {\color{red}1 \cdots k}$ at the vertex $\mathrm{beg}_{k-1}(\mathbb{S}(\gamma)) = {\color{red}1 \cdots k-1}$ (recall that we use the colour red to denote the colour one). For a coloured permutation, we define the number of *trailing ones* to be the number of consecutive elements that are coloured one in the end of the permutation. In this way, for any edge $e \in \mathfrak{C}\mathcal{E}^k(k) \setminus \{\mathbb{S}(\gamma)\}$, the number of trailing ones of $\mathrm{beg}_{k-1}(e)$ is strictly smaller than the number of trailing ones of $\mathrm{end}_{k-1}(e)$ (by definition of completion process). We conclude that there is no cycle in $\mathfrak{C}\mathcal{E}^k(k) \setminus \{\mathbb{S}(\gamma)\}$.

We now determine the set $\mathcal{V}$ of rows of the matrix A. On $\mathrm{Av}_k(\searrow_n) \uplus \mathcal{C}_{n-1}(k-1)$, consider the set

$$\mathcal{V} = \mathcal{N}\mathcal{E}^k(k) \uplus \{\gamma\} \uplus \left(\mathcal{C}_{n-1}(k-1) \setminus \{\mathrm{beg}_{k-1}(\mathbb{S}(\gamma))\}\right),$$

where we note that $\mathrm{beg}_{k-1}(\mathbb{S}(\gamma))$ is an inherited permutation. Observe that $|\mathcal{V}| = |\,\mathrm{Av}_k(\searrow_n)| - |\,\mathrm{Av}_{k-1}(\searrow_n)| + |\mathcal{C}_{n-1}(k-1)|$.

Proof that the minor is non-singular. We establish now that the minor of A determined by $\mathfrak{C}\mathcal{E}$ and $\mathcal{V}$ is non-singular, by presenting two orders on these sets so that the corresponding minor becomes upper-triangular with non-zero entries in the diagonal. Recall that

$$\mathfrak{C}\mathcal{E} = \{\mathbb{S}(\gamma)\} \uplus (\mathfrak{C}\mathcal{E}^k(k) \setminus \{\mathbb{S}(\gamma)\}) \uplus \mathfrak{C}\mathcal{N}\mathcal{E}^k(k)$$

and that

$$\mathcal{V} = \{\gamma\} \uplus \left(\mathcal{C}_{n-1}(k-1) \setminus \{\mathrm{beg}_{k-1}(\mathbb{S}(\gamma))\}\right) \uplus \mathcal{N}\mathcal{E}^k(k).$$

We will define two total orders in these sets that preserves the order described by the decompositions above.

Let us denote by $\leq_{\mathbb{S}(\gamma)}$ and $\leq_\gamma$ the trivial orders in $\{\mathbb{S}(\gamma)\}$ and $\{\gamma\}$. Consider a total order $\leq_{\mathcal{N}\mathcal{E}^k(k)}$ in the set $\mathcal{N}\mathcal{E}^k(k)$, and construct the corresponding total order $\leq_{\mathfrak{C}\mathcal{N}\mathcal{E}^k(k)}$ in $\mathfrak{C}\mathcal{N}\mathcal{E}^k(k)$ according to the bijection described above between $\mathcal{N}\mathcal{E}^k(k)$ and $\mathfrak{C}\mathcal{N}\mathcal{E}^k(k)$.

Additionally, in $\mathcal{C}_{n-1}(k-1) \setminus \{\mathrm{beg}_{k-1}(\mathbb{S}(\gamma))\}$ define the partial order $\leq_{\mathcal{C}}$ by setting $v_1 \leq_{\mathcal{C}} v_2$ if there is an edge $e \in \mathfrak{C}\mathcal{E}^k(k)$ such that $e = v_2 \to v_1$. Equivalently, $v_1 \leq_{\mathcal{C}} v_2$ if $\mathrm{end}_{k-1}(\mathrm{comp}(v_2)) = v_1$, that is, if the completion process of v_2 gives an edge pointing to v_1. We extend transitively $\leq_{\mathcal{C}}$ to become a partial order. This is a partial order because the edges in $\mathfrak{C}\mathcal{E}^k(k) \setminus \{\mathbb{S}(\gamma)\}$ do not form any cycle, as explained above. We fix an extension of the partial order $\leq_{\mathcal{C}}$ into a total order on $\mathcal{C}_{n-1}(k-1) \setminus \{\mathrm{beg}_{k-1}(\mathbb{S}(\gamma))\}$ and we still denote it by $\leq_{\mathcal{C}}$.

Finally, by identifying the edges $e \in \mathfrak{C}\mathcal{E}^k(k) \setminus \{\mathbb{S}(\gamma)\}$ with the vertices in the set $\mathcal{C}_{n-1}(k-1) \setminus \{\mathrm{beg}_{k-1}(\mathbb{S}(\gamma))\}$ via the map $e \mapsto \mathrm{beg}_{k-1}(e)$, the total order $\leq_{\mathcal{C}}$ on $\mathcal{C}_{n-1}(k-1) \setminus \{\mathrm{beg}_{k-1}(\mathbb{S}(\gamma))\}$ induces a total order also on the set $\mathfrak{C}\mathcal{E}^k(k) \setminus \{\mathbb{S}(\gamma)\}$ that we denote $\leq_{\tilde{\mathcal{C}}}$.

If A, B are two disjoint sets, equipped with the partial orders $\leq_A, \leq_B$, respectively, we denote by $\leq_{A,B}$ the partial order on $A \cup B$ which restricts to $\leq_A$ in A, which restricts to $\leq_B$ in B and which has $a \leq_{A,B} b$ for any $a \in A, b \in B$. Note that this operation on partial orders is associative. Define the following two total orders:

$$\begin{aligned} \leq_{\mathfrak{C}\mathcal{E}} &:= \leq_{\mathbb{S}(\gamma),\, \tilde{\mathcal{C}},\, \mathfrak{C}\mathcal{N}\mathcal{E}^k(k)}, \quad \text{on } \mathfrak{C}\mathcal{E}, \\ \leq_{\mathcal{V}} &:= \leq_{\gamma,\, \mathcal{C},\, \mathcal{N}\mathcal{E}^k(k)}, \quad \text{on } \mathcal{V}, \end{aligned}$$

where we recall that the operation $\bullet$ was defined after Eq. (3), page 8. Under these total orders, one can see that the minor $\mathcal{V} \times \mathfrak{C}\mathcal{E}$ of the matrix A becomes

$$A|_{\mathcal{V}\times\mathfrak{C}\mathcal{E}} = \begin{array}{c} \begin{array}{ccc} \mathbb{S}(\gamma) & \mathfrak{C}\mathcal{E}^k(k) \setminus \{\mathbb{S}(\gamma)\} & \mathfrak{C}\mathcal{N}\mathcal{E}^k(k) \end{array} \\ \begin{bmatrix} 1 & A_1 & A_2 \\ Z_1 & B & A_3 \\ Z_2 & Z_3 & C \end{bmatrix} \end{array} \begin{array}{c} \\ \gamma \\ \mathcal{C}_{n-1}(k-1) \setminus \{\mathrm{beg}_{k-1} \mathbb{S}(\gamma)\} \\ \mathcal{N}\mathcal{E}^k(k) \end{array} .$$

It is immediate to argue that Z_1 and Z_2 are zero matrices. That Z_3 is a zero matrix follows from the observation that for any edge $(\sigma, \mathfrak{c}) \in \mathfrak{C}\mathcal{E}^k(k)$ we have that $\sigma(k) = k$, so $\sigma \notin \mathcal{N}\mathcal{E}^k(k)$. The matrix C is the identity matrix by definition of the two orders $\leq_{\mathcal{N}\mathcal{E}^k(k)}$ on $\mathcal{N}\mathcal{E}^k(k)$ and $\leq_{\mathfrak{C}\mathcal{N}\mathcal{E}^k(k)}$ on $\mathfrak{C}\mathcal{N}\mathcal{E}^k(k)$.

We finally claim that the matrix B is upper triangular. Recall that the matrix B is a minor of the incidence matrix A_S of the graph $\mathfrak{C}\mathcal{O}v^{\mathcal{A}v(\searrow_n)}(k)$. Consider a non-zero off-diagonal entry $B_{e,v}$. Since it is off-diagonal then $\mathrm{beg}_{k-1}(e) \neq v$ by definition of the orders $\leq_{\mathcal{C}}, \leq_{\tilde{\mathcal{C}}}$. Moreover, since it is non-zero, we must have $\mathrm{end}_{k-1}(e) = v$, and so $v \leq_{\mathcal{C}} \mathrm{beg}_{k-1}(e)$ by definition of $\leq_{\mathcal{C}}$. We conclude, by definition of $\leq_{\tilde{\mathcal{C}}}$, that the entry $B_{e,v}$ is above the diagonal. Conversely, if $B_{e,v}$ is a diagonal entry, then $\mathrm{beg}_{k-1}(e) = v$ and so $B_{e,v} = 1$ is non-zero.

We conclude that $A|_{\mathcal{V}\times\mathfrak{C}\mathcal{E}}$ is an upper triangular matrix with non-zero entries on the diagonal. This concludes the proof that $\mathrm{rk}\, A \geq |\mathrm{Av}_k(\searrow_n)| + |\mathcal{C}_{n-1}(k-1)| - |\mathrm{Av}_{k-1}(\searrow_n)|$. $\square$

Example 4.23 (The case $n = 3$ and $k = 3$)**.** As alluded to above, we present the matrix A, introduced in the proof of Theorem 1.12 for the case $n = 3$ and $k = 3$:

$$A := \begin{array}{c} \\ 123 \\ 132 \\ 213 \\ 231 \\ 312 \\ 12 \\ 12 \\ 12 \\ 21 \end{array} \begin{array}{c} \begin{array}{ccccccccc} 123 & 123 & 123 & 123 & 132 & 132 & 213 & 231 & 312 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 & 1 & -1 & 1 \end{bmatrix} \end{array} .$$

We also present the corresponding upper triangular minor of size $d \times d$ with $d = |\mathrm{Av}_3(321)| + |\mathcal{C}_2(2)| - |\mathrm{Av}_2(321)| = 7$. Some choices were made to obtain this matrix that we clarify here. Recall that given a permutation ρ with a colouring $\mathfrak{c}$ we set $\mathrm{per}(\rho, \mathfrak{c}) = \rho$. The set $\mathfrak{C}\mathcal{N}\mathcal{E}^3(3)$ of edges in bijection with $\mathcal{N}\mathcal{E}^3(3) = \{231, 132, 312\}$ via the map per was chosen to be $\mathfrak{C}\mathcal{N}\mathcal{E}^3(3) = \{231, 132, 312\}$, but could have been, for instance, $\{231, 132, 312\}$. The ordering on these sets must be consistent with the map per, thus we fix

$$\{132 < 231 < 312\}, \quad \{132 < 231 < 312\}.$$

For the order on the set $\mathcal{C}_{n-1}(k-1) \setminus \{\text{beg}_{k-1} \mathbb{S}(\gamma)\} = \{12, 12, 21\}$, we have to choose a linear order such that $12 \leq 12$ and $12 \leq 21$, thus the following works:

$$\{12 < 12 < 21\} .$$

Finally, the corresponding order in $\mathfrak{C}\mathcal{E}^k(k) \setminus \{\mathbb{S}(\gamma)\} = \{123, 123, 213\}$ is

$$\{\text{comp}(12) < \text{comp}(12) < \text{comp}(21)\} = \{123 < 123 < 213\} .$$

In this way, the matrix $A|_{\mathcal{V}\times\mathfrak{C}\mathcal{E}}$ is upper triangular:

$$A|_{\mathcal{V}\times\mathfrak{C}\mathcal{E}} \coloneqq \begin{array}{c} \\ 123 \\ 12 \\ 12 \\ 21 \\ 132 \\ 231 \\ 312 \end{array} \begin{array}{c} \begin{array}{ccccccc} 123 & 123 & 123 & 213 & 132 & 231 & 312 \end{array} \\ \left[\begin{array}{c|ccc|ccc} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array} .$$

Acknowledgements

The authors are very grateful to Valentin Féray and Mathilde Bouvel for some precious discussions during the preparation of this paper.

$$
\begin{array}{c}
 \\ 123 \\ 132 \\ 213 \\ 231 \\ 312 \\ 321 \\ 12 \\ 12 \\ 12 \\ 12 \\ 12 \\ 12 \\ 21 \\ 21 \\ 21
\end{array}
\begin{array}{c}
\begin{array}{ccccccccccccccccccccccccccccc}
123 & 123 & 123 & 123 & 123 & 123 & 123 & 123 & 123 & 123 & 132 & 132 & 132 & 132 & 132 & 132 & 213 & 213 & 213 & 213 & 231 & 231 & 231 & 231 & 312 & 312 & 312 & 312 & 321
\end{array} \\
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1
\end{bmatrix}
\end{array}.
$$

References

[AAH$^+$02] M. H. Albert, M. D. Atkinson, C. C. Handley, D. A. Holton, and W. Stromquist. On packing densities of permutations. *Electron. J. Combin.*, 9(1):Research Paper 5, 20, 2002.

[Bar04] R. W. Barton. Packing densities of patterns. *Electron. J. Combin.*, 11(1):Research Paper 80, 16, 2004.

[BBF$^+$19] F. Bassino, M. Bouvel, V. Féray, L. Gerin, M. Maazoun, and A. Pierrot. Universal limits of substitution-closed permutation classes. *Journal of the European Mathematical Society, to appear*, 2019.

[Bev20] D. Bevan. Independence of permutation limits at infinitely many scales. *arXiv preprint:2005.11568*, 2020.

[BKL$^+$18] M. Bukata, R. Kulwicki, N. Lewandowski, L. Pudwell, J. Roth, and T. Wheeland. Distributions of statistics over pattern-avoiding permutations. *arXiv preprint:1812.07112*, 2018.

[Bón12] M. Bóna. *Combinatorics of permutations.* Discrete Mathematics and its Applications (Boca Raton). CRC Press, Boca Raton, FL, second edition, 2012. With a foreword by Richard Stanley.

[Bor20a] J. Borga. Asymptotic normality of consecutive patterns in permutations encoded by generating trees with one-dimensional labels. *arXiv preprint:2003.08426*, 2020.

[Bor20b] J. Borga. Local convergence for permutations and local limits for uniform ρ-avoiding permutations with $|\rho| = 3$. *Probab. Theory Related Fields*, 176(1-2):449–531, 2020.

[BP19] J. Borga and R. Penaguiao. The feasible region for consecutive patterns of permutations is a cycle polytope. *arXiv preprint:1910.02233*, 2019.

[DHW03] E. Deutsch, A. J. Hildebrand, and H. S. Wilf. Longest increasing subsequences in pattern-restricted permutations. *arXiv preprint:math/0304126*, 2003.

[Gre74] C. Greene. An extension of Schensted's theorem. *Advances in Mathematics*, 14(2):254–265, 1974.

[HKM$^+$13] C. Hoppen, Y. Kohayakawa, C. G. Moreira, B. Ráth, and R. Menezes Sampaio. Limits of permutation sequences. *J. Combin. Theory Ser. B*, 103(1):93–113, 2013.

[Jan17] S. Janson. Patterns in random permutations avoiding the pattern 132. *Combinatorics, Probability and Computing*, 26(1):24–51, 2017.

[Jan18a] S. Janson. Patterns in random permutations avoiding some sets of multiple patterns. *ArXiv preprint:1804.06071*, 2018.

[Jan18b] S. Janson. Patterns in random permutations avoiding the pattern 321. *Random Structures Algorithms,*, 2018.

[MY19] T. Mansour and G. Yildirim. Longest increasing subsequences in involutions avoiding patterns of length three. *Turkish Journal of Mathematics*, 43(5):2183–2192, 2019.

[Pri97] A. L. Price. *Packing densities of layered patterns.* ProQuest LLC, Ann Arbor, MI, 1997. Thesis (Ph.D.)–University of Pennsylvania.

[Pud20] L. Pudwell. Packing patterns in restricted permutations. *(in preparation, for some slides see http://faculty.valpo.edu/lpudwell/slides/PP2019Pudwell.pdf)*, 2020+.

[Slo96] N. J. A. Sloane. *The On-line Encyclopedia of Integer Sequences (OEIS).* Published electronically at https://oeis.org, 1996.