ON THE NUMBER OF PARTITIONS OF n WHOSE PRODUCT OF THE SUMMANDS IS AT MOST n

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Abstract. We prove an explicit formula to count the partitions of n whose product of the summands is at most n. In the process, we also deduce a result to count the multiplicative partitions of n.

1. Introduction

A partition of a non-negative integer n is a representation of n as a sum of unordered positive integers which are called parts or summands of that partition. The number of partitions of n is denoted by $p(n)$. For example, the partitions of 7 are:

1+1+1+1+1+1+1,
1+1+1+1+1+2, 1+1+1+1+3, 1+1+1+4, 1+1+5, 1+6, 7,
1+1+1+2+2, 1+1+2+3, 1+2+4, 2+5, 1+3+3, 3+4,
1+2+2+2, 2+2+3.
So, $p(7) = 15$.

Many restricted partitions such as partitions with only odd summands, partitions with distinct summands, partitions whose summands are divisible by a certain number, partitions with restricted number of summands, partitions with designated summands, etc. have been studied over the last two and half centuries. The partitions of n whose product of the summands is at most n are another kind of restricted partition. We denote the total number of such partitions of n by $p_{\le n}(n)$. Also, we denote by $p_{=n}(n)$ the number of partitions of n whose product of the summands is equal to n. Similarly we use the notations $p_{<n}(n)$, $p_{\ge n}(n)$, etc.

The value of the product of the summands of a partition depends on the summands of that partition which are greater than one. We call these summands the non-one summands or non-one parts. Let n be a positive integer and the canonical decomposition of n as a product of distinct primes be

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r},$$

where $\alpha_i \in N$ for all $i = 1, 2, \ldots, r$.

Proposition 1.1. *The product of the summands of a partition of n is equal to n if and only if*

(1) *each summand is a divisor of n, and*
(2) *p_i appears as a factor of the summands exactly α_i times, for all $i = 1, 2, \ldots, r$.*

Proposition 1.2. *Let the product of the summands of a partition of n be n, and each non-one summand of the partition has one prime factor. The total number of such partitions of n is*

$$\prod_{i=1}^{r} p(\alpha_i).$$

A multiplicative partition of a positive integer n is a representation of n as a product of unordered positive non-one integers. The multiplicative partition function was introduced by

2020 *Mathematics Subject Classification.* 11P81, 05A17.
Key words and phrases. partition function, product of summands, multiplicative partition, unordered factorization, floor function.

MacMahon [6], [7] in 1923. Since then many properties of this function have been studied. Some works can be found in [1], [2], [3], [4], [5] and [8]. $p_{=n}(n)$ is equal to the total number of multiplicative partitions of n. In this paper, we introduce and prove formulas for $p_{\leq n}(n)$ and $p_{<n}(n)$, and using both we find $p_{=n}(n)$.

2. Formulas for $p_{\leq n}(n)$, $p_{<n}(n)$ and $p_{=n}(n)$

The floor function of x, denoted by $\lfloor x \rfloor$, is the greatest integer less than or equal to x, where x is any real number. In this section this function appears many times.

Theorem 2.1. *We have*

$$p_{\leq n}(n) = n + \sum_{k=2}^{l} \sum_{i_1=2}^{\lfloor \sqrt[k]{n} \rfloor} \sum_{i_2=i_1}^{\left\lfloor \sqrt[k-1]{\frac{n}{i_1}} \right\rfloor} \sum_{i_3=i_2}^{\left\lfloor \sqrt[k-2]{\frac{n}{i_1 i_2}} \right\rfloor} \cdots \sum_{i_{k-1}=i_{k-2}}^{\left\lfloor \sqrt{\frac{n}{i_1 i_2 \cdots i_{k-2}}} \right\rfloor} \left(\left\lfloor \frac{n}{i_1 i_2 \cdots i_{k-1}} \right\rfloor - i_{k-1} + 1 \right), \tag{2.1}$$

where $2^l \leq n < 2^{l+1}$.

Proof. $1 + 1 + 1 + \cdots + 1$ is the only partition of n where there is no non-one summand. The partitions of n where there is only one non-one summand are

$$1 + \cdots + 1 + 1 + 2, 1 + \cdots + 1 + 3, \ldots, 1 + 1 + (n-2), 1 + (n-1) \text{ and } n.$$

So we get n partitions whose product of summands $\leq n$ and one partition whose product of summands $= n$.

Now we count the partitions where there are only two non-one summands.

$$\begin{aligned} 1 + \cdots + 1 + 1 + 2 + 2, 1 + \cdots + 1 + 2 + 3, 1 + \cdots + 2 + 4, \ldots, 1 + 2 + (n-3), 2 + (n-2), \\ 1 + \cdots + 1 + 3 + 3, 1 + \cdots + 3 + 4, \ldots, 1 + 3 + (n-4), 3 + (n-3), \\ \ldots\ldots, \\ \ldots\ldots, \\ \text{and so on.} \end{aligned}$$

Here, in the first row, where the initial non-one summand is 2, the total number of partitions whose product of summands $\leq n$ is $\left\lfloor \frac{n}{2} \right\rfloor - 1$. That of second row is $\left\lfloor \frac{n}{3} \right\rfloor - 2$. In this way we can count up to the row where initial non-one summand is $\lfloor \sqrt{n} \rfloor$; since $(\lfloor \sqrt{n} \rfloor + 1)(\lfloor \sqrt{n} \rfloor + 1) > n$. Thus, the total number of such partitions having exactly two non-one summands is

$$\sum_{i_1=2}^{\lfloor \sqrt{n} \rfloor} \left(\left\lfloor \frac{n}{i_1} \right\rfloor - i_1 + 1 \right).$$

To see the clear picture, we now count such partitions where there are four non-one summands.

$$\begin{aligned} 1+\cdots+1+1+2+2+2+2, 1+\cdots+1+2+2+2+3, 1+\cdots+2+2+2+4, \ldots, 2+2+2+(n-6), \\ 1 + \cdots + 1 + 2 + 2 + 3 + 3, 1 + \cdots + 2 + 2 + 3 + 4, \ldots, 2 + 2 + 3 + (n-7), \\ 1 + \cdots + 1 + 2 + 2 + 4 + 4, \ldots, 2 + 2 + 4 + (n-8), \\ \ldots\ldots, \\ \ldots\ldots, \\ 1 + \cdots + 1 + 2 + 3 + 3 + 3, 1 + \cdots + 2 + 3 + 3 + 4, 1 + \cdots + 2 + 3 + 3 + 5, \ldots, 2 + 3 + 3 + (n-8), \\ 1 + \cdots + 2 + 3 + 4 + 4, 1 + \cdots + 2 + 3 + 4 + 5, \ldots, 2 + 3 + 4 + (n-9), \\ 1 + \cdots + 2 + 3 + 5 + 5, \ldots, 2 + 3 + 5 + (n-10), \\ \ldots\ldots, \\ \ldots\ldots, \\ 1 + \cdots + 1 + 3 + 3 + 3 + 3, 1 + \cdots + 3 + 3 + 3 + 4, 1 + \cdots + 3 + 3 + 3 + 5, \ldots, 3 + 3 + 3 + (n-9), \\ \ldots\ldots, \\ \ldots\ldots, \end{aligned}$$

and so on.

In the first row, the number of partitions whose product of summands $\le n$ is $\left\lfloor \frac{n}{2\cdot 2\cdot 2} \right\rfloor - 1$. So, in the first group of rows, the number of partitions whose product of summands $\le n$ is

$$\sum_{i_3=2}^{\left\lfloor \sqrt{\frac{n}{2\cdot 2}} \right\rfloor} \left(\left\lfloor \frac{n}{2\cdot 2\cdot i_3} \right\rfloor - i_3 + 1 \right).$$

In the second group of rows it equals to

$$\sum_{i_3=3}^{\left\lfloor \sqrt{\frac{n}{2\cdot 3}} \right\rfloor} \left(\left\lfloor \frac{n}{2\cdot 3\cdot i_3} \right\rfloor - i_3 + 1 \right).$$

When the initial non-one summand is 2, then we can count up to the partitions where the second non-one summand is $\left\lfloor \sqrt[3]{\frac{n}{2}} \right\rfloor$. Thus, when the initial non-one summand is 2, then the number of partitions whose product of summands $\le n$ is

$$\sum_{i_2=2}^{\left\lfloor \sqrt[3]{\frac{n}{2}} \right\rfloor} \sum_{i_3=i_2}^{\left\lfloor \sqrt{\frac{n}{2\cdot i_2}} \right\rfloor} \left(\left\lfloor \frac{n}{2\cdot i_2\cdot i_3} \right\rfloor - i_3 + 1 \right).$$

In this way, we can count up to the group of rows, where the initial non-one summand is $\lfloor \sqrt[4]{n} \rfloor$. Therefore, the total number of such partitions having exactly four non-one summands is

$$\sum_{i_1=2}^{\lfloor \sqrt[4]{n} \rfloor} \sum_{i_2=i_1}^{\left\lfloor \sqrt[3]{\frac{n}{i_1}} \right\rfloor} \sum_{i_3=i_2}^{\left\lfloor \sqrt{\frac{n}{i_1\cdot i_2}} \right\rfloor} \left(\left\lfloor \frac{n}{i_1\cdot i_2\cdot i_3} \right\rfloor - i_3 + 1 \right).$$

Thus, when there are exactly k non-one summands, then the number of partitions whose product of summands $\le n$ is

$$\sum_{i_1=2}^{\lfloor \sqrt[k]{n} \rfloor} \sum_{i_2=i_1}^{\left\lfloor \sqrt[k-1]{\frac{n}{i_1}} \right\rfloor} \sum_{i_3=i_2}^{\left\lfloor \sqrt[k-2]{\frac{n}{i_1 i_2}} \right\rfloor} \cdots \sum_{i_{k-1}=i_{k-2}}^{\left\lfloor \sqrt{\frac{n}{i_1 i_2 \cdots i_{k-2}}} \right\rfloor} \left(\left\lfloor \frac{n}{i_1 i_2 \cdots i_{k-1}} \right\rfloor - i_{k-1} + 1 \right).$$

If $2^l \le n < 2^{l+1}$, then a partition of n, whose product of summands is $\le n$, must have at most l non-one summands. This completes the proof. □

Corollary 2.2.

$$p_{<n}(n) = n - 1 + \sum_{k=2}^{l} \sum_{i_1=2}^{\lfloor \sqrt[k]{n} \rfloor} \sum_{i_2=i_1}^{\left\lfloor \sqrt[k-1]{\frac{n}{i_1}} \right\rfloor} \sum_{i_3=i_2}^{\left\lfloor \sqrt[k-2]{\frac{n}{i_1 i_2}} \right\rfloor} \cdots \sum_{i_{k-1}=i_{k-2}}^{\left\lfloor \sqrt{\frac{n}{i_1 i_2 \cdots i_{k-2}}} \right\rfloor} \left(\left\lfloor \frac{n-1}{i_1 i_2 \cdots i_{k-1}} \right\rfloor - i_{k-1} + 1 \right), \quad (2.2)$$

where $2^l \le n < 2^{l+1}$.

Corollary 2.3. $p_{=n}(n) = p_{\le n}(n) - p_{<n}(n)$.

Corollary 2.4. *We have*

(1) $p_{\le n}(n) = p_{<n+1}(n+1)$.
(2) $p_{\le n}(n) = p_{=n}(n) + p_{\le n-1}(n-1)$.

Proof. If $2^l \le n < 2^{l+1}$, then one of the following two cases must be satisfied:

- $2^l \le n - 1 < 2^{l+1}$
- $2^{l-1} \le n - 1 < 2^l$

In the first case $\left\lfloor \sqrt[k]{\frac{n}{m}} \right\rfloor = \left\lfloor \sqrt[k]{\frac{n-1}{m}} \right\rfloor$ for all $2 \leq k \leq l$ and $1 \leq m \leq n-1$. So, from (2.2) we get

$$\begin{aligned} p_{<n}(n) &= n - 1 + \sum_{k=2}^{l} \sum_{i_1=2}^{\lfloor \sqrt[k]{n-1} \rfloor} \sum_{i_2=i_1}^{\left\lfloor \sqrt[k-1]{\frac{n-1}{i_1}} \right\rfloor} \sum_{i_3=i_2}^{\left\lfloor \sqrt[k-2]{\frac{n-1}{i_1 i_2}} \right\rfloor} \cdots \sum_{i_{k-1}=i_{k-2}}^{\left\lfloor \sqrt{\frac{n-1}{i_1 i_2 \cdots i_{k-2}}} \right\rfloor} \left(\left\lfloor \frac{n-1}{i_1 i_2 \cdots i_{k-1}} \right\rfloor - i_{k-1} + 1 \right) \\ &= p_{\leq n-1}(n-1). \end{aligned}$$

In the second case $\left\lfloor \sqrt[k]{\frac{n}{m}} \right\rfloor = \left\lfloor \sqrt[k]{\frac{n-1}{m}} \right\rfloor$ for all $2 \leq k \leq l-1$ and $1 \leq m \leq n-1$. And $\lfloor \sqrt[l]{n} \rfloor = \lfloor \sqrt[l]{n-1} \rfloor = 1$. Therefore, in this case, when $k = l$ then the terms of the right hand side of (2.2) vanish. Hence in this case also we get the same result.

This completes the proof of the first result of the corollary. The second result follows from the first result and Corollary 2.3. □

Corollary 2.5. *If n is a prime, then $p_{\leq n}(n) = p_{\leq n-1}(n-1) + 1$.*

Proof. If n is a prime, then $\left\lfloor \frac{n}{m} \right\rfloor = \left\lfloor \frac{n-1}{m} \right\rfloor$ for all $2 \leq m \leq n-1$. Therefore, from (2.1) we get

$$\begin{aligned} p_{\leq n}(n) &= 1 + n - 1 + \sum_{k=2}^{l} \sum_{i_1=2}^{\lfloor \sqrt[k]{n-1} \rfloor} \sum_{i_2=i_1}^{\left\lfloor \sqrt[k-1]{\frac{n-1}{i_1}} \right\rfloor} \sum_{i_3=i_2}^{\left\lfloor \sqrt[k-2]{\frac{n-1}{i_1 i_2}} \right\rfloor} \cdots \sum_{i_{k-1}=i_{k-2}}^{\left\lfloor \sqrt{\frac{n-1}{i_1 i_2 \cdots i_{k-2}}} \right\rfloor} \left(\left\lfloor \frac{n-1}{i_1 i_2 \cdots i_{k-1}} \right\rfloor - i_{k-1} + 1 \right) \\ &= 1 + p_{\leq n-1}(n-1). \end{aligned}$$

□

3. Concluding remarks

A few values of $p_{=n}(n)$, $p_{\leq n}(n)$, $p_{\geq n}(n)$ and $p_{>n}(n)$ can be found in the OEIS sequences A001055, A096276, A319005 and A114324 respectively. The first result of Corollary 2.4 implies that the sequence A096276 gives the values of $p_{<n}(n)$ too.

It seems that an explicit formula for $p_{\geq n}(n)$ or $p_{>n}(n)$ in the spirit of Theorem 2.1 can be found. We leave this as an open problem.

Acknowledgements

The author is grateful to Manjil P. Saikia for his helpful comments on an earlier version of the paper.

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