Gray codes for Fibonacci q-decreasing words

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Abstract. An n-length binary word is q-decreasing, $q \geqslant 1$, if every of its length maximal factor of the form 0^a1^b satisfies $a = 0$ or $q{\cdot}a > b$. We show constructively that these words are in bijection with binary words having no occurrences of 1^{q+1}, and thus they are enumerated by the $(q+1)$-generalized Fibonacci numbers. We give some enumerative results and reveal similarities between q-decreasing words and binary words having no occurrences of 1^{q+1} in terms of frequency of 1 bit. In the second part of our paper, we provide an efficient exhaustive generating algorithm for q-decreasing words in lexicographic order, for any $q \geqslant 1$, show the existence of 3-Gray codes and explain how a generating algorithm for these Gray codes can be obtained. Moreover, we give the construction of a more restrictive 1-Gray code for 1-decreasing words, which in particular settles a conjecture stated recently in the context of interconnection networks by Eğecioğlu and Iršič.

1 Introduction and preliminaries

The Fibonacci sequence origins have been traced back to the works of ancient Indian mathematician Ācārya Piṅgala dealing with rhythmic structure patterns in Sanskrit poetry [18,11, p. 50]. Over the time, the study of words and patterns became more abstract and systematic (see for instance Lothaire's books [12,13,14] and [3]). An important amount of questions concerning efficient enumeration and generation of words respecting certain properties (including pattern avoidance) were mathematically formulated and answered only relatively recently, the works closely related to the present study include [1,2,4,5,7,20,21,22].

In this paper we introduce q-decreasing words, a novel class of run-restricted binary words enumerated by the $(q + 1)$-generalized Fibonacci numbers, $q \geqslant 1$. For $q = 1$ the subclass of such words that start with 0 was recently considered in the context of induced subgraphs of hypercubes [4,5]. In Section 2 we present a bijection between this novel class of words and Fibonacci words, i.e. binary words avoiding consecutive 1s. Section 3 is devoted to the presentation of several generating functions and enumeration results. Finally, in Section 4, we show the existence of a 3-Gray code for any $q \geqslant 1$, give an efficient exhaustive generating algorithms and a much more involved construction for a 1-Gray code in the special case $q = 1$. In particular, the latter Gray code gives a Hamiltonian path in Fibonacci-run graphs whose existence is conjectured in [4].

The following set of notations is adopted. Let $\mathcal{B}$ denote the set of all finite-length binary words, i.e. strings over alphabet $\{0, 1\}$, and $\mathcal{B}_n$, $n \geqslant 0$, be the set of all binary words of length n. For a given binary word w we use the notation w_i to mean the letter at position i. A non empty sequence of adjacent letters inside a word is called *factor*. A factor x repeated k times is denoted by x^k, for instance $(00)^2 1^2 = 000011$. For a given length n, the notation x^* is used to repeat factor x as many times as possible, until the length n is reached, possibly trimming extra bits at the end; and the length n will be understood from the context. For example, if a word w of length $n = 7$ is equal to $(001)^*$ it means $w = 0010010$.

The set of all n-length binary words containing no occurrences of factor x is denoted by $\mathcal{B}_n(x)$. The concatenation of two words w and x is denoted by $w \cdot x$ or simply by wx. If x is a binary word and $\mathcal{W}$ is a set of binary words, let $\mathcal{W} \cdot x = \{w \cdot x : w \in \mathcal{W}\}$, and $x \cdot \mathcal{W}$ is defined similarly. Whenever $\mathcal{A}$ and $\mathcal{B}$ are two subsets of $\mathcal{B}$, we define $\mathcal{A} \cdot \mathcal{B} = \{a \cdot b : a \in \mathcal{A}, b \in \mathcal{B}\}$.

Following [15] the nth k-generalized Fibonacci number is defined as

$$f_{n,k} = \begin{cases} 0 & \text{if } 0 \leqslant n \leqslant k-2, \\ 1 & \text{if } n = k-1, \\ \sum_{i=1}^{k} f_{n-i,k} & \text{otherwise.} \end{cases} \tag{1}$$

Classical fact. *The number of words in $\mathcal{B}_n(1^k)$ equals $f_{n+k,k}$ for $k \geqslant 2$, moreover*

$$\mathcal{B}_n(1^k) = \begin{cases} \mathcal{B}_n & \text{if } n < k, \\ \bigcup_{i=0}^{k-1} 1^i 0 \cdot \mathcal{B}_{n-i-1}(1^k) & \text{otherwise.} \end{cases} \tag{2}$$

The classical fact comes, for instance, from [10, p. 286]. The binary words avoiding consecutive 1s are counted by Fibonacci numbers, words without factor 111 are counted by Tribonacci numbers, etc. We call such words *(generalized) Fibonacci words*. The On-line Encyclopedia of Integer Sequences founded by N.J.A. Sloane [19] contains several corresponding sequences (see for example A000045 and A000073, after taking a binary complement). Gray codes for Fibonacci words are discussed in [20], and in [21].

The *Hamming distance* between two same length binary words equals the number of positions at which they differ. A k-Gray code for a set $\mathcal{A} \subset \mathcal{B}_n$ is an ordered list, denoted by $\mathbf{A}$, for $\mathcal{A}$, such that the Hamming distance between any two consecutive words in $\mathbf{A}$ is at most k, and we say that words in $\mathbf{A}$ are listed in *Gray code order*. Frank Gray's patent [7] discusses an early example and application of such a code for the set of n-length binary words. The concatenation of two ordered lists $\mathbf{L}_1$ and $\mathbf{L}_2$ is denoted by $\mathbf{L}_1 \circ \mathbf{L}_2$, and $\overline{\mathbf{L}}$ designates the reverse of the list $\mathbf{L}$. If $\mathbf{L}$ is a list of words, then $\mathbf{L}^i = \mathbf{L}$ whenever i is even, and $\mathbf{L}^i = \overline{\mathbf{L}}$ otherwise. First and last elements of $\mathbf{L}$ are denoted respectively by $\text{first}(\mathbf{L})$ and $\text{last}(\mathbf{L})$. Also, we denote by $\widehat{\mathbf{L}}$ the list obtained from $\mathbf{L}$ by deleting its last element.

Definition 1. *A binary word is called* q-decreasing*, for $q \geqslant 1$, if any of its length maximal factors of the form $0^a 1^b$, $a > 0$, satisfies $q \cdot a > b$.*

The set of q-decreasing words of length n is denoted by $\mathcal{W}_n^q$. For example we have $\mathcal{W}_4^1 = \{0000, 0001, 0010, 1000, 1001, 1100, 1110, 1111\}$. See also Table 1 for the sets $\mathcal{W}_4^2$ and $\mathcal{W}_6^1$.

2 Bijection with classical Fibonacci words

In this section we prove that q-decreasing words, $q \geqslant 1$, are enumerated by $(q+1)$-generalized Fibonacci numbers defined in relation (1). We start with a definition and several propositions.

Definition 2. *For any $q \geqslant 1$, we define the map ψ^q from $\mathcal{B}_n$ to $\mathcal{B}_{n+q+1}$ as*

$$\psi^q(w) = \begin{cases} v001^{k+q} & \text{if } w = v01^k, v \in \mathcal{B}, k \geqslant 0, \\ 1^{n+q+1} & \text{otherwise.} \end{cases}$$

Less formally, ψ^q inserts a factor 01^q immediately after the last occurrence of 0, and it adds the suffix 1^{q+1} to the word containing no 0. For example $\psi^1(0) = 0\mathbf{01}$, $\psi^1(00011) = 000\mathbf{01}11$, $\psi^2(0011101) = 001110\mathbf{011}1$ and $\psi^5(1) = 1\mathbf{111111}$. The value of q will be clear from the context, so by slight abuse of notation ψ^q will be denoted ψ throughout the paper.

Proposition 1. *ψ is an injective map.*

Proof. For two n-length words $w \neq w'$ we show that $\psi(w) \neq \psi(w')$. It is clear that if at least one of the given words contains no 0 the injectivity holds. Otherwise we have two cases. If $w = v01^k$ and $w' = v'01^k$ then we have necessary $v \neq v'$ and $v001^{k+q} \neq v'001^{k+q}$, so the images are different. If $w = v01^k$ and $w' = v'01^\ell$ with $k \neq \ell$, then $v001^{k+q} \neq v'001^{\ell+q}$ and again $\psi(w) \neq \psi(w')$. □

In the following, we will use the restriction of ψ to the set of q-decreasing words, namely $\psi : \mathcal{W}_n^q \to \mathcal{W}_{n+q+1}^q$. It is possible due to Proposition 2 below.

Proposition 2. *For $n, q > 0$, $\psi(\mathcal{W}_n^q)$ consists of all q-decreasing words of length $n+q+1$ ending with at least q ones.*

Proof. If $w = 1^n$, then $\psi(w) = 1^{n+q+1}$. Otherwise, we write $w = v0^a1^b$ where $a > b/q \geqslant 0$ and the word v is either empty or ends with 1. So $\psi(v0^a1^b) = v0^{a+1}1^{q+b}$. As we have $1 + a > 1 + b/q = (q+b)/q$, $\psi(w)$ is a q-decreasing word ending with at least q 1s. Similarly, any $(n+q+1)$-length q-decreasing word ending with at least q 1s can be obtained from a (unique) word in $\mathcal{W}_n^q$ by ψ. □

Now, we present a one-to-one correspondence between Fibonacci and q-decreasing words. Recall that, for $q \geqslant 1$, the set $\mathcal{B}(1^{q+1})$ of $(q+1)$-generalized Fibonacci words is the set of binary words with no 1^{q+1} factors, see relation (2) for the recursive definition of these words according to their length.

Definition 3. *We define the length-preserving map* $\phi : \mathcal{B}(1^{q+1}) \to \mathcal{W}^q$ *as*

$$\phi(w) = \begin{cases} 1^k & \text{if } w = 1^k \text{ and } k \in [0, q], \\ \psi\big(\phi(v)\big) & \text{if } w = 1^q 0 v, \\ \phi(v) 0 1^k & \text{if } w = 1^k 0 v \text{ and } k \in [0, q-1]. \end{cases} \tag{3}$$

See Table 1(a) for the images of the words in $\mathcal{B}_4(111)$ through ϕ.

Theorem 1. *The map* ϕ *is a bijection between* $\mathcal{B}(1^{q+1})$ *and* $\mathcal{W}^q$.

Proof. We proceed by induction on n. The classical decomposition in relation (2) gives rise to three cases. (i) Any word of the form 1^k is sent by ϕ to 1^k for any $k \in [0, q]$. (ii) Words of the form $1^q 0 v$, where $v \in \mathcal{B}(1^{q+1})$ are sent to words ending by at least q 1s. (iii) Words of the form $1^k 0 v$, $k \in [0, q-1]$, where $v \in \mathcal{B}(1^{q+1})$ are sent to words ending by at most $q-1$ 1s. Using the bijectivity of ψ (see Proposition 1 and Proposition 2) and induction hypothesis, one can easily show that for any two different words $w \neq w'$ we have $\phi(w) \neq \phi(w')$.

Similarly, by induction on n, any word in $\mathcal{W}^q$ can be obtained by ϕ from a word in $\mathcal{B}(1^{q+1})$, and the statement holds. □

It follows that Fibonacci words of order $(q+1)$ and q-decreasing words are equinumerous.

3 Some enumeration results

Here we provide a bivariate generating function $W_q(x, y) = \sum_{n,k \geqslant 0} w_{n,k} x^n y^k$, where $w_{n,k}$ is the number of n-length q-decreasing words having k 1s. This bivariate generating function is of a particular interest since it will help us (see Corollary 1) to prove a necessary condition for the existence of 1-Gray code, called *parity condition*. More precisely, if a set $\mathcal{A}$ of binary words admits a 1-Gray code, and $\mathcal{A}^+$ (resp. $\mathcal{A}^-$) denotes the subset of $\mathcal{A}$ having even (resp. odd) number of 1s, then the *parity difference* $|\mathcal{A}^+| - |\mathcal{A}^-|$ must be equal to either 0, 1, or -1. This parity condition is used for instance in [20] to investigate the possibility of 1-Gray code for a set of words avoiding a given factor.

In order to derive the expression of $W_q(x, y)$, we use the following decomposition of the set $\mathcal{W}^q$:

$$\mathcal{W}^q = \mathbb{1} \cup \mathcal{W}^q \cdot \mathcal{S}^q,$$

where $\mathbb{1} = \cup_{n=0}^{\infty} \{1^n\}$ and $\mathcal{S}^q$ corresponds to the set of all factors of the form $0^a 1^b$ respecting q-decreasing property (i.e. $a > b/q \geqslant 0$) such that none of them is a concatenation of others factors from $\mathcal{S}^q$. More precisely, a is the smallest integer strictly greater than b/q, i.e. $a = \lfloor b/q \rfloor + 1$. A factor from $\mathcal{S}^q$ will be called *q-prime factor*, and thus $\mathcal{S}^q$ is the set of such factors. For instance: $\mathcal{S}^1 = \{0, 001, 00011, 0000111, \ldots\}$, $\mathcal{S}^2 = \{0, 01, 0011, 00111, 0001111, 00011111, \ldots\}$.

Lemma 1. *The bivariate generating function* $S_q(x,y)=\sum_{n,k\geqslant 0} s_{n,k}x^ny^k$ *where the coefficient* $s_{n,k}$ *is the number of* q*-prime factors of length* n *having exactly* k 1*s is:*

$$S_q(x,y)=\frac{x\,(1-(xy)^q)}{(xy-1)(x^{q+1}y^q-1)}.$$

Proof. Any q-prime factor is of the form 0^a1^b with $a=\lfloor b/q\rfloor+1$. So, if $b=kq+r$ with $k\geqslant 0$ and $r\in[0,q-1]$, then $a+b=k(q+1)+r+1$ and we can write:

$$S_q(x,y)=\sum_{k=0}^{\infty}\sum_{r=0}^{q-1} x^{k(q+1)+r+1}y^{kq+r}.$$

A simple calculation results to the claimed formula. □

Theorem 2. *The bivariate generating function* $W_q(x,y)=\sum_{n,k\geqslant 0} w_{n,k}x^ny^k$ *where the coefficient* $w_{n,k}$ *is the number of* n*-length* q*-decreasing words containing exactly* k 1*s is given by:*

$$W_q(x,y)=\frac{1-x^{q+1}y^q}{1-xy-x+x^{q+2}y^{q+1}}.$$

Proof. Due to the decomposition $\mathcal{W}^q=\mathbb{1}\cup\mathcal{W}^q\cdot\mathcal{S}^q$, we have $W_q(x,y)=\frac{1}{1-xy}\cdot\frac{1}{1-S_q(x,y)}$, and the result hold after applying Lemma 1. □

Corollary 1. *For any* $n,q\geqslant 1$*, the set* $\mathcal{W}^q_n$ *satisfies the parity condition.*

Proof. The generating function $D_q(x)=\sum_{n\geqslant 0} d_nx^n$ where the coefficient d_n is the parity difference corresponding to the set $\mathcal{W}^q_n$ is obtained by making the substitution $y=-1$ in $W_q(x,y)$:

$$D_q(x)=\frac{(-1)^qx^{q+1}-1}{(-1)^qx^{q+2}-1}.$$

When q is even, $D_q(x)=\frac{x^{q+1}-1}{x^{q+2}-1}=\sum_{n=0}^{\infty}\left(x^{n(q+2)}-x^{n(q+2)+q+1}\right)$, otherwise $D_q(x)=\frac{x^{q+1}+1}{x^{q+2}+1}=\sum_{n=0}^{\infty}(-1)^n\left(x^{n(q+2)}+x^{n(q+2)+q+1}\right)$. All involved coefficients are from $\{-1,0,1\}$, and the parity condition holds.

The following two corollaries are obtained by respectively calculating the expressions: $W_q(x,y)\big|_{y=1}$, $\frac{\partial W_q(x,y)}{\partial y}\big|_{y=1}$ and $\frac{\partial W_q(xy,1/y)}{\partial y}\big|_{y=1}$.

Corollary 2. *The generating function* $F_q(x)=\sum_{n\geqslant 0} f_nx^n$ *where the coefficient* f_n *is the number of* n*-length* q*-decreasing words is given by:*

$$F_q(x)=\frac{1-x^{q+1}}{1-2x+x^{q+2}}.$$

Note that, as predicted by Theorem 1, $F_q(x)$ is the generating function for the integer sequence $(f_{n+q+1,q+1})_{n\geqslant 0}$, see relation (1) and the classical fact following it.

The *popularity* of a symbol in a set of words is the overall number of the symbol in the words of the set.

Corollary 3. *The generating function $P_{q,1}(x) = \sum_{n\geqslant 0} p_n x^n$ where the coefficient p_n is the popularity of* 1*s in all n-length q-decreasing words is:*

$$P_{q,1}(x) = \frac{x\left(1 - qx^q + qx^{q+1} - 2x^{q+1} + x^{2q+2}\right)}{\left(1 - 2x + x^{q+2}\right)^2}.$$

Similarly, the generating function for the popularity of 0*s in all n-length q-decreasing words is:*

$$P_{q,0}(x) = \frac{x\left(1 - x^q\right)}{\left(1 - 2x + x^{q+2}\right)^2}.$$

The popularity of 1s in $\mathcal{B}_n(11)$ is equal to the number of edges in the Fibonacci cube [8] of order n, see [9] and comments to the sequence A001629 in [19]. The generating function $P_{1,0}(x)$ allows us to show that the popularity of 0s in $\mathcal{W}_n^1$ is a shift of the sequence A006478 enumerating the number of edges in the *Fibonacci hypercube* [16], i.e. in a polytope determined by the convex hull of the Fibonacci cube.

Despite the q-decreasing words and Fibonacci words have quite different definitions, they are equinumerous and share some common features. We end this section by showing that the 1s frequency (define formally below) of both sets have the same limit when n tends to infinity.

If u_n (resp. v_n) is the ratio between the popularity of 1s and that of 0s in the words in $\mathcal{W}_n^1$ (resp. in $\mathcal{B}_n(11)$), then $\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n$. Indeed, extracting the coefficients of x^n in both $P_{1,1}$ and of $P_{1,0}$, their ratio tends to $2 - \varphi \approx 0.3819660113$ when n tends to infinity, where φ is the golden ratio; and this is also the limit of v_n.

The *1s frequency* of a set of binary words is the ratio between the popularity of 1s and the overall number of bits in the words of the set. Alternatively, it is the expected value when a bit is randomly chosen in the words of the set. With the notations above, we have that the 1s frequency of $\mathcal{W}_n^1$ is $\frac{1}{1+\frac{1}{u_n}}$ and that of $\mathcal{B}_n(11)$ is $\frac{1}{1+\frac{1}{v_n}}$, and we have the next result.

Corollary 4. *The 1s frequency of $\mathcal{W}_n^1$ and of $\mathcal{B}_n(11)$ both tend to $\frac{2-\varphi}{3-\varphi}$ when n tends to infinity, where φ is $\frac{1+\sqrt{5}}{2}$.*

More generally, for any $q \geqslant 1$, the overall number of bits in both sets $\mathcal{W}_n^q$ and $\mathcal{B}_n(1^{q+1})$ is $n \cdot f_{n+q+1,q+1}$, and due to the second rule in relation (3) defining the bijection $\phi : \mathcal{B}(1^{q+1}) \to \mathcal{W}^q$ we have that in $\mathcal{W}_n^q$ there are more 1s than in $\mathcal{B}_n(1^{q+1})$. However, the next corollary shows that the difference between the 1s frequency of $\mathcal{W}_n^q$ and that of $\mathcal{B}_n(1^{q+1})$ tends to zero when n tends to infinity.

Corollary 5. *For any $q \geqslant 1$, if $u_{n,q}$ (resp. $v_{n,q}$) is the popularity of* 1*s in $\mathcal{W}_n^q$ (resp. in $\mathcal{B}_n(1^{q+1})$), then we have*

$$\lim_{n\to\infty} \frac{u_{n,q} - v_{n,q}}{n \cdot f_{n+q+1,q+1}} = 0.$$

Proof. Since, for any $q \geqslant 1$, $u_{n,q} - v_{n,q} \geqslant 0$, it suffices to prove that we have $u_{n,q} - v_{n,q} \leqslant f_{n+q+1,q+1}$. Alternative to relation (2), the set $\mathcal{B}(1^{q+1})$ of (any length) binary words avoiding 1^{q+1} can be defined recursively as

$$\mathcal{B}(1^{q+1}) = \mathbb{1}_q \cup \bigcup_{i=0}^{q} 1^i 0 \cdot \mathcal{B}(1^{q+1})$$

where $\mathbb{1}_q = \bigcup_{i=0}^{q}\{1^i\}$. It follows that the bivariate generating function $F_q(x,y)$ where the coefficient of $x^n y^k$ is the number of Fibonacci words having $k \leqslant q$ 1s in $\mathcal{B}_n(1^{q+1})$ satisfies the functional equation

$$F_q(x,y) = \sum_{i=0}^{q} x^i y^i + F_q(x,y) \sum_{i=0}^{q} x^{i+1} y^i,$$

and we have $F_q(x,y) = \frac{y\left(1-(xy)^{q+1}\right)}{y-xy^2-xy+(xy)^{q+2}}$. Using Corollary 3, the generating function $H(x)$ where the coefficient of x^n is $f_{n+q+1,q+1} + v_{n,q+1} - u_{n,q}$ is

$$\begin{aligned} H(x) &= F_q(x,1) + \left.\frac{\partial F_q(x,y)}{\partial y}\right|_{y=1} - P_{q,1}(x) \\ &= \frac{1-2x^{q+1}}{1-2x+x^{q+2}}, \end{aligned}$$

which satisfies the functional equation $H(x) = 1 - 2x^{q+1} + 2xH(x) - x^{q+2}H(x)$. By a simple observation, $H(x)$ is also the generating function with respect to the length of binary words different from 0^{q+1} and 1^{q+1} and that do not start with 0^{q+2}. Then we have $u_{n,q} - v_{n,q} \leqslant f_{n+q+1,q+1}$. Dividing by $nf_{n+q+1,q+1}$, and taking the limit when n tends to infinity, we obtain the expected result. □

Corollary 4 says that, for $q = 1$, the 1s frequency of $\mathcal{W}_n^q$ and that of $\mathcal{B}_n(1^{q+1})$ have a common limit when n tends to infinity. For $q \geqslant 2$, Corollary 5 does not ensure that each of the 1s frequency of $\mathcal{W}_n^q$ (that is $\frac{u_{n,q}}{n \cdot f_{n+q+1,q+1}}$) and that of $\mathcal{B}_n(1^{q+1})$ (that is $\frac{v_{n,q}}{n \cdot f_{n+q+1,q+1}}$) has a limit when n tends to infinity. However, using asymptotic analysis (see for instance [6]) it can be shown that $\frac{v_{n,q}}{n \cdot f_{n+q+1,q+1}}$ converges to a non-zero value when n tends to infinity, and the limit can be approximated by numerical methods. From Corollary 5 it follows that so does $\frac{u_{n,q}}{n \cdot f_{n+q+1,q+1}}$, and the two limits are equal.

Since the proof of this result is beyond the scope of the present paper we state it (including the case $q = 1$ in Corollary 4) without proof.

Corollary 6. *For $q \geqslant 1$ the 1s frequency of $\mathcal{W}_n^q$ and of $\mathcal{B}_n(1^{q+1})$ have a common non-zero limit when n tends to infinity.*

4 Exhaustive generation and Gray codes for q-decreasing words

Here we show that q-decreasing words can be efficiently generated in lexicographical order and explain how the obtained generating algorithm can be turned into a 3-Gray code generating one. Then, we give a more intricate construction of a 1-Gray code for the particular case $q = 1$. As a byproduct, this construction gives a positive answer for the existence a Hamiltonian path in Fibonacci-run graph conjectured in [4].

4.1 3-Gray codes and exhaustive generation

Algorithm in Figure 1 generates prefixes of q-decreasing words in lexicographical order, and eventually all n-length q-decreasing words. The size n, parameter q and the array w of length $n + 1$ are global variables and the main call is LexFib(1, n). For convenience, $w[0]$ is initialized by 1 and the parameter *delta* is the number of consecutive 1s that can be added to the current generated prefix without violating the q-decreasingness. It can be seen that this algorithm satisfies Frank Ruskey's *constant amortized time* principle [17], and thus it is an efficient exhaustive generating algorithm.

```
procedure LexFib(pos, delta: integer)
  if (pos = n + 1) print w;
  else  w[pos] := 0;
        if (w[pos − 1] = 1) d := q − 1; else d:=delta + q; endif
        LexFib(pos + 1, d);
        if (delta > 0)
           w[pos] := 1; LexFib(pos + 1, delta − 1);
        endif
  endif
end procedure
```

Fig. 1: Lexicographic generation algorithm for q-decreasing words.

The bijection ϕ in relation (3) does not preserve *Graycodeness*: for instance, when $n = 2k + 1$ and $q = 1$ the Gray code for Fibonacci words in [21] always contains two consecutive words $u = (10)^k 1$ and $v = (10)^k 0$, but their images $\phi(u) = 1^{2k+1}$ and $\phi(v) = 0^{k+1}1^k$ have arbitrarily large Hamming distance for enough large n. A similar phenomenon happens when $n = 2k$ and $q = 1$ with $u = (10)^{k-2}10$ and $v = (10)^{k-2}00$: $\phi(u) = 1^{2k}$ and $\phi(v) = 0^{k+1}1^{k-1}$. See also Table 1(a) for the image through ϕ of the 1-Gray code in [21] for $\mathcal{B}_4(111)$.

Below we show that BRGC order (that is, the order induced by Binary Reflected Gray Code in [7]) yields a 3-Gray code on $\mathcal{W}_n^q$. Much more interestingly,

$u \in \mathcal{B}_4(111)$	$\phi(u) \in \mathcal{W}_4^2$
1100	0011
1101	1111
1001	1001
1000	0001
1010	0101
1011	1101
0011	1100
0010	0100
0000	0000
0001	1000
0101	1010
0100	0010
0110	1110

(a)

Words in $\mathcal{W}_6^1$ in BRGC order

1	000000		*12*	111100	3
2	000001	1	*13*	111111	2
3	000011	1	*14*	111110	1
4	000010	1	*15*	111001	3
5	000110	1	*16*	111000	1
6	000100	1	*17*	100100	3
7	001001	3	*18*	100010	2
8	001000	1	*19*	100011	1
9	110000	3	*20*	100001	1
10	110001	1	*21*	100000	1
11	110010	2			

(b)

Table 1: (a) The images of words in $\mathcal{B}_4\,(111)$ under the bijection ϕ. Words in $\mathcal{B}_4\,(111)$ are listed in a BRGC-like order, called local reflected order in [21], which yields a 1-Gray code order. (b) The set $\mathcal{W}_6^1$ in BRGC order together with the Hamming distance between consecutive words.

thanks to Corollary 1, the necessary condition for the existence of a 1-Gray code is satisfied, and we will provide such a Gray code for $\mathcal{W}_n^1$ in the following part.

In [22] the author introduces the notion of *absorbent set*, which (up to complement) is defined as: a binary word set $\mathcal{X} \subset \{0,1\}^n$ is called absorbent if for any $u \in \mathcal{X}$ and any k, $1 \leqslant k < n$, $u_1u_2 \ldots u_{k-1}0^{n-k}$ is also a word in $\mathcal{X}$. Corollary 1 from the same paper proves that any absorbent set listed in BRGC order yields a 3-Gray. Clearly, $\mathcal{W}_n^q$ is an absorbent set and we have the following consequence.

Corollary 7. *The restriction of BRGC order yields a 3-Gray code for* $\mathcal{W}_n^q$.

Reversing lists technique [17] allows to turn the algorithm in Figure 1 into one generating the same class of words in BRGC order, so producing a 3-Gray code for $\mathcal{W}_n^q$. See for an example Table 1(b).

4.2 1-Gray code for $\mathcal{W}_n^1$

In this part, we construct a 1-Gray code for the set $\mathcal{W}_n^q$, $n \geqslant 0$, when $q = 1$, which in particular gives a positive answer to a conjecture in [4]. For this purpose, we decompose $\mathcal{W}_n^1$, $n \geqslant 1$, as

$$\mathcal{W}_n^1 = \mathcal{Z}_n \cup 1 \cdot \mathcal{W}_{n-1}^1$$

where $\mathcal{W}_0^1 = \varnothing$ and $\mathcal{Z}_n$ is the subset of words starting with 0 in $\mathcal{W}_n^1$. In turn, we decompose $\mathcal{Z}_n$ as

$$\mathcal{Z}_n = \{0^n\} \cup \bigcup_{r=3}^{n} \mathcal{D}_n^r$$

where $\mathcal{D}_n^r = \bigcup_{j=1}^{\lfloor \frac{r-1}{2} \rfloor} 0^{r-j}1^j \cdot \mathcal{Z}_{n-r}$. We refer to Figure 2(a) for a graphical illustration of the decomposition of $\mathcal{Z}_n$ for $n = 17$ where the point at coordinates (i, j) corresponds to the set $0^i1^j \cdot \mathcal{Z}_{n-i-j}$, $1 \leqslant j < i \leqslant n-1$, $3 \leqslant i + j \leqslant n$, except the lowest point which corresponds to $\{0^n\}$. The sets $\mathcal{D}_n^r$, $3 \leqslant r \leqslant n$, correspond to the southwest-northeast diagonals of the graphic.

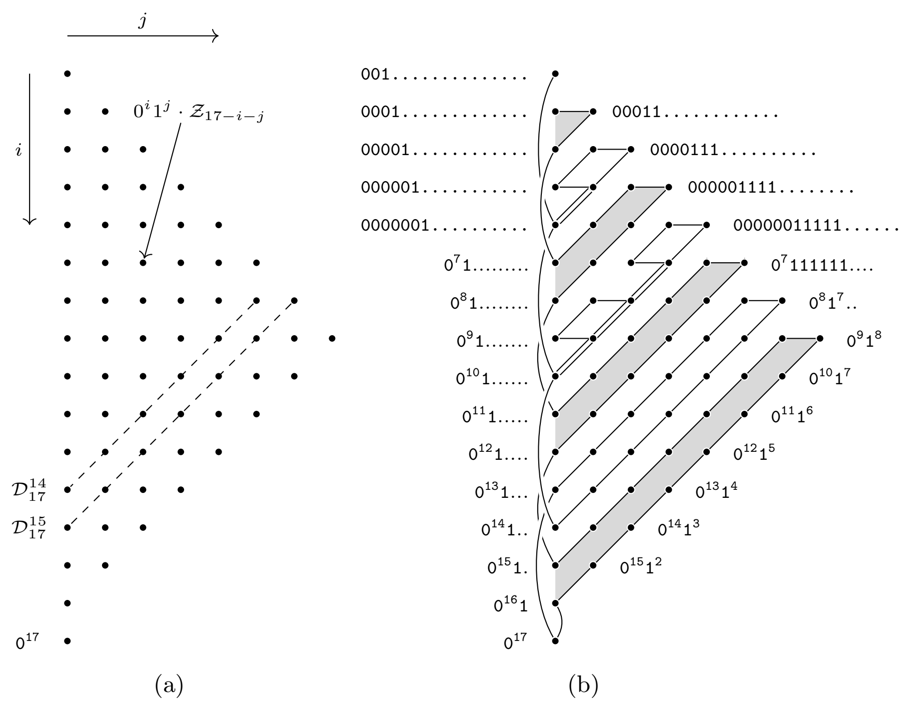

Fig. 2: (a) Decomposition of $\mathcal{Z}_{17}$ as a union of subsets $0^i1^j \cdot \mathcal{Z}_{17-i-j}$ (or equivalently a union of diagonals $\mathcal{D}_{17}^r$). (b) Illustration of the 1-Gray code $\mathbf{Z}_{17}$. The pairs of consecutive diagonals dealt with Lemma 3 are shown in gray-filled area; the other pairs are dealt with Lemma 4. A point labelled 0^91....... (that is 0^91 followed by seven dots) corresponds to the set of words in $0^91 \cdot \mathcal{Z}_7$.

According to the above definitions, it is straightforward to check the following lemma.

Lemma 2. *For any $k < n$, we suppose that $\mathbf{Z}_k$ is a 1-Gray code for $\mathcal{Z}_k$ with* $\mathrm{first}(\mathbf{Z}_k) = 0(001)^\star$ *and* $\mathrm{last}(\mathbf{Z}_k) = (001)^\star$. *Given i and j such that $1 \leqslant j < i \leqslant n$ and $3 \leqslant i + j \leqslant n$, then*

(i) the list $\mathbf{L} = 0^i1^j \cdot \mathbf{Z}_{n-i-j}$ *is a* 1-*Gray code for* $0^i1^j \cdot \mathcal{Z}_{n-i-j}$ *with* $\mathrm{first}(\mathbf{L}) = 0^i1^j0(001)^\star$ *and* $\mathrm{last}(\mathbf{L}) = 0^i1^j(001)^\star$;

(ii) for $i + j \neq n$, *the list* $\mathbf{L} = 0^i1^{j+1} \cdot \mathbf{Z}_{n-i-j-1} \circ 0^i1^j \cdot \mathbf{Z}_{n-i-j}$ *is a* 1-*Gray code for* $0^i1^j \cdot \mathcal{Z}_{n-i-j} \cup 0^i1^{j+1} \cdot \mathcal{Z}_{n-i-j-1}$ *with* $\mathrm{first}(\mathbf{L}) = 0^i1^{j+1}0(001)^\star$ *and* $\mathrm{last}(\mathbf{L}) = 0^i1^j(001)^\star$;

(iii) the list $\mathbf{L} = 0^i1^j \cdot \mathbf{Z}_{n-i-j} \circ \overline{0^{i-1}1^{j+1} \cdot \mathbf{Z}_{n-i-j}}$ *is a* 1-*Gray code for* $0^i1^j \cdot \mathcal{Z}_{n-i-j} \cup 0^{i-1}1^{j+1} \cdot \mathcal{Z}_{n-i-j}$ *with* $\mathrm{first}(\mathbf{L}) = 0^i1^j0(001)^\star$ *and* $\mathrm{last}(\mathbf{L}) = 0^{i-1}1^{j+1}0(001)^\star$;

(iv) the list $\mathbf{L} = \overline{0^i1^j \cdot \mathbf{Z}_{n-i-j}} \circ 0^{i-1}1^{j+1} \cdot \mathbf{Z}_{n-i-j}$ *is a* 1-*Gray code for* $0^i1^j \cdot \mathcal{Z}_{n-i-j} \cup 0^{i-1}1^{j+1} \cdot \mathcal{Z}_{n-i-j}$ *with* $\mathrm{first}(\mathbf{L}) = 0^i1^j(001)^\star$ *and* $\mathrm{last}(\mathbf{L}) = 0^{i-1}1^{j+1}(001)^\star$.

Lemma 3. *Let us consider $r = 1 \mod 4$, $3 \leqslant r \leqslant n$. For any $k < n$, we suppose that $\mathbf{Z}_k$ is a 1-Gray code for $\mathcal{Z}_k$ with* $\mathrm{first}(\mathbf{Z}_k) = 0(001)^\star$ *and* $\mathrm{last}(\mathbf{Z}_k) = (001)^\star$.

(i) If $r \neq n - 1$, then there is a 1-Gray code Δ_n^r for $\mathcal{D}_n^r \cup \mathcal{D}_n^{r-1}$ such that $\mathrm{first}(\Delta_n^r) = 0^{r-2}1(001)^\star$ *and* $\mathrm{last}(\Delta_n^r) = 0^{r-1}1(001)^\star$.

(ii) If $r = n-1$, then there is a 1-Gray code Δ_n^{n-1} for $\mathcal{D}_n^n \cup \mathcal{D}_n^{n-1} \cup \mathcal{D}_n^{n-2}$ such that $\mathrm{first}(\Delta_n^{n-1}) = 0^{n-3}100$ *and* $\mathrm{last}(\Delta_n^{n-1}) = 0^{n-1}1$.

Proof. For the first assertion (i), it suffices to consider the list

$$\Delta_n^r = \bigcirc_{j=1}^{\frac{r-3}{2}} 0^{r-1-j}1^j\mathbf{Z}_{n-r+1}^j \circ \overline{\bigcirc_{j=1}^{\frac{r-1}{2}} 0^{r-j}1^j\mathbf{Z}_{n-r}^j}.$$

After considering assertions of Lemma 2, it remains to examine the transition between $w_0 = 0^{r-1-j_0}1^{j_0}\mathbf{Z}_{n-r+1}^{j_0}$ for $j_0 = \frac{r-3}{2}$ and $w_1 = 0^{r-j_1}1^{j_1}\mathbf{Z}_{n-r}^{j_1}$ for $j_1 = \frac{r-1}{2}$. Since $r = 1 \mod 4$, we have necessarily $j_0 + 1 = j_1$ which implies that w_0 and w_1 differ by exactly one bit.

For the second assertion (ii), we consider the list:

$$\Delta_n^{n-1} = \bigcirc_{j=1}^{\frac{n-4}{2}} 0^{n-2-j}1^j00 \circ 0^{\frac{n}{2}}1^{\frac{n-2}{2}}0 \circ \overline{\bigcirc_{j=1}^{\frac{n-4}{2}} \left(0^{n-j-1}1^{j+1} \circ 0^{n-1-j}1^j0\right)^{j-1}} \circ 0^{n-1}1.$$

A simple study of each kind of transitions allows us to see that Δ_n^{n-1} is a 1-Gray code for $\mathcal{D}_n^n \cup \mathcal{D}_n^{n-1} \cup \mathcal{D}_n^{n-2}$ satisfying $\mathrm{first}(\Delta_n^{n-1}) = 0^{n-3}100$ and $\mathrm{last}(\Delta_n^{n-1}) = 0^{n-1}1$. An illustration of this Gray code for $n = 10$ (and thus $r = 9$) can be found in the last sketch of Figure 4. □

Lemma 4. *Let us consider $r = 3 \mod 4$, $3 \leqslant r \leqslant n$. For any $k < n$, we suppose that $\mathbf{Z}_k$ is a 1-Gray code for $\mathcal{Z}_k$ with* $\mathrm{first}(\mathbf{Z}_k) = 0(001)^\star$ *and* $\mathrm{last}(\mathbf{Z}_k) = (001)^\star$.

(i) If $r = 3$, then there is a 1-Gray code Δ_n^3 for $\mathcal{D}_n^3$ such that $\mathrm{first}(\Delta_n^3) = 0010(001)^\star$ *and* $\mathrm{last}(\Delta_n^3) = (001)^\star$.

(ii) *If* $r = n-2$*, then there is a* 1*-Gray code* Δ_n^{n-2} *for* $\mathcal{D}_n^{n-2} \cup \mathcal{D}_n^{n-3}$ *such that* $\text{first}(\Delta_n^{n-2}) = 0^{n-4}1000$ *and* $\text{last}(\Delta_n^{n-2}) = 0^{n-3}100$*.*

(iii) *If* $r = n-1$*, then there is a* 1*-Gray code* Δ_n^{n-1} *for* $\mathcal{D}_n^n \cup \mathcal{D}_n^{n-1} \cup \mathcal{D}_n^{n-2} \backslash \{0^{n-1}1\}$ *such that* $\text{first}(\Delta_n^{n-1}) = 0^{n-3}100$ *and* $\text{last}(\Delta_n^{n-1}) = 0^{n-2}10$*.*

(iv) *If* $r = n$*, then there is a* 1*-Gray code* Δ_n^n *for* $\mathcal{D}_n^n \cup \mathcal{D}_n^{n-1}$ *such that* $\text{first}(\Delta_n^n) = 0^{n-2}10$ *and* $\text{last}(\Delta_n^n) = 0^{n-1}1$*.*

(v) *If* $r \notin \{3, n-2, n-1, n\}$*, then there is a* 1*-Gray code* Δ_n^r *for* $\mathcal{D}_n^r \cup \mathcal{D}_n^{r-1}$ *such that* $\text{first}(\Delta_n^r) = 0^{r-1}10(001)^\star$ *and* $\text{last}(\Delta_n^r) = 0^{r-1}1(001)^\star$*.*

Proof. For the case (i), we set: $\Delta_3 = 0^21\mathbf{Z}_{n-3}$.

For the case (ii), we set: $\Delta_n^{n-2} = \bigcirc_{j=1}^{\frac{n-5}{2}} 0^{n-3-j}1^j\mathbf{Z}_3^{j-1} \circ \overline{\bigcirc_{j=1}^{\frac{n-3}{2}} 0^{n-2-j}1^j\mathbf{Z}_2}$. Since we have $\mathbf{Z}_2 = 00$ and $\mathbf{Z}_3 = 000, 001$, it is straightforward to see that Δ_n^{n-2} is a 1-Gray code with $\text{first}(\Delta_n^{n-2}) = 0^{n-4}1000$ and $\text{last}(\Delta_n^{n-2}) = 0^{n-3}100$.

For the case (iii), we set:

$$\Delta_n^{n-1} = \bigcirc_{j=1}^{\frac{n-4}{2}} 0^{n-2-j}1^j\mathbf{Z}_2 \circ 0^{\frac{n}{2}}1^{\frac{n-2}{2}}\mathbf{Z}_1 \circ \overline{\bigcirc_{j=1}^{\frac{n-4}{2}} \left(0^{n-1-j}1^{j+1} \circ 0^{n-1-j}1^j\mathbf{Z}_1\right)^j}.$$

Knowing that $\mathbf{Z}_2 = 00$ and $\mathbf{Z}_1 = 0$, we can easily check that any pair of consecutive words differ by exactly one bit, which proves that Δ_n^{n-1} is a 1-Gray code.

For the case (iv), we set: $\Delta_n^n = \bigcirc_{j=1}^{\frac{n-3}{2}} 0^{n-1-j}1^j0 \circ \overline{\bigcirc_{j=1}^{\frac{n-1}{2}} 0^{n-j}1^j}$. As previously the result can be obtained easily.

The case (v) is more challenging to handle. The set $\mathcal{D}_n^r \cup \mathcal{D}_n^{r-1}$ consists of the union of the following subsets: $K_1 = 0^{r-2}1\mathcal{Z}_{n-r+1}$, $K_2 = 0^{r-3}11\mathcal{Z}_{n-r+1}$, …, $K_a = 0^{\frac{r-1}{2}}1^{\frac{r-1}{2}}\mathcal{Z}_{n-r+1}$ and $L_1 = 0^{r-1}1\mathcal{Z}_{n-r}$, $L_2 = 0^{r-2}11\mathcal{Z}_{n-r}$, …, $L_b = 0^{\frac{r+1}{2}}1^{\frac{r-1}{2}}\mathcal{Z}_{n-r}$ with $a = \lfloor \frac{r-2}{2} \rfloor = \frac{r-3}{2}$ and $b = \lfloor \frac{r-1}{2} \rfloor = a+1$. Let us denote by $\mathbf{K}_1, \mathbf{K}_2, \ldots, \mathbf{K}_a$ and $\mathbf{L}_1, \mathbf{L}_2, \ldots, \mathbf{L}_b$ the associated Gray codes obtained by replacing $\mathcal{Z}_k$ with the Gray code $\mathbf{Z}_k$.

Remark that for $1 \leqslant i \leqslant a-1$ (resp. $1 \leqslant i \leqslant a$) and for a given j, the jth word of $\mathbf{K}_i$ (resp. $\mathbf{L}_i$) and the jth word of $\mathbf{K}_{i+1}$ (resp. $\mathbf{L}_{i+1}$) differ by exactly one bit; the words $\text{last}(\mathbf{K}_i)$ and $\text{first}(\mathbf{L}_{i+1})$ differ by one bit. Since $r = 3 \mod 4$, a is even, and thus $\text{last}((\mathbf{K}_a \circ \widehat{\mathbf{L}_{a+1}})^a) = \text{last}(\widehat{\mathbf{L}_{a+1}})$ differs by one bit from $\text{last}(\mathbf{L}_{a+1})$. Taking into account all these remarks, the list Δ_n^r defined below is a Gray code:

$$\Delta_n^r = \widehat{\mathbf{L}_1} \circ \bigcirc_{i=1}^{a} \left(\mathbf{K}_i \circ \widehat{\mathbf{L}_{i+1}}\right)^i \circ \overline{\bigcirc_{i=1}^{a+1} \text{last}(\mathbf{L}_i)}.$$

We refer to Figure 3 for a graphical representation of this Gray code. $\square$

Theorem 3. *For any* $n \geqslant 0$*, there exists a* 1*-Gray code* $\mathbf{Z}_n$ *for* $\mathcal{Z}_n$ *such that* $\text{first}(\mathbf{Z}_n) = 0(001)^\star$ *and* $\text{last}(\mathbf{Z}_n) = (001)^\star$*.*

Proof. We define recursively the 1-Gray code $\mathbf{Z}_n$ as follows:

$$\mathbf{Z}_n = \begin{cases} \Delta_n^5 \circ \Delta_n^9 \circ \cdots \circ \Delta_n^n \circ 0^n \circ \Delta_n^{n-2} \circ \cdots \circ \Delta_n^7 \circ \Delta_n^3 & \text{if } n = 1 \mod 4, \\ \Delta_n^5 \circ \Delta_n^9 \circ \cdots \circ \Delta_n^{n-1} \circ 0^n \circ \Delta_n^{n-3} \circ \cdots \circ \Delta_n^7 \circ \Delta_n^3 & \text{if } n = 2 \mod 4, \\ \Delta_n^5 \circ \Delta_n^9 \circ \cdots \circ \Delta_n^{n-2} \circ 0^n \circ \Delta_n^n \circ \cdots \circ \Delta_n^7 \circ \Delta_n^3 & \text{if } n = 3 \mod 4. \\ \Delta_n^5 \circ \Delta_n^9 \circ \cdots \circ \Delta_n^{n-3} \circ 0^{n-1}1 \circ 0^n \circ \Delta_n^{n-1} \circ \cdots \circ \Delta_n^7 \circ \Delta_n^3 & \text{if } n = 0 \mod 4. \end{cases}$$

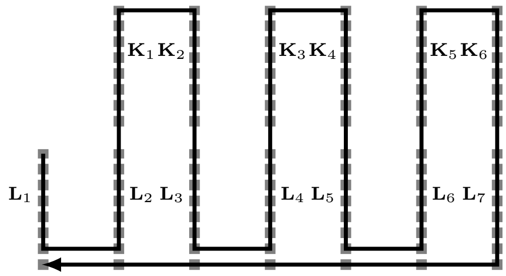

Fig. 3: Illustration of the Gray code Δ_n^r for the case (v) in the proof of Lemma 4 (we consider $a = 6$). Vertical sequences of squares are Gray codes $\mathbf{K}_i$, $1 \leqslant i \leqslant a$, and $\mathbf{L}_i$, $1 \leqslant i \leqslant a+1$, so that the first and the last elements are respectively the top and bottom squares of the segments. The walk illustrates the Gray code Δ_n^r that starts with $\text{first}(\mathbf{L}_1)$ and ends with $\text{last}(\mathbf{L}_1)$.

Due to Lemmas 2-4, $\text{last}(\Delta_n^{4i+1})$ differ by one bit from $\text{last}(\Delta_n^{4i+5})$ and $\text{last}(\Delta_n^{4i+3})$ differ by one bit from $\text{last}(\Delta_n^{4i+7})$ which ensure that $\mathbf{Z}_n$ is a 1-Gray code. □

We refer to Figure 4 for a graphical representation of $\mathbf{Z}_n$ for $4 \leqslant n \leqslant 10$, see also Figure 2(b) for $n = 17$.

An immediate consequence of Theorem 3 is the following.

Theorem 4. *For any $n \geqslant 1$, $\mathbf{W}_n^1 = 1 \cdot \mathbf{W}_{n-1}^1 \circ \mathbf{Z}_n$ is a 1-Gray code for $\mathcal{W}_n^1$ such that* $\text{first}(\mathbf{W}_n^1) = 1^n$ *and* $\text{last}(\mathbf{W}_n^1) = (001)^\star$.

1 111111	*8* 110010	*15* 000110
2 111110	*9* 100010	*16* 000010
3 111100	*10* 100011	*17* 000011
4 111000	*11* 100001	*18* 000001
5 111001	*12* 100000	*19* 000000
6 110001	*13* 100100	*20* 001000
7 110000	*14* 000100	*21* 001001

Table 2: The Gray code $\mathbf{W}_6^1$ for the set $\mathcal{W}_6^1$. The Hamming distance between two consecutive words is one.

Fibonacci-run graph introduced in [4] is the induced subgraphs of the hypercube on the run-length restricted words as vertices. It turns out that run-length restricted words are precisely the reverse of 1-decreasing words beginning by 0.

In this light, the Gray code $\mathbf{Z}_n$ in Theorem 3 gives a Hamiltonian path in the Fibonacci-run graph. The next corollary settles a conjecture in [4].

Corollary 8. *The Fibonacci-run graph has a Hamiltonian path.*

Finally, the validity of parity condition stated in Corollary 1 and experimental investigations for small q suggest the following extension of Theorem 4 .

Conjecture 1 *For any $n \geqslant 1$ and $q \geqslant 1$, there is a 1-Gray code for $\mathcal{W}_n^q$.*

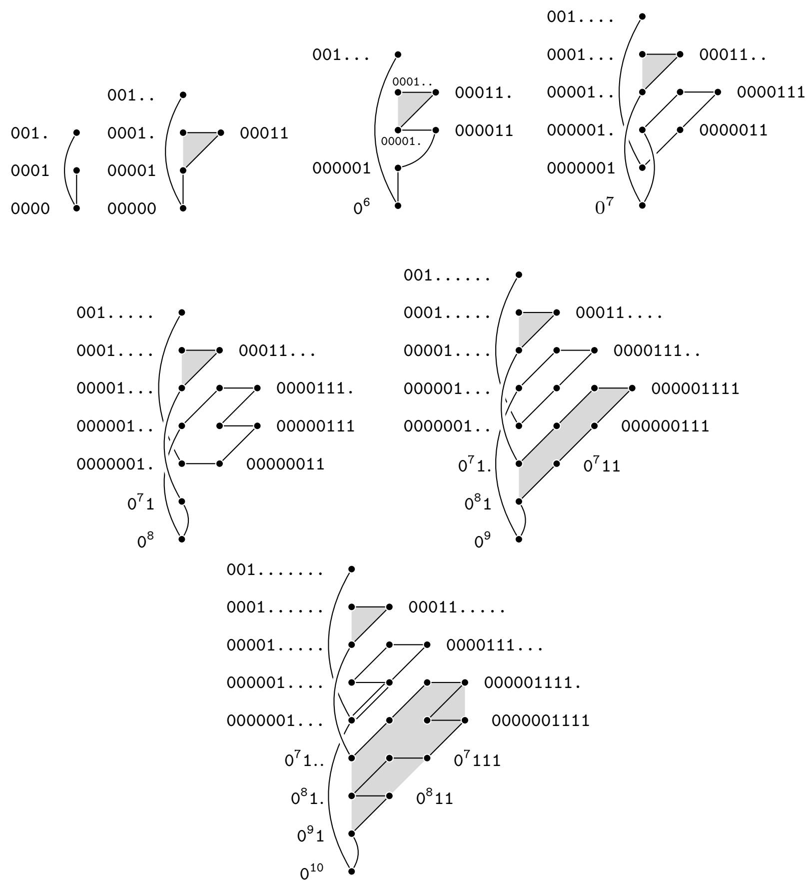

Fig. 4: Illustration of the recursive definition for the Gray codes $\mathbf{Z}_n$, $4 \leqslant n \leqslant 10$.

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