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ASYMPTOTICS ON A CLASS OF LEGENDRE FORMULAS

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Abstract. Let $f : \mathbb{R} \to \mathbb{R}$ be positive over $\mathbb{R}$ and let $n!_f$ denote its associated Legendre formula, defined as

$$n!_f = \prod_{p\in\mathbb{P}} p^{\sum_{k\geq 0}\left\lfloor \frac{n}{f(p)p^k}\right\rfloor},$$

where $\lfloor\cdot\rfloor$ denotes the floor function. In this paper, we will show, subject to certain criteria, that $n!_f$ is an abstract factorial that satisfies a weak Stirling approximation. As an application, we will give weak approximations to the Bhargava factorial over the set of primes and to a less well-known Legendre formula.

1. Introduction

The factorial function is a fundamental object of utmost importance, finding itself in various disciplines, including combinatorics, number theory, ring theory, and many others. These include, but are not limited to, being a building block for binomial coefficients, used to count the number of permutations of a collection of objects, and contributes a major part to Taylor series expansions. The study of obtaining asymptotics to $n!$ date back to the 17th and 18th century, where, in particular, Abraham De Moivre showed that [1]

$$n! \sim Ke^{-n}n^{n+1/2}$$

for some constant K. Later, in the 18th century, James Stirling [2] was able to specify the constant: $K = \sqrt{2\pi}$.

At the turn of the 21st century, Manjul Bhargava introduced a generalization of the factorial in [3], $n!_S$ with $S \subseteq \mathbb{Z}$, that satisfies four properties analogous to the factorial:

(1) For any nonnegative integers k and l, $(k + l)!_S$ is a multiple of $k!_S l!_S$.
(2) Let f be a primitive polynomial of degree k and let $d(S, f) = \gcd\{f(a)\,|\,a \in S\}$. Then $d(S, f)$ divides $k!_S$.
(3) Let $a_0, a_1, \ldots, a_n \in S$ be any $n + 1$ integers. Then the product

$$\prod_{i<j}(a_i - a_j)$$

is a multiple of $0!_S 1!_S \cdots n!_S$.
(4) The number of polynomial functions from S to $\mathbb{Z}/n\mathbb{Z}$ is given by

$$\prod_{k=0}^{n-1} \frac{n}{\gcd(n, k!_S)}.$$

Manjul Bhargava in [3] remarks the following:

Question 31: What are analogues of Stirling's formula for generalized factorials?

Upon further investigation, each of the Bhargava factorials discussed in [3] yield a Legendre formula that can be expressed as

$$\prod_{p\in\mathbb{P}} p^{\sum_{k\geq 0}\left\lfloor \frac{n}{f(p)p^k}\right\rfloor}, \tag{1}$$

for some real-valued map f. In addition, other classes of Legendre formulas fall under this pattern: for example, the sequence

$$1, 6, 360, 45360, 5443200, 359251200, \dots$$

which can be found under A202367 in [4]. While this sequence may not be obviously related to the Bhargava factorial, interestingly in [5], N. Mathur devised an algorithm to show, for any sequence under specific criteria, we can determine a set $S' \subseteq \mathbb{Z}$ associated with the sequence. From this, we can determine a set S' associated with sequence A202367. As a consequence of this pattern, this prompted an investigation in (1).

In Section 2, we will define the associated Legendre formula, or alternatively f-factorial for a real-valued map f and prove specific properties about them: this includes showing when the f-factorial takes on a natural number and show that, analogous to the factorial and under specific conditions, the f-factorial is an abstract factorial, as defined in [6].

In Section 3, we will prove, under certain criteria for f, an asymptotic formula for the f-factorial. Due to the lack of assumptions on f, f may be continuous or discontinuous, so we consider classes of functions f that can be approximated "well-enough" by polynomials, which we will explain further in this section.

In Section 4, we will apply our results to certain classes of Legendre formulas obtained from the Bhargava factorial, including over $\mathbb{P}$ and S' discussed earlier.

2. Preliminaries

Before we proceed, we let $\lfloor x\rfloor$ and $\lceil x\rceil$ denote the floor function and ceiling function for a real number x, respectively; $\mathbb{P}$ denote the set of primes; log denote the natural logarithm; $\{x\}$ denote the fractional part of a real number x; $\mathbb{N}$ denote the natural numbers (or, alternatively, the positive integers $\mathbb{Z}^+$) $\{1, 2, 3, \dots\}$; and $P(s)$ denote the prime zeta function, which is defined as

$$P(s) = \sum_{p\in\mathbb{P}} \frac{1}{p^s}$$

and converges for $\Re(s) > 1$ (see [7]). For the precise definition of the f-factorial, we require f to be positive over $\mathbb{P}$.

Definition 1. *Let $f : \mathbb{R} \to \mathbb{R}$ such that f is positive over $\mathbb{P}$. The associated Legendre formula, or f-factorial, is defined as*

$$n!_f = \prod_{p\in\mathbb{P}} p^{\sum_{k\geq 0}\left\lfloor \frac{n}{f(p)p^k}\right\rfloor}.$$

Example 2.1. *Let $f(p) = p$ be the identity map; by Legendre's formula, we have*

$$n!_p = n! = \prod_{p\in\mathbb{P}} p^{\sum_{k\geq 0}\left\lfloor \frac{n}{p^{k+1}} \right\rfloor},$$

with the sequence $\{n!_p\}_{n\geq 0} = \{1, 1, 2, 6, 24, 120, \dots\}$.

Example 2.2. *Let $f(p) = \log p$; we have*

$$n!_{\log p} = \prod_{p\in\mathbb{P}} p^{\sum_{k\geq 0}\left\lfloor \frac{n}{p^k \log p} \right\rfloor},$$

with the sequence $\{n!_{\log p}\}_{n\geq 0} = \{1, 2, 840, 1862340480, \dots\}$.

While the above examples produce interesting and known-examples of Legendre formulas, there are other examples for which $n!_f$ is not well-defined.

Example 2.3. *Let $f(p) = \sin p$; we have*

$$n!_{\sin p} = \prod_{p\in\mathbb{P}} p^{\sum_{k\geq 0}\left\lfloor \frac{n}{p^k \sin p} \right\rfloor}.$$

However, since $\csc p \geq 1$, we have

$$n!_{\sin p} \geq \prod_{p\in\mathbb{P}} p^n,$$

where the product diverges for $n \geq 1$. Thus, $n!_{\sin p}$ is not defined for all n.

Example 2.4. *Let $f(p) = p$ for $p \equiv 1 \bmod 4$ and $1/p$ for $p \equiv 3 \bmod 4$. Denote $\mathbb{P}_{1,4}$ and $\mathbb{P}_{3,4}$ for primes p that are $1 \bmod 4$ and $3 \bmod 4$, respectively; we have*

$$n!_f = \prod_{p\in\mathbb{P}} p^{\sum_{k\geq 0}\left\lfloor \frac{n}{p^k f(p)} \right\rfloor} = \prod_{p\in\mathbb{P}_{1,4}} p^{\sum_{k\geq 0}\left\lfloor \frac{n}{p^{k+1}} \right\rfloor} \prod_{p\in\mathbb{P}_{3,4}} p^{\sum_{k\geq 0}\left\lfloor \frac{n}{p^{k-1}} \right\rfloor}$$

Observe that we have,

$$n!_f \geq \prod_{p\in\mathbb{P}_{3,4}} p^{pn},$$

where the product diverges for $n \geq 1$, since there are infinitely many primes that are $3 \bmod 4$. Thus, $n!_f$ is not defined for all n.

Based on this, we find, whenever f is unbounded over $\mathbb{P}$, except on a finite subset of $\mathbb{P}$, we have its f-factorial $n!_f$ is natural. As a matter fact, one of the interesting facts about $n!_f$ is the equivalence between its naturality and the first exponential term, in addition to showing $n!_f$ is natural whenever we consider the classes of functions f mentioned previously.

Theorem 1. *Let $f : \mathbb{R} \to \mathbb{R}$ such that it is positive over $\mathbb{P}$. The following are equivalent:*

(1) *the sum* $\displaystyle\sum_{p\in\mathbb{P}} \log p \left\lfloor \frac{n}{f(p)} \right\rfloor$ *converges to* $\log k$ *for some natural* k.

(2) $n!_f$ *is a natural number.*

Proof. For (1) $\implies$ (2), suppose that the sum

$$\sum_{p\in\mathbb{P}} \log p \left\lfloor \frac{n}{f(p)} \right\rfloor$$

converges to $\log k$ for some natural k. We will appeal to the following lemma:

Lemma 1. *Let $b_k \geq 0$ denote a strictly decreasing sequence of non-negative integers; then the sequence defined by*

$$a_k = \prod_{p\in\mathbb{P}} p^{b_k},$$

with a_0 assumed to be finite, is a strictly decreasing sequence of naturals such that $a_N = a_{N+1} = \cdots = 1$ for some N.

Since b_k is strictly decreasing, we have a_k is strictly decreasing; furthermore, since b_k is a strictly decreasing sequence of non-negative integers, by the pigeon-hole principle, there exists an N such $b_N = b_{N+1} = \cdots = 0$. Then, consequently, we have $a_N = a_{N+1} = \cdots = 1$, as desired.

Let $b_k = \left\lfloor \frac{n}{f(p)p^k} \right\rfloor$; then

$$a_k = \prod_{p\in\mathbb{P}} p^{\left\lfloor \frac{n}{f(p)p^k} \right\rfloor},$$

and a_0 is finite by our assumption. By Lemma 1, we have then $n!_f = a_1 a_2 \dots a_{N-1}$, which is a product of naturals. Thus, $n!_f$ is natural, as desired.

For (2) $\implies$ (1), since $n!_f$ is natural, $n!_f$ admits the unique factorization $n!_f = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$; this implies the f-factorial has the closed form

$$\sum_{p\in\mathbb{P}} \log p \sum_{k=0}^{\infty} \left\lfloor \frac{n}{f(p)p^k} \right\rfloor = \log \prod_{1\leq i\leq r} p_i^{a_i}. \tag{2}$$

Since the sum converges, and the right-sum hand sum in (2) is a logarithm of a natural, all of the terms in the right-sum are finite, which implies the sum

$$\sum_{p\in\mathbb{P}} \log p \left\lfloor \frac{n}{f(p)} \right\rfloor$$

is finite. As a consequence of convergence, the terms must converge to 0; in other words, for any $\epsilon > 0$, there is some natural, call it n_f, such that for all primes $p \geq n_f$, we have $\left\lfloor \dfrac{n}{f(p)} \right\rfloor < \epsilon / \log p$. We choose n_f large enough such that there is an ϵ satisfying $\epsilon < \log p$; then $\left\lfloor \dfrac{n}{f(p)} \right\rfloor = 0$, so that the sum is

$$\sum_{p\in\mathbb{P}} \log p \left\lfloor \frac{n}{f(p)} \right\rfloor = \log \prod_{p<n_f} p^{\left\lfloor \frac{n}{f(p)} \right\rfloor},$$

which implies (1), as desired. □

As mentioned previously, we have the following relationship between the classes of unbounded functions f and their f-factorials.

Theorem 2. *If f is unbounded over $\mathbb{P}$ except on a finite subset, then $n!_f$ is a natural number for all n.*

Proof. Since f is unbounded over $\mathbb{P}$ except on a finite subset, denoted as A, for every n, there exists a prime p' such that for primes $p \geq p'$, $f(p) > n$. Partition $\mathbb{P} = \mathbb{P}_f \cup \mathbb{P}'_f$, where

$$\mathbb{P}_f = \{p \in \mathbb{P} \,|\, f(p) \leq n\},$$

and $\mathbb{P}'_f$ is its complement. Observe that $\mathbb{P}_f$ is finite, i.e. $|\mathbb{P}_f| \leq p'$. Since it may or may not be the case $A \subseteq \mathbb{P}_f$, partition $A = A_1 \cup A_2$ such that $A_1 \subseteq \mathbb{P}_f$ and $A_2 \subseteq \mathbb{P}'_f$; then our original sum can be decomposed as

$$\sum_{p\in\mathbb{P}} \log p \left\lfloor \frac{n}{f(p)} \right\rfloor = \log \prod_{p\in\mathbb{P}_f \cup A_2} p^{\left\lfloor \frac{n}{f(p)} \right\rfloor}.$$

Since $|\mathbb{P}_f|$ and $|A|$ are finite, $|\mathbb{P}_f \cup A_2|$ is finite, so that our sum converges to $\log k$ for some natural k. By Theorem 2, we have $n!_f$ is natural, as desired. □

For the rest of the paper, we will assume that $n!_f$ is natural and that f is positive over $\mathbb{P}$. An important property of the f-factorial $n!_f$ is that it is an abstract factorial whenever $f(p) \leq p$; for an extensive discussion on abstract factorials, see [6]. Abstract factorials obey certain properties that a factorial satisfies; in particular, the definition of an abstract factorial is the following:

Definition 2. *An abstract (or generalized) factorial is a function $!_a : \mathbb{N} \to \mathbb{Z}^+$ that satisfies the following conditions:*

1. $0!_a = 1$.
2. *For every non-negative integers n, k, $0 \leq k \leq n$, the generalized binomial coefficients*
$$\binom{n}{k}_a = \frac{n!_a}{(n-k)!_a \, k!_a} \in \mathbb{Z}^+.$$
3. *For every natural n, $n!$ divides $n!_a$.*

We will now prove $n!_f$ is an abstract factorial:

Theorem 3. *Let $n!_f$ be the f-factorial for f such that $0 < f(p) \leq p$. Then $n!_f$ is an abstract factorial.*

Proof. For property (1), by construction, we have $0!_f = 1$. For property (2), let n, k be arbitrary integers satisfying $0 \leq k \leq n$; by the subadditivity of the floor function, we have

$$\left\lfloor \frac{n}{f(p)} \right\rfloor - \left\lfloor \frac{n-k}{f(p)} \right\rfloor - \left\lfloor \frac{k}{f(p)} \right\rfloor \geq 0,$$

so that the sequence a_l, defined as

$$a_l = \prod_{p\in\mathbb{P}} p^{\left\lfloor \frac{n}{f(p)p^l} \right\rfloor - \left\lfloor \frac{n-k}{f(p)p^l} \right\rfloor - \left\lfloor \frac{k}{f(p)p^l} \right\rfloor},$$

gives us, by Lemma 1, $n!_f = m\, k!_f (n-k)!_f$, i.e.

$$\binom{n}{k}_a = \frac{n!_a}{(n-k)!_a \, k!_a} \in \mathbb{Z}^+,$$

as desired. For property (3), since $f(p) \leq p$, we have

$$\left\lfloor \frac{n}{f(p)p^k} \right\rfloor - \left\lfloor \frac{n}{p^{k+1}} \right\rfloor \geq 0,$$

so that the sequence a_k, defined as

$$a_k = \prod_{p \in \mathbb{P}} p^{\left\lfloor \frac{n}{f(p)p^k} \right\rfloor - \left\lfloor \frac{n}{p^{k+1}} \right\rfloor},$$

gives us, by Lemma 1, $n! \,|\, n!_f$, as desired. □

In Lemma 13 in [3], M. Bhargava noted that if T, S are two subsets of $\mathbb{Z}$ and $n!_T$, $n!_S$ are their respective Bhargava factorials such that $T \subseteq S$, then $n!_S \,|\, n!_T$ for every $n \geq 0$. We derive a similar result for the f-factorials.

Theorem 4. *Let f,g satisfy $f(p) \leq g(p)$ and let $n!_f$, $n!_g$ denote their respective f-factorials; then $n!_g \,|\, n!_f$.*

Proof. By the subadditivity of $\lfloor \cdot \rfloor$, we have

$$\left\lfloor \frac{n}{f(p)} \right\rfloor - \left\lfloor \frac{n}{g(p)} \right\rfloor \geq 0,$$

so that the sequence a_k, defined as

$$a_k = \prod_{p \in \mathbb{P}} p^{\left\lfloor \frac{n}{f(p)p^k} \right\rfloor - \left\lfloor \frac{n}{g(p)p^k} \right\rfloor},$$

gives us, by Lemma 1, $n!_g \,|\, n!_f$, as desired. □

As an example, if $f(p) = p - 1$ and $g(p) = p$, we have $n!_f = (n+1)!_{\mathbb{P}}$ and $n!_g = n!$, so that $n! \,|\, (n+1)!_{\mathbb{P}}$. We conclude this section to note, since we have shown $n!_f$ is an abstract factorial for $0 \leq f(p) \leq p$, various properties follow, including the irrationality of the sum

$$\sum_{n=0}^{\infty} \frac{1}{n!_f},$$

all of which are extensively discussed in [6].

3. Asymptotic Formula

The question of when $n!_f$ would admit an asymptotic expansion for arbitrary f seems out of reach due to the various classes of functions f that we would need to consider, as they can potentially be discontinuous; as an example, the set S' discussed earlier has an f-factorial corresponding to $f(p) = \lceil (p-1)/2 \rceil$, which is a discontinuous map. For our purposes, we will consider maps that can be approximated "well enough" by polynomials in the following sense:

Theorem 5. *Let f be real-valued and positive over $\mathbb{P}$. Assume that f satisfies the following condition:*

(1) *there exists a polynomial $q(x)$, with zero constant term and coefficients $b_1, b_2, \cdots, b_m$, such that f satisfies the inequality*

$$0 \leq \frac{1}{f(p)} - q\left(\frac{1}{p}\right) \leq \frac{C}{p^\delta},$$

for some $C > 0$, $\delta > 1$.

Then $\log n!_f$, up to $o(n)$, is asymptotic to

$$\beta_f n + \sum_{p\in\mathbb{P}} \sum_{k\geq 0} \log p \left(b_1 \left\lfloor \frac{n}{p^{k+1}} \right\rfloor + b_2 \left\lfloor \frac{n}{p^{k+2}} \right\rfloor + \cdots + b_m \left\lfloor \frac{n}{p^{k+m}} \right\rfloor \right),$$

for some non-negative constant β_f.

Proof. Let $q(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x$. Let μ denote the counting measure and define a sequence of functions $\{h_n(p,k)\}_{n\geq 1}$ as

$$h_n(p,k) = \frac{\log p}{n} \left(\left\lfloor \frac{n}{f(p)p^k} \right\rfloor - b_1 \left\lfloor \frac{n}{p^{k+1}} \right\rfloor - b_2 \left\lfloor \frac{n}{p^{k+2}} \right\rfloor - \cdots - b_m \left\lfloor \frac{n}{p^{k+m}} \right\rfloor \right).$$

Since $\lfloor nx \rfloor / n \to x$ pointwise for any real x, it follows that $h_n(p,k) \to h(p,k)$ pointwise, where

$$h(p,k) = \frac{\log p}{p^k} \left(\frac{1}{f(p)} - q\left(\frac{1}{p}\right) \right).$$

We are interested in passing the limit under the integrals

$$\lim_{n\to\infty} \int_{\mathbb{P}} \int_{\mathbb{Z}_{\geq 0}} h_{n,k}(p)\, d\mu(k)\, d\mu(p) = \int_{\mathbb{P}} \int_{\mathbb{Z}_{\geq 0}} h_k(p)\, d\mu(k)\, d\mu(p),$$

which will require a doubly application of the dominated convergence theorem. By our assumption, we have the series of inequalities

$$\begin{aligned} h_n(p,k) &\leq \frac{\log p}{n} \left(\left\lfloor \frac{b_1 n}{p^{k+1}} + \cdots + \frac{b_m n}{p^{k+m}} + \frac{Cn}{p^{k+\delta}} \right\rfloor - b_1 \left\lfloor \frac{n}{p^{k+1}} \right\rfloor - \cdots - b_m \left\lfloor \frac{n}{p^{k+m}} \right\rfloor \right) \\ &\leq \frac{\log p}{n} \left(\frac{b_1 n}{p^{k+1}} - b_1 \left\lfloor \frac{n}{p^{k+1}} \right\rfloor + \cdots + \frac{b_m n}{p^{k+m}} - b_m \left\lfloor \frac{n}{p^{k+m}} \right\rfloor + \frac{Cn}{p^{k+\delta}} \right) \\ &\leq \frac{\log p}{n} \left(b_1 \left\{ \frac{n}{p^{k+1}} \right\} + \cdots + b_m \left\{ \frac{n}{p^{k+m}} \right\} + \frac{Cn}{p^{k+\delta}} \right) \\ &\leq \frac{\log p}{p^k} \left(q\left(\frac{1}{p}\right) + \frac{C}{p^\delta} \right), \end{aligned}$$

which implies

$$h_{n,k}(p) \leq \frac{\log p}{p^k} \left(q\left(\frac{1}{p}\right) + \frac{C}{p^\delta} \right). \tag{3}$$

This upper bound in (3) is integrable; in particular, we have

$$\int_{\mathbb{Z}_{\geq 0}} \frac{\log p}{p^k} \left(q\left(\frac{1}{p}\right) + \frac{C}{p^\delta} \right) d\mu(k) = \sum_{k\geq 0} \frac{\log p}{p^k} \left(q\left(\frac{1}{p}\right) + \frac{C}{p^\delta} \right) = \frac{p\log p}{p-1} \left(q\left(\frac{1}{p}\right) + \frac{C}{p^\delta} \right),$$

so that, by the dominated convergence theorem, we can pass the limit under the inner integral. For the outer integral, we consider

$$\int_{\mathbb{Z}_{\geq 0}} h_{n,k}(p)\, d\mu(k) = \frac{\log p}{n} \sum_{k\geq 0} \left(\left\lfloor \frac{n}{f(p)p^k} \right\rfloor - b_1 \left\lfloor \frac{n}{p^{k+1}} \right\rfloor - \cdots - b_m \left\lfloor \frac{n}{p^{k+m}} \right\rfloor \right) \tag{4}$$

By our assumption, (4) has the upper bound

$$\begin{aligned}\int_{\mathbb{Z}_{\geq 0}} h_{n,k}(p)\,d\mu(k) &\leq \frac{\log p}{n}\sum_{k\geq 0}\left\lfloor \frac{n}{f(p)p^k}\right\rfloor\\ &\leq \frac{\log p}{n}\sum_{k\geq 0}\left\lfloor \frac{n}{p^k}\left(\frac{b_1}{p}+\frac{b_2}{p^2}+\cdots+\frac{b_m}{p^m}+\frac{C}{p^\delta}\right)\right\rfloor\\ &\leq \frac{\log p}{n}\sum_{k\geq 0}\left\lfloor \frac{n}{p^k}\left(\frac{b_1p^{m-1}+b_2p^{m-2}+\cdots+b_m+C}{p^m}\right)\right\rfloor\\ &\leq \frac{\log p}{n}\sum_{k\geq 0}\left\lfloor \frac{n(b_1+b_2+\cdots+b_m+C)}{p^{k+1}}\right\rfloor\\ &\leq \frac{\log p}{n}\sum_{k\geq 0}\left\lfloor \frac{nK}{p^{k+1}}\right\rfloor, \qquad (5)\end{aligned}$$

where K is a natural bounding $b_1+b_2+\cdots+b_m+C$. The upper bound in (5) is integrable, since, by Legendre's formula, we have

$$\int_{\mathbb{P}} \frac{\log p}{n}\sum_{k\geq 0}\left\lfloor \frac{nK}{p^{k+1}}\right\rfloor\, d\mu(p) = \frac{1}{n}\sum_{p\in\mathbb{P}}\log p\sum_{k\geq 0}\left\lfloor \frac{nK}{p^{k+1}}\right\rfloor = \frac{1}{n}\log\,(nK)!.$$

Since $\int_{\mathbb{Z}_{\geq 0}} h_{n,k}(p) \to \int_{\mathbb{Z}_{\geq 0}} h_k(p)$ pointwise, as we have previously shown, by the dominated convergence theorem, we have

$$\lim_{n\to\infty}\int_{\mathbb{P}}\int_{\mathbb{Z}_{\geq 0}} h_{n,k}(p)\,d\mu(k)\,d\mu(p) = \int_{\mathbb{P}}\int_{\mathbb{Z}_{\geq 0}} h_k(p)\,d\mu(k)\,d\mu(p),$$

where

$$\int_{\mathbb{P}}\int_{\mathbb{Z}_{\geq 0}} h_k(p)\,d\mu(k)\,d\mu(p) = \sum_{p\in\mathbb{P}}\frac{p\log p}{p-1}\left(\frac{1}{f(p)} - q\left(\frac{1}{p}\right)\right). \qquad (6)$$

Denote (6) as β_f; to see β_f is a non-negative constant, from our assumption, $\beta_f \geq 0$. Furthermore, we have

$$\sum_{p\in\mathbb{P}}\frac{p\log p}{p-1}\left(\frac{1}{f(p)} - q\left(\frac{1}{p}\right)\right) \leq C\sum_{p\in\mathbb{P}}\frac{\log p}{p^{\delta-1}(p-1)} \leq 2C\sum_{p\in\mathbb{P}}\frac{\log p}{p^\delta} = -2CP'(\delta),$$

where $P'(\delta)$ is the derivative of the prime zeta function, which converges for $\delta > 1$. By the monotone convergence theorem, β_f is finite. Thus, we find $\log n!_f$ is asymptotic, up to $o(n)$, to

$$\beta_f n + \sum_{p\in\mathbb{P}}\sum_{k\geq 0}\log p\left(b_1\left\lfloor\frac{n}{p^{k+1}}\right\rfloor + b_2\left\lfloor\frac{n}{p^{k+2}}\right\rfloor + \cdots + b_m\left\lfloor\frac{n}{p^{k+m}}\right\rfloor\right),$$

as desired. □

4. Applications

We are interested in discussing applications of Theorem 4 to various Legendre formulas, in particular focusing on the Bhargava factorials over the set of primes and to the Legendre formula discussed under A202357.

4.1. **Bhargava factorial over the set of primes.** Recall the Bhargava factorial $n!_S$ as defined in [3]; M. Bhargava found an analogous Legendre's formula for $n!_{\mathbb{P}}$, in particular deriving

$$n!_{\mathbb{P}} = \prod_{p\in\mathbb{P}} p^{\sum_{k\geq 0}\left\lfloor \frac{n-1}{(p-1)p^k}\right\rfloor}.$$

Let $f(p) = p - 1$; then, choosing $q(p) = p$, the criteria for Theorem 4 is satisfied, with $n!_f = (n+1)!_{\mathbb{P}}$. We have

$$\log(n+1)!_{\mathbb{P}} = \log n! + Cn + o(n),$$

where C is

$$C = \sum_{p\in\mathbb{P}} \frac{\log p}{(p-1)^2} = 1.2269688\ldots.$$

n	$\log(n+1)!_{\mathbb{P}}$	$\log n! + Cn$
1	.6931..	1.2269...
2	3.1780...	3.1470...
3	3.8712...	5.4726...
4	8.6586...	8.0859...
5	9.3518...	10.9223...
6	14.8812...	13.9410...
7	15.5744...	17.1139...
8	21.0550...	20.4203...
9	21.7482...	23.8445...
10	26.6310...	27.3741...
100	471.9704...	480.6040...
1,000	7,119.5084...	7,130.9600...
5,000	43,759.7980...	43,726.0000...
10,000	94,417.8375...	94,378.6000...

TABLE 1. Table of values comparing $\log(n+1)!_{\mathbb{P}}$ and $\log n! + Cn$.

Interestingly, the constant C occurs in the study of the Carmichael function $\lambda(n)$, which is defined to be the smallest positive integer satisfying $a^{\lambda(n)} \equiv 1 \pmod n$ for every integer a between 1 and n coprime to n. In particular, for all numbers m but $o(m)$ positive integers such that $n \leq m$, we have

$$\lambda(n) = \frac{n}{(\log n)^{\log\log\log n + A + o(1)}},$$

where $A = C - 1$ (see [8]).

4.2. **LCM of a class of a polynomials.** Consider the least common multiple of the denominators of the coefficients of polynomials $p_m(n)$ defined by the recursion

$$p_m(n) = \sum_{i=1}^{n} i^2 p_{(m-1)}(i),$$

for $m \geq 1$, with $p_0(n) = 1$; this generates the sequence

$$1, 6, 360, 45360, 5443200, \ldots, \tag{7}$$

which is sequence A202367. In [4], Vladimir Shevelev and Peter Moses noted that it is conjectured the sequence in (7), which we will denote as $a(n)$, is

$$a(n) = \prod_{p \in \mathbb{P}} p^{\sum_{k \geq 0} \left\lfloor \frac{n-1}{\lceil (p-1)/2 \rceil p^k} \right\rfloor}.$$

Let $f(p) = \lceil (p-1)/2 \rceil$; choosing $q(p) = 2p$, the criteria for Theorem 4 is satisfied, with $n!_f = a(n+1)$. We have

$$\log a(n+1) = 2 \log n! + \beta_f n + o(n),$$

where

$$\beta_f = \sum_{p \in \mathbb{P}} \frac{\log p}{(p-1)} \left(\frac{p}{\lceil (p-1)/2 \rceil} - 2 \right) = 1.0676431....$$

n	$\log a(n+1)$	$2 \log n! + \beta_f n$
1	1.7917...	1.0676...
2	5.8861...	3.5215...
3	10.7223...	6.7864...
4	15.5098...	10.6266...
5	19.6995...	14.9131...
6	29.4033...	19.5643...
7	31.1951...	24.5238...
8	39.5089...	29.7503...
9	48.3882...	35.21244...
10	56.4899...	40.88525...
100	982.0880...	834.243...
1,000	14,288.7934...	12,891.9000...
5,000	87,486.3657...	80,520.5000...
10,000	188,805.0729...	174,894.0000...

TABLE 2. Table of values comparing $\log a(n+1)$ and $2 \log n! + \beta_f n$.

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