Smooth multisections of odd-dimensional tori and other manifolds

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ABSTRACT. After Gay–Kirby extended the classical notion of Heegaard splittings of 3-manifolds by introducing *trisections* of smooth 4-manifolds, Rubinstein–Tillmann defined *multisections* of PL n-manifolds. We consider smooth multisections, which are decompositions into $k = \lfloor n/2 \rfloor + 1$ n-dimensional 1-handlebodies with nice intersection properties. We prove that a manifold X of even dimension $n \geq 6$ admits no smooth multisection if $H_i(X) \neq 0$ for any odd $i \neq 1, n-1$. By contrast, every manifold admits a PL multisection, and every manifold of dimension $n \leq 5$ admits a smooth multisection.

What about odd dimensions $n \geq 7$? We construct, for each odd-dimensional torus T^n, a smooth multisection which is efficient in the sense that each 1-handlebody has genus n, which we prove is optimal; each multisection is symmetric with respect to both the permutation action of S_n on the indices and the $\mathbb{Z}_k$ translation action along the main diagonal. We also construct such a trisection of T^4, lift all symmetric multisections of tori to certain cubulated manifolds, and obtain combinatorial identities as corollaries.

1. Introduction

Every closed 3-manifold[1] X admits a decomposition into two 3-dimensional 1-handlebodies[2] glued along their boundaries. Gay–Kirby (resp. Lambert-Cole–Miller) extended this classical notion of *Heegaard splittings* by proving that every closed smooth 4- (resp. 5-) manifold admits a decomposition $X = \bigcup_{i \in \mathbb{Z}_3} X_i$ where each X_i is a 4- (resp. 5-) dimensional 1-handlebody, each $X_i \cap X_{i+1}$ is a 3-dimensional 1-handlebody (resp. 4-dimensional 2-handlebody), and $X_0 \cap X_1 \cap X_2$ is a closed surface (resp. 3-manifold). Each of these *smooth trisections* gives a handle decomposition of X in which X_0 contributes all the 0- and 1-handles, X_1 all the 2- and $(n-2)$-handles, and X_2 all the $(n-1)$- and n-handles.

In the *PL category*, Rubinstein-Tillmann proved that every closed manifold of arbitrary dimension $n = 2k - 1$ (resp. $2k - 2$) admits a decomposition

[1]All **manifolds** throughout, whether PL or smooth, are compact, connected, and orientable. A manifold X is **closed** if $\partial X = \varnothing$.

[2]A d-dimensional h-handlebody is a d-manifold obtained by gluing d-dimensional r-handles for various $r = 0, \ldots, h$.

$X = \bigcup_{i\in\mathbb{Z}_k} X_i$ where each X_i is an n-dimensional 1-handlebody; for each $I \subsetneqq \mathbb{Z}_k$ with $|I| \geq 2$, $\bigcap_{i\in I} X_i$ is an $(n+1-|I|)$-dimensional submanifold with an $|I|$- (resp. $(|I|-1)$-) dimensional spine;[3] and the central intersection $\bigcap_{i\in\mathbb{Z}_k} X_i$ is a k- (resp. $(k-1)$-) dimensional manifold. We consider the *smooth* analog of these *PL multisections*:

Definition 1.1. Let X be a closed manifold of dimension $n = 2k-1$ (resp. $2k-2$). A *smooth multisection* is a decomposition $X = \bigcup_{i\in\mathbb{Z}_k} X_i$, where:

- Each X_i is an n-dimensional 1-handlebody.
- For each $I \subset \mathbb{Z}_k$ with $2 \leq |I| \leq k-1$, $\bigcap_{i\in I} X_i$ is an $(n+1-|I|)$-dimensional $|I|$- (resp. $(|I|-1)$-) handlebody.
- The central intersection $\bigcap_{i\in\mathbb{Z}_k} X_i$ is a closed k- (resp. $(k-1)$-) dimensional submanifold.

Section 2 shows that any combinatorial description of a smooth multisection $X = \bigcup_{i\in\mathbb{Z}_k} X_i$ gives a handle decomposition of X, and thus a smooth structure (in which each X_I is smoothly embedded, generally with corners).

Theorem 2.5. *Let $X = \bigcup_{i\in\mathbb{Z}_k} X_i$ be a smooth multisection of a closed manifold of dimension $n = 2k-1$. Then X has a handle decomposition in which each X_i provides all the $2i$- and $(2i+1)$-handles.*

Theorem 2.7. *Let $k \geq 3$. If a closed $(2k-2)$-manifold has a smooth multisection $X = \bigcup_{i\in\mathbb{Z}_k} X_i$, then X has a handle decomposition in which each X_i provides all the $2i$-handles, and the only handles with odd-dimensional cores are the 1-handles from X_0 and the $(n-1)$-handles from X_{k-1}.*

In even dimension n, this handle decomposition reveals that any nontrivial homology group in odd dimension $i \neq 1, n-1$ obstructs the existence of smooth multisections. In general, the handle decomposition bounds the *efficiency* of smooth multisections. Let $g(X_i)$ denote the **genus** of X_i; that is, since X_i is an n-dimensional 1-handlebody, $X_i \approx \natural^g(S^1 \times D^{n-1})$ for some $g = g(X_i)$.

Definition 1.2. The **efficiency** of a smooth multisection $X = \bigcup_{i\in\mathbb{Z}_k} X_i$ is

$$\begin{cases} \dfrac{1}{1+\max_i g(X_i)} & H_1(X) = 0 \\[2ex] \dfrac{\text{rank } \pi_1(X)}{\max_i g(X_i)} & H_1(X) \neq 0 \end{cases}$$

A multisection is **efficient** if its efficiency is 1.

[3] A spine of a PL manifold N is a subpolyhedron $P \subset \text{int}(N)$ onto which N collapses.

Corollary 2.10. *Any smooth multisection in dimension $n \neq 2$ has efficiency at most 1; if $X = \bigcup_{i \in \mathbb{Z}_k} X_i$ is efficient, then all X_i have the same genus.*[4]

Section 3 begins a detailed investigation of smooth multisections of *odd-dimensional tori*. Roughly stated, the main result of that investigation is:

Theorem 7.11. *Each $n = (2k-1)$-torus admits an efficient smooth multisection which is symmetric with respect to the permutation action by S_n on the indices and the translation action by $\mathbb{Z}_k$ along the main diagonal.*

The full version of Theorem 7.11 gives a simple expression (1) for each piece X_i of this multisection. The hard part will be describing, in arbitrary odd dimension, a handle decomposition of $X_I = \bigcap_{i \in I} X_i$ for arbitrary $I \subsetneq \mathbb{Z}_k$.

Section 3 describes the multisections of T^n for $n = 3, 4, 5$ in detail.

Section 4 introduces three types of building blocks; each handle of each X_I in arbitrary odd dimension will be a product of such blocks.

Section 5 describes further examples, each featuring a new complication in the handle decomposition of X_I.

Section 6 proves several combinatorial facts, including that $T^n = \bigcup_{i \in \mathbb{Z}_k} X_i$, and obtains two combinatorial identities as corollaries. In particular, §6.4 establishes a closed expression (2) for arbitrary X_I.

Section 7 describes the handle decomposition of arbitrary X_I, confirms the details of this decomposition, shows that the central intersection $\bigcap_{i \in \mathbb{Z}_k} X_i$ is a closed k-manifold, and puts everything together to prove Theorem 7.11

Section 8 extends Theorem 7.11 to certain cubulated manifolds.

Appendix 1 features tables, several detailing follow-up examples for the complications introduced in §§3 and 5, others detailing aspects of the handle decomposition described in §7.2. Appendix 2 describes three other ways that one might attempt to multisect T^n. Two attempts fail in instructive ways; the status of the other attempt is uncertain.

Thank you to Mark Brittenham, Charlie Frohman, Peter Lambert-Cole for helpful discussions. Special thank you to Alex Zupan for helpful discussions throughout the project, especially during its early stages, when we collaborated to find efficient trisections of T^4 and T^5.

[4]Any surface of positive even genus, however, admits a smooth multisection with efficiency 2. This is the maximum possible efficiency for any multisection of any surface.

2. Smooth multisections and their efficiency

Example 2.1. For $n = 2k-1$, the n-sphere

$$S^n = \partial \prod_{i=0}^{k-1} D^2 = \bigcup_{i=0}^{k-1} \left(\prod_{j=0}^{i-1} D^2 \times S^1 \times \prod_{j=i+1}^{k-1} D^2 \right)$$

admits a smooth multisection in which each

$$X_i = \prod_{j=0}^{i-1} D^2 \times S^1 \times \prod_{j=i+1}^{k-1} D^2$$

is an n-dimensional 1-handlebody of genus 1. In dimension 3, this is the genus 1 Heegaard splitting of S^3, with central surface $S^1 \times S^1$. In arbitrary dimension n, the central intersection is the k-torus $\prod_{j=0}^{k-1} S^1$, and more generally, for each $I \subset \mathbb{Z}_k$ with $1 \leq |I| = \ell \leq k-1$, the intersection

$$X_I = \bigcap_{j \in I} X_i = \prod_{j=0}^{k-1} \begin{Bmatrix} S^1 & j \in I \\ D^2 & j \notin I \end{Bmatrix} \approx \prod_{j=0}^{\ell-1} S^1 \times \prod_{j=\ell}^{k-1} D^2 \approx T^\ell \times D^{2(k-\ell)}$$

is a thickened ℓ-torus. In dimension 5, Lambert-Cole–Miller use this construction and a second trisection of S^5, whose central intersection is a 3-sphere rather than a 3-torus, to show that, unlike Heegaard splittings of 3-manifolds and trisections of 4-manifolds, trisections of a given 5-manifold need not be stably equivalent [3].

Example 2.2. Using homogeneous coordinates $[z_0 : \cdots : z_{k-1}]$ on $\mathbb{CP}^{k-1}$, one can define a smooth multisection by [7]

$$X_i = \{[z_0 : \cdots : z_{k-1}] \mid |z_i| \geq |z_j| \text{ for } j = 0, \ldots, k-1\}.$$

Then each X_I with $|I| = \ell$ is related by permutation to a thickened torus

$$\begin{aligned} \bigcap_{i=0}^{\ell-1} X_i &= \left\{ [1 : z_1 : \cdots : z_{k-1}] \;\middle|\; \begin{matrix} |z_j| = 1 \text{ for } j = 1, \ldots, \ell-1, \\ |z_j| \leq 1 \text{ for } j = \ell, \ldots, k-1 \end{matrix} \right\} \\ &\approx T^{\ell-1} \times D^{2(k-\ell)}. \end{aligned}$$

In particular, the central intersection is

$$\{[1 : z_1 : \cdots : z_k] : \ |z_1| = \cdots = |z_k| = 1\}.$$

These symmetric multisections are also efficient, since each X_i has genus 0.

Proposition 2.3. *For $i = 1, 2$, let Z_i be an n-manifold admitting an h_i-handle decomposition, and let $\phi : Y_1 \to Y_2$ glue compact $Y_i \subset \partial Z_i$, such that $Y_1 \approx Y_2$ admit h-handle decompositions. Then $Z = Z_1 \cup_\phi Z_2$ admits an h'-handle decomposition for $h' = \max\{h_1, h_2, h+1\}$.*

Proof. By taking a bicollared neighborhood N of $Y = \phi(Y_1) = \phi(Y_2)$ in Z, where $N \equiv [-1,1] \times Y$, we may identify $Z \setminus \mathrm{int}(N)$ with $Z_1 \sqcup Z_2$, which admits an h''-handle decomposition where $h'' = \max\{h_1, h_2\}$. Then, for each i-handle $H \equiv D^i \times D^{n-1-i}$ in Y, $0 \leq i \leq h$, we can glue on $[-1,1] \times H$ along $\partial([-1,1] \times D^i) \times D^{n-1-i} \approx S^i \times D^{n-1-i}$, and so attaching $[-1,1] \times H$ is the same as attaching an $(i+1)$-handle, where $i+1 \leq h+1$. □

Proposition 2.4. *Let $X = \bigcup_{i \in \mathbb{Z}_k} X_i$ be a smooth multisection of a closed manifold of dimension $n = 2k-1$. Then for each $1 \leq j \leq i \leq k-1$:*

$$\bigcup_{t=0}^{j-1} X_t \cap \bigcap_{t=j}^{i} X_t$$

admits an $(i+j)$-handle decomposition.

In particular, taking $j = i$, $(X_0 \cup \cdots \cup X_{i-1}) \cap X_i$ admits a $2i$-handle decomposition. Hence, each $X_0 \cup \cdots \cup X_i$ admits a $(2i+1)$-handle decomposition.

Proof. We argue by lexicographical induction on (i,j). When $(i,j) = (1,1)$, proposition is true by definition, since $X_0 \cap X_1$ is a 2-handlebody. The last claim then follows from Proposition 2.3.

Let $(i,j) > (1,1)$. Assume for each $(r,s) < (i,j)$ that $(X_0 \cup \cdots \cup X_{s-1}) \cap X_s \cap \cdots \cap X_r$ admits an $(r+s)$-handle decomposition. Assume also that $X_0 \cup \cdots \cup X_{i-1}$ admits a $(2i-1)$-handle decomposition. Let

$$Z_1 = \bigcup_{t=0}^{j-2} X_t \cap \bigcap_{t=j}^{i} X_t$$

and

$$Z_2 = \bigcap_{t=j-1}^{i} X_t,$$

so that

$$\bigcup_{t=0}^{j-1} X_t \cap \bigcap_{t=j}^{i} X_t = Z_1 \cup Z_2$$

Then Z_2 admits an $(i+1-j)$-handle decomposition. So does Z_1, by symmetry and the induction hypothesis. Further,

$$Z_1 \cap Z_2 = \bigcup_{t=0}^{j-2} X_t \cap \bigcap_{t=j-1}^{i} X_t,$$

which, by induction, admits an $(i+j-1)$-handle decomposition. Therefore, $Z_1 \cup Z_2$ admits a h-handle decomposition, where

$$h = \max\{i+j-1, i+1-j, i+j\} = i+j.$$

Finally, consider $W = X_0 \cup \cdots \cup X_i$. Then $W = W_1 \cup X_i$ where $W_1 = X_0 \cup \cdots \cup X_{i-1}$, which, by induction, admits a $2(i-1)$-handle decomposition. We just showed that $W_1 \cap X_i$ has a $2i$-handle decomposition. Therefore, by Proposition 2.3, W admits an h-handle decomposition where

$$h = \max\{2(i-1), 1, 2i+1\} = 2i+1. \qquad \square$$

Flipping X upside down reveals:

Theorem 2.5. *Let $X = \bigcup_{i \in \mathbb{Z}_k} X_i$ be a smooth multisection of a closed manifold of dimension $n = 2k-1$. Then X has a handle decomposition in which each X_i provides all the $2i$- and $(2i+1)$-handles.*

Proof. Given such a multisection, Proposition 2.4 implies that X admits an n-handle decomposition in which each X_i contributes only h-handles for various $h \leq 2i+1$. After flipping X upside down, Proposition 2.4 implies that each X_i contributes only $(n-h)$-handles for various $n-h \leq 2(k-1-i)+1$. Combining these two-sided bounds gives $2i \leq h \leq 2i+1$. $\square$

Proposition 2.4 and its proof adapt directly to even dimensions:

Proposition 2.6. *Let $X = \bigcup_{i \in \mathbb{Z}_k} X_i$ be a smooth multisection of a closed manifold of dimension $n = 2k-2$. Then for each $1 \leq j \leq i \leq k-1$:*

$$\bigcup_{t=0}^{j-1} X_t \cap \bigcap_{t=j}^{i} X_t$$

admits an $(i+j-1)$-handle decomposition. Hence, each $X_0 \cup \cdots \cup X_i$ admits a $2i$-handle decomposition.

Proof. Argue by lexicographical induction on (i,j). The base case holds, since $X_0 \cap X_1$ has a core of dimension 1. The induction step follows exactly as in the proof of Proposition 2.4. $\square$

In odd dimensions, X_1 contributes 2- *and* 3-handles, but in even dimensions, X_1 contributes no 3-handles. The ramifications of this difference are striking:

Theorem 2.7. *Let $k \geq 3$. If a closed $(2k-2)$-manifold has a smooth multisection $X = \bigcup_{i \in \mathbb{Z}_k} X_i$, then X has a handle decomposition in which each X_i provides all the $2i$-handles, and the only handles with odd-dimensional cores are the 1-handles from X_0 and the $(n-1)$-handles from X_{k-1}.*

Proof. By the reasoning from the proof of Theorem 2.5, each X_i contributes only h-handles for various $h \leq 2i$ and only $(n-h)$-handles for various $n-h \leq 2(k-1-i)$, hence contributes only $2i$-handles. $\square$

Corollary 2.8. *Let X be a closed smooth manifold of even dimension $n \geq 6$. If $H_i(X) \neq 0$ for any odd $i \neq 1, n-1$, then X admits no smooth multisection. In particular, for even $n \geq 6$, T^n admits no smooth multisection.*

Corollary 2.9. *When $n \neq 2$, any smooth multisection $X = \bigcup_{i \in \mathbb{Z}_k} X_i$ obeys*

$$\min_{i \in \mathbb{Z}_k} g(X_i) \geq \operatorname{rank} \pi_1(X).$$

Corollary 2.10. *No smooth multisection of any manifold of any dimension $n \neq 2$ has efficiency greater than 1. In any efficient smooth multisection $X = \bigcup_{i \in \mathbb{Z}_k} X_i$, all X_i have the same genus.*

Question 2.11. Does every smooth odd-dimensional manifold have a smooth multisection?

3. Motivating examples

Figure 1 illustrates an efficient Heegaard splitting of the 3-torus, which suggests viewing T^3 as $(\mathbb{R}/2\mathbb{Z})^3$; then the splitting is determined by a partition of the eight unit cubes with vertices in the lattice $(\mathbb{Z}/2\mathbb{Z})^3$. Moreover, *this* partition satisfies two symmetry properties: first, the permutation action of S_3 on the indices in T^3 fixes each piece of the slitting, and second, the $\mathbb{Z}_2$ translation action along the main diagonal of T^3 satisfies $X_i + (1, 1, 1, 1) = X_{i+1}$. Note that, while this splitting *looks PL* it (as with any Heegaard splitting) qualifies as *smooth*, since both X_i are handlebodies.

How might one construct efficient smooth trisections of T^n, $n = 4, 5$, with symmetry properties analogous to Figure 1's splitting of T^3? To begin, one might view these T^n as $(\mathbb{R}/3\mathbb{Z})^n$—rather than, say, $(\mathbb{R}/2\mathbb{Z})^n$, because we seek a trisection rather than a splitting—and seek an appropriate partition of the 3^n unit cubes with vertices in the lattice $(\mathbb{Z}/3\mathbb{Z})^n$. From now on, for brevity, we will refer to these unit cubes as *subcubes* of T^n.

To start forming this partition, one might assign each subcube $[i, i+1]^n$ to X_i (because of the translation action). Next, one might assign each subcube of the form $[i, i+1]^{n-1}[i+1, i+2]$, $[i, i+1]^{n-1}[i-1, i]$ to X_i as well, and

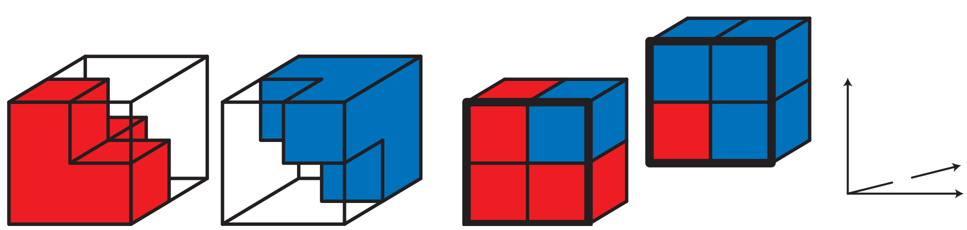

FIGURE 1. A Heegaard splitting of T^3.

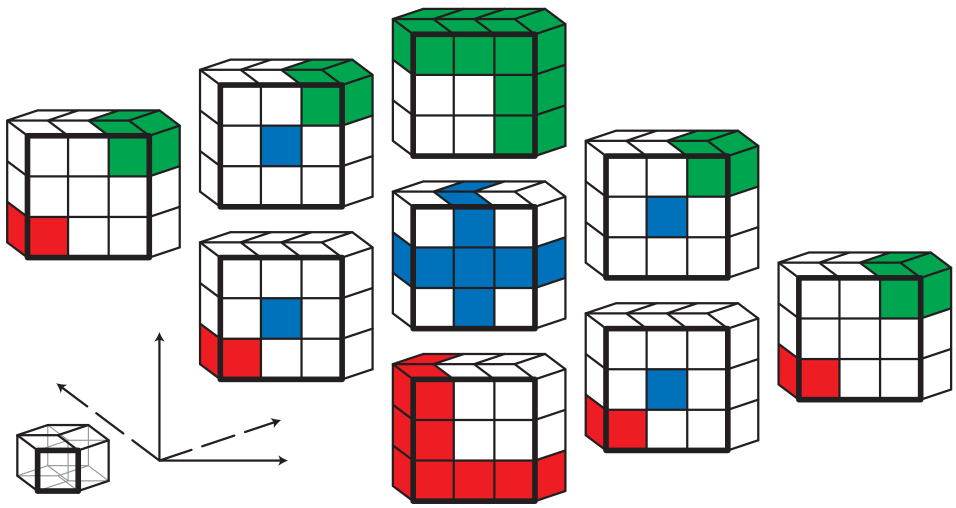

Figure 2. Start partitioning the subcubes of $T^4 = (\mathbb{R}/3\mathbb{Z})^4$ like this, giving three 4-dimensional 1-handlebodies.

extend these assignments using the permutation action on the indices. At this point, each X_i is indeed an n-dimensional 1-handlebody, and so the rest of the partition should be constructed in a way that preserves this fact, while also giving rise to the needed pairwise intersection properties. Figure 2 illustrates this intermediate stage in the case of T^4.

For T^4, the symmetry properties imply that the remaining partition is determined by the assignments of the subcubes $[0,1]^2[1,2][2,3]$ and $[0,1]^2[1,2]^2$. Assigning both subcubes to X_0 and extending symmetrically gives the decomposition of T^4 illustrated in Figures 3 and 4. Section 3.2 will confirm that this decomposition is indeed a trisection.

A similar approach leads to the decomposition of T^5 shown in Figure 5. Section 3.3 will confirm that this, too, is a trisection.

3.1. Notation.

Notation 3.1. Let $X, Y \subset Z$ be compact subspaces of a topological space. Denote "X cut along Y" by $X \setminus\setminus Y$. In every example where we use this notation, $X \setminus\setminus Y$ equals the closure in Z of $X \setminus Y$. (The general construction is somewhat more complicated.)

Given $n = 2k-1, 2k-2$, view the n-torus T^n as $(\mathbb{R}/k\mathbb{Z})^n$. Let S_n denote the permutation group on n elements.

Notation 3.2. Given $\vec{x} = (x_1, \ldots, x_n) \in T^n$ and $\sigma \in S_n$, denote

$$\vec{x}_\sigma = (x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

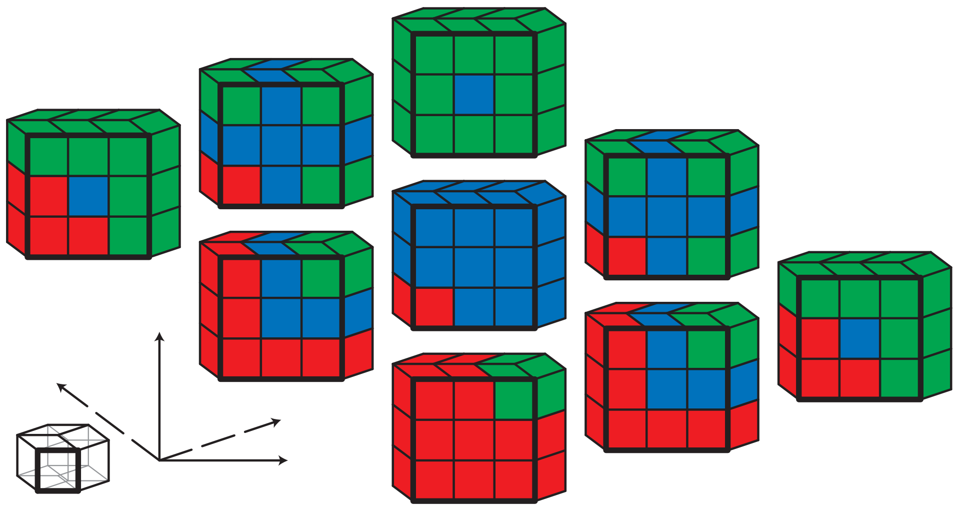

FIGURE 3. Partitioning the 3^4 subcubes of $T^4 = (\mathbb{R}/3\mathbb{Z})^4$ like this gives a symmetric efficient smooth trisection of T^4.

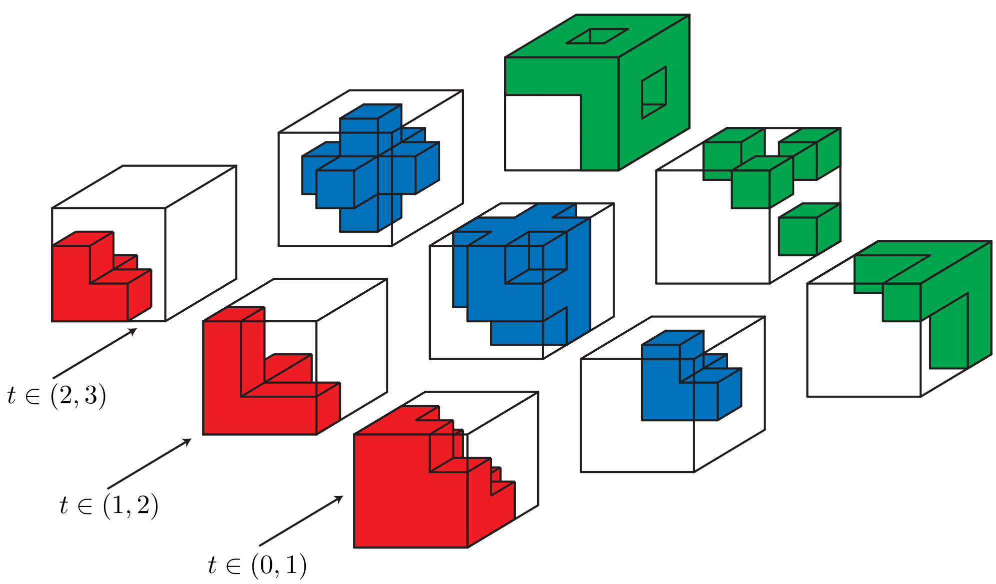

FIGURE 4. In the multisection of T^4 from Figure 3, each slice $T^3 \times \{t\}$, $t \in (\mathbb{R}/3\mathbb{Z}) \setminus \mathbb{Z}_3$, intersects X_0, X_1, X_2 like this.

Also, given $U \subset T^n$ and $\vec{v} \in T^n$, denote

$$U + \vec{v} = \{\vec{u} + \vec{v} : \vec{u} \in U\}.$$

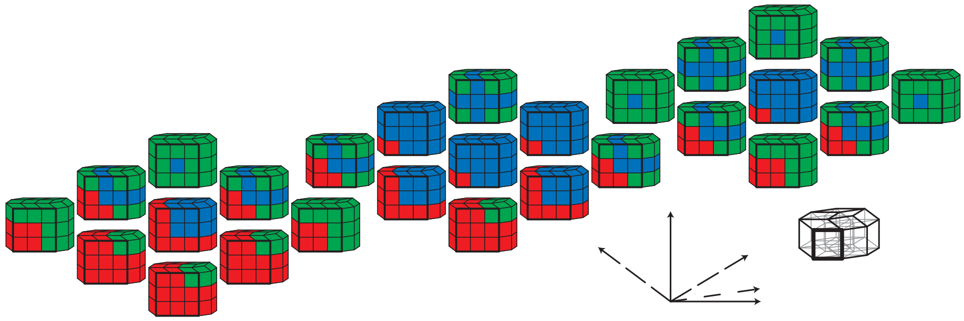

FIGURE 5. Partitioning the 3^5 subcubes of $T^5 = (\mathbb{R}/3\mathbb{Z})^5$ like this gives a symmetric efficient smooth trisection of T^5.

The symmetric group S_n acts on T^n by permuting the indices, $\sigma : \vec{x} \mapsto \vec{x}_\sigma$. Because we are interested in subsets of T^n which are fixed by this action:

Notation 3.3. For any subset $U \subset T^n$, denote

$$\langle U \rangle = \{\vec{x}_\sigma : \vec{x} \in U,\ \sigma \in S_n\} \subset T^n.$$

Note, for any $U \subset T^n$, that $\langle U \rangle$ is fixed by the action of S_n on T^n.

Theorem 7.11. *For $n = 2k-1$, the n-torus $T^n = (\mathbb{R}/k\mathbb{Z})^n = [0,k]^n/\sim$ admits an efficient smooth multisection $T^n = \bigcup_{i \in \mathbb{Z}_k} X_i$ defined by*

$$\begin{aligned} X_0 &= \left\langle [0,1]^2 \cdots [0,k-1]^2 [0,k] \right\rangle, \\ X_i &= X_0 + (i,\dots,i),\ i \in \mathbb{Z}_k. \end{aligned} \tag{1}$$

By construction, the decomposition is symmetric with respect to the permutation action on the indices and the translation action on the main diagonal.

Anticipating the concrete and (somewhat) low-dimensional nature of §§3, 5 and Appendix 1, give the first few intervals $[i, i+1]$, $i \in \mathbb{Z}_k$, special notations:

Notation 3.4. Denote

$$[0,1] = \alpha,\ [1,2] = \beta,\ [2,3] = \gamma, [3,4] = \delta, [4,5] = \varepsilon,\ [5,6] = \zeta,\ [6,7] = \eta.$$

To further abbreviate our notation, we will often omit $\times$ symbols and use exponents to denote repeated factors. For example, we can describe the two pieces of the Heegaard splitting of T^3 from Figure 1 like this:

$$X_0 = \alpha^3 \cup \alpha^2\beta \cup \alpha\beta\alpha \cup \beta\alpha^2, \qquad X_1 = \beta^3 \cup \beta^2\alpha \cup \beta\alpha\beta \cup \alpha\beta^2.$$

Using Notation 3.3, we can further abbreviate this notation:

$$\begin{aligned} X_0 &= \alpha^3 \cup \left\langle \alpha^2\beta \right\rangle & X_1 &= \beta^3 \cup \left\langle \beta^2\alpha \right\rangle \\ &= \left\langle \alpha^2[0,2] \right\rangle & &= \left\langle \beta^2[1,3] \right\rangle. \end{aligned}$$

We often omit the braces around singleton factors. For example, in T^3:

$$X_0 \cap X_1 = \langle [0,1] \times [1,2] \times \{0\} \rangle \cup \langle [0,1] \times [1,2] \times \{1\} \rangle$$
$$= \langle \alpha\beta 0 \rangle \cup \langle \alpha\beta 1 \rangle\,.$$

We also extend Notation 3.3 in the way suggested by the following example:

$$\langle 0\alpha \rangle\, \beta^2 = (\{0\} \times \alpha \times \beta \times \beta) \cup (\alpha \times \{0\} \times \beta \times \beta)\,.$$

More precisely, if we decompose T^n as a product $T^n = T^{n_1} \times \cdots \times T^{n_p}$ and $U_i \subset T^{n_i}$ for $i = 1, \ldots, p$, then

$$\langle U_1 \rangle \cdots \langle U_p \rangle = \Big\{ (\vec{x}^1_{\sigma_1}, \vec{x}^2_{\sigma_2}, \ldots, \vec{x}^p_{\sigma_p}) : \ \vec{x}^i \in T^{n_i},\ \sigma_i \in S_{n_i},\ i = 1, \ldots, p \Big\}$$

where, extending Notation 3.2, denoting each $\vec{x}^i = (x^i_1, \ldots, x^i_{n_i})$, each

$$\vec{x}^i_{\sigma_i} = \left(x^i_{\sigma_i(1)}, \ldots, x^i_{\sigma_i(n_i)} \right).$$

Starting in dimension 7, some handle decompositions will require subdividing unit subintervals $\alpha, \beta, \gamma, \delta, \ldots$ into halves or thirds. Anticipating this:

Notation 3.5. Denote

$$\alpha^- = \left[0, \tfrac{1}{2}\right],\ \alpha^+ = \left[\tfrac{1}{2}, 1\right], \ldots, [\eta^-] = \left[6, \tfrac{13}{2}\right], \eta^+ = \left[\tfrac{13}{2}, 7\right]$$

and

$$\alpha_3^- = \left[0, \tfrac{1}{3}\right], \alpha_3^\circ = \left[\tfrac{1}{3}, \tfrac{2}{3}\right],\ \alpha_3^+ = \left[\tfrac{2}{3}, 1\right], \ldots, \eta_3^\circ = \left[\tfrac{19}{3}, \tfrac{20}{3}\right],\ \eta_3^+ = \left[\tfrac{20}{3}, 7\right].$$

Because of the symmetry of the construction under the $\mathbb{Z}_k$ translation action on T^n, it will suffice, when considering X_I, to allow I to be arbitrary *only up to cyclic permutation*. In order to take advantage of this convenience:

Notation 3.6. Given $I \subset \mathbb{Z}_k$ with $|I| = \ell > 0$, denote $X_I = \bigcap_{i \in I} X_i$, and denote $I = \{i_s\}_{s \in \mathbb{Z}_\ell}$ such that

$$0 \le i_0 < i_1 < \cdots < i_{\ell-1} \le k-1.$$

Definition 3.7. Let $I = \{i_s\}_{s \in \mathbb{Z}_\ell}$ as in Notation 3.6. For each $r \in \mathbb{Z}_\ell$, define $I^r = \{i + r :\ i \in I\} \subset \mathbb{Z}_k$. Denote each $I^r = \{i^r_s\}_{s \in \mathbb{Z}_\ell}$ with $0 \le i^r_0 < i^r_1 < \cdots < i^r_{\ell-1} \le k-1$. Say that I is **simple** if, for each $r \in \mathbb{Z}_\ell$, we have $I \le I^r$ under the lexicographical ordering of their elements, i.e. if each $I_r \neq I$ has some $s \in \mathbb{Z}_\ell$ with $i_t = i^r_t$ for each $t = 0, \ldots, s-1$ and $i_s < i^r_s$.

Notation 3.8. Given simple $I = \{i_s\}_{s \in \mathbb{Z}_\ell} \subset \mathbb{Z}_k$ as in Notation 3.6, define

$$T = \{r \in \mathbb{Z}_k :\ i_r - 1 \notin I\}.$$

Denote $T = \{t_r\}_{r \in \mathbb{Z}_m}$ with $0 = t_0 < \cdots < t_m < \ell$. For each $r \in \mathbb{Z}_m$, denote $I_r = \{i_{t_r}, \ldots, i_{t_{r+1}-1}\}$. Then

$$I = I_1 \sqcup \cdots \sqcup I_m,$$

and for each $r = 0, \ldots, m-1$, we have $|I_r| = \max I_r + 1 - \min I_r$ (each block I_r is comprised of consecutive indices) and $\min I_{r+1} \ge \max I_r + 2$ (the blocks are nonconsecutive).

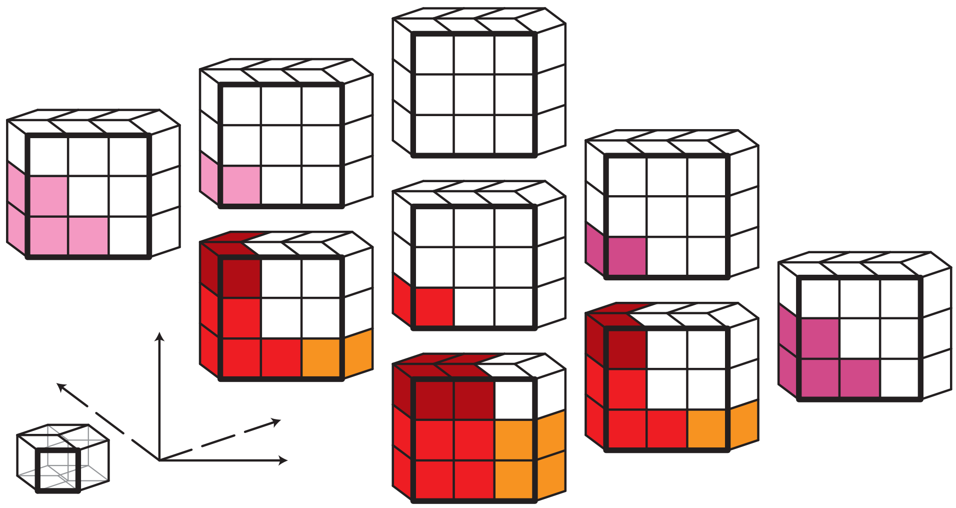

FIGURE 6. A handle decomposition of X_0 in Figure 3's trisection of T^4: the 0-handle consists of 11 subcubes; each of four 1-handles consists of four subcubes.

Given $i_* \in I$ (denoted specifically as i_*), denote the I_r containing i_* by I_*.

Convention 3.9. Throughout, reserve the notations n, k, $\alpha, \ldots, \eta$, $\alpha^-, \ldots$, η^+, $\alpha_3^-, \ldots, \eta_3^+$, I, X_I, ℓ, T, and m for the way they are used in Notations 3.4-3.8. Assume, unless otherwise stated, that $I \subset \mathbb{Z}_k$ is simple. Also reserve, for any $s \in \mathbb{Z}_\ell$ or $r \in \mathbb{Z}_m$, the notations i_s, t_r, I_r, i_*, and I_* for the way they are used in Notations 3.6 and 3.8.

Observation 3.10. *Given $I \subsetneq \mathbb{Z}_k$, we have $i_0 = 0$, $i_{\ell-1} \leq k-2$, and $|I_0| \geq |I_r|$ for each $r \in \mathbb{Z}_m$; if $|I_0| = |I_r|$, then $|I_1| \geq |I_{r+1}|$.*

Given $I \subset \mathbb{Z}_k$ and $s \in \mathbb{Z}_\ell$, denote

$$(i_1, \ldots, \widehat{i_s}, \ldots, i_\ell) = (i_1, \ldots, i_{s-1}, i_{s+1}, \ldots, i_\ell) \subset T^{\ell-1}.$$

Lemma 6.13. *Given nonempty $I \subseteqq \mathbb{Z}_k$, X_I is given by:*

$$\bigcup_{i_* \in I} \left\langle (i_1, \ldots, \widehat{i_*}, \ldots, i_\ell) \prod_{r \in \mathbb{Z}_\ell} [i_r, i_r+1]^2 \cdots [i_r, i_{r+1}-1]^2 [i_r, i_{r+1}] \right\rangle. \tag{2}$$

In particular,

$$\bigcap_{i \in \mathbb{Z}_k} X_i = \bigcup_{i_* \in \mathbb{Z}_k} \left\langle (0, \ldots, \widehat{i_*}, \ldots, k-1) \prod_{i \in \mathbb{Z}_k} [i, i+1] \right\rangle. \tag{3}$$

We will prove Lemma 6.13 in §6.4.

3.2. Trisection of T^4. The decomposition of T^4 from Figure 3 is given by

$$X_0 = \left\langle \alpha^2[0,2][0,3] \right\rangle = \left\langle \alpha^4 \right\rangle \cup \left\langle \alpha^3\beta \right\rangle \cup \left\langle \alpha^3\gamma \right\rangle \cup \left\langle \alpha^2\beta^2 \right\rangle \cup \left\langle \alpha^2\beta\gamma \right\rangle$$
$$X_i = X_0 + (i,i,i,i). \tag{4}$$

It is evident from Figure 3 that $X_0 \cup X_1 \cup X_2 = T^4$. Also, $I = \{0\}$ and $I = \{0,1\}$ are the only subsets of $\{0,1,2\}$ which are simple. Therefore, in order to check that (4) determines a trisection of T^4, it suffices to prove that X_0 is a 4-dimensional 1-handlebody and $X_0 \cap X_1$ is a 3-dimensional 1-handlebody with $\partial(X_0 \cap X_1) = X_0 \cap X_1 \cap X_2$.

Indeed, Figure 6 shows a handle decomposition of X_0 in which $\left\langle \alpha^2[0,2]^2 \right\rangle$ is a 0-handle and $\left\langle \alpha^2[0,2]\gamma \right\rangle$ supplies four 1-handles, each a permutation of $\left\langle \alpha^2[0,2] \right\rangle \gamma$. More precisely, each 1-handle is given, in terms of some permutation $\sigma \in S_4$ (using Notation 3.2), by

$$\left\{ \vec{x}_\sigma : \ \vec{x} \in \left\langle \alpha^2[0,2] \right\rangle \gamma \right\}.$$

Now consider

$$X_0 \cap X_1 = \langle \alpha 1 \beta [1,3] \rangle \cup \left\langle 0 \alpha \beta^2 \right\rangle.$$

We claim that this is a 3-dimensional 1-handlebody in which:

- $Y_1 = \left\langle \alpha 1 \beta^2 \right\rangle$ is the 0-handle;
- $Y_2 = \left\langle 0 \alpha \beta^2 \right\rangle$ gives six 1-handles, all permutations of $Y_2^* = \langle 0\alpha \rangle \, \beta^2$;
- $Y_3 = \langle \alpha 1 \beta \gamma \rangle$ gives four 1-handles, all permutations of $Y_3^* = \langle \alpha 1 \beta \rangle \, \gamma$.

Figure 7 shows this decomposition of $X_0 \cap X_1$:

- The shape in the center (which looks like a truncated tetrahedron) is the 0-handle $\left\langle \alpha 1 \beta^2 \right\rangle$, comprised of 12 cubes, each a permutation of $\alpha 1 \beta^2$. The interior lattice point is $(1,1,1,1)$, and each triangular-looking face is a permutation of $0 \left\langle 1 \beta^2 \right\rangle$. Each blue segment on $\partial \left\langle \alpha 1 \beta^2 \right\rangle$ is a permutation of $\langle \alpha 1 \rangle \, 2^2$.
- Each of the four three-pronged pieces is a permutation of $0 \left\langle \alpha \beta^2 \right\rangle$, glued to the 0-handle along $0 \left\langle 1 \beta^2 \right\rangle$. The twelve cubes comprising these pieces are then glued in pairs: $0\alpha\beta^2$ and $\alpha 0 \beta^2$, e.g., meet along the face $00\beta^2$, and the other pairs are permutations of this. The union of each pair of cubes, (a permutation of) $Y_2^* = \langle 0\alpha \rangle \, \beta^2$, is a 1-handle which is glued to the 0-handle along (the corresponding permutation of) $\langle 01 \rangle \, \beta^2$. Note that Y_2^* intersects other permutations of Y_2^*, but only within $Y_2^* \cap Y_1$. Therefore, attaching Y_2^* to Y_1 amounts to attaching six 1-handles.
- Each of the four remaining pieces is a permutation of $Y_3^* = \langle \alpha 1 \beta \rangle \, \gamma$, attaching to Y_1 along (a permutation of) $\langle \alpha 1 \beta \rangle \, 2$ and to Y_2 along $\langle \alpha 1 \beta \rangle \, 0 \subset \left\langle \alpha \beta^2 \right\rangle 0$.

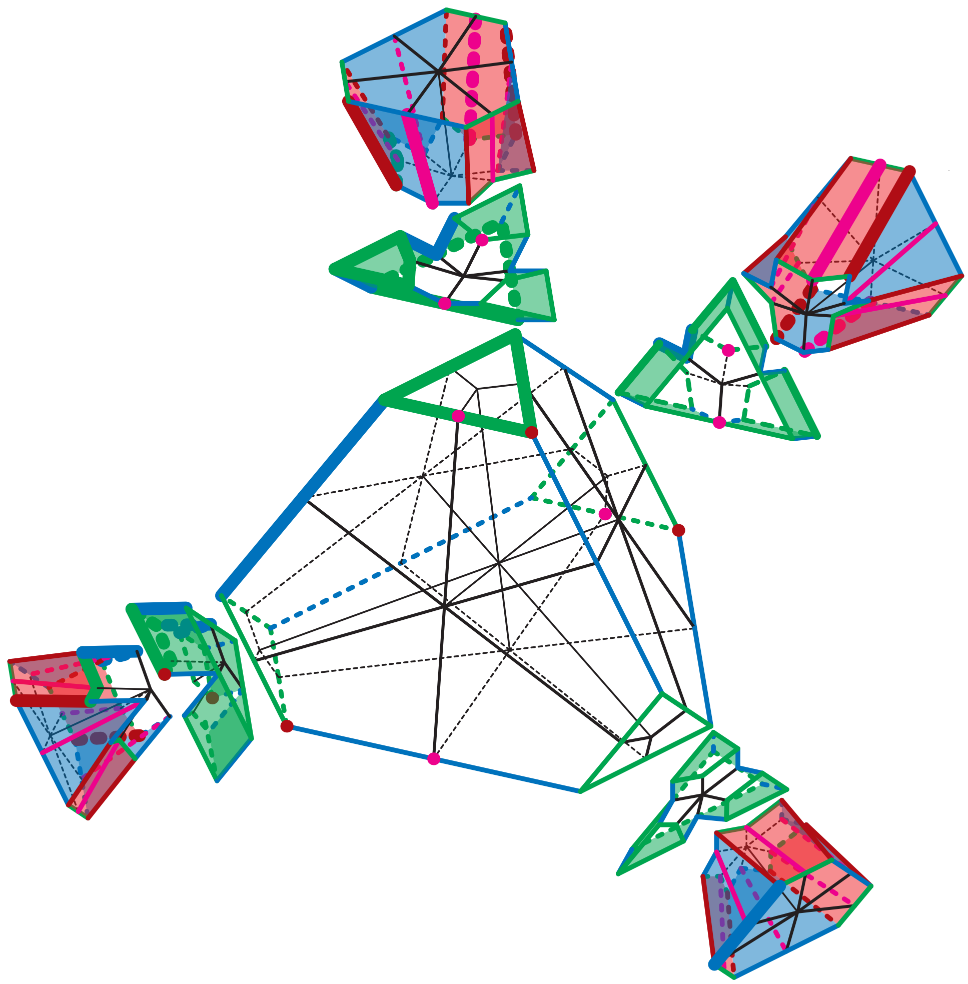

FIGURE 7. A handle decomposition of $X_0 \cap X_1$ in Figure 3's trisection of T^4. The trisection diagram on $\partial(X_0 \cap X_1) = X_0 \cap X_1 \cap X_2 = \langle \alpha\beta 02\rangle \cup \langle \alpha\gamma 12\rangle \cup \langle \beta\gamma 01\rangle$ has two types of red curves; one of each is in bold. Same with blue and green.

For emphasis, here are some key details of this decomposition which will be instructive toward the odd-dimensional case:

$$Y_1 = Y_1^* = \left\langle \alpha 1 \beta^2 \right\rangle \approx D^3,$$

so Y_1 is a 0-handle;

$$Y_2^* = \langle 0\alpha\rangle\, \beta^2 \approx D^1 \times D^2 \text{ and}$$
$$Y_2^* \cap (Y_2 \setminus \backslash Y_2^*) \subset Y_2^* \cap Y_1 = (\partial \langle 0\alpha\rangle) \times \beta^2 = \langle 01\rangle\, \beta^2 \approx S^0 \times D^2,$$

so attaching Y_2 to Y_1 amounts to attaching a collection of 1-handles; and

$$Y_3^* = \langle\alpha 1\beta\rangle\,\gamma \approx D^2 \times D^1 \text{ and}$$

$$Y_3^* \cap (Y_3 \setminus \backslash Y_3^*) \subset Y_3^* \cap (Y_1 \cup Y_2) = \langle\alpha 1\beta\rangle \times \partial\gamma \approx D^2 \times S^0,$$

so attaching Y_3 to $Y_1 \cup Y_2$ amounts to attaching a collection of 1-handles. Thus, $X_0 \cap X_1$ is a 4-dimensional 1-handlebody. Note in Figure 7 that $\partial(X_0 \cap X_1)$ is the central surface

$$X_0 \cap X_1 \cap X_2 = \langle\alpha\beta 02\rangle \cup \langle\alpha\gamma 12\rangle \cup \langle\beta\gamma 01\rangle\,, \tag{5}$$

which is colored in Figure 7 according to the color scheme from (5). Moreover, the red (resp. blue, green) line segments in Figure 7 comprise the "red (resp. blue, green) curves" in a trisection diagram for this trisection, and so Figure 7 contains the information of a trisection diagram (see [1, 4]).

3.3. Trisection of T^5. The decomposition of T^5 from Figure 5 is given by

$$X_0 = \left\langle\alpha^2[0,2]^2[0,3]\right\rangle,\ X_i = X_0 + (i,i,i,i,i).$$

The handle decompositions of X_I, $I = \{0\}, \{0,1\}$, are quite similar to those from T^4. Focusing first on $I = \{0\}$, compare Tables 1 and 2.

Y_z	Y_z^*	h	z	glue to
$\left\langle\alpha^2[0,2]^2\right\rangle$	$\left\langle\alpha^2[0,2]^2\right\rangle$	0	1	
$\left\langle\alpha^2[0,2]\gamma\right\rangle$	$\left\langle\alpha^2[0,2]\right\rangle\gamma$	**1**	2	1

TABLE 1. X_0 from the trisection of T^4

J	Y_z	Y_z^*	h	z	glue to
$\varnothing$	$\left\langle\alpha^2[0,2]^3\right\rangle$	$\left\langle\alpha^2[0,2]^3\right\rangle$	0	1	
$\{0\}$	$\left\langle\alpha^2[0,2]^2\gamma\right\rangle$	$\left\langle\alpha^2[0,2]^2\right\rangle\gamma$	**1**	2	1

TABLE 2. X_0 from the trisection of T^5

Note in each case that $Y_1 = Y_1^*$ is star-shaped in a particularly nice way. In §4, we will formalize and generalize this, giving one of three basic types of building blocks for our general construction. Notice in both cases that the handle decomposition of X_0 comes from the decomposition of the interval

$$[0,3] = [0,2] \cup \gamma.$$

To observe the other two types building blocks, consider X_I, $I = \{0,1\}$ from T^4 and T^5, whose handle decompositions are summarized in Tables 3, 4. The factor $\langle 0\alpha\rangle \approx D^1$ of Y_2^* is an example of the second type of building block. The factor $\langle\gamma 0\alpha\rangle \approx D^2$ from Y_4^* is an example of the third type (from Y_3^*, $\langle\alpha 1\beta\rangle \approx D^2$ from T^4 and $\left\langle\alpha 1\beta^2\right\rangle \approx D^3$ from T^5 are further examples).

Y_z	Y_z^*	h	z	glue to
$\langle \alpha 1 \beta^2 \rangle$	$\langle \alpha 1 \beta^2 \rangle$	0	1	
$\langle 0 \alpha \beta^2 \rangle$	$\langle 0 \alpha \rangle\, \beta^2$	1	2	1
$\langle \alpha 1 \beta \gamma \rangle$	$\langle \alpha 1 \beta \rangle\, \gamma$	1	3	1,2

TABLE 3. From the trisection of T^4: $X_0 \cap X_1 = \langle \alpha 1 \beta [1,3] \rangle \cup \langle 0 \alpha \beta^2 \rangle$.

J	i_*	Y_z	Y_z^*	h	z	glue to
$\varnothing$	0	$\langle \alpha 1 \beta^3 \rangle$	$\langle \alpha 1 \beta^3 \rangle$	0	1	
	1	$\langle 0 \alpha \beta^3 \rangle$	$\langle 0 \alpha \rangle\, \beta^3$	1	2	1
$\{0\}$	0	$\langle \alpha 1 \beta^2 \gamma \rangle$	$\langle \alpha 1 \beta^2 \rangle\, \gamma$	1	3	1,2
	1	$\langle \gamma 0 \alpha \beta^2 \rangle$	$\langle \gamma 0 \alpha \rangle\, \beta^2$	**2**	4	2,3

TABLE 4. From the trisection of T^5: $X_0 \cap X_1 = \left\langle \alpha 1 \beta^2 [1,3] \right\rangle \cup \left\langle 0 \alpha \beta^2 [1,3] \right\rangle$.

The most striking difference between X_0 and $X_{\{0,1\}}$ is the presence of the singletons in the expressions for the latter. Note the consistency between the singleton $(i_1, \dots, \widehat{i_*}, \dots, i_\ell)$ in the general expression (2) for X_I, and the specific singleton in each row of Table 4, depending on i_*. The most striking difference between $X_{\{0,1\}}$ in T^4 versus T^5 is the appearance of the 2-handles Y_4 in the latter. Other than this, most of the structure described for T^4 applies (by analogy) to T^5 as well.

In the handle decomposition of $X_{\{0,1\}}$ from T^5, each Y_z is characterized as follows by a pair (J, i_*), where $J \subset T = \{0\}$ and $i_* \in I = \{0,1\}$. (The reader may wish to skip these formalities for now and may find this more useful when considering these formalities in §7.2 in the context of the handle decomposition of arbitrary X_I.) Define $\rho_1 = \alpha$ and $\widehat{C}_0 = \beta^2$. Given $J \subset T$ and $i_* \in I$, let i be the element of $I \setminus \{i_*\}$, and define:

$$\rho_0 = \begin{cases} \beta & J = \varnothing \\ \gamma & J = \{0\}, \end{cases}$$
$$Y_z = \left\langle \{i\} \times \rho_0 \times \rho_1 \times \widehat{C}_0 \right\rangle.$$

Order the four possibilities for (J, i_*) lexicographically, with $(J, i_*) \prec (J', i'_*)$ if $J \subsetneq J'$ or if $J = J'$ and $i_* < i'_*$, and index Y_1, Y_2, Y_3, Y_4 according to this order. For each $z = 1,2,3,4$ corresponding to some (J, i_*), define $\xi_i(z)$,

$\xi_{i_*}(z)$, and Y_z^* as follows:

$$\xi_i(z) = \begin{cases} \{i\} \times \widehat{C}_0 \times \prod_{t\in I:\ \rho_t \ni i} \rho_t & i = 1 \\ \{i\} \times \prod_{t\in I:\ \rho_t \ni i} \rho_t & i = 0, \end{cases}$$

$$\xi_{i_*}(z) = \begin{cases} \widehat{C}_0 \times \prod_{t\in I:\ \rho_t \ni i} \rho_t & i_* = 1 \\ \prod_{t\in I:\ \rho_t \ni i} \rho_t & i_* = 0, \end{cases}$$

$$Y_z^* = \langle \xi_0(z) \rangle \times \langle \xi_1(z) \rangle .$$

Note that when $J = \varnothing$ and $i_* = 0$, $i \in \rho_0 \cap \rho_1$, so ξ_0 is an empty product. In that case, we regard ξ_0 not as the empty set but rather as "no factor", so

$$Y_1^* = \langle \xi_0(1) \rangle \times \langle \xi_1(1) \rangle = \langle \xi_1(1) \rangle = \left\langle \alpha 1 \beta^3 \right\rangle = Y_1.$$

4. Star-shaped building blocks

Given $\vec{p}, \vec{q} \in \mathbb{R}^n$, denote

$$[\vec{p}, \vec{q}] = \{ t\vec{p} + (1-t)\vec{q} :\ 0 \le t \le 1 \} .$$

Let $Y \subset \mathbb{R}^n$. Given $\vec{p} \in Y$, the *link* of $\vec{p}$ in Y is

$$\mathrm{lk}_Y(\vec{p}) = \{ \vec{v} \in \mathbb{R}^n :\ |\vec{v}| = 1,\ [\vec{p}, \vec{p} + \varepsilon\vec{v}] \subset Y \text{ for some } \varepsilon > 0 \} .$$

Then Y is a d-dimensional submanifold near a point $\vec{p} \in Y$ if either

- $\mathrm{lk}_Y(\vec{p}) \approx S^{d-1}$, in which case $\vec{p}$ is in the *interior* of Y; or
- $\mathrm{lk}_Y(\vec{p}) \approx D^{d-1}$, in which case $\vec{p} \in \partial Y$.

If $\mathrm{lk}_Y(\vec{p}) \approx S^{d-1}$, define the *scope* of $\vec{p}$ in Y to be

$$\mathrm{scope}(Y; \vec{p}) = \{ \vec{q} \in Y :\ [\vec{p}, \vec{q}] \subset Y \}.$$

If $\mathrm{lk}_Y(\vec{p}) \approx S^{d-1}$, say that Y is *strongly star-shaped* about $\vec{p}$ if, for every point $\vec{q} \in \mathrm{scope}(Y; \vec{p})$, every point $\vec{x} \in [\vec{p}, \vec{q}] \setminus \{\vec{q}\}$ satisfies $\mathrm{lk}_Y(\vec{x}) \approx S^{d-1}$.

Proposition 4.1. *If $Y \subset \mathbb{R}^n$ is strongly star-shaped about $\vec{p} \in Y$, then $A = \mathrm{scope}(Y; \vec{p})$ is homeomorphic to the compact d-ball, $A \approx D^d$.*

Proof. There is a homeomorphism $\phi : S^{d-1} \to \mathrm{lk}_Y(\vec{p})$, since $\vec{p} \in \mathrm{int}\, Y$, and another $\psi : \partial A \to \mathrm{lk}_Y(\vec{p})$ given by $\psi : \vec{q} \mapsto \frac{\vec{q}-\vec{p}}{|\vec{q}-\vec{p}|}$, because Y is strongly star-shaped about $\vec{p}$. Define a polar coordinate system $\Phi : A \to D^q$ by $\Phi : \vec{p} \mapsto \vec{0}$ and, for $\vec{q} \neq \vec{p}$, with $\vec{\theta} = \frac{\vec{q}-\vec{p}}{|\vec{q}-\vec{p}|} \in \mathrm{lk}_Y(\vec{p})$, by

$$\Phi : \vec{q} \mapsto \frac{|\vec{q} - \vec{p}|}{|\psi^{-1}(v) - \vec{p}|} \cdot \vec{\theta}.$$

This is a homeomorphism, because the inverse map $D^d \to A$ is

$$\Phi^{-1} : r\vec{\theta} \mapsto \vec{p} + r|\psi^{-1} \circ \phi(\vec{\theta}) - \vec{p}| \cdot \phi(\vec{\theta}). \qquad \square$$

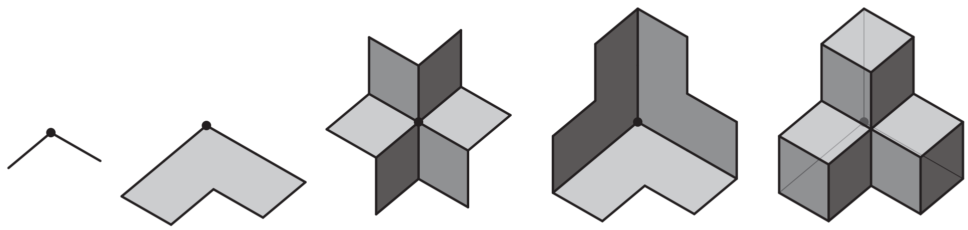

FIGURE 8. Left to right: $\langle 0\alpha\rangle$, $\langle\alpha[0,2]\rangle$, $\langle\alpha 1\beta\rangle$, $\langle 0\alpha[0,2]\rangle$, $\langle\alpha^2[0,2]\rangle$.

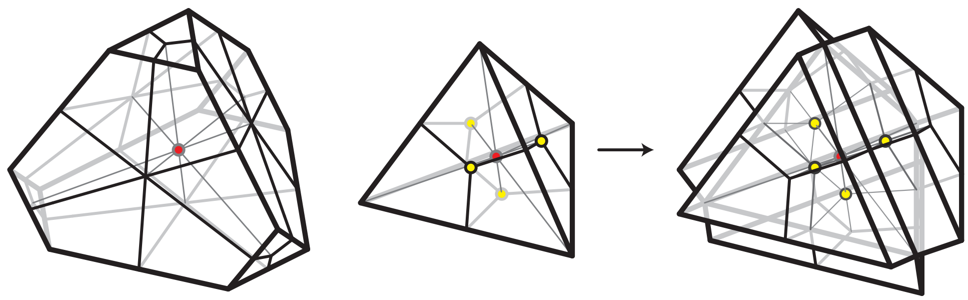

FIGURE 9. Left to right: $\left\langle\alpha 0\beta^2\right\rangle$ and $\left\langle 0\alpha^3\right\rangle \to \left\langle 0\alpha^2[0,2]\right\rangle$.

In $T^n = (\mathbb{R}/k\mathbb{Z})^n$, for $d \le n-1$, identify $T^d = (\mathbb{R}/k\mathbb{Z})^d$ with $(\mathbb{R}/k\mathbb{Z})^d \times \{\vec{0}\} \subset T^n$, and likewise for T^{d+1}. With $0 < a_1 \le \cdots \le a_d < k$, define

$$C_1 = \left\langle \prod_{r=1}^{d} [0, a_r] \right\rangle \subset T^d, \tag{6}$$

$$C_2 = \left\langle \{0\} \times \prod_{r=1}^{d} [0, a_r] \right\rangle \subset T^{d+1}. \tag{7}$$

Also, assuming that $k - a_1 > a_d$, define

$$C_3 = \left\langle [0, a_1] \times \{a_1\} \times \prod_{r=2}^{d} [a_1, a_r] \right\rangle \subset T^{d+1}. \tag{8}$$

Figures 8 and 9 show low-dimensional examples of these building blocks.

Lemma 4.2. *C_1, C_2, and C_3 from (6)-(8) are homeomorphic to D^d.*

Proof. Let $a = \frac{a_1}{2}$, $\vec{a} = (a, \ldots, a, 0, \ldots, 0)$, $b = \frac{1}{2}(k + a_d)$, and $U = [0, b]^d \subset T^d$. Then $C_1 \subset U \approx D^d$ and $C_2, C_3 \subset U \times [0, b] \approx D^{d+1}$, so we may view C_1 in $\mathbb{R}^d$ and C_2, C_3 in $\mathbb{R}^{d+1}$.

C_1 is strongly star-shaped about $\vec{a}$, and the scope of $\vec{a}$ in C_1 is all of C_1, so $C_1 \approx D^d$, by Proposition 4.1.

Consider $U \times [0,b] \approx D^{d+1}$ with $\partial(U \times [0,b]) = Y_0 \cup Y_b$, where, for $t = 0, b$:

$$Y_t = \left\langle \{t\}[0,b]^d \right\rangle .$$

Proposition 4.1 gives $D^d \approx \operatorname{scope}(Y_0, \vec{0}) = C_2$, as Y_0 is strongly star-shaped about $\vec{0}$. Let $\vec{a}' = (a_1, \ldots, a_1)$, and consider

$$\begin{aligned} D^{d+1} &\approx Y_1 = \left\langle [0,a_1]^2 \prod_{r=2}^{d} [0,a_r] \right\rangle , \\ D^d &\approx Y_2 = \left\langle \{0\} \prod_{r=1}^{d} [0,a_r] \right\rangle \subset \partial D^{d+1} \approx S^d , \\ D^d &\approx Y_3 = \partial Y_1 \setminus \setminus Y_2 \\ &= C_3 \cup \bigcup_{s=2}^{d} \left\langle [0,a_1]^2 \prod_{r=2}^{s-1} \{a_s\} \prod_{r=s+1}^{d} [a_s, a_d] \right\rangle . \end{aligned}$$

Proposition 4.1 implies that $D^d \approx \operatorname{scope}(Y_3; \vec{a}') = C_3$, since Y_3 is strongly star-shaped about $\vec{a}'$. □

5. Further examples

5.1. Quadrisection of T^7. Consider the decomposition of T^7 given by $X_0 = \langle \alpha^2[0,2]^2[0,3]^2[0,4] \rangle$ and $X_i = X_0 + (i,i,i,i,i,i,i)$.

5.1.1. X_I, $I = \{0\}, \{0,1\}$. The handle decompositions of X_I, $I = \{0\}, \{0,1\}$, summarized in Tables 5 and 6, respectively, follow the same pattern in dimension seven (and all higher odd dimensions) as in dimension five (recall Tables 2 and 4 and the attending discussions).

J	Y_z	Y_z^*	h	z	glue to
$\varnothing$	$\langle \alpha^2[0,2]^2[0,3]^2[0,4]^3 \rangle$	$\langle \alpha^2[0,2]^2[0,3]^2[0,4]^3 \rangle$	0	1	
$\{0\}$	$\langle \alpha^2[0,2]^2[0,3]^2[0,4]^2\varepsilon \rangle$	$\langle \alpha^2[0,2]^2[0,3]^2[0,4]^2 \rangle \varepsilon$	**1**	2	1

TABLE 5. X_0 from the quadrisection of T^7

5.1.2. X_I when $I = \{0,2\}$. Consider

$$X_0 \cap X_2 = \langle \alpha^2[0,2]\gamma^2[2,4]^2[2,5] \rangle \cup \langle \alpha^2[0,2]2\gamma^2[2,4]^2[2,5] \rangle .$$

Table 7 summarizes a handle decomposition $X_I = Y_1 \cup \cdots \cup Y_{12}$. As with X_I, $I = \{0,1\}$, the decomposition of X_I, $I = \{0,2\}$ is organized largely

J	i_*	Y_z	Y_z^*	h	z	glue to
$\varnothing$	0	$\langle\alpha 1\beta^2[1,3]^3\rangle$	$\langle\alpha 1\beta^2[1,3]^3\rangle$	0	1	
	1	$\langle 0\alpha\beta^2[1,3]^3\rangle$	$\langle 0\alpha\rangle\langle\beta^2[1,3]^3\rangle$	1	2	1
$\{0\}$	0	$\langle\alpha 1\beta^2[1,3]^2\delta\rangle$	$\langle\alpha 1\beta^2[1,3]^2\rangle\,\delta$	1	3	1,2
	1	$\langle\delta 0\alpha\beta^2[1,3]^2\rangle$	$\langle\delta 0\alpha\rangle\langle\beta^2[1,3]^2\rangle$	**2**	4	2,3

Table 6. X_I, $I = \{0,1\}$ from the quadrisection of T^7

J	i_*	V	V^-	Y_z	Y_z^*	h	z	glue to
$\varnothing$	0	$\varnothing$	$\varnothing$	$\langle\alpha^3 2\gamma^3\rangle$	$\alpha^3\langle 2\gamma^3\rangle$	0	1	
	2	$\varnothing$		$\langle 0\alpha^3\gamma^3\rangle$	$\langle 0\alpha^3\rangle\,\gamma^3$	0	2	
$\{0\}$	0	$\varnothing$	$\varnothing$	$\langle\alpha^3 2\gamma^2\delta\rangle$	$\alpha^3\langle 2\gamma^2\rangle\,\delta$	1	3	1,2
	2	$\{0\}$	$\varnothing$	$\langle\delta^+ 0\alpha^3\gamma^2\rangle$	$\langle\delta^+ 0\alpha^3\rangle\,\gamma^2$	0	4	
			$\{0\}$	$\langle\delta^- 0\alpha^3\gamma^2\rangle$	$\delta^-\langle 0\alpha^3\rangle\,\gamma^2$	1	5	2,4
$\{2\}$	0	$\{2\}$	$\varnothing$	$\langle\alpha^2\beta^+ 2\gamma^3\rangle$	$\alpha^2\langle\beta^+ 2\gamma^3\rangle$	0	6	
			$\{2\}$	$\langle\alpha^2\beta^- 2\gamma^3\rangle$	$\alpha^2\beta^-\langle 2\gamma^3\rangle$	1	7	1,6
	2	$\varnothing$	$\varnothing$	$\langle 0\alpha^2\beta\gamma^3\rangle$	$\langle 0\alpha^2\rangle\,\beta\gamma^3$	1	8	1,2
$\{0,2\}$	0	$\{2\}$	$\varnothing$	$\langle\alpha^2\beta^+ 2\gamma^2\delta\rangle$	$\alpha^2\langle\beta^+ 2\gamma^2\rangle\,\delta$	1	9	6,8
			$\{2\}$	$\langle\alpha^2\beta^- 2\gamma^2\delta\rangle$	$\alpha^2\beta^-\langle 2\gamma^2\rangle\,\delta$	**2**	10	3,7,8,9
	2	$\{0\}$	$\varnothing$	$\langle\delta^+ 0\alpha^2\beta\gamma^2\rangle$	$\langle\delta^+ 0\alpha^2\rangle\,\beta\gamma^2$	1	11	3,4
			$\{0\}$	$\langle 0\alpha^2\beta\gamma^2\delta^-\rangle$	$\langle 0\alpha^2\rangle\,\beta\gamma^2\delta^-$	**2**	12	3,5,8,11

Table 7. X_I, $I = \{0,2\}$ from the quadrisection of T^7

according to $\{(J,i_*) : J \subset \{\min I_r\},\ i_* \in I\}$. Here, Y_4 and Y_5 provide the first instance where $J \setminus \{\min I_*\} \neq \varnothing$, requiring us to split a unit interval into subintervals, in this case halves.

5.1.3. X_I when $I = \{0,1,2\}$.

Consider

$$X_0 \cap X_1 \cap X_2 = \langle\alpha^2[0,2]\gamma^2[2,4]^2[2,5]\rangle \cup \langle\alpha^2[0,2]2\gamma^2[2,4]^2[2,5]\rangle .$$

Table 8 summarizes a handle decomposition $X_I = Y_1 \cup \cdots \cup Y_{12}$. Again, the decomposition of X_I, $I = \{0,2\}$ is organized largely according to $\{(J,i_*) : J \subset \{\min I_r\},\ i_* \in I\}$. Here, we have the first instance where a block I_r (in this case $I_r = I$) has $|I_r| \geq 3$, requiring us to split a unit interval at times into thirds, seen here in $Y_6 - Y_8$ and $Y_{14} - Y_{16}$. Also, $Y_1 - Y_4$ and $Y_9 - Y_{12}$ provide the first instances where $i_* + 2 \in I_*$, requiring us to split certain unit intervals into halves according to a different rule than in §5.1.2.

5.2. X_I, $I = \{0,1,2,4\}$ from T^{11}.

There is one more complication, which arises, first in dimension 11, whenever X_I, $I = I_1 \sqcup \cdots \sqcup I_m$, has some $I_r \not\ni i_*$ with $|I_r| \geq 3$. Consider X_I in the sexasection of T^{11} where

J	i_*	U	V	V^-	Y_z^*	h	z	glue to
∅	0	∅	{1,2}	{1}	$\alpha^- 1 \left\langle \beta^+ 2\gamma^3 \right\rangle$	0	1	
				∅	$\langle \alpha^+ 1\rangle \left\langle \beta^+ 2\gamma^3 \right\rangle$	1	2	1
				{1,2}	$\alpha^- \langle 1\beta^- \rangle \left\langle 2\gamma^3 \right\rangle$	1	3	1
				{2}	$\langle \alpha^+ 1\beta^- \rangle \left\langle 2\gamma^3 \right\rangle$	2	4	2,3
	1	∅	∅	∅	$\langle 0\alpha \rangle \left\langle \beta 2\gamma^3 \right\rangle$	1	5	1,3
	2	{1}	∅	∅	$0\alpha_3^\circ \langle 1\beta \rangle \gamma^3$	1	6	5
					$\left\langle 0\alpha_3^- \right\rangle \langle 1\beta \rangle \gamma^3$	2	7	5,6
					$0 \left\langle \alpha_3^+ 1\beta \right\rangle \gamma^3$	2	8	5,6
{0}	0	∅	{1,2}	{1}	$\delta\alpha^- 1 \left\langle \beta^+ 2\gamma^2 \right\rangle$	1	9	1,6,7
				∅	$\delta \langle \alpha^+ 1 \rangle \left\langle \beta^+ 2\gamma^2 \right\rangle$	2	10	2,6,8
				{1,2}	$\delta\alpha^- \langle 1\beta^- \rangle \left\langle 2\gamma^2 \right\rangle$	2	11	3,6,7
				{2}	$\delta \langle \alpha^+ 1\beta^- \rangle \left\langle 2\gamma^2 \right\rangle$	**3**	12	4,6,8
	1	∅	∅	∅	$\langle \delta 0\alpha \rangle \left\langle \beta 2\gamma^2 \right\rangle$	2	13	5,9,11
	2	{1}	∅	∅	$\langle \delta 0 \rangle \alpha_3^\circ \langle 1\beta \rangle \gamma^2$	2	14	6,13
					$\left\langle \delta 0\alpha_3^- \right\rangle \langle 1\beta \rangle \gamma^2$	**3**	15	7,13,14
					$\langle \delta 0 \rangle \left\langle \alpha_3^+ 1\beta \right\rangle \gamma^2$	**3**	16	8,13,14

TABLE 8. X_I, $I = \{0,1,2\}$ from the quadrisection of T^7

$I = \{0,1,2,4\}$, which is given by

$$\left\langle \alpha 1\beta 2\gamma^2 [2,4] 4\delta^2 [4,6] \right\rangle \cup \left\langle 0\alpha\beta 2\gamma^2 [2,4] 4\delta^2 [4,6] \right\rangle$$
$$\cup \left\langle 0\alpha 1\beta\gamma^2 [2,4] 4\delta^2 [4,6] \right\rangle \cup \left\langle 0\alpha 1\beta 2\gamma^2 [2,4] \delta^2 [4,6] \right\rangle .$$

Tables 18 and 19 in Appendix 1 detail the handle decomposition. The new complication arises when $s = 4$, i.e in the part of X_I given by

$$\left\langle 0\alpha 1\beta 2\gamma^2 [2,4] \delta^2 [4,6] \right\rangle .$$

The difficult part of this complication is the question of how to order the pieces Y_z (when some $|I_r| \geq 3$ has $I_r \not\ni i_*$). To highlight that difficulty and its solution, Table 9 summarizes the first several Y_z in the handle decomposition of X_I, $I = \{0,1,2,3,5\}$, from the septisection of T^{13}. In that table, $J = \varnothing$, $s = 5$, $U = \varnothing$, and $V = \{1,2,3\}$.

Also see Table 20 in Appendix 1, which summarizes the start of the handle decomposition of X_I, $I = \{0,1,2,3,4,6\}$, from T^{15}. In that table, $J = \varnothing$, $s = 6$, $U = \varnothing$, and $V = \{1,2,3,4\}$.

6. Combinatorics

6.1. Notation. Because each X_i is symmetric under the permutation action of S_n on the indices in T^n, it will often suffice, when considering an arbitrary point $\vec{x} = (x_1, \ldots, x_n) \in (\mathbb{R}/k\mathbb{Z})^n = T^n$, to assume that $\vec{x}$ is **monotonic** in the sense that $x_1 \leq x_2 \leq \cdots \leq x_n \leq k + x_1$.

V^-	Y_z^*	h	z	glue to
$\varnothing$	$0\,\langle\alpha^+1\rangle\,\langle\beta^+2\rangle\,\langle\gamma^+3\delta^3\rangle\,\zeta^3$	0	1	
$\{1\}$	$\langle 0\alpha^-\rangle\,1\,\langle\beta^+2\rangle\,\langle\gamma^+3\delta^3\rangle\,\zeta^3$	1	2	1
$\{1,2\}$	$\langle 0\alpha^-\rangle\,\langle 1\beta^-\rangle\,2\,\langle\gamma^+3\delta^3\rangle\,\zeta^3$	1	3	2
$\{2\}$	$0\,\langle\alpha^+1\beta^-\rangle\,2\,\langle\gamma^+3\delta^3\rangle\,\zeta^3$	2	4	1,3
$\{2,3\}$	$0\,\langle\alpha^+1\beta^-\rangle\,\langle 2\gamma^-\rangle\,\langle 3\delta^3\rangle\,\zeta^3$	1	5	4
$\{1,2,3\}$	$\langle 0\alpha^-\rangle\,\langle 1\beta^-\rangle\,\langle 2\gamma^-\rangle\,\langle 3\delta^3\rangle\,\zeta^3$	2	6	3,5
$\{1,3\}$	$\langle 0\alpha^-\rangle\,1\,\langle\beta^+2\gamma^-\rangle\,\langle 3\delta^3\rangle\,\zeta^3$	2	7	2,6
$\{3\}$	$0\,\langle\alpha^+1\rangle\,\langle\beta^+2\gamma^-\rangle\,\langle 3\delta^3\rangle\,\zeta^3$	3	8	1,5,7

TABLE 9. From the septisection of T^{13}: the start of the handle decomposition of X_I when $I = \{0,1,2,3,5\}$.

Denoting the main diagonal of T^n by Δ, note that each monotonic point $\vec{x} = (x_1,\dots,x_n) \in T^n \setminus \Delta$ corresponds to a unique point $(x_1,\dots,x_n) \in \mathbb{R}^n$ with $0 \le x_1 \le x_2 \le \cdots \le x_n \le x_1 + k < 2k$. For such $\vec{x}$, extend the point $(x_1,\dots,x_n) \in \mathbb{R}^n$ to a point $\vec{x}_\infty = (x_r)_{r\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$ by defining for each $r \in \mathbb{Z}_k$ and $m \in \mathbb{Z}$:

$$x_{r+mn} = x_r + mk.$$

We will mainly be interested in $0 \le x_1 \le \cdots \le x_{2n}$, where

$$x_{2n} = x_n + k \le x_1 + 2k \le 3k.$$

With this setup for any monotonic $\vec{x} \in T^n \setminus \Delta$, define the following **cutoff indices** $a_r(\vec{x}), b_r(\vec{x}) \in \mathbb{Z}$ for each $r \in \mathbb{Z}$:

$$a_r(\vec{x}) = \min\{a : x_{a+1} \ge r\} \text{ and}$$
$$b_r(\vec{x}) = \min\{b : x_{b+1} > r\}.$$

Note that, in all cases, we have $a_0(\vec{x}) \le 0$, with equality if and only if $x_n \ne k \equiv 0 \in \mathbb{R}/k\mathbb{Z}$. The main point is:

Observation 6.1. *Let $\vec{x} \in T^n \setminus \Delta$ be monotonic. Then $\vec{x} \in [0,1]^2 \cdots [0,k-1]^2[0,k]$ if and only if $b_s(\vec{x}) \ge 2s$ for every $s = 0,\dots,k-1$.*

Note that $b_0(\vec{x}) \ge 0$ in all cases. In order to apply the principle of Observation 6.1 more broadly, denote for each $r \in \mathbb{Z}$:

$$\vec{x}_r = (x_{1+a_r(\vec{x})}, x_{2+a_r(\vec{x})}, \dots, x_{a_r(\vec{x})})$$

The point regarding monotonic points off the main diagonal is:

Observation 6.2. *If $\vec{x} \in T^n \setminus \Delta$ is monotonic and $r \in \mathbb{Z}$, then*

$$r \le x_{1+a_r(\vec{x})} \le \cdots \le x_{a_r(\vec{x})} < r + k,$$

and the following conditions are equivalent:

- $\vec{x}_r \in [r,r+1]^2 \cdots [r,r+k-1]^2[r,r+k]$;
- $b_{r+s}(\vec{x}_r) \ge 2s$ *for every* $s = r+1,\dots,r+k$;

- $b_{r+s}(\vec{x}) \geq a_r(\vec{x}) + 2s$ *for every* $s = r+1, \ldots, r+k$

The point more generally is:

Observation 6.3. *If* $\vec{x} \in X_r \subset T^n \setminus \Delta$*, then there is a permutation* $\sigma \in S_n$ *such that* $\vec{x}_\sigma \in [r, r+1]^2 \cdots [r, r+k-1]^2[r, r+k]$ *is monotonic.*

Note also that each class of cutoff indices provides two-sided bounds for the other class:

Observation 6.4. *If* $\vec{x} \in T^n$ *is nonzero and monotonic and* $r \in \mathbb{Z}$*, then*

$$\cdots \leq a_r(\vec{x}) \leq b_r(\vec{x}) \leq a_{r+1}(\vec{x}) \leq b_{r+1}(\vec{x}) \leq \cdots$$

with $a_r(\vec{x}) = b_r(\vec{x})$ *if and only if* $x_{a_r(\vec{x})+1} \notin \mathbb{Z}_k$*, and* $b_r(\vec{x}) = a_{r+1}(\vec{x})$ *if and only if* $x_{b_r(\vec{x})+1} \geq r+1$*.*

Note that $x_{b_r(\vec{x})+1}$ is the first coordinate in $\vec{x}$ that exceeds r. Here is another convenient property:

Observation 6.5. *Any nonzero monotonic* $\vec{x} \in T^n$*,* $r \in \mathbb{Z}_{\geq} 0$ *satisfy*

$$\begin{aligned} a_{r+k}(\vec{x}) &= n + a_r(\vec{x}), \\ b_{r+k}(\vec{x}) &= n + b_r(\vec{x}). \end{aligned} \tag{9}$$

Noting that $X_r \cap \Delta = \{(x, \ldots, x) : \ x \in [r, r+1]\}$, we can express each X_r in terms of cutoff indices as follows.

Proposition 6.6. *Let* $\vec{x} \in T^n \setminus \Delta$ *be monotonic, and let* $r \in \mathbb{Z}_k$*. Then* $\vec{x} \in X_r$ *if and only if* $\vec{x}_r \in [r, r+1]^2 \cdots [r, r+k-1]^2[r, r+k]$*. In particular,*

$$X_r \setminus \Delta = \langle \textit{monotonic } \vec{x} : \ b_{r+s}(\vec{x}) \geq a_r(\vec{x}) + 2s \textit{ for } s = 0, \ldots, k-1 \rangle . \tag{10}$$

Proof. Write $\vec{x}_r = (x_1, \ldots, x_n$. Note that $r \leq x_1 \leq \cdots \leq x_n < r+k$. To show that $\vec{x}_r \in [r, r+1]^2 \cdots [r, r+k-1]^2[r, r+k]$ if and only if $\vec{x}_r \in X_r$, we will prove both containments. One is trivial. For the other, suppose that $\vec{x}_r \notin [r, r+1]^2 \cdots [r, r+k-1]^2[r, r+k]$. Then Observation 6.2 implies that $b_{r+s}(\vec{x}_r) < 2s$ for some $s = 0, \ldots, k-1$, so

$$r + s < x_{2s}, \ldots, x_n < r + k.$$

Thus, at least $n+1-2s$ of the coordinates of $\vec{x}$ lie in the open interval $(r+s, r+k)$. Yet, $2s$ of the n factors of $[r, r+1]^2 \cdots [r, r+k-1]^2[r, r+k]$ are disjoint from that open interval. Contradiction. Observation 6.3 now implies that $\vec{x} \in X_r$ if and only if $\vec{x}$ is an element of the RHS of (10). $\square$

6.2. The X_r have disjoint interiors and cover T^n.

Proposition 6.7. *With the setup from Theorem 7.11, X_r and X_s have disjoint interiors whenever $0 \leq r < s \leq k-1$.*

This will follow from Lemma 6.13, but the following proof is much easier than that of the lemma; we include it for expository reasons.

Proof. By the symmetry of the construction, we may assume that $r = 0$. Assume for contradiction that the interiors of X_r and X_s intersect. Then there is a monotonic point $\vec{x} = (x_1, \dots, x_n) \in X_0 \cap X_j$ such that for every $i = 1, \dots, n$ we have $x_i \notin \mathbb{Z}_k$. (This is not to say that every interior point has this property.)

This implies that $a_i(\vec{x}) = b_i(\vec{x})$ for each $i = 1, \dots, n$, by Observation 6.4. In particular, since $\vec{x} \in X_0$, we have $a_0 = b_0 = 0$, and $a_s = b_s \geq 2s$ by Proposition 6.6. But then, since $\vec{x} \in X_s$ and $a_s \geq 2s$, Observation 6.5 and Proposition 6.6 give the following contradiction:

$$\begin{aligned}
n &= n + b_0 = b_k = b_{s+(k-s)} \\
n &\geq a_s + 2(k-s) \\
n &\geq 2k.
\end{aligned}$$

□

Lemma 6.8. *We have $X_0 \cup \cdots \cup X_{k-1} = T^n$.*

Proof. Let $\vec{x} \in T^n$. We will prove that $\vec{x} \in X_s$ for some s. If $\vec{x} = (x, \dots, x) \in \Delta$, then $\vec{x} \in X_{\lfloor x \rfloor}$. Assume instead that $\vec{x} \in T^n \backslash \Delta$. Also assume without loss of generality that $\vec{x}$ is monotonic with $a_0(\vec{x}) = 0$. Throughout this proof, denote each $a_s(\vec{x})$ by a_s and each $b_s(\vec{x})$ by b_s.

Let $s_0 = 0$, so that $a_{s_0} = a_0 = 0$. If $b_s \geq 2s = 2s - a_{s_0}$ for all $s = 1, \dots, k-1$, then $\vec{x} \in X_0 = X_{s_0}$. Otherwise, choose the smallest s_1 such that $b_{s_1} < 2s_1$. Thus, $b_s \geq 2s$ whenever $s < s_1$, so by Observation 6.4:

$$2s_1 - 2 \leq b_{s_1 - 1} \leq a_{s_1} \leq b_{s_1} \leq 2s_1 - 1.$$

Continue in this way: for each s_t, choose the minimum $s_{t+1} = s_t + 1, \dots, k-1$ such that $b_{s_{t+1}} < a_{s_t} + 2(s_{t+1} - s_t)$, if such s_{t+1} exists. Eventually this process terminates with some s_u, so that:

- $b_s \geq a_{s_t} + 2(s - s_t)$ whenever $s_t \leq s \leq s_{t+1}$ for $t = 0, \dots, u-1$,
- $b_s \geq a_{s_t} + 2(s - s_u)$ whenever $s_u \leq s \leq k-1$, and
- $b_{s_{t+1}} < a_{s_t} + 2(s_{t+1} - s_t)$ for each $t = 0, \dots, u-1$.

Hence, for each $t = 0, \dots, u-1$, Observation 6.4 gives:

$$a_{s_t} + 2(s_{t+1} - 1 - s_t) \leq b_{s_{t+1}-1} \leq a_{s_{t+1}} \leq b_{s_{t+1}} \leq a_{s_t} + 2(s_{t+1} - s_t) - 1.$$

Subtracting a_{s_t} from the first, middle, and last expressions gives:

$$2(s_{t+1} - s_t) - 2 \leq a_{s_{t+1}} - a_{s_t} \leq 2(s_{t+1} - s_t) - 1.$$

Therefore, for any $t = 0, \ldots, u-1$:

$$\begin{aligned} a_{s_u} - a_{s_t} &= \sum_{r=t}^{u-1} (a_{s_{r+1}} - a_{s_r}) \\ &\leq \sum_{r=t}^{u-1} \left(2(s_{r+1} - s_r) - 1\right) \\ &= 2(s_u - s_t) - (u-t) \\ a_{s_u} - a_{s_t} &\leq 2(s_u - s_t) - 1. \end{aligned}$$

Rearranging gives

$$a_{s_u} - 2s_u \leq a_{s_t} - 2s_t - 1. \tag{11}$$

We claim that $\vec{x} \in X_{s_u}$. This is true if (and only if) $b_s \geq a_{s_u} + 2(s - s_u)$ for each $s = s_u, \ldots, s_u + k - 1$. Fix some $s = k, \ldots, s_u + k - 1$. Then $s_t \leq s - k \leq s_{t+1} - 1$ for some $t = 0, \ldots, u-1$. By construction, we have $b_{s-k} \geq a_{s_t} + 2(s - k - s_t)$. Together with (9) and (11), this gives:

$$\begin{aligned} b_s &= b_{s-k} + n \\ &\geq a_{s_t} + 2(s - k - s_t) + 2k - 1 \\ &= (a_{s_t} - 2s_t - 1) + 2s \\ &\geq (a_{s_u} - 2s_u) + 2s \\ &= a_{s_u} + 2(s - s_u). \end{aligned}$$

□

6.3. Combinatorial corollaries. We have proven that the pieces X_r of the multisection of T^n have disjoint interiors and cover T^n. Also, each $X_r = X_0 + (r, \ldots, r)$, so all X_r have the same number of unit cubes. Since there are k^n unit cubes in $T^n = (\mathbb{R}/k\mathbb{Z})^n$, each X_r contains k^{n-1} unit cubes. By counting these unit cubes a different way, we obtain the following.[5]

Corollary 6.9. *For any $n = 2k-1$, we have:*

$$\begin{aligned} k^{n-1} = &\sum_{i_1=2}^{n} \binom{n}{i_1} \sum_{i_2=4-i_1}^{n-i_1} \binom{n-i_1}{i_2} \sum_{i_3=6-i_1-i_2}^{n-i_1-i_2} \binom{n-i_1-i_2}{i_3} \cdots \\ &\cdots \sum_{i_{k-1}=2k-2-\sum_{j=1}^{k-2} i_j}^{n-\sum_{j=1}^{k-2} i_j} \binom{n-\sum_{j=1}^{k-2} i_j}{i_{k-1}}. \end{aligned} \tag{12}$$

Proof. Each X_i consists of k^{n-1} subcubes, each of the form $\prod_{r=1}^{n}[w_r, w_r+1]$ for some $w_1, \ldots, w_n \in \mathbb{Z}_k$. For each $s = 1, \ldots, k-1$, there are at least $2s$ indices $r = 1, \ldots, n$ with $w_r \in \{i, \ldots, i+s-1\}$, and conversely any subcube of that form with this property will be in X_i. □

[5]Note that by definition, if $a, b \in \mathbb{Z}$ with $b < 0$, then $\binom{a}{b} = 0$.

Note that (12) is also the number of spanning trees of the complete bipartite graph $K_{j,j}$ where $j = k$ [6]. Counting combinatorial *cube types* in three different ways yields:

Corollary 6.10. *For any $n = 2k-1$, we have:*

$$\begin{aligned}
&k \sum_{i_1=2}^{n} \sum_{i_2=\max\{0,4-i_1\}}^{n-i_1} \sum_{i_3=\max\{0,6-i_1-i_2\}}^{n-i_1-i_2} \cdots \sum_{i_{k-1}=\max\left\{0,2k-2-\sum_{j=1}^{k-2} i_j\right\}}^{n-\sum_{j=1}^{k-2} i_j} 1 \\
&\qquad = \sum_{i_1=0}^{n} \sum_{i_2=0}^{n-i_1} \sum_{i_3=0}^{n-i_1-i_2} \cdots \sum_{i_{k-1}=0}^{n-\sum_{j=1}^{k-2} i_j} 1 \\
&\qquad = \binom{3k-2}{k-1}.
\end{aligned} \tag{13}$$

Proof. The first expression is k times the number of *cube types* in X_i, counted by the same principle as in Corollary 6.9. The second counts the number of cube types in T^n, each of the form $\prod_{r=0}^{k-1}[r, r+1]^{v_r}$, characterized by a tuple $(v_0, \dots, v_{k-1})$ with $\sum_{r=0}^{k-1} v_r = n$. The third counts the number of cube types in T^n by denoting $a_0 = 0, a_k = 3k-1$ and associating to each $A = \{a_1, \dots, a_{k-1}\} \subset \{1, \dots, 3k-2\}$ with $a_1 < \cdots < a_{k-1}$ the cube type

$$\prod_{i=1}^{k} \prod_{j=a_{i-1}+1}^{a_i-1} [i-1, i]. \qquad \square$$

See [6] for other interpretations of (13).

6.4. Verification of the formula $X_I =$ (2).

Next, we will use the cutoff indices $a_r(\vec{x}), b_r(\vec{x})$ to verify (2). To prepare this, we define subsets $C_{I,s} \subset T^n$ as follows. Let $I \subset \mathbb{Z}_k$ following Convention 3.9, $i_* = i_s \in I$. Then define:

$$\begin{aligned}
C_{I,s} = &\left(\prod_{t=0}^{s-1} \{i_t\} \times [i_t, i_t+1]^2 \times \cdots \times [i_t, i_{t+1}-1]^2 \times [i_t, i_{t+1}] \right) \\
&\times [i_*, i_*+1]^2 \times \cdots \times [i_*, i_{s+1}-1]^2 \times [i_*, i_{s+1}] \\
&\times \left(\prod_{t=s+1}^{\ell-1} \{i_t\} \times [i_t, i_t+1]^2 \times \cdots \times [i_t, i_{t+1}-1]^2 \times [i_t, i_{t+1}] \right)
\end{aligned} \tag{14}$$

Note the "missing" $\{i_*\}$ at the start of the second line; this corresponds to the $\widehat{i_*}$ in (2). Observe that the expression on the RHS of (2) equals

$$\bigcup_{s \in \mathbb{Z}_\ell} \langle C_{I,s} \rangle .$$

Proposition 6.11. *Let $I \subset \mathbb{Z}_k$ follow Convention 3.9, $s \in \mathbb{Z}_\ell$, and $C_{I,s}$ as in (14). Suppose $\vec{x} \in T^n \setminus \Delta$ is monotonic. Then $\vec{x} \in C_{I,s}$ if and only if all of the following conditions hold:*

- $b_t(\vec{x}) \geq 2t+1$ *for* $0 \leq t < i_*$,
- $b_t(\vec{x}) \geq 2t$ *for* $i_* \leq t \leq k-1$,
- $a_t(\vec{x}) \leq 2t$ *for* $t = i_0, \dots, i_*$, *and*
- $a_t(\vec{x}) \leq 2t-1$ *for* $t = i_{s+1}, \dots, i_{\ell-1}$.

Proof. This follows immediately from the definitions, upon consideration of each entry in $\vec{x}$. □

Also note the following generalization of Observation 6.3:

Observation 6.12. *Let $I \subset \mathbb{Z}_k$ follow Convention 3.9, $s \in \mathbb{Z}_\ell$, and $C_{I,s}$ as in (14). Suppose $\vec{x} \in \langle C_{I,s} \rangle$. Then there is a permutation $\sigma \in S_n$ such that $\vec{x}_\sigma \in C_{I,s}$ is monotonic.*

Lemma 6.13. *Given nonempty $I \subset \mathbb{Z}_k$ (following Convention 3.9),*

$$X_I = \bigcup_{s \in \mathbb{Z}_\ell} \langle C_{I,s} \rangle . \tag{15}$$

In particular,

$$\bigcap_{i_* \in \mathbb{Z}_k} X_i = \bigcup_{i_* \in I} \left\langle (i_1, \dots, \widehat{i_*}, \dots, i_\ell) \prod_{i \in \mathbb{Z}_k} [i, i+1] \right\rangle . \tag{3}$$

Note that the formula (15) is equivalent to (2).

Proof. We argue by induction on ℓ. When $\ell = 1$, $X_I = X_0 = \langle C_{I,0} \rangle = (2)$.

Assume now that $\ell > 1$. First, we will show that

$$X_I \subset \bigcup_{s \in \mathbb{Z}_\ell} \langle C_{I,s} \rangle . \tag{16}$$

Let $\vec{x} \in X_I$, and define $I' = I \setminus \{i_{\ell-1}\}$. Note that I' is simple and $X_I = X_{I'} \cap X_{i_{\ell-1}}$. Since $\vec{x} \in X_{I'}$, the induction hypothesis implies that $\vec{x} \in \langle C_{I',s_0} \rangle$ for some $s_0 \in \mathbb{Z}_{\ell-1}$. By Observation 6.12, there exists $\sigma \in S_n$ such that $\vec{x}_\sigma$ is monotonic and $\vec{x}_\sigma \in C_{I',s_0}$. Proposition 6.11 implies that:

- $b_t(\vec{x}_\sigma) \geq 2t+1$ for $0 \leq t \leq i_{s_0} - 1$,
- $b_t(\vec{x}_\sigma) \geq 2t$ for $i_{s_0} \leq t \leq k-1$,
- $a_t(\vec{x}_\sigma) \leq 2t$ for $t = i_0, \dots, i_{s_0}$, and
- $a_t(\vec{x}_\sigma) \leq 2t-1$ for $t = i_{s_0+1}, \dots, i_{\ell-2}$.

If also $a_{i_{\ell-1}}(\vec{x}_\sigma) \leq 2i_{\ell-1} - 1$, then Proposition 6.11 implies that $\vec{x}_\sigma \in C_{I,s_0}$. In that case, we are done proving the forward containment. Assume instead that $a_{i_{\ell-1}}(\vec{x}_\sigma) \geq 2i_{\ell-1}$. We now split into two cases:

Case 1: Assume that $a_{i_{\ell-1}}(\vec{x}_\sigma) = 2i_{\ell-1}$. We claim that $\vec{x}_\sigma \in C_{I,\ell-1}$. By Proposition 6.11, since $\vec{x}_\sigma$ is monotonic, it will suffice to show:

(a) $b_t(\vec{x}_\sigma) \geq 2t + 1$ for $0 \leq t \leq i_{\ell-1} - 1$,
(b) $b_t(\vec{x}_\sigma) \geq 2t$ for $i_{\ell-1} \leq t \leq k - 1$, and
(c) $a_t(\vec{x}_\sigma) \leq 2t$ for $t = i_0, \ldots, i_{\ell-1}$.

Observation 6.5, Proposition 6.6, and the facts that $\vec{x}_\sigma \in X_{i_{\ell-1}}$ and $a_{i_{\ell-1}}(\vec{x}_\sigma) = 2i_{\ell-1}$ imply for each $t = 0, \ldots, i_{\ell-1} - 1$ that:

$$\begin{aligned} b_t(\vec{x}_\sigma) &= b_{t+k}(\vec{x}_\sigma) - n \\ &\geq 2(t+k) + a_{i_{\ell-1}}(\vec{x}_\sigma) - 2i_{\ell-1} - n \\ &\geq 2t + 1. \end{aligned}$$

This verifies (a). Taking $t = i_{\ell-1}, \ldots, k-1$, similar reasoning confirms (b):

$$b_t(\vec{x}_\sigma) \geq a_{i_{\ell-1}}(\vec{x}_\sigma) + 2(t - i_{\ell-1}) \geq 2t.$$

Finally, we have $a_t(\vec{x}_\sigma) \leq 2t$ for each $t = i_0, \ldots, i_{\ell-1}$. For $t = i_0, \ldots, i_{\ell-2}$, this is because $\vec{x}_\sigma \in X_{I'}$; for $t = i_{\ell-1}$, it is our assumption in Case 1. Thus, in Case 1, (a), (b), and (c) hold, and so $\vec{x}_\sigma \in C_{I,\ell-1}$.

Case 2: Assume instead that $a_{i_{\ell-1}}(\vec{x}_\sigma) \geq 2i_{\ell-1}+1$. Denoting $\vec{x}_\sigma = (x_1, \ldots, x_n)$, we claim in this case that $x_1 = x_2 = 0 \equiv k$ and that $\vec{y} = (x_2, \ldots, x_n, x_1) \in C_{I,\ell-1}$. By similar reasoning to Case 1, we have:

$$\begin{aligned} b_0(\vec{x}_\sigma) &= b_k(\vec{x}_\sigma) - n \\ &\geq a_{i_{\ell-1}}(\vec{x}_\sigma) + 2(k - i_{\ell-1}) - n \\ &\geq 2. \end{aligned}$$

Thus, $x_1 = x_2 = 0 \equiv k$. Define $\vec{y}$ as above. Note that, since $\vec{x}_\sigma$ is monotonic, $\vec{y}$ is also monotonic. It remains to show that $\vec{y} \in C_{I,\ell-1}$. The arguments are almost identical to those in Case 1, except that we need to check that $a_{i_{\ell-1}}(\vec{y}) \leq 2i_{\ell-1}$. Using Observations 6.4 and 6.5 and the fact that $a_{i_{\ell-1}}(\vec{y}) = a_{i_{\ell-1}}(\vec{x}_\sigma) - 1$, we compute:

$$\begin{aligned} a_{i_{\ell-1}}(\vec{y}) &= a_{i_{\ell-1}}(\vec{x}_\sigma) - 1 \\ &\leq b_{k-1}(\vec{x}_\sigma) - 2(k - 1 - i_{\ell-1}) - 1 \\ &\leq a_k(\vec{x}_\sigma) - 2k + 1 + 2i_{\ell-1} \\ &= a_0(\vec{x}_\sigma) + (n + 1 - 2k) + 2i_{\ell-1} \\ &= a_0(\vec{x}_\sigma) + 2i_{\ell-1} \\ &\leq 2i_{\ell-1}. \end{aligned}$$

This completes the proof of the forward containment (16). For the reverse containment, keep the same subset $I \subset \mathbb{Z}_k$ from the start of the induction step of the proof, fix some $s \in \mathbb{Z}_\ell$, and let $\vec{x} \in C_{I,s}$ be monotonic, $t \in I = \{i_0, \dots, i_{\ell-1}\}$. We will show for each $r = 0, \dots, k-1$ that $b_{t+r}(\vec{x}) \geq a_t(\vec{x}) + 2r$. Proposition 6.6 will then imply that $\vec{x} \in X_t$. Since t is arbitrary, this will imply that $\vec{x} \in X_I$, completing the proof. We will split into cases, but first note, since $\vec{x}$ is monotonic, that Proposition 6.11 implies:

- $b_t(\vec{x}) \geq 2t + 1$ for $0 \leq t \leq i_s - 1$,
- $b_t(\vec{x}) \geq 2t$ for $i_s \leq t \leq k - 1$,
- $a_t(\vec{x}) \leq 2t$ for $t = i_0, \dots, i_s$, and
- $a_t(\vec{x}) \leq 2t - 1$ for $t = i_{s+1}, \dots, i_{\ell-2}$.

Case 1: If $t + r \leq k - 1$, then $b_{t+r}(\vec{x}) \geq 2(t + r) \geq a_t(\vec{x}) + 2r$.

Case 2: If instead $t + r \geq k$ and $t + r \leq k + i_s - 1$, then

$$\begin{aligned} b_{t+r}(\vec{x}) &= n + b_{t+r-k}(\vec{x}) \geq n + 2(t + r - k) + 1 = 2t + 2r + (n + 1 - 2k) \\ b_{t+r}(\vec{x}) &\geq a_t(\vec{x}) + 2r \end{aligned}$$

Case 3: Similarly, if $t + r \geq k$ and $t \geq i_{s+1}$, then

$$\begin{aligned} b_{t+r}(\vec{x}) &= n + b_{t+r-k}(\vec{x}) \geq n + 2(t + r - k) = (2t - 1) + 2r + (n + 1 - 2k) \\ b_{t+r}(\vec{x}) &\geq a_t(\vec{x}) + 2r \end{aligned}$$

Are there other cases? If there were, they would satisfy $t + r \geq k + i_s$ and $t \leq i_s$, giving

$$\begin{aligned} k + i_s \leq t + r &\leq i_s + r \\ k &\leq r. \end{aligned}$$

Yet $r \leq k - 1$ by assumption. Therefore, in every case, $b_{t+r}(\vec{x}) \geq a_t(\vec{x}) + 2r$, and so $\vec{x} \in X_{i_t}$ for arbitrary $t \in I$. Thus, $\vec{x} \in X_I$. This completes the proof of the reverse containment, and thus of the equality in (2)=(15). $\square$

7. General construction

7.1. Notation. Section 7 uses Notations 3.3, 3.6, 3.8, and Convention 3.9.

Notation 7.1. Denote the symmetric difference of sets R and S by

$$R \ominus S = (R \setminus S) \cup (S \setminus R).$$

7.2. Handle decompositions: the general case. Let $I = \bigsqcup_{r \in \mathbb{Z}_m} I_r$ be arbitrary, following Convention 3.9. Recall that $T = \{\min I_r : r \in \mathbb{Z}_m\}$. Decompose X_I into handles in several steps as follows. First, decompose X_I

$$X_I = \bigcup_{J \subset T,\ i_* \in I} X_{I,J,i_*}$$

as follows. Fix arbitrary $r \in \mathbb{Z}_m$. Denote $a = \min I_r$, $b = \max I_r$, $c = \min I_{r+1}$. Define

$$\widehat{C}_r = \prod_{j=b+1}^{c-1} [b,j]^2, \tag{17}$$

$$\begin{aligned} C_r = &\begin{Bmatrix} [a-1,a] & i_* = a \in J \\ [a-1,a]\times\{a\} & i_* \neq a \in J \\ \{a\} & i_* \neq a \notin J \\ \text{(no factor)} & i_* = a \notin J \end{Bmatrix} \\ &\times \prod_{i=a+1}^{b} \begin{Bmatrix} [i-1,i]\times\{i\} & i \neq i_* \\ [i-1,i] & i = i_* \end{Bmatrix} \times \begin{Bmatrix} \widehat{C}_r \times [b,c-1] & c \notin J \\ \widehat{C}_r & c \in J \end{Bmatrix}. \end{aligned} \tag{18}$$

Now the piece of X_I corresponding to the pair (J, i_*) is given by

$$X_{I,J,i_*} = \left\langle \prod_{r\in\mathbb{Z}_m} C_r \right\rangle.$$

	$i_* \notin I_r$, $a \notin J$	$i_* \notin I_r$, $a \in J$	$i_* \in I_r$, $i_* \leq b-2$	$i_* \in I_r$, $i_* \geq b-1$
U_r	$\varnothing$	$I_r \setminus \{a,b\}$	$I_r \setminus \{a, i_*, i_*+1, b\}$	$I_r \setminus \{a, i_*, b\}$
V_r	$I_r \setminus \{a\}$	$\{b\}$	$\{i_*+1, b\}$	$\varnothing$
$I_r \setminus (U_r \cup V_r)$	$\{a\}$	$\{a\} \setminus \{b\}$	$\{a, i_*\}$	$\{a, i_*, b\}$

TABLE 10. The index subsets $U_r, V_r \subset I_r$ when $I_r = \{a, \dots, b\}$.

Second, fix arbitrary $J \subset T$, $i_* \in I$, and $r \in \mathbb{Z}_m$. For each $r \in \mathbb{Z}_m$, define subsets $U_r, V_r \subset I_r$ following Table 10 (depending on J and i_*). Tables 11 and 12 in Appendix 1 present U_r and V_r more explicitly. Then define

$$U = \bigcup_{r\in\mathbb{Z}_m} U_r \text{ and } V = \bigcup_{r\in\mathbb{Z}_m} V_r.$$

Note that $\min I_r \notin (U_r \cup V_r)$ unless $I_r \neq I_*$ and $\min I_r = \max I_r \in J$. See Table 7 for an example of this exceptional case: X_I, $I = \{0, 2\}$, from T^7.

Third, decompose each X_{I,J,i_*} into pieces $X_{I,J,i_*,V^-,U^\circ,U^-}$ as follows. Denote

$$2^V = \{V^- \subset V\},$$
$$2^U = \{U^\circ \subset U\},$$

and given $U^\circ \subset U$, denote

$$2^{U\setminus U^\circ} = \{U^- \subset U \setminus U^\circ\}.$$

Given $V^- \subset V$, denote $V^+ = V \setminus V^-$, and given $U^\circ \subset U$ and $U^- \subset U \setminus U^\circ$, denote $U^+ = U \setminus (U^\circ \cup U^-)$. Then $V = V^- \sqcup V^+$ and $U = U^- \sqcup U^\circ \sqcup U^+$. For each $r \in \mathbb{Z}_m$, denote $\widehat{C}_r$ as in (17). For each $i \in I_r$, define

$$\rho_i = \begin{cases} [i-1, i-\frac{2}{3}] & i \in U^- \\ [i-\frac{2}{3}, i-\frac{1}{3}] & i \in U^\circ \\ [i-\frac{1}{3}, i] & i \in U^+ \\ [i-1, i-\frac{1}{2}] & i \in V^- \\ [i-\frac{1}{2}, i+1] & i \in V^+ \\ [\max I_{r-1}, i-1] & i \in T \setminus (J \cup V). \\ [i-1, i] & \text{else} \end{cases}$$

Fix arbitrary $r \in \mathbb{Z}_m$. Denote $a = \min I_r$, $b = \max I_r$, $c = \min I_{r+1}$, and $\widehat{C}_r$ as in (17). Define:

$$X_{I,J,i_*,V^-,U^\circ,U^-,r} = \begin{Bmatrix} \rho_a & i_* = a \in J \\ \rho_a \times \{a\} & i_* \neq a \in J \\ \{a\} & i_* \neq a \notin J \\ \text{(no factor)} & i_* = a \notin J \end{Bmatrix} \times \prod_{i=a+1}^{b} \begin{Bmatrix} \rho_i \times \{i\} & i \neq i_* \\ \rho_i & i = i_* \end{Bmatrix} \times \begin{Bmatrix} \widehat{C}_r \times \rho_c & c \notin J \\ \widehat{C}_r & c \in J \end{Bmatrix}.$$

Note that $\rho_i \subset [i-1, i]$ for each $i = a+1, \ldots, b$, $\rho_a \subset [a-1, a]$ if $a \in J$, and $\rho_c = [b, c-1]$ if $c \notin J$. Define

$$X_{I,J,i_*,V^-,U^\circ,U^-} = \left\langle \prod_{r \in \mathbb{Z}_m} X_{I,J,i_*,V^-,U^\circ,U^-,r} \right\rangle$$

Note that $X_{I,J,i_*} = \bigcup_{V^-,U^\circ,U^-} X_{I,J,i_*,V^-,U^\circ,U^-}$. Fourth, order the pieces $X_{I,J,i_*,V^-,U^\circ,U^-}$ according to the lexicographical order on

$$\big\{(J, i_*, V^-, U^\circ, U^-)\big\}_{J \subset T,\ i_* \in I,\ V^- \subset V,\ U^\circ \subset U,\ U^- \subset U \setminus U^\circ} \tag{19}$$

determined by the following orders $\prec$ on $\{J \subset T\}$, I, 2^V, 2^U, and $2^{U \setminus U^\circ}$. Order $\{J \subset T\}$ and 2^U partially by inclusion, so that $J' \prec J$ if $J' \subsetneqq J$ and $U'^\circ \prec U^\circ$ if $U'^\circ \subsetneqq U^\circ$; extend these partial orders arbitrarily to total orders. Define an arbitrary total order $\prec$ on $2^{U \setminus U^\circ}$. Partially order I such that $i < i'$, with $i \in I_r$ and $i' \in I_s$, if $i - \min I_r < i_s - \min I_s$; extend arbitrarily to a total order on I.

It remains to order 2^V. This will be slightly more complicated. To do this, we first define a total order $\prec_r$ on 2^{V_r} for each $r \in \mathbb{Z}_m$. First consider the $I_r \ni i_*$. If $i_* \geq \max I_* - 1$, we have $V_r = \varnothing$, so there is nothing to do. Otherwise, we have $i_* \leq \max I_* - 2$ and $V_r = \{i_* + 1, \max I_*\}$; in this case, order 2^{V_r} as follows:

$$\{i_*+1\} \prec_r \varnothing \prec_r \{i_*+1, \max I_*\} \prec_r \{\max I_*\}.$$

Now consider $I_r \not\ni i_*$. Define $\prec_r$ on 2^{V_r} recursively by $V_r^- \prec_r V_r'^-$ if:

- $\max V_r^- < \max V_r'^-$, or
- $\max V_r^- = \max V_r'^-$ and $V_r'^- \setminus \{\max V_r'^-\} \prec_r V_r^- \setminus \{\max V_r^-\}$.

Explicitly, denoting $V_r = \{a, \dots, b\}$:

$$\begin{aligned} &\varnothing \prec_r \{a\} \prec_r \{a, a+1\} \prec_r \{a+1\} \prec_r \{a+1, a+2\} \\ &\quad \prec_r \{a, a+1, a+2\} \prec_r \{a, a+2\} \prec_r \{a+2\} \prec_r \cdots \\ &\quad \prec_r \{a+1, b\} \prec_r \{a, a+1, b\} \prec_r \{a, b\} \prec_r \{b\}. \end{aligned} \tag{20}$$

For more examples, see Tables 9 and 20, and the parts of Tables 18 and 19 where $s = 4$. Use the orderings $\prec_r$ on 2^{V_r} to define a partial order on 2^V by declaring $V^- \prec V'^-$ if

- $V^- \cap I_r \prec_r V'^- \cap I_r$ for some r, and
- there is no r for which $V'^- \cap I_r \prec_r V^- \cap I_r$.

Extend $\prec$ arbitrarily to a total order on 2^V. This determines a total order on (19), and thus on the pieces $X_{I,J,i_*,V^-,U^\circ,U^-}$. Relabel these pieces as Y_z, $z = 1, \dots, \#(19)$, according to this order.

Fourth and finally, for each z, we will define $Y_z^* \subset Y_z$. In §7.3, we will see that attaching Y_z^* to $\bigcup_{w<z} Y_w$ amounts to attaching an $(n+1-|I|)$-dimensional $h(z)$-handle for some $h(z) \leq |I|$, and moreover that attaching all of Y_z to $\bigcup_{w<z} Y_w$ amounts to attaching several such handles.

Consider arbitrary $Y_z = X_{I,J,i_*,V^-,U^\circ,U^-}$. Note, for each $r \in \mathbb{Z}_\ell$, that ρ_r contains at most one point in $I \setminus \{i_*\}$.[6] Moreover, if $\rho_r \cap I \setminus \{i_*\} = \varnothing$, then, with $i_r \in I_s$, either (i) $i_r \in U^\circ$, (ii) $i_r \in J \cap V^-$ (hence $|I_s| = 1$), (iii) $i_r = \min I_s$ with either $i_r = i_* \in J$ or $i_{r-1} = i_*$ and $i_r \notin J$, or (iv) $i_r = i_* \leq \max I_s - 2$ with $i_r + 1 \in V^-$. Note also, for each $s \in \mathbb{Z}_m$, that $\widehat{C}_s \cap I = \{\max I_s\}$, so $\widehat{C}_s \cap (I \setminus \{i_*\}) = \varnothing$ only if $i_* = \max I_s$. For each $r = 0, \dots, \ell-1$ with $i_* \neq i_r \in I_s$ and $i_r \neq \max I_s$, define

$$\xi_r(z) = \left\langle \{i_r\} \times \prod_{t \in \mathbb{Z}_\ell:\ \rho_t \ni i_r} \rho_t \right\rangle. \tag{21}$$

[6]To see why, consider the last row of Table 10. It shows for each $s \in \mathbb{Z}_m$ with $I_s \neq I_*$ that $I_s \setminus (U \cup V) \subset \{\min I_s\}$. Therefore, each ρ_r coming from such I_s contains no full unit interval $[i, i+1]$ where $i, i+1 \in I_s$. Similar reasoning applies to those ρ_r coming from I_*, where $I_* \setminus (U \cup V \cup i_*) \subsetneqq \{\max I_*, \min I_*\}$ with equality only when $\min I_* < i_* = \max I_* - 1$.

Similarly, for each $r = 0, \dots, \ell - 1$ with $i_* \neq i_r = \max I_s$, define

$$\xi_r(z) = \left\langle \{i_r\} \times \widehat{C}_t \times \prod_{t \in \mathbb{Z}_\ell:\ \rho_t \ni i_r} \rho_t \right\rangle . \tag{22}$$

For $r = \ell, \dots, \ell + |U^\circ| - 1$, denote $\xi_r(z) = \left[i - \frac{2}{3}, i - \frac{1}{3}\right]$, $i \in U^\circ$, so that each $\xi_r(z)$ is distinct. For $r = \ell + |U^\circ|, \dots, \ell + |U^\circ| + |J \cap V^-| - 1$, denote $\xi_r(z) = \left[i - 1, i - \frac{1}{2}\right]$, $i \in J \cap V^-$, so that each $\xi_r(z)$ is distinct. Define

$$p = \ell + |U^\circ| + |J \cap V^-| - \begin{Bmatrix} 0 & i_* = \max I_s,\ \min I_{s+1} \in J \\ 1 & \text{else} \end{Bmatrix} .$$

If $i_* = \max I_s$ for some $s \in \mathbb{Z}_m$ and $\min I_{s+1} \in J$, denote $\xi_p(z) = \widehat{C}_s$.

Notation 7.2. For each r, let $\min \xi_r(z)$ (resp. $\max \xi_r(z)$) denote the supremum (resp. infimum) of all coordinates in $(0, k)$ among all points in $\xi_r(z)$.

Reorder $\xi_0(z), \dots, \xi_p(z)$ as follows. If $i_* \neq 0$, do this such that $\xi_0(z)$ (with $\{0\}$ as a factor) remains $\xi_0(z)$ and $\max \xi_r(z) \leq \min \xi_{r+1}(z)$ for $r = 1, \dots, p - 1$. If $i_* = 0$, do this such that ρ_1 is a factor of $\xi_0(z)$ and $\max \xi_r(z) \leq \min \xi_{r+1}(z)$ for $r = 1, \dots, p-1$. Now define

$$Y_z^* = \prod_{r=0}^{p} \xi_r(z).$$

Observe that:

$$Y_z = \left\langle \rho_{i_*} \times \prod_{i \in I \setminus i_*} (\rho_i \times \{i\}) \times \prod_{r \in \mathbb{Z}_m} \widehat{C}_r \right\rangle = \left\langle \prod_{r=0}^{p} \xi_r(z) \right\rangle = \langle Y_r^* \rangle .$$

Example 7.3. Consider $X_I \subset T^9$ where $I = \{0, 1, 2, 3\}$, which is detailed in Tables 14 and 17. Note that $T = \{0\}$. In particular, consider the first and twelfth rows of Table 14 (after the headings), where $J = \varnothing$, $s = 0$, $U = \{2\}$, and $V = \{1, 3\}$. The first row of Table 14 corresponds to

$$Y_1 = X_{I,J,s,V^-,U^\circ,U^-} = \left\langle \alpha^- 1 \beta_3^\circ 2 \gamma^+ 3 \delta^3 \right\rangle , \tag{23}$$

where $V^- = \{1\}$, $U^\circ = \{2\}$, and $U^- = \varnothing$. Note, comparing (21), (22),(23), and Notation 3.5, that

$$\xi_0 = \alpha^- = \left[0, \frac{1}{2}\right],\ \xi_1 = \{1\},\ \xi_2 = \beta_3^\circ = \left[\frac{4}{3}, \frac{5}{3}\right],\ \xi_3 = \{2\},$$
$$\xi_4 = \gamma^+ = \left[\frac{5}{2}, 3\right],\ \xi_5 = \{3\},\ \text{and } \xi_6 = \xi_7 = \xi_8 = \delta = [3, 4].$$

Thus:

$$Y_1^* = \alpha^- 1 \beta_3^\circ 2 \left\langle \gamma^+ 3 \delta^3 \right\rangle .$$

The twelfth row of Table 14 corresponds to

$$Y_{12} = X_{I,J,s,V^-,U^\circ,U^-} = \left\langle \alpha^+ 1 \beta_3^+ 2 \gamma^- 3 \delta^3 \right\rangle ,$$

where $V^- = \{3\}$, $U^\circ = \varnothing = U^-$. Thus:

$$Y_{12}^* = \langle \alpha^+ 1 \rangle \langle \beta_3^+ 2\gamma^- \rangle \langle 3\delta^3 \rangle .$$

7.3. Properties of handle decompositions.

Proposition 7.4. *Let $i \in V^- \subset V$ for some $i \in I_s$, $s \in \mathbb{Z}_m$, where $i_* \notin I_s$. Denote $b = \max I_s$, $c = \max(I_s \cap V^-)$. Let $V'^- = V^- \setminus \{i\}$. Then $V'^- \prec V^-$ if and only if $|V^- \cap \{i+1, \dots, b\}|$ is even.*

Proof. We argue by induction on $c - i$. When $c - i = 0$, we have $c = i > \max(I_s \cap V^- \setminus \{i\})$ and $I_r \cap V^- = I_r \cap V^- \setminus \{i\}$ for all $r \neq s$, so $V'^- \prec V^-$.

Now assume that $c - i = t > 0$, and assume that the claim is true whenever $\max(I_s \cap V^-) - i < t$. Let $W^- = V^- \setminus \{c\}$ and $W'^- = V'^- \setminus \{c\}$. Then $|V^- \cap \{i+1, \dots, b\})$ and $|W^- \cap \{i+1, \dots, b\}|$ have opposite parities. Also, by construction, $V^- \prec V'^-$ if and only if $W'^- \prec W^-$. The result now follows by induction. $\square$

Proposition 7.5. *Let $A \subset V^- \subset V$ such that $V^- \prec V^- \ominus \{a\}$ for each $a \in A$. Then $V^- \prec V^- \setminus A$.*

Proof. Suppose first that $A \subset I_s$ for some $s \in \mathbb{Z}_m$. Denote $A = (a_1, \dots, a_q)$ with $\min I_s \leq a_1 \leq \cdots \leq a_q \leq \max I_s = b$. Assume that $i_* \notin I_s$ and $|I_s| \geq 3$ (the other cases are trivial). Note that Proposition 7.4 implies, for each $a \in A$, that $|V^- \cap \{a+1, \dots, b\}|$ is odd if and only if $a \in V^-$. For each $r = 1, \dots, q$, denote the symmetric difference $V_r^- = V^- \ominus \{a_1, \dots, a_r\}$. Then, $|V_a^- \cap \{a+1, \dots, b\}| = |V^- \cap \{a+1, \dots, b\}|$ for each $a = 0, \dots, q-1$. Since this quantity is odd if and only if $a \in V^-$, Proposition 7.4 implies:

$$V^- \prec V_1^- \prec \cdots \prec V_q^- = V^- \setminus A.$$

For the general case, apply this argument repeatedly for each $s \in \mathbb{Z}_m$. $\square$

Observation 7.6. *If Y_z comes from $J, i_*, V^-, U^\circ, U^-$ and Y_w comes from $J, i_*, V^-, U^\circ, U'^-$, then $Y_z \cap Y_w = \varnothing$ unless $U^- = U'^-$.*

Lemma 7.7. *For each $r = 0, \dots, p$, $\xi_r(z)$ has one of the forms described in Lemma 4.2, and thus is homeomorphic to $D^{d(r)}$ for some $d(r) \geq 0$.*

Moreover, $\sum_{r=0}^p d(r) = n + 1 - |I|$, so $Y_z^* \approx D^{n+1-|I|}$.

Proof. The first claim follows by construction, since each ρ_i contains at most one point in $I \setminus \{i_*\}$. (For a more explicit accounting, see Tables 21–23.)

Moreover, for each $r = 0, \dots, p$, $d(r)$ equals the number of intervals in the expression for $\xi_r(z)$, which equals the number of coordinates in that expression minus the number of singleton factors. Since there are n factors and $|I| - 1$ singletons among $\xi_0(z), \dots, \xi_p(z)$ all together, it follows that $\sum_{r=0}^p d(r) = n + 1 - |I|$. Thus, $Y_z^* \approx D^{n+1-|I|}$. $\square$

In the following way, classify each $\xi_r(z)$ into one of two classes, (A) or (B). Say that $\xi_r(z)$ is in class (B) if:

- Some $\widehat{C}_s$ appears in the expression for $\xi_r(z)$;
- $\xi_r(z) = \left[i - \frac{2}{3}, i - \frac{1}{3}\right]$ for some $i \in I$;
- $\xi_r(z) = \left[i_*, i_* + \frac{1}{2}\right]$; or
- Some $\{i\}$ appears in the expression for $\xi_r(z)$, with $i \in I_s$ and:
  - $i \in V^+$ and $i+1 \in U^\circ \cup U^+ \cup V^+$, or $i \in U^- \cup U^\circ \cup V^-$ and $i+1 \in V^-$; and
  - $|V^- \cap \{i+1, \ldots, \max I_s\}|$ is even.

All other types of $\xi_r(z)$ are of class (A). Tables 21, 22, and 23 in Appendix 1 list the possibilities explicitly.

Lemma 7.8. *Let $Y_z^* = \prod_{r=0}^p \xi_r(z)$ come from some $J, i_*, V^-, U^\circ, U^-$. If, for some $a = 0, \ldots, p$, $\xi_a(z)$ is of class (A) and*

$$\vec{x} = (x_1, \ldots, x_n) \in \prod_{r=0}^{a-1} \xi_r(z) \times \partial \xi_a(z) \times \prod_{r=a+1}^{p} \xi_r(z),$$

then $\vec{x} \in Y_w$ for some $w < z$.

Proof. Suppose first that some $\{i\}$ appears in the expression for $\xi_a(z)$, with $i \in I_s$; $i \in V^+$ and $i+1 \in U^\circ \cup U^+ \cup V^+$, or $i \in U^- \cup U^\circ \cup V^-$ and $i+1 \in V^-$; and $|V^- \cap \{i+1, \ldots, \max I_s\}|$ is odd. Then $\vec{x}$ is in the Y_w coming from $J, i_*, V'^-, U^\circ, U^-$ where V'^- is either $V^- \cup \{i\}$ or $V^- \setminus \{i+1\}$. In either case, Proposition 7.4 implies that $V'^- \prec V$ and thus $w < z$.

Next, suppose that $\xi_a(z)$ has no singleton factors. There are two possibilities. If $\xi_a(z) = [i_* - 1, i_*]$ with $i_* \in J$, then $\vec{x}$ is in some Y_w coming from $J \setminus \{i_*\} \prec J$. Otherwise, $\xi_a(z) = \left[i - 1, i - \frac{1}{2}\right]$ for some $i \in J \cap V^-$; in this case, $i + 1 \notin I$, and so $\vec{x}$ is in some Y_w coming either from $J \setminus \{i\} \prec J$ or the from same J and i_* and $V'^- = V^- \setminus \{i\}$, where Proposition 7.4 implies that $V'^- \prec V^-$ because $i + 1 \notin I$.

The remaining cases follow by similar reasoning. The interested reader may find Table 23 useful for this. $\square$

Lemma 7.9. *Let $Y_z^* = \prod_{r=0}^p \xi_r(z)$ come from some $J, i_*, V^-, U^\circ, U^-$. If*

$$\vec{x} = (x_1, \ldots, x_n) \in Y_z^* \cap \bigcup_{w<z} Y_w,$$

then

$$\vec{x} \in \prod_{r=0}^{a-1} \xi_r(z) \times \partial \xi_a(z) \times \prod_{r=a+1}^{p} \xi_r(z)$$

for some $a = 0, \ldots, p$, such that $\xi_a(z)$ is of class (A).

Proof. Let $\vec{x} = (x_1, \dots, x_n) \in Y_z^* \cap Y_{w'}$ for some $w' < z$. Choose the smallest $w < z$ such that $\vec{x} \in Y_w$, and assume that Y_w comes from some $J', i'_*, V'^-, U'^\circ, U'^-$ with $V'^- \subset V'$ and $U'^\circ \subset U'$, whereas Y_z comes from some $J, i_*, V^-, U^\circ, U^-$ with $V^- \subset V$ and $U^\circ \subset U$. Denote

$$S = \left\{ a = 0, \dots, p : \ \vec{x} \in \prod_{r=0}^{a-1} \xi_r(z) \times \partial\xi_a(z) \times \prod_{r=a+1}^{p} \xi_r(z) \right\}.$$

Assume for contradiction that $\xi_a(z)$ is of class (B) for every $a \in S$. If $S = \varnothing$, then no coordinate of $\vec{x}$ equals i_*, so $i'_* = i_*$. Also, in that case, no coordinate of $\vec{x}$ equals $\min I_s - 1$ for any $s \in \mathbb{Z}_m$, and so J and J' completely determine the number of coordinates that $\vec{x}$ has in each open interval $(\min I_s - 1, \min I_{s+1} - 1)$. It follows that either $J' = J$ or $J' = T \setminus J$. If $J' = T \setminus J$, then considering the coordinates of $\vec{x}$ in $[\min I_*, \max I_*]$ yields a contradiction. If $J' = J$, then the fact that $S = \varnothing$ implies that $V^- = V'^-$, $U^\circ = U'^\circ$, and $U^- = U'^-$, contradicting the fact that $w < z$.

Therefore, $S \neq \varnothing$. If no coordinate of $\vec{x}$ equals i_*, then $i'_* = i_*$, so again either $J' = J$ or $J' = T \setminus J$. The latter case gives the same contradiction as before. Therefore $J' = J$, and so $V' = V$.

For each $i \in V^- \ominus V'^-$, $\vec{x}$ has a coordinate $x_t = i - \frac{1}{2}$ (using the fact that $i'_* = i_*$ and $J' = J$). The corresponding $\xi_r(z)$ has $r \in S$, and so by assumption $\xi_r(z)$ is of class (B). Therefore, $V^- \prec V^- \ominus \{i\}$ for each $i \in V^- \ominus V'^-$. Proposition 7.5 implies that $V^- \prec V'^-$ unless $V^- = V'^-$. Since $w < z$, we must have $V^- = V'^-$.

Each $i \in U'^\circ$ must also be in U°, or else the corresponding coordinate of $\vec{x}$ would equal $i - \frac{1}{3}$ or $i - \frac{2}{3}$, and the corresponding $\xi_a(z)$ would be of class (A) with $a \in S$, contrary to assumption. Thus, $U^\circ \subset U'^\circ$. Similarly, each $i \in U^\circ$ must also be in U'°, or else the $Y_{w'}$ coming from $J, i_*, V, U'^\circ \cup \{i\}, U^- \setminus \{i\}$ would still contain $\vec{x}$ but with $w' < w$, contrary to assumption. Thus, $U'^\circ = U^\circ$.

Finally, we must have $U'^- = U^-$, by Observation 7.6. This implies, contrary to assumption, that $Y_w = Y_z$. □

Lemma 7.10. *Let $Y_z^* = \prod_{r=0}^{p} \xi_r(z)$ come from some $J, i_*, V^-, U^\circ, U^-$. If*

$$\vec{x} = (x_1, \dots, x_n) \in Y_z^* \cap (Y_z \setminus \backslash Y_z^*),$$

then

$$\vec{x} \in \prod_{r=0}^{a-1} \xi_r(z) \times \partial\xi_a(z) \times \prod_{r=a+1}^{p} \xi_r(z)$$

for some $a = 0, \dots, p$, such that $\xi_a(z)$ is of class (A).

Proof. This follows from a case analysis, for which the interested reader may find Tables 21, 23, and 22 useful. It comes down to this. Consider two pieces $\xi_a(z)$ and $\xi_b(z)$ where $\max \xi_a(z) = \min \xi_b(z) = c$ (recall Notation 7.2). Then $c \in \mathbb{Z}_k$. If c equals $i-1$ for some $i \in T$, then $i \in J$ and $\xi_b(z)$ is of class (A). Otherwise, $c = i_*$ and $\xi_a(z)$ is of class (A). □

The results of §§6, 7.3 provide all the details we need to prove:

Theorem 7.11. *For $n = 2k-1 \in \mathbb{Z}_+$, the n-torus admits a smooth multisection $T^n = \bigcup_{r \in \mathbb{Z}_k} X_r$ defined by*

$$
\begin{aligned}
X_0 &= \left\{ \vec{x}_\sigma : \ \vec{x} \in [0,1]^2 \cdots [0,k-1]^2[0,k]/\sim, \ \sigma \in S_n \right\}, \\
X_i &= \{ \vec{x} + (i, \dots, i) : \ \vec{x} \in X_0 \}.
\end{aligned} \tag{1}
$$

Proof. Lemma 6.8 implies that $X = \bigcup_{i \in \mathbb{Z}_k} X_i$, so it remains only to prove for each nonempty proper subset $I \subset \mathbb{Z}_k$, that $X_I = \bigcap_{i \in I} X_i$ is an $(n+1-|I|)$-dimensional submanifold of X with a spine of dimension $|I|$.

Fix some such I. Assume WLOG that I is simple. Then $X_I =$ (2), by Lemma 6.13. Decompose $X_I = \bigcup_z Y_z$ as described in §7.2. Lemmas 7.7 and 7.8 imply that Y_1^* is an $(n+1-|I|)$-dimensional 0-handle with no pieces $\xi_r(1)$ of class (A); Lemma 7.10 and the symmetry of the construction imply further that Y_1 is a union of $(n+1-|I|)$-dimensional 0-handles.

Similarly, for each z, Lemmas 7.7, 7.8, and 7.9 imply that attaching Y_z^* to $\bigcup_{w<z} Y_z$ amounts to attaching an $(n+1-|I|)$-dimensional h-handle, where $h(z)$ is the sum of the dimensions of those $\xi_r(z)$ of class (A):

$$h = \{ i \in I : \ \xi_r(z) \text{ is of class (A)} \} \leq |I|.$$

Again, Lemma 7.10 and the symmetry of the construction imply further that attaching all of Y_z to $\bigcup_{w<z} Y_w$ amounts to attaching several such handles. Thus, X_I is an $(n+1-|I|)$-dimensional $|I|$-handlebody in T^n.

It remains to check that $X_{\mathbb{Z}_k} = \bigcap_{i \in \mathbb{Z}_k} X_i$ is a closed k-manifold. We know from Lemma 6.13 that $X_{\mathbb{Z}_k}$ is given by (3).

Since $X_{\mathbb{Z}_k \setminus \{k-1\}}$ is $(k+1)$-manifold, it suffices to check that $X_{\mathbb{Z}_k}$ equals $\partial X_{\mathbb{Z}_k \setminus \{k-1\}}$, which is the union of those k-faces of the Y_z from the handle decomposition of $X_{\mathbb{Z}_k \setminus \{k-1\}}$ that are not glued to any other Y_w. Case analysis confirms that this union equals the expression from 3. (The reader may find Tables 21-21 useful.)

Alternatively, construct a handle decomposition of $X_{\mathbb{Z}_k}$ as follows. Cut each unit interval $[i, i+1]$ into thirds and, for each $i_* \in \mathbb{Z}_k$, further cut $\left[i_* - \frac{1}{3}, i_*\right]$ and $\left[i_*, i_* + \frac{1}{3}\right]$ into halves. Then, for each $i_* \in \mathbb{Z}_k$, $U^\circ \subset \mathbb{Z}_k$, $U^- \subset \mathbb{Z}_k \setminus U^\circ$, and $U^* \subset (\{i_*+1\} \cap U^-) \cup (\{i_*\} \setminus (U^\circ \cup U^-)$, define

$$\rho_i = \begin{cases} [i-\frac{2}{3}, i-\frac{1}{3}] & i \in U^\circ \\ [i-1, i-\frac{2}{3}] & i_*+1 \neq i \in U^- \\ [i-\frac{1}{3}, i] & i_* \neq i \in \mathbb{Z}_k \setminus (U^\circ \cup U^-) \\ [i_*, i_*+\frac{1}{6}] & i_*+1 = i \in U^* \\ [i_*+\frac{1}{6}, i_*+\frac{1}{3}] & i_*+1 = i \in U^- \setminus U^* \\ [i_*-\frac{1}{6}, i_*] & i_* = i \in U^* \\ [i_*-\frac{1}{3}, i_*-\frac{1}{6}] & i_* = i \in U^+ \setminus U^*, \end{cases}$$

$$X_{\mathbb{Z}_k, i_*, U^\circ, U^-, U^*} = \prod_{i \in \mathbb{Z}_k} \begin{Bmatrix} \rho_i \times \{i\} & i \neq i_* \\ \rho_i & = i_* \end{Bmatrix}.$$

Order the pieces $X_{\mathbb{Z}_k, i_*, U^\circ, U^-, U^*}$ as Y_z, $z = 1, 2, 3, \dots$, lexicographically according to the following orders on the possibilities for (i_*, U°, U^-, U^*). Order $\{i_* \in I\}$ and $U^- \subset U^\circ$ arbitrarily. Partially order $\{U^\circ \subset \mathbb{Z}_k\}$ by inclusion, with $U^\circ \prec U'^\circ$ if $U^\circ \subset U'^\circ$, and extend arbitrarily to a total order. Order the possibilities for U^* the same way. Then

$$\bigcup_{i=1,\dots,k} Y_z = \bigcup_{i_* \in \mathbb{Z}_k} X_{\mathbb{Z}_k, i_*, \mathbb{Z}_k, \varnothing, \varnothing}$$

is a union of 0-handles, and to attach each $Y_z = X_{\mathbb{Z}_k, i_*, U^\circ, U^-, U^*}$ to $\bigcup_{w<z} Y_w$ is to attach a collection of $h(z)$-handles for $h(z) = k - |U^\circ| - |U^*|$. $\square$

8. Cubulated manifolds of odd dimension

Consider a covering space $p : M \to T^n$, where $n = 2k-1$. Multisect $T^n = \bigcup_{i \in \mathbb{Z}_k} X_i$ as in Theorem 7.11. Then, by Corollary 17 of [7], $M = \bigcup_{i \in \mathbb{Z}_k} p^{-1}(X_i)$ determines a PL multisection of M. In general, one expects such multisections to be less efficient than those from Theorem 7.11. Also, there seems to be no reason to expect that one can extend the main construction to cubulated odd-dimensional manifolds in general. There is, however, an intermediate case to which our construction does extend.

First, we propose the following modest generalization of the usual notion of a cubulation. The generalization is similar to Hatcher's Δ-complexes vis a vis simplicial complexes [2]. A *cube* is a homeomorphic copy of I^n for some $n \geq 0$, with the usual cell structure; its *faces* are defined in the traditional way.

Definition 8.1. A **⧉-complex** is a quotient space of a collection of disjoint cubes obtained by identifying certain of their faces via homeomorphisms.

Note that, by definition, a ⧉-complex comes equipped with a cell structure.

Definition 8.2. A **generalized cubulation** of a manifold is a homeomorphism to a ⧉-complex.

In other words, a generalized cubulation of an n-manifold M imposes a cell structure on M in which every cell "looks like" an n-cube. The point of generalizing the usual definition is that the usual cell structure on T^n counts as a generalized cubulation, but not as a cubulation in the traditional sense.

Consider an arbitrary edge of I^n, joining $\vec{a} = (a_1, \ldots, a_{i-1}, 0, a_{i+1}, \ldots, a_n)$ and $\vec{b} = (a_1, \ldots, a_{i-1}, 1, a_{i+1}, \ldots, a_n)$. Orient this edge so that it runs from $\vec{a}$ to $\vec{b}$. Do the same with every edge of the n-cube. Call these the *standard orientations* on the edges of the n-cube.

Definition 8.3. A ⧉-complex K is **directable** if it is possible to orient the edges in K in such a way that, for each n-cell C in K, there is a continuous map h to K from the n-cube with the standard orientations on its edges, such that h respects these orientations and maps the interior of the n-cube homeomorphically to the interior of C. A **directed** ⧉-complex is one whose edges have been oriented in this way.

Definition 8.4. A **directed cubulation** of a manifold is a homeomorphism to a directed ⧉-complex.

Fix some $n = 2k - 1$. Let $g : I^n = [0, k]^n \to T^n = (\mathbb{R}/k\mathbb{Z})^n = [0, k]^n / \sim$ be the quotient map. Let $f : M \to K$ be a directed cubulation of an n-manifold. For each n-cell C in K, denote $h : I^n \to C$ as in Definition 8.3. Multisect $T^n = \bigcup_{i \in \mathbb{Z}_k} X_i$ as in Theorem 7.11. For each $i \in \mathbb{Z}_k$, define $X_{i,C} = h(g^{-1}(X_i)) \subset C$. Then, for each $i \in \mathbb{Z}_k$, define

$$X_i' = \bigcup_{n\text{-cubes } C \text{ in } K} f^{-1}(X_{i,C}).$$

Observation 8.5. *With the setup above, $M = \bigcup_{i \in \mathbb{Z}_k} X_i'$ determines a PL multisection of M.*

This follows from the fact that the multisection of T^n is fixed by the permutation action on the indices and K is a **directed** cubulation. Another way to see this is by noting that, because M admits a directed cubulation, then there is a well-defined map map from M to the symmetric space T^n/S_n, which one can use to define this multisection.

Appendix 1: Additional tables detailing handle decompositions

Tables 11 and 12 explicitly detail $U_r, V_r \subset I_r$ for arbitrary I_r (following Notation 3.8). For simplicity, these tables have $I_r = I_0 = \{0, \ldots, w\}$, listing

U_0, V_0; this is not necessarily consistent with Convention 3.9. To adapt $U_0, V_0 \subset I_0$ to the general case $U_r, V_r \subset I_r$, add $\min I_r$ in each coordinate.

I_0	$0 \notin J$		$0 \in J$	
	U_0	V_0	U_0	V_0
$\{0\}$	$\varnothing$	$\varnothing$	$\varnothing$	$\{0\}$
$\{0,1\}$	$\varnothing$	$\{1\}$	$\varnothing$	$\{1\}$
$\{0,1,2\}$	$\varnothing$	$\{1,2\}$	$\{1\}$	$\{2\}$
$\{0,1,2,3\}$	$\varnothing$	$\{1,2,3\}$	$\{1,2\}$	$\{3\}$
$\{0,1,2,3,4\}$	$\varnothing$	$\{1,2,3,4\}$	$\{1,2,3\}$	$\{4\}$
$\{0,\dots,w\}$	$\varnothing$	$\{1,\dots,w\}$	$\{1,\dots,w-1\}$	$\{w\}$

TABLE 11. The index subsets $U_0, V_0 \subset I_0$ when $i_* \notin I_0$.

I_0	U_0	V_0
$\{0\}$	$\varnothing$	$\varnothing$
$\{0,1\}$	$\varnothing$	$\varnothing$
$\{0,1,2\}$	$\left\{\begin{array}{ll}\{1\} & s=2\\ \varnothing & s\neq 2\end{array}\right\}$	$\left\{\begin{array}{ll}\{1,2\} & s=0\\ \varnothing & s\neq 0\end{array}\right\}$
$\{0,1,2,3\}$	$\left\{\begin{array}{ll}\{2\} & s=0\\ \varnothing & s=1\\ \{1\} & s=2\\ \{1,2\} & s=3\end{array}\right\}$	$\left\{\begin{array}{ll}\{i_*+1,3\} & s\leq 1\\ \varnothing & s\geq 2\end{array}\right\}$
$\{0,\dots,w\}$	$I_0 \setminus \{0, i_*, i_*+1, w\}$	$\left\{\begin{array}{ll}\{i_*+1,w\} & i_*\leq w-2\\ \varnothing & i_*\geq w-1\end{array}\right\}$

TABLE 12. The index subsets $U_0, V_0 \subset I_0$ when $i_* \in I_0$.

Table 13 details the handle decomposition of X_I from T^9 with $I = \{0,1,3\} = I_1 \sqcup I_2$, $I_1 = \{0,1\}$, $I_2 = \{3\}$. The interesting feature of this example is how the two blocks of indices I_1, I_2 interact.

J	i_*	U	V	V^-	Y_z^*	h	z	glue to
$\varnothing$	0	$\varnothing$	$\varnothing$		$\langle\alpha 1\beta^3\rangle\langle 3\delta^3\rangle$	0	1	
$\varnothing$	1	$\varnothing$	$\varnothing$	$\varnothing$	$\langle 0\alpha\rangle\beta^3\langle 3\delta^3\rangle$	1	2	1
$\varnothing$	3	$\varnothing$	{1}	$\varnothing$	$0\langle\alpha^+ 1\beta^3\rangle\delta^3$	0	3	
$\varnothing$	3	$\varnothing$	{1}	{1}	$\langle 0\alpha^-\rangle\langle 1\beta^3\rangle\delta^3$	1	4	3
{0}	0	$\varnothing$	$\varnothing$	$\varnothing$	$\langle\alpha 1\beta^3\rangle\langle 3\delta^2\rangle\varepsilon$	1	5	1,3,4
{0}	1	$\varnothing$	$\varnothing$	$\varnothing$	$\langle\varepsilon 0\alpha\rangle\beta^3\langle 3\delta^2\rangle$	2	6	2,5
{0}	3	$\varnothing$	{1}		$\langle\varepsilon 0\rangle\langle\alpha^+ 1\beta^3\rangle\delta^2$	1	7	3
{0}	3	$\varnothing$	{1}		$\langle\varepsilon 0\alpha^-\rangle\langle 1\beta^3\rangle\delta^2$	2	8	4,7
{3}	0	$\varnothing$	{3}	$\varnothing$	$\langle\alpha 1\beta^2\rangle\langle\gamma^+ 3\delta^3\rangle$	0	9	
{3}	0	$\varnothing$	{3}	{3}	$\langle\alpha 1\beta^2\rangle\gamma^-\langle 3\delta^3\rangle$	1	10	1,9
{3}	1	$\varnothing$	{3}	$\varnothing$	$\langle 0\alpha\rangle\beta^2\langle\gamma^+ 3\delta^3\rangle$	1	11	9
{3}	1	$\varnothing$	{3}	{3}	$\langle 0\alpha\rangle\beta^2\gamma^-\langle 3\delta^3\rangle$	2	12	2,10,11
{3}	3	$\varnothing$	{1}	$\varnothing$	$0\langle\alpha^+ 1\beta^2\rangle\gamma\delta^3$	1	13	2,3
{3}	3	$\varnothing$	{1}	{1}	$\langle 0\alpha^-\rangle\langle 1\beta^2\rangle\gamma\delta^3$	2	14	2,4,13
{0,3}	0	$\varnothing$	{3}	$\varnothing$	$\langle\alpha 1\beta^2\rangle\langle\gamma^+ 3\delta^2\rangle\varepsilon$	1	15	9,13,14
{0,3}	0	$\varnothing$	{3}	{3}	$\langle\alpha 1\beta^2\rangle\gamma^-\langle 3\delta^2\rangle\varepsilon$	2	16	10,13,14,15
{0,3}	1	$\varnothing$	{3}	$\varnothing$	$\langle\varepsilon 0\alpha\rangle\beta^2\langle\gamma^+ 3\delta^2\rangle$	2	17	11,15
{0,3}	1	$\varnothing$	{3}	{3}	$\langle\varepsilon 0\alpha\rangle\beta^2\gamma^-\langle 3\delta^2\rangle$	**3**	18	6,12,16,17
{0,3}	3	$\varnothing$	{1}	$\varnothing$	$\langle\varepsilon 0\rangle\langle\alpha^+ 1\beta^2\rangle\gamma\delta^2$	2	19	6,7,13
{0,3}	3	$\varnothing$	{1}	{1}	$\langle\varepsilon 0\alpha^-\rangle\langle 1\beta^2\rangle\gamma\delta^2$	**3**	20	6,8,14,19

TABLE 13. A genus 9 quintisection of T^9: X_I when $I = \{0, 1, 3\}$

Tables 14-17 detail the handle decomposition of X_I, $I = \{0, 1, 2, 3\}$, from the quintisection of T^9. Note that, since $I = I_1$ consists of a single block in this example, we always have $I_1 = I_*$.

i_*	U	V	V^-	Y_z^*	h	z	glue to
0	{2}	{1,3}	$\varnothing$	$\alpha^- 1\beta_3^\circ 2\langle\gamma^+ 3\delta^3\rangle$	0	1	
				$\alpha^-\langle 1\beta_3^-\rangle 2\langle\gamma^+ 3\delta^3\rangle$	1	2	1
				$\alpha^- 1\langle\beta_3^+ 2\rangle\langle\gamma^+ 3\delta^3\rangle$	1	3	1
			{1}	$\langle\alpha^+ 1\rangle\beta_3^\circ 2\langle\gamma^+ 3\delta^3\rangle$	1	4	1
				$\langle\alpha^+ 1\beta_3^-\rangle 2\langle\gamma^+ 3\delta^3\rangle$	2	5	2,4
				$\langle\alpha^+ 1\rangle\langle\beta_3^+ 2\rangle\langle\gamma^+ 3\delta^3\rangle$	2	6	3,4
			{3}	$\alpha^- 1\beta_3^\circ\langle 2\gamma^-\rangle\langle 3\delta^3\rangle$	1	7	1
				$\alpha^-\langle 1\beta_3^-\rangle\langle 2\gamma^-\rangle\langle 3\delta^3\rangle$	2	8	2,7
				$\alpha^- 1\langle\beta_3^+ 2\gamma^-\rangle\langle 3\delta^3\rangle$	2	9	3,7
			{1,3}	$\langle\alpha^+ 1\rangle\beta_3^\circ\langle 2\gamma^-\rangle\langle 3\delta^3\rangle$	2	10	4,7
				$\langle\alpha^+ 1\beta_3^-\rangle\langle 2\gamma^-\rangle\langle 3\delta^3\rangle$	3	11	5,8,10
				$\langle\alpha^+ 1\rangle\langle\beta_3^+ 2\gamma^-\rangle\langle 3\delta^3\rangle$	3	12	6,9,10

TABLE 14. X_I, $I = \{0, 1, 2, 3\}$, from T^9. Part 1: $J = \varnothing$.

i_*	U	V	V^-	Y_z^*	h	z	glue to
1	$\varnothing$	{2,3}	$\varnothing$	$\langle 0\alpha\rangle\, \beta^- 2 \left\langle \gamma^+ 3\delta^3\right\rangle$	1	13	1,2
			{2}	$\langle 0\alpha\rangle \langle \beta^+ 2\rangle \left\langle \gamma^+ 3\delta^3\right\rangle$	2	14	1,3,13
			{3}	$\langle 0\alpha\rangle\, \beta^- \langle 2\gamma^-\rangle \left\langle 3\delta^3\right\rangle$	2	15	7,8,13
			{2,3}	$\langle 0\alpha\rangle \langle \beta^+ 2\gamma^-\rangle \left\langle 3\delta^3\right\rangle$	3	16	7,9,14,15
2	{1}	$\varnothing$	$\varnothing$	$0\alpha_3^\circ \langle 1\beta\rangle \left\langle \gamma 3\delta^3\right\rangle$	1	17	13
				$\left\langle 0\alpha_3^-\right\rangle \langle 1\beta\rangle \left\langle \gamma 3\delta^3\right\rangle$	2	18	13,17
				$0\left\langle \alpha_3^+ 1\beta\right\rangle \left\langle \gamma 3\delta^3\right\rangle$	2	19	13,17
3	{1,2}	$\varnothing$	$\varnothing$	$0\alpha_3^\circ 1\beta_3^\circ \langle 2\gamma\rangle\, \delta^3$	1	20	17
				$\left\langle 0\alpha_3^-\right\rangle 1\beta_3^\circ \langle 2\gamma\rangle\, \delta^3$	2	21	18,20
				$0\left\langle \alpha_3^+ 1\right\rangle \beta_3^\circ \langle 2\gamma\rangle\, \delta^3$	2	22	19,20
				$0\alpha_3^\circ \left\langle 1\beta_3^-\right\rangle \langle 2\gamma\rangle\, \delta^3$	2	23	17,20
				$\left\langle 0\alpha_3^-\right\rangle \left\langle 1\beta_3^-\right\rangle \langle 2\gamma\rangle\, \delta^3$	3	24	18,20,23
				$0\left\langle \alpha_3^+ 1\beta_3^-\right\rangle \langle 2\gamma\rangle\, \delta^3$	3	25	19,22,23
				$0\alpha_3^\circ 1\left\langle \beta_3^+ 2\gamma\right\rangle \delta^3$	2	26	17,20
				$\left\langle 0\alpha_3^-\right\rangle 1\left\langle \beta_3^+ 2\gamma\right\rangle \delta^3$	3	27	18,21,26
				$0\left\langle \alpha_3^+ 1\right\rangle \left\langle \beta_3^+ 2\gamma\right\rangle \delta^3$	3	28	19,22,26

TABLE 15. X_I, $I = \{0,1,2,3\}$, from T^9. Part 2: $J = \varnothing$.

i_*	U	V	V^-	Y_z^*	h	z	glue to
0	{2}	{1,3}	$\varnothing$	$\alpha^- 1\beta_3^\circ 2\left\langle \gamma^+ 3\delta^2\right\rangle \varepsilon$	1	29	1,19,20
				$\alpha^- \left\langle 1\beta_3^-\right\rangle 2\left\langle \gamma^+ 3\delta^2\right\rangle \varepsilon$	2	30	2,22,23,29
				$\alpha^- 1\beta_3^+ 2\left\langle \gamma^+ 3\delta^2\right\rangle \varepsilon$	2	31	3,25,26,29
			{1}	$\left\langle \alpha^+ 1\right\rangle \beta_3^\circ 2\left\langle \gamma^+ 3\delta^2\right\rangle \varepsilon$	2	32	4,19,21
				$\left\langle \alpha^+ 1\beta_3^-\right\rangle 2\left\langle \gamma^+ 3\delta^2\right\rangle \varepsilon$	3	33	5,22,24,30,32
				$\left\langle \alpha^+ 1\right\rangle \beta_3^+ 2\left\langle \gamma^+ 3\delta^2\right\rangle \varepsilon$	3	34	6,25,27,31,32
			{3}	$\alpha^- 1\beta_3^\circ \langle 2\gamma^-\rangle \left\langle 3\delta^2\right\rangle \varepsilon$	2	35	7,19,20,29
				$\alpha^- \left\langle 1\beta_3^-\right\rangle \langle 2\gamma^-\rangle \left\langle 3\delta^2\right\rangle \varepsilon$	3	36	8,22,23,30,35
				$\alpha^- 1\left\langle \beta_3^+ 2\gamma^-\right\rangle \left\langle 3\delta^2\right\rangle \varepsilon$	3	37	9,25,26,31,35
			{1,3}	$\left\langle \alpha^+ 1\right\rangle \beta_3^\circ \langle 2\gamma^-\rangle \left\langle 3\delta^2\right\rangle \varepsilon$	3	38	10,19,21,32,35
				$\left\langle \alpha^+ 1\beta_3^-\right\rangle \langle 2\gamma^-\rangle \left\langle 3\delta^2\right\rangle \varepsilon$	**4**	39	11,22,24,33,36,38
				$\left\langle \alpha^+ 1\right\rangle \left\langle \beta_3^+ 2\gamma^-\right\rangle \left\langle 3\delta^2\right\rangle \varepsilon$	**4**	40	12,25,27,34,37,38
1	$\varnothing$	{2,3}	$\varnothing$	$\langle \varepsilon 0\alpha\rangle\, \beta^- 2\left\langle \gamma^+ 3\delta^2\right\rangle$	2	41	13,29,30
			{2}	$\langle \varepsilon 0\alpha\rangle \langle \beta^+ 2\rangle \left\langle \gamma^+ 3\delta^2\right\rangle$	3	42	14,29,31,41
			{3}	$\langle \varepsilon 0\alpha\rangle\, \beta^- \langle 2\gamma^-\rangle \left\langle 3\delta^2\right\rangle$	3	43	15,35,36,41
			{2,3}	$\langle \varepsilon 0\alpha\rangle \langle \beta^+ 2\gamma^-\rangle \left\langle 3\delta^2\right\rangle$	**4**	44	16,35,37,42,43
2	{1}	$\varnothing$	$\varnothing$	$\langle \varepsilon 0\rangle\, \alpha_3^\circ \langle 1\beta\rangle \left\langle \gamma 3\delta^2\right\rangle$	2	45	17,41,43
				$\left\langle \varepsilon 0\alpha_3^-\right\rangle \langle 1\beta\rangle \left\langle \gamma 3\delta^2\right\rangle$	3	46	18,41,43,45
				$\langle \varepsilon 0\rangle \left\langle \alpha_3^+ 1\beta\right\rangle \left\langle \gamma 3\delta^2\right\rangle$	3	47	19,41,43,45

TABLE 16. X_I, $I = \{0,1,2,3\}$, from T^9. Part 3: $J = \{0\}$.

i_*	U	V	V^-	Y_z^*	h	z	glue to
3	{1,2}	∅	∅	$\langle\varepsilon 0\rangle\, \alpha_3^\circ 1 \beta_3^\circ\, \langle 2\gamma\rangle\, \delta^2$	2	48	20,45
				$\langle\varepsilon 0\alpha_3^-\rangle\, 1\beta_3^\circ\, \langle 2\gamma\rangle\, \delta^2$	3	49	21,46,48
				$\langle\varepsilon 0\rangle\, \langle\alpha_3^+ 1\rangle\, \beta_3^\circ\, \langle 2\gamma\rangle\, \delta^2$	3	50	22,47,48
				$\langle\varepsilon 0\rangle\, \alpha_3^\circ\, \langle 1\beta_3^-\rangle\, \langle 2\gamma\rangle\, \delta^2$	3	51	23,45,48
				$\langle\varepsilon 0\alpha_3^-\rangle\, \langle 1\beta_3^-\rangle\, \langle 2\gamma\rangle\, \delta^2$	**4**	52	24,46,49,51
				$\langle\varepsilon 0\rangle\, \langle\alpha_3^+ 1\beta_3^-\rangle\, \langle 2\gamma\rangle\, \delta^2$	**4**	53	25,47,50,51
				$\langle\varepsilon 0\rangle\, \alpha_3^\circ 1\, \langle\beta_3^+ 2\gamma\rangle\, \delta^2$	3	54	26,45,48
				$\langle\varepsilon 0\alpha_3^-\rangle\, 1\, \langle\beta_3^+ 2\gamma\rangle\, \delta^2$	**4**	55	27,46,49,54
				$\langle\varepsilon 0\rangle\, \langle\alpha_3^+ 1\rangle\, \langle\beta_3^+ 2\gamma\rangle\, \delta^2$	**4**	56	28,47,50,54

Table 17. X_I, $I = \{0,1,2,3\}$, from T^9. Part 4: $J = \{0\}$.

Tables 18 and 19 detail handle decompositions of X_I, $I = \{0,1,2,4\}$ from the sexasection of T^{11}. The parts of these tables with $i_* = 4$ feature a "new complication," i.e. one that does not appear in dimensions $n \leq 5$.

J	i_*	U	V	V^-	Y_z^*	h	z	glue to
∅	4	∅	{1,2}	∅	$0\,\langle\alpha^+ 1\rangle\, \langle\beta^+ 2\gamma^3\rangle\, \varepsilon^3$	0	1	
				{1}	$\langle 0\alpha^-\rangle\, 1\, \langle\beta^+ 2\gamma^3\rangle\, \varepsilon^3$	1	2	1
				{1, 2}	$\langle 0\alpha^-\rangle\, \langle 1\beta^-\rangle\, \langle 2\gamma^3\rangle\, \varepsilon^3$	1	3	2
				{2}	$0\,\langle\alpha^+ 1\beta^-\rangle\, \langle 2\gamma^3\rangle\, \varepsilon^3$	2	4	1,3
	0	∅	{1,2}	∅	$\alpha^- 1\, \langle\beta^+ 2\gamma^3\rangle\, \langle 4\varepsilon^3\rangle$	0	5	
				{1}	$\langle\alpha^+ 1\rangle\, \langle\beta^+ 2\gamma^3\rangle\, \langle 4\varepsilon^3\rangle$	1	6	5
				{2}	$\alpha^-\, \langle 1\beta^-\rangle\, \langle 2\gamma^3\rangle\, \langle 4\varepsilon^3\rangle$	1	7	5
				{1,2}	$\langle\alpha^+ 1\beta^-\rangle\, \langle 2\gamma^3\rangle\, \langle 4\varepsilon^3\rangle$	2	8	5,6
	1	∅	∅	∅	$\langle 0\alpha\rangle\, \langle\beta 2\gamma^3\rangle\, \langle 4\varepsilon^3\rangle$	1	9	5,7
	2	{1}	∅	∅	$0\alpha_3^\circ\, \langle 1\beta\rangle\, \gamma^3\, \langle 4\varepsilon^3\rangle$	1	10	9
					$\langle 0\alpha_3^-\rangle\, \langle 1\beta\rangle\, \gamma^3\, \langle 4\varepsilon^3\rangle$	2	11	9,10
					$0\,\langle\alpha_3^+ 1\beta\rangle\, \gamma^3\, \langle 4\varepsilon^3\rangle$	2	12	9,10
{4}	4	∅	{1,2}	∅	$0\,\langle\alpha^+ 1\rangle\, \langle\beta^+ 2\gamma^2\rangle\, \delta\varepsilon^3$	1	13	1,10,12
					$\langle 0\alpha^-\rangle\, 1\, \langle\beta^+ 2\gamma^2\rangle\, \delta\varepsilon^3$	2	14	2,10,11,13
					$\langle 0\alpha^-\rangle\, \langle 1\beta^-\rangle\, \langle 2\gamma^2\rangle\, \delta\varepsilon^3$	2	15	3,10,11,14
					$0\,\langle\alpha^+ 1\beta^-\rangle\, \langle 2\gamma^2\rangle\, \delta\varepsilon^3$	3	16	4,10,12,13,15
	0	∅	{1,2}	∅	$\alpha^- 1\, \langle\beta^+ 2\gamma^2\rangle\, \langle\delta^+ 4\varepsilon^3\rangle$	0	17	
					$\alpha^- 1\, \langle\beta^+ 2\gamma^2\rangle\, \delta^-\, \langle 4\varepsilon^3\rangle$	1	18	5,17
				{1}	$\langle\alpha^+ 1\rangle\, \langle\beta^+ 2\gamma^2\rangle\, \langle\delta^+ 4\varepsilon^3\rangle$	1	19	17
					$\langle\alpha^+ 1\rangle\, \langle\beta^+ 2\gamma^2\rangle\, \delta^-\, \langle 4\varepsilon^3\rangle$	2	20	6,18,19
				{2}	$\alpha^-\, \langle 1\beta^-\rangle\, \langle 2\gamma^2\rangle\, \langle\delta^+ 4\varepsilon^3\rangle$	1	21	19
					$\alpha^-\, \langle 1\beta^-\rangle\, \langle 2\gamma^2\rangle\, \delta^-\, \langle 4\varepsilon^3\rangle$	2	22	7,20,21
				{1,2}	$\langle\alpha^+ 1\beta^-\rangle\, \langle 2\gamma^2\rangle\, \langle\delta^+ 4\varepsilon^3\rangle$	2	23	19,21
					$\langle\alpha^+ 1\beta^-\rangle\, \langle 2\gamma^2\rangle\, \delta^-\, \langle 4\varepsilon^3\rangle$	3	24	8,20,22,23

Table 18. Part 1 of X_I, $I = \{0,1,2,4\}$, from T^{11}. The pattern for $s = 4$ is new: also see Tables 9, 20.

J	i_*	U	V	V^-	Y_z^*	h	z	glue to
$\{4\}$	1	$\varnothing$	$\varnothing$	$\varnothing$	$\langle 0\alpha\rangle \langle \beta 2\gamma^2\rangle \langle \delta^+ 4\varepsilon^3\rangle$	1	25	17,21
					$\langle 0\alpha\rangle \langle \beta 2\gamma^2\rangle \delta^- \langle 4\varepsilon^3\rangle$	2	26	9,18,22,25
	2	$\{1\}$	$\varnothing$	$\varnothing$	$0\alpha_3^\circ \langle 1\beta\rangle \gamma^2 \langle \delta^+ 4\varepsilon^3\rangle$	1	27	25
					$0\alpha_3^\circ \langle 1\beta\rangle \gamma^2\delta^- \langle 4\varepsilon^3\rangle$	2	28	10,26,27
					$\langle 0\alpha_3^-\rangle \langle 1\beta\rangle \gamma^2 \langle \delta^+ 4\varepsilon^3\rangle$	2	29	25,27
					$\langle 0\alpha_3^-\rangle \langle 1\beta\rangle \gamma^2\delta^- \langle 4\varepsilon^3\rangle$	3	30	11,26,28,29
					$0 \langle \alpha_3^+ 1\beta\rangle \gamma^2 \langle \delta^+ 4\varepsilon^3\rangle$	2	31	25,27
					$0 \langle \alpha_3^+ 1\beta\rangle \gamma^2\delta^- \langle 4\varepsilon^3\rangle$	3	32	12,26,28,31
$\{0\}$	4	$\{1\}$	$\{2\}$	$\varnothing$	$\langle \zeta 0\rangle \alpha_3^\circ 1 \langle \beta^+ 2\gamma^3\rangle \varepsilon^2$	1	33	1,2
					$\langle \zeta 0\rangle \langle \alpha_3^+ 1\rangle \langle \beta^+ 2\gamma^3\rangle \varepsilon^2$	2	34	1,33
					$\langle \zeta 0\alpha_3^-\rangle 1 \langle \beta^+ 2\gamma^3\rangle \varepsilon^2$	2	35	2,33
				$\{2\}$	$\langle \zeta 0\rangle \alpha_3^\circ \langle 1\beta^-\rangle \langle 2\gamma^3\rangle \varepsilon^2$	2	36	3,4,33
					$\langle \zeta 0\rangle \langle \alpha_3^+ 1\beta^-\rangle \langle 2\gamma^3\rangle \varepsilon^2$	3	37	3,34,36
					$\langle \zeta 0\alpha_3^-\rangle \langle 1\beta^-\rangle \langle 2\gamma^3\rangle \varepsilon^2$	3	38	4,35,36
	0	$\varnothing$	$\{1{,}2\}$	$\varnothing$	$\zeta\alpha^- 1 \langle \beta^+ 2\gamma^3\rangle \langle 4\varepsilon^2\rangle$	1	39	2,5
				$\{1\}$	$\zeta \langle \alpha^+ 1\rangle \langle \beta^+ 2\gamma^3\rangle \langle 4\varepsilon^2\rangle$	2	40	1,6,39
				$\{2\}$	$\zeta\alpha^- \langle 1\beta^-\rangle \langle 2\gamma^3\rangle \langle 4\varepsilon^2\rangle$	2	41	3,7,39
				$\{1{,}2\}$	$\zeta \langle \alpha^+ 1\beta^-\rangle \langle 2\gamma^3\rangle \langle 4\varepsilon^2\rangle$	3	42	4,8,40
	1	$\varnothing$	$\varnothing$		$\langle \zeta 0\alpha\rangle \langle \beta 2\gamma^3\rangle \langle 4\varepsilon^2\rangle$	2	43	9,39,41
	2	$\{1\}$	$\varnothing$	$\varnothing$	$\langle \zeta 0\rangle \alpha_3^\circ \langle 1\beta\rangle \gamma^3 \langle 4\varepsilon^2\rangle$	2	44	10,43
					$\langle \zeta 0\alpha_3^-\rangle \langle 1\beta\rangle \gamma^3 \langle 4\varepsilon^2\rangle$	3	45	11,43,44
					$\langle \zeta 0\rangle \langle \alpha_3^+ 1\beta\rangle \gamma^3 \langle 4\varepsilon^2\rangle$	3	46	12,43,44
$\{0, 4\}$	4	$\{1\}$	$\{2\}$	$\varnothing$	$\langle \zeta 0\rangle \alpha_3^\circ 1 \langle \beta^+ 2\gamma^2\rangle \delta\varepsilon^2$	2	47	13,14,33,44
					$\langle \zeta 0\rangle \langle \alpha_3^+ 1\rangle \langle \beta^+ 2\gamma^2\rangle \delta\varepsilon^2$	3	48	13,34,45,47
					$\langle \zeta 0\alpha_3^-\rangle 1 \langle \beta^+ 2\gamma^2\rangle \delta\varepsilon^2$	3	49	14,35,46,47
				$\{2\}$	$\langle \zeta 0\rangle \alpha_3^\circ \langle 1\beta^-\rangle \langle 2\gamma^2\rangle \delta\varepsilon^2$	3	50	15,16,36,44,47
					$\langle \zeta 0\rangle \langle \alpha_3^+ 1\beta^-\rangle \langle 2\gamma^2\rangle \delta\varepsilon^2$	4	51	16,37,45,48,50
					$\langle \zeta 0\alpha_3^-\rangle \langle 1\beta^-\rangle \langle 2\gamma^2\rangle \delta\varepsilon^2$	4	52	15,38,46,49,50
	0	$\varnothing$	$\{1{,}2\}$	$\varnothing$	$\zeta\alpha^- 1 \langle \beta^+ 2\gamma^2\rangle \langle \delta^+ 4\varepsilon^2\rangle$	1	53	14,17
					$\zeta\alpha^- 1 \langle \beta^+ 2\gamma^2\rangle \delta^- \langle 4\varepsilon^2\rangle$	2	54	14,18,39,53
				$\{1\}$	$\zeta \langle \alpha^+ 1\rangle \langle \beta^+ 2\gamma^2\rangle \langle \delta^+ 4\varepsilon^2\rangle$	2	55	13,19,53
					$\zeta \langle \alpha^+ 1\rangle \langle \beta^+ 2\gamma^2\rangle \delta^- \langle 4\varepsilon^2\rangle$	3	56	13,20,40,54,55
				$\{2\}$	$\zeta\alpha^- \langle 1\beta^-\rangle \langle 2\gamma^2\rangle \langle \delta^+ 4\varepsilon^2\rangle$	2	57	15,21,53
					$\zeta\alpha^- \langle 1\beta^-\rangle \langle 2\gamma^2\rangle \delta^- \langle 4\varepsilon^2\rangle$	3	58	15,22,41,54,57
				$\{1{,}2\}$	$\zeta \langle \alpha^+ 1\beta^-\rangle \langle 2\gamma^2\rangle \langle \delta^+ 4\varepsilon^2\rangle$	3	59	16,23,55,57
					$\zeta \langle \alpha^+ 1\beta^-\rangle \langle 2\gamma^2\rangle \delta^- \langle 4\varepsilon^2\rangle$	4	60	16,24,42,56,58,59
	1	$\varnothing$	$\varnothing$		$\langle \zeta 0\alpha\rangle \langle \beta 2\gamma^2\rangle \langle \delta^+ 4\varepsilon^2\rangle$	2	61	25,53,57
					$\langle \zeta 0\alpha\rangle \langle \beta 2\gamma^2\rangle \delta^- \langle 4\varepsilon^2\rangle$	3	62	26,43,54,58,61
	2	$\{1\}$	$\varnothing$	$\varnothing$	$\langle \zeta 0\rangle \alpha_3^\circ \langle 1\beta\rangle \gamma^2 \langle \delta^+ 4\varepsilon^2\rangle$	2	63	27,61
					$\langle \zeta 0\alpha_3^-\rangle \langle 1\beta\rangle \gamma^2 \langle \delta^+ 4\varepsilon^2\rangle$	3	64	29,61,63
					$\langle \zeta 0\rangle \langle \alpha_3^+ 1\beta\rangle \gamma^2 \langle \delta^+ 4\varepsilon^2\rangle$	3	65	31,61,63
					$\langle \zeta 0\rangle \alpha_3^\circ \langle 1\beta\rangle \gamma^2\delta^- \langle 4\varepsilon^2\rangle$	3	66	28,44,62,63
					$\langle \zeta 0\alpha_3^-\rangle \langle 1\beta\rangle \gamma^2\delta^- \langle 4\varepsilon^2\rangle$	4	67	30,44,62,64,66
					$\langle \zeta 0\rangle \langle \alpha_3^+ 1\beta\rangle \gamma^2\delta^- \langle 4\varepsilon^2\rangle$	4	68	32,45,62,65,66

TABLE 19. Part 3 of X_I, $I = \{0,1,2,4\}$, from T^{11}. The pattern for $s = 4$ is new: also see Tables 9, 20.

Table 20 details the start of the handle decomposition of X_I from T^{15} with $I = \{0,1,2,4,6\}$, focusing on the first few pieces Y_z. Those pieces have $J = \varnothing$, $i_* = 6$, $U = \varnothing$, $V = \{1,2,3,4\}$. The interesting feature of this example is the ordering of these pieces. Compare to (20).

V^-	Y_z^*	h	z	glue to
$\varnothing$	$0\langle\alpha^+1\rangle\langle\beta^+2\rangle\langle\gamma^+3\rangle\langle\delta^+4\varepsilon^3\rangle\eta^3$	0	1	
$\{1\}$	$\langle 0\alpha^-\rangle 1\langle\beta^+2\rangle\langle\gamma^+3\rangle\langle\delta^+4\varepsilon^3\rangle\eta^3$	1	2	1
$\{1,2\}$	$\langle 0\alpha^-\rangle\langle 1\beta^-\rangle 2\langle\gamma^+3\rangle\langle\delta^+4\varepsilon^3\rangle\eta^3$	1	3	2
$\{2\}$	$0\langle\alpha^+1\beta^-\rangle 2\langle\gamma^+3\rangle\langle\delta^+4\varepsilon^3\rangle\eta^3$	2	4	1,3
$\{2,3\}$	$0\langle\alpha^+1\beta^-\rangle\langle 2\gamma^-\rangle 3\langle\delta^+4\varepsilon^3\rangle\eta^3$	1	5	4
$\{1,2,3\}$	$\langle 0\alpha^-\rangle\langle 1\beta^-\rangle\langle 2\gamma^-\rangle 3\langle\delta^+4\varepsilon^3\rangle\eta^3$	2	6	3,5
$\{1,3\}$	$\langle 0\alpha^-\rangle 1\langle\beta^+2\gamma^-\rangle 3\langle\delta^+4\varepsilon^3\rangle\eta^3$	2	7	2,6
$\{3\}$	$0\langle\alpha^+1\rangle\langle\beta^+2\gamma^-\rangle 3\langle\delta^+4\varepsilon^3\rangle\eta^3$	3	8	1,5,7
$\{3,4\}$	$0\langle\alpha^+1\rangle\langle\beta^+2\gamma^-\rangle\langle 3\delta^-\rangle\langle 4\varepsilon^3\rangle\eta^3$	1	9	8
$\{1,3,4\}$	$\langle 0\alpha^-\rangle 1\langle\beta^+2\gamma^-\rangle\langle 3\delta^-\rangle\langle 4\varepsilon^3\rangle\eta^3$	2	10	7,9
$\{1,2,3,4\}$	$\langle 0\alpha^-\rangle\langle 1\beta^-\rangle\langle 2\gamma^-\rangle\langle 3\delta^-\rangle\langle 4\varepsilon^3\rangle\eta^3$	2	11	6,10
$\{2,3,4\}$	$0\langle\alpha^+1\beta^-\rangle\langle 2\gamma^-\rangle\langle 3\delta^-\rangle\langle 4\varepsilon^3\rangle\eta^3$	3	12	5,9,11
$\{2,4\}$	$0\langle\alpha^+1\beta^-\rangle 2\langle\gamma^+3\delta^-\rangle\langle 4\varepsilon^3\rangle\eta^3$	2	13	4,12
$\{1,2,4\}$	$\langle 0\alpha^-\rangle\langle 1\beta^-\rangle 2\langle\gamma^+3\delta^-\rangle\langle 4\varepsilon^3\rangle\eta^3$	3	14	3,11,13
$\{1,4\}$	$\langle 0\alpha^-\rangle 1\langle\beta^+2\rangle\langle\gamma^+3\delta^-\rangle\langle 4\varepsilon^3\rangle\eta^3$	3	15	2,10,14
$\{4\}$	$0\langle\alpha^+1\rangle\langle\beta^+2\rangle\langle\gamma^+3\delta^-\rangle\langle 4\varepsilon^3\rangle\eta^3$	4	16	1,9,13,15

TABLE 20. Start of the handle decomposition from T^{15} with $I = \{0,1,2,4,6\}$, $J = \varnothing$, $i_* = 6$, $U = \varnothing$, $V = \{1,2,3,4\}$.

Tables 21, 22, and 23 list the possible forms for $\xi_r(z)$. Table 21 lists those *with no* singleton factor. Table 22 lists those *with* a singleton factor $\{i\}$, where $i \in V^+$ and $i+1 \in U^\circ \cup U^+ \cup V^+$, or $i \in U^- \cup U^\circ \cup V^-$ and $i+1 \in V^-$; the class of this case depends on the parity of $\#(V^- \cap \{i+1, \dots, \max I_s\})$, where $i \in I_s$. Table 22 lists the remaining possibilities for $\xi_r(z)$.

class	$\xi_r(z)$	conditions
(A)	$[i_* - 1, i_*]$	$i_* \in J$
(A)	$\left[i-1, i-\frac{1}{2}\right]$	$i \in J \cap V^- \implies i \neq i_*,\ i+1 \notin I$
(B)	$\left[i_*, i_* + \frac{1}{2}\right]$	$a \le i_* \le b-2,\ i_*+1 \in V^-$
(B)	$\left[i - \frac{2}{3}, i - \frac{1}{3}\right]$	$i \in U^\circ$
(B)	$\prod_{j=i_*+1}^{c-1} [i_*, j]^2$	$i_* = b,\ c \in J$
(B)	$\prod_{j=i_*+1}^{c-2} [i_*, j]^2 [i_*, c-1]^3$	$i_* = b,\ c \notin J$

TABLE 21. The possible forms for $\xi_r(z)$ with no singleton factor, where $i_* \in I_s$, $a = \min I_s$, $b = \max I_s$, $c = \min I_{s+1}$.

class	$\xi_r(z)$	conditions on i	conditions on $i+1$	parity
(A)	$\left[i-\frac{1}{2},i\right]\{i\}$	$i\in V^+$	$i+1\in U^\circ\cup U^+\cup V^+$	odd
(A)	$\{i\}\left[i,i+\frac{1}{2}\right]$	$i\in U^-\cup U^\circ\cup V^-$	$i+1\in V^-$	odd
(A)	$\left[i-\frac{1}{2},i\right]\{i\}\left[i,i+\frac{1}{2}\right]$	$i\in V^+$	$i+1\in V^-$	odd
(B)	$\left[i-\frac{1}{2},i\right]\{i\}$	$i\in V^+$	$i+1\in U^\circ\cup U^+\cup V^+$	even
(B)	$\{i\}\left[i,i+\frac{1}{2}\right]$	$i\in U^-\cup U^\circ\cup V^-$	$i+1\in V^-$	even
(B)	$\left[i-\frac{1}{2},i\right]\{i\}\left[i,i+\frac{1}{2}\right]$	$i\in V^+$	$i+1\in V^-$	even

TABLE 22. The possible forms for each $\xi_r(z)$ containing a singleton factor $\{i\}$, where $i\in V^+$ and $i+1\in U^\circ\cup U^+\cup V^+$, or $i\in U^-\cup U^\circ\cup V^-$ and $i+1\in V^-$; the class depends on the parity of $\#(V^-\cap\{i+1,\dots,\max I_s\})$, where $i\in I_s$.

class	$\xi_r(z)$	conditions on i
(A)	$[i-1,i]\{i\}$	$i\in J,\ i+1\in U^\circ\cup U^+\cup V^+$
(A)	$[i-1,i]\{i\}[i,i+1]$	$i\in J,\ i_*=i+1$
(A)	$[i-1,i]\{i\}\left[i,i+\frac{1}{3}\right]$	$i\in J,\ i+1\in U^-$
(A)	$[i-1,i]\{i\}\left[i,i+\frac{1}{2}\right]$	$i\in J,\ i+1\in V^-$
(A)	$\left[i-\frac{1}{3},i\right]\{i\}$	$i\in U^+,\ i+1\in U^\circ\cup U^+\cup V^+$
(A)	$\{i\}\left[i,i+\frac{1}{3}\right]$	$i+1\in U^-,\ i\in U^-\cup U^\circ\cup V^-$
(A)	$\left[i-\frac{1}{3},i\right]\{i\}\left[i,i+\frac{1}{3}\right]$	$i\in U^+,\ i+1\in U^-$
(A)	$\left[i-\frac{1}{3},i\right]\{i\}\left[i,i+\frac{1}{2}\right]$,	$i\in U^+,\ i+1\in V^-$ $\implies i+1=\max I_s\neq i_*$
(A)	$\left[i-\frac{1}{2},i\right]\{i\}\left[i,i+\frac{1}{3}\right]$,	$i\in V^+,\ i+1\in U^-$ $\implies i=i_*+1\leq\max I_s-1$
(A)	$\{i\}$	$i\in(T\setminus J)\cup U^-\cup U^\circ\cup V^-$, $i+1\in U^\circ\cup U^+\cup V^+$
(B)	$[i-1,i]\{i\}\prod_{j=i+1}^{c-2}[i,j]^2[i,c-1]^q$	$i_*=\min I_s=i-1=\max I_s-1$
(B)	$\left[i-\frac{1}{2},i\right]\{i\}\prod_{j=i+1}^{c-2}[i,j]^2[i,c-1]^q$	$i=\max I_s\in V^+$
(B)	$\{i\}\prod_{j=i+1}^{c-2}[i,j]^2[i,c-1]^q$	$i=\max I_s\in V^-$

TABLE 23. The possible forms for each $\xi_r(z)$ not listed in Tables 21, 22. Each contains a singleton factor $\{i\}$, $i_*\neq i\in I_s$, $s\in\mathbb{Z}_m$. Denote $c=\min I_{s+1}$ with $q\in\{2,3\}$.

Appendix 2: Three other attempts to multisect T^n for n odd

From the handle decomposition. The n-torus has a natural handle decomposition, with $\binom{n}{r}$ h-handles for each $h=0,\dots,n$. Viewing T^n as $(\mathbb{R}/2\mathbb{Z})^n$, each unit cube with vertices in $(\mathbb{Z}/2\mathbb{Z})^n$ corresponds to a handle; more precisely, each permutation of $\alpha^{n-h}\beta^h$ corresponds to an h-handle.

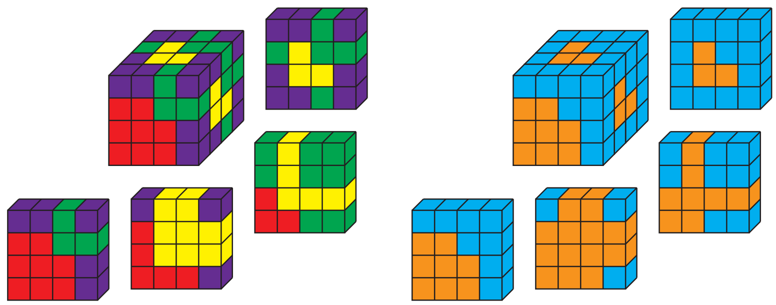

FIGURE 10. Another construction of the minimal genus Heegaard splitting of S^3

One might hope that $X_i = \langle \alpha^{n-i}\beta^i \rangle \cup \langle \alpha^{n+1-i}\beta^{i-1} \rangle$ determines a multisection. Indeed, in dimension 3, this is the Heegaard splitting shown in Figure 1. Yet, the construction does not work beyond dimension 3, as one can see by noting, e.g., that $X_0 \cap X_{k-1} = \langle \alpha\beta 1^{n-2} \rangle$ is always 2-dimensional.

Partition cubes into pairs of balls. Instead, at least in odd dimensions, one might attempt to generalize the following construction. See Figure 10.

First, somewhat like the approach taken throughout the paper, view $T^n = [0, 2k]^n/{\sim} = (\mathbb{R}/2k\mathbb{Z})^n$. Partition the $(2k)^n$ unit cubes with vertices in the lattice $(\mathbb{Z}/2k\mathbb{Z})^n$ so as to form $V_0, \dots, V_n$ subject to the following conditions:[7]

- If $\vec{x} \in V_0$, then $\vec{x} + (r, \dots, r) \in V_r$;
- The permutation action on the indices fixes each V_r;
- V_0 contains $[0, 1]^n$, is star-shaped about $(0, \dots, 0)$, and contains no points with any coordinate in $(n-1, n)$.

For $i = 0, \dots, k = \frac{n+1}{2}$, let $X_i = V_{2i} \cup V_{2i+1}$. This construction does in fact give a genus 3 Heegaard splitting of T^3. See Figure 10.

In higher dimensions, this construction is promising for many of the same reasons as the construction behind Theorem 7.11. This construction has at least one additional advantage, namely that each V_i is a ball. This makes it easy to check that each X_i is indeed an n-dimensional handlebody of genus n. Unfortunately, the complexity of this construction grows much more rapidly than the construction behind Theorem 7.11, making it hard to check the other details, even in dimension 5. Indeed, see Figure 11.

Question 8.6. Does this construction also give a (PL or smooth) trisection of T^5? Does it give a multisection of T^n for arbitrary $n = 2k - 1$?

[7]These conditions uniquely determine $V_0, \dots, V_n$.

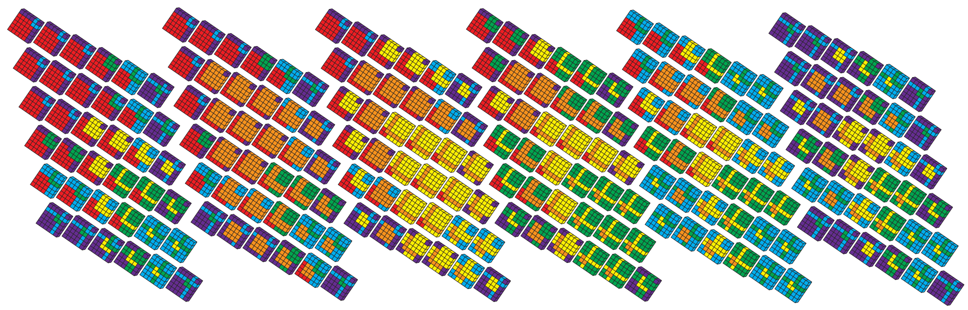

FIGURE 11. Decomposing $T^5 = [0,6]^5$ as $V_0 \cup \cdots \cup V_5$. Does $(V_0 \cup V_1, V_2 \cup V_3, V_4 \cup V_5)$ determine a trisection?

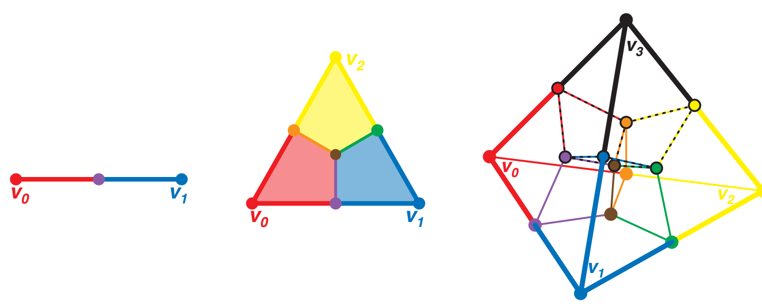

FIGURE 12. The decompositions $\Delta_{k-1} = \bigcup_{i\in\mathbb{Z}_k} Z_i$ of the 1-, 2-, and 3-simplices following Rubinstein–Tillmann.

Using the symmetric space T^n/S_n. Given a triangulation K of an n-manifold X, Rubinstein–Tillmann multisect X by mapping each n-simplex of K to the standard $(k-1)$-simplex

$$\Delta_{k-1} = [\vec{v}_0, \ldots, \vec{v}_{k-1}] = \left\{ \sum_{j\in\mathbb{Z}_k} a_j\vec{v}_j : \ 0 \leq a_j, \ \sum_{j\in\mathbb{Z}_k} a_j = 1 \right\},$$

decomposing $\Delta_{k-1} = \bigcup_{i\in\mathbb{Z}_k} Z_i$ where each

$$Z_i = \{\vec{x} \in \Delta_{k-1} : \ |\vec{x} - \vec{v}_i| \leq |\vec{x} - \vec{v}_j| \ \forall j \in \mathbb{Z}_k\}, \tag{24}$$

(see Figure 12), and pulling back. Their maps from the n-simplices of K to Δ_{k-1} are simplest to construct in odd dimension $n = 2k-1$. Namely:

- map the barycenter of each r-face to $\vec{v}_j \in \Delta_{k-1}$, $j = 2r, 2r+1$; and
- extend linearly in the first barycentric subdivision of K.

The even-dimensional case is similar, but with an extra move.

For example, the triangulation of S^3 with two 3-simplices gives a genus 3 Heegaard splitting, as shown in Figure 13.

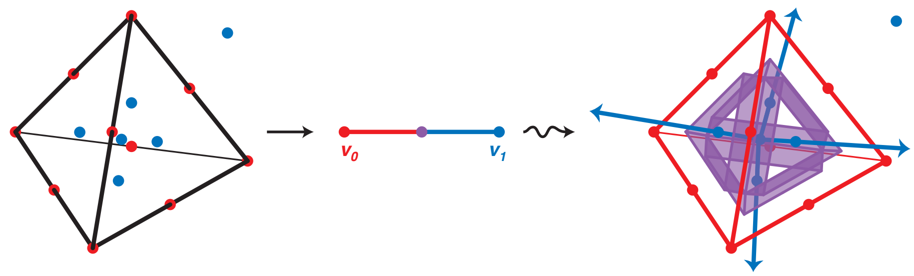

FIGURE 13. A genus 3 Heegaard splitting (right) of S^3, following Rubinstein–Tillmann's construction.

Following Rubinstein-Tillmann, one might try to construct a, say PL, multisection of T^n using the symmetric space T^n/S_n, which is homeomorphic to a disk-bundle over the circle; this bundle is twisted when n is even and untwisted when n is odd.

One can also view the symmetric space T^n/S_n as an n-simplex $\Delta_n = [\vec{v}_0, \ldots, \vec{v}_n]$ with certain faces identified. When $n = 2k-1$, one can also view Δ_n as an iterated join of intervals,

$$\Delta_n = [\vec{v}_0, \vec{v}_1] * \cdots * [\vec{v}_{2(k-1)}, \vec{v}_{2k-1}].$$

Hence, there is a map $\phi : \Delta_n \to \Delta_{k-1} = [\vec{v}_0, \ldots, \vec{v}_n]$ given by

$$\phi : \vec{x} = \sum_{i=0}^{k-1} w_i(c_i\vec{v}_{2i} + (1-c_i)\vec{v}_{2i+1}) \mapsto \sum_{i=0}^{k-1} w_i\vec{v}_i.$$

One can then decompose Δ_{k-1} symmetrically into k pieces using barycentric coordinates as in (24) and Figure 14. Following Rubinstein–Tillmann's construction of PL multisections from triangulations [7], one might attempt to construct a multisection of T^n by pulling back each X_i via ϕ, mapping forward by the quotient map $\Delta_n \to T^n/S_n$, and pulling back by the quotient map $T^n \to T^n/S_n$.

This construction works for T^3 and cuts any T^n into k 1-handlebodies of genus n. Unfortunately, the needed intersection properties fail, even for T^5, so the decomposition is not a multisection. Note that by writing

$$\Delta_n = [\vec{v}_0, \vec{v}_1] * \cdots * [\vec{v}_{2(k-1)}, \vec{v}_{2k-1}]$$

we made an asymmetric choice, and that the resulting decomposition is generally different than the one obtained by writing

$$\Delta_n = [\vec{v}_{\sigma(0)}, \vec{v}_{\sigma(1)}] * \cdots * [\vec{v}_{\sigma(2k-2)}, \vec{v}_{\sigma(2k-1)}]$$

for arbitrary $\sigma \in S_n$ and then following the same procedure.

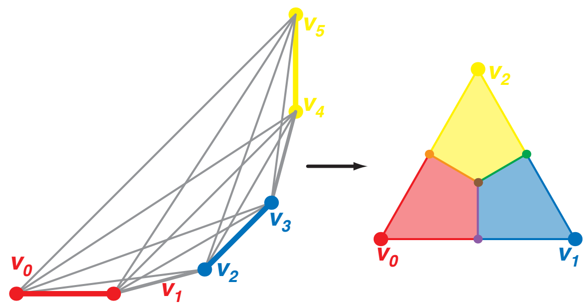

FIGURE 14. Try viewing T^n/S_n as $\Delta_n/\sim$ and Δ_n as an iterated join of k intervals. Then map $\Delta_n \to \Delta_{k-1}$, decompose Δ_{k-1}, and pull back. It fails, even for $n = 5$, shown.

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