Non-Hermitian $N-$state degeneracies: unitary realizations via antisymmetric anharmonicities

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Abstract

In a way inspired by the phenomenon of quantum phase transitions two mutually interrelated aspects of their theory are addressed. The first one is pragmatic: a non-numerical picture of the unitary processes of the loss of the observability is offered. A specific anharmonic-oscillator (AHO) class of the real N by N matrix toy-model Hamiltonians is recalled and adapted for the purpose. The second aspect is theoretical: the non-Hermitian degeneracy of an $N-$plet of the stable bound states (a.k.a. the Kato's exceptional-point - EPN - degeneracy) is realized in the language of unitary (called, sometimes, $\mathcal{PT}-$symmetric) quantum physics of closed systems. What is obtained is a classification pattern involving an auxiliary integer K (prescribing a "clusterization index" *alias* geometric multiplicity of the EPN limit) and the choice of one of the partitionings of the equidistant unperturbed spectrum into equidistant and centered unperturbed subspectra.

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1 Introduction

The practical, experiment-oriented descriptions of processes of quantum phase transitions mediated by non-Hermitian degeneracies [1] *alias* exceptional points (EP, [2]) are usually based, in the Feshbach's open-system spirit [3], on an *ad hoc* effective Hamiltonian H_{eff}. In these applications of the idea the systems in question are usually next-to-prohibitively complicated so that their analysis usually requires various forms of a simplification of mathematics. For this reason the criteria of the trial-and-error choice of H_{eff} are mostly purely pragmatic and not too deeply motivated. A more ambitious theoretical background, if any, is provided *a posteriori*. Typically, the necessary omission of all of the irrelevant degrees of freedom is made via perturbation-theory-based tests of their negligibility [4].

The dominance of the underlying open-system philosophy has been shattered by Bender and Boettcher [5]. They proposed that at least some of the processes of the quantum phase transitions (and, in particular, the loss-of-the-observability processes) might find a more natural description and explanation in the alternative, closed-system theoretical framework (see some of its details in reviews [6, 7, 8]). The Bender- and Boettcher-inspired change of the paradigm has two roots. One is that even in many closed quantum systems the unitary evolution can be controlled by the Hamiltonian (with real spectrum) which is non-Hermitian (i.e., admitting the EPs). Although such a conjecture might sound like a paradox, the Bender's and Boettcher's secret trick is that the latter Hamiltonian can be, via an appropriate amendment of the Hilbert space of states, hermitized, fitting suddenly all of the standard postulates of textbooks (see, e.g., the older review paper [9] for explanation).

The second root of the change of the paradigm immediately concerns the problem of the generic quantum phase transitions, opening one of its possible innovative treatments. The point is that in [5] the phase transitions were sampled by the spontaneous breakdown of parity-time (i.e., $\mathcal{PT}$) symmetry of elementary models. In such a specific class of exemplifications of the phenomenon it was not too difficult to relate the collapse of the system to the coincidence of parameter λ in Hamiltonian $H(\lambda)$ with its EP value $\lambda^{(EP)}$. The difference in behavior between the open and closed systems proves inessential. The closed, unitary systems do encounter, in the phase-transition limit $\lambda \to \lambda^{(EP)}$, a "catastrophe" realized as a merger followed by an abrupt complexification of at least some (or, in general, of an $N-$plet [10]) of the bound-state energies, i.e., by an abrupt loss of the observability of the system.

Within the framework of the new hermitizable-Hamiltonian paradigm, one of the most important ingredients in the upgraded theory would be an exhaustive classification of the possible new evolution scenarios. In particular, any type of such a classification seems needed for our understanding of the onset-of-the-phase-transition dynamical regime. Unfortunately, the absence of a similar classification in the literature reflects the existence of the numerous technical obstacles. *Pars pro toto* let us mention the scepticism (formulated, e.g., in [11]) concerning the very feasibility of analysis of alternative loss-of-the-observability mechanisms, or the fairly discouraging results of several numerical experiments simulating, in [12], the non-Hermitian $N-$level degeneracies at the values of N as small as $N = 6$.

In our present paper we are going to oppose the scepticism. Our return to optimism has several independent roots. The most important one is that we imagined that there exist enough toy-model matrices which could help forming certain building blocks of a sufficiently rich descriptive, qualitative theory. In this sense we will construct a family of non-numerical benchmark models exhibiting the explicit loss-of-the-observability behavior.

The presentation of our results will start, in section 2, by an identification of an arbitrary non-Hermitian degeneracy of N levels with a sudden loss of the diagonalizability of the Hamiltonian. The main goal of our paper may be formulated as a classification of these EPs of the $N-$th order (or, more explicitly, EPNs) which would take into account the so called geometric multiplicity K of the $N-$plet of levels in the EPN phase-transition dynamical extreme.

In section 3 a key concept of our considerations, viz., a certain specific class of anharmonic oscillators is introduced. In the model the conventional matrix form of the (initially dominant) harmonic oscillator Hamiltonian is diagonal (with equidistant elements) while the anharmonicity (i.e., initially, a small perturbation) is a real matrix which is antisymmetric. The reasons for such a choice are given. Subsequently, in section 4, our present systematic search for the models with anomalous non-Hermitian degeneracies (i.e., EPNs with $K \neq 1$) is initiated. We start there from a five-diagonal N by N matrix ansatz for $H(\lambda)$, and we illustrate the emerging picture of the EPN-related physics via a specific $N = 7$ model. On a methodical level we finally add and explain the last technical ingredient of our present analysis, viz., the idea of a systematic reduction of the EPN-supporting Schrödinger equations based on suitable multidiagonal versions of the anharmonic oscillator Hamiltonians into a composition of decoupled (or weakly coupled) sub-equations: this is done in sections 5 and 6 followed by the last section 7 containing a concise discussion and a brief summary.

2 Mechanisms of the loss of diagonalizability

The above-outlined danger of a non-Hermitian degeneracy and collapse concerns, in particular, quantum systems living in a small vicinity of their EP boundary where $\lambda = \lambda^{(EP)}$. The current state of our understanding of these systems has recently been reviewed in the immediate predecessor [12] of our present paper. First of all it has been emphasized there that near the EP-mediated loss-of-observability limit the quantum bound-state problem becomes, in a way well known to mathematicians [13, 14], numerically ill-conditioned. As a consequence, whenever $\lambda \approx \lambda^{(EP)}$, the spectra of the bound-state energies become extraordinarily sensitive even to the smallest random numerical errors [15].

Fortunately, the apparent paradox and conceptual problem immediately disappears when we recall any formulation of quantum mechanics which is applicable to the situation (let us recommend, e.g., the extensive but well-written review paper [7]). For a clarification of the situation it is necessary to distinguish between the computer-generated random numerical errors (which are defined as small in a conventional "working" Hilbert space $\mathcal{K}$) and the real-world perturbations which must be realized and specified as small in the physics-representing Hilbert space $\mathcal{H}$ [16].

In paper [12] the necessity of a reliable control of numerical errors led to the restriction of the scope of the paper to a few toy-model matrices with $N = 6$. Even within such a reduced project the practical control of the numerical precision remained nontrivial. In similar situations one must mostly rely upon the results obtained by non-numerical means.

2.1 Elementary two by two matrix example

One of the most efficient simplifications of the implementation of quantum theory is provided by the finite-matrix N by N truncation of the Hamiltonians. Their finite-dimensional *alias* separable-interaction representations prove methodically useful even at $N = 2$. One of the most persuasive demonstrations of such a remark may be found in the Kato's book on perturbation theory [2]. In it, the mathematically rigorous definition of the EPs (see p. 64 in *loc. cit.*) is immediately followed by an elementary illustrative example

$$T^{(2)}(\kappa) = \left[\begin{array}{cc} 1 & \kappa \\ \kappa & -1 \end{array} \right] \tag{1}$$

This is a traceless two-by-two matrix with eigenvalues $E_{\pm}(\kappa) = \pm\sqrt{1+\kappa^2}$ [cf. Example 1.1.(a) in *loc. cit.*] which could be perceived as the simplest possible truncated version of our present anharemonic-oscillator models (cf. Eq. (7) below). By Kato [2], in contrast, the model was recalled as yielding the complex conjugate pair of the EPs characterized by the degeneracy of the levels,

$$\lim_{\kappa \to \kappa^{(EP)}} E_+(\kappa) = \lim_{\kappa \to \kappa^{(EP)}} E_-(\kappa) = 0\,, \qquad \kappa^{(EP)} = \mathrm{i} \;\; \text{and/or} \;\; \kappa^{(EP)} = -\mathrm{i}\,. \tag{2}$$

In the context of physics of closed, unitary quantum systems (which will be of primary interest in the present paper) one has to add that the benchmark matrix (1) is Hermitian (i.e., from the point of view of conventional textbooks [17], acceptable as a toy-model Hamiltonian) if and only if κ is real. From such a perspective, the unitarity of the system becomes lost near all of its EPs. In model (1) this makes the EPs much less interesting for physicists of course.

2.2 Physics behind EPs

The inaccessibility of the EPs is not a model-dependent anomaly. Its validity extends to any generic Hermitian N by N matrix $T^{(N)}(\kappa)$ without *ad hoc* symmetries. In the past this made the concept of EPs, from the phenomenological point of view, useless. From the present point of view, the scepticism was undeserved, contradicting multiple subsequent discoveries of relevance of the non-Hermitian dynamics and of the related EPs. Multiple emerging applications range from the condensed-matter theory [18] and from the diverse branches of experimental physics [9, 19] up to the quantum analogues of the classical theory of catastrophes [20].

Any introductory list of the concrete samples of the existence and experimental or theoretical relevance of quantum phenomena connected with the presence and proximity of the Kato's EPs in Hamiltonians $H(\lambda)$ at $\lambda = \lambda^{(EP)}$ should probably start from the recollection of the EP-related spontaneous breakdown of the parity-time symmetry as mentioned by Bender and Milton [21] (in the context of quantum field theory), and by Bender with Boettcher [5] (in a mathematically much better understood context of quantum mechanics using non-selfadjoint operators with the real and discrete bound-state spectra [7, 8, 9]). At present an updated list of the samples of the use of EPs in quantum physics would certainly involve various quantum phase transitions [22] and catastrophes [20]. What they would all share is a connection, direct or indirect, with the formal limit

$$H(\lambda^{(EP)}) = \lim_{\lambda \to \lambda^{(EP)}} H(\lambda) \tag{3}$$

of the Hamiltonian.

In the subset of samples of the applicability of EPs of our present interest (dealing, exclusively, with the unitary, *closed* quantum systems) the most important mathematical feature of operator $H(\lambda)$ should be seen in the reality of its spectrum (in an "interval of unitarity" with $\lambda < \lambda^{(EP)}$), guaranteed up to the very EP limit (3). At the same time, the limit $H(\lambda^{(EP)})$ itself cannot be perceived as a valid quantum Hamiltonian anymore. The main reason is that this operator is, by definition, non-diagonalizable. This means that in an immediate vicinity of the EP singularity one should expect the emergence of multiple interesting phenomena.

A firm theoretical ground for their description should be sought in the above-mentioned amendment $\mathcal{H}$ of the conventional Hilbert space $\mathcal{K}$. Whenever the spectrum of any given $H(\lambda)$ is kept real, such a Hamiltonian can be non-Hermitian in $\mathcal{K}$ but still Hermitian in the other, *ad hoc* Hilbert space $\mathcal{H}$. As a consequence, the evolution generated by the Hamiltonian may always be interpreted as unitary (cf., e.g., the thorough review [7] for details).

The amended Hilbert space will be $\lambda-$dependent, $\mathcal{H} = \mathcal{H}(\lambda)$. Thus, Hamiltonian $H(\lambda)$ itself may be interpreted as Hermitian in a corridor of λs having a non-empty overlap with an arbitrarily small vicinity of $\lambda^{(EP)}$ [23]. The system is able to reach the instant of phase transition via unitary evolution.

2.3 Exceptional point degeneracies and their classification

Most of the quantitative analyses of the EP-influenced dynamical scenarios have been found technically difficult, especially when one tries to move beyond the most elementary models with the smallest matrix dimensions N. There exist several sources of difficulties. First, for a given non-Hermitian Hamiltonian $H^{(N)}(\lambda)$, even the proof of the reality of the spectrum ceases to be an elementary task. Even at $N = 3$, the availability of the energies in the exact and closed form of Cardano formulae is often marred by a typical occurrence of the mutually canceled complex components in the formula causing the emergence of numerical errors [12]. Second, once we manage to keep the numerical uncertainties of the energies under control, this control becomes more and more difficult when the parameter λ moves closer to its EP value. Third, even when we succeed in the precise localization of the position of the EP parameter, we have to keep in mind that the size of difficulties will increase with the growth of dimension N, especially when we study the EPs of maximal order N (abbreviated as EPNs).

Along all of these stages of development an efficient help can be provided by a variable-length computer arithmetics (see, e.g., [24]). Even then, one still has to complement the necessary condition of the EPN confluence of *all* of the N energy levels,

$$\lim_{\kappa \to \kappa^{(EPN)}} E_n(\kappa) = \eta\,, \qquad n = 1, 2, \ldots, N \tag{4}$$

by a more detailed characteristics of the structure of the parallel confluence of the wave functions. The general form of the latter confluence will depend on an integer (say, K) called the geometric multiplicity of the EPN degeneracy [2]. It will characterize the following $K-$centered clusterization of the eigenstates near the EPN singularity,

$$\lim_{\lambda \to \lambda^{(EPN)}} |\psi^{(N)}_{n_k}(\lambda)\rangle = |\chi^{(N)}_k(\lambda)\rangle\,, \qquad n_k \in S_k\,, \qquad k = 1, 2, \ldots, K\,. \tag{5}$$

In this formula the $N-$plet of integer subscripts $\{1, 2, \ldots, N\}$ counting the states gets partitioned into a $K-$plet of its disjoint subsets S_k formed by the N_k-plets of the separate indices. Nontriviality of the situation requires that $N_k \geq 2$ (indeed, the singlets can be ignored as belonging to an irrelevant, decoupled part of the Hilbert space). Finally, once the overall dimension N is fixed, the separate non-equivalent EPN scenarios may be numbered by the set of all possible partitions $P[N] = P_j[N]$ of

$$N = N_1 + N_2 + \ldots + N_K\,.$$

A short table of all realizations of these partitions may be found in [25, 26].

3 Antisymmetrically anharmonic oscillators

In the area of non-Hermitian quantum mechanics, a number of methodical challenges occurred, recently, in connection with the mathematically rigorous studies of the onset of instabilities due to perturbations [27]. In parallel, physicists newly reopened the questions of a systematic qualitative understanding of the EP-related quantum phase transitions [5, 22, 28]. Both of these tendencies in research are mutually interconnected of course.

3.1 Conventional anharmonic oscillators

In the most convenient harmonic-oscillator (HO) basis, the general anharmonic-oscillator (AHO) Hamiltonian

$$H(\lambda) = H^{(HO)} + \lambda\, V\,, \qquad \lambda \geq 0 \tag{6}$$

acquires, in the weak-coupling regime, the diagonally dominated matrix form

$$H(\lambda) = \left[\begin{array}{ccccc} 1+\lambda\, V_{0,0} & \lambda\, V_{0,1} & \lambda\, V_{0,2} & \ldots \\ \lambda\, V_{1,0} & 3+\lambda\, V_{1,1} & \lambda\, V_{1,2} & \ddots \\ \lambda\, V_{2,0} & \lambda\, V_{2,1} & 5+\lambda\, V_{2,2} & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{array} \right]\,, \qquad \lambda = {\rm small}\,. \tag{7}$$

The study of the AHO models was initially motivated by their methodical perturbation-analysis implications. The unperturbed Hamiltonian itself is represented, in the well known harmonic-oscillator basis, by the diagonal matrix with equidistant elements $H^{(HO)}_{0,0} = 1$, $H^{(HO)}_{1,1} = 3$, etc. The couplings λ remain, in this setting, small and positive, $\lambda \in (0, \lambda_{\max})$.

The most popular choice of the coordinate-dependent quartic anharmonicity $V \sim x^4$ converts the Hamiltonian into a particularly computation-friendly real pentadiagonal matrix. In [29], this encouraged Bender and Wu to give a detailed account of the positions of EPs . The motivation of this early study was still formal, based on the fact that one of the most important mathematical prerequisities of perturbation expansions, viz., a guarantee of convergence finds a clarification after an extension of the range of the admissible couplings to complex plane [2]. The value of the radius of convergence of the most common version of the Rayleigh-Schrödinger perturbation series can be then identified with the distance of the reference coupling λ_0 from the nearest EP singularity.

For the general and popular Hermitian Hamiltonians, *none* of its EP singularities $\lambda^{(EP)}$ can be real. Some of them may be made real via an analytic continuation of the model in λ. Naturally, what seems to be inadvertently lost is the self-adjointness of the Hamiltonian. Still, surprisingly enough, there exist certain specific models in which it is possible to recover the self-adjointness. One of the best proofs of the existence of such an option is even provided by the above-mentioned quartic anharmonic model with $V \sim x^4$. In it, the non-Hermiticity of $H(\lambda)$ can most easily be achieved via a counterintuitive choice of a negative $\lambda = -\kappa^2 < 0$. Indeed, after such a change of the sign, utterly unexpectedly, the spectrum $\{E_n(-\kappa^2)\}$ is found real again [5, 30, 31].

3.2 Hiddenly Hermitian Hamiltonians

In 1993, the occurrence of the latter surprise was related, in a marginal remark [30], to the $\mathcal{PT}-$symmetry of the non-Hermitian model. A few years later, Bender with coauthors [5, 6] converted the remark into a new paradigm. An innovated, more flexible formulation of quantum theory has been born admitting the existence of unitary quantum evolution generated by non-Hermitian Hamiltonians [7]. The status of EPs has thoroughly been changed. It evolved from the mere mathematical curiosity to the concept of a central phenomenological interest [8].

There exist at least two keys to the disentanglement of these developments. The main one makes use of the correspondence between the conventional Hermitian Hamiltonians and their upgraded versions called quasi-Hermitian [9], pseudo-Hermitian [7], crypto-Hermitian [32], or non-Hermitian but hermitizable [6]. Within this framework, the second key to a correct description of the EP-related quantum physics is more pragmatic, aimed at a simplification of the underlying mathematics. In this spirit, several authors suggested to circumvent some of the technical obstacles via an *ad hoc*, apparently redundant assumption like, say, a pseudo-bosonization of the operators of observables [33] (for a few reviews of some further possibilities see, e.g., [8, 34]).

The second key to the necessary amendment of the conventional model-building strategy is model-dependent, emphasizing the feasibility of the calculations. Its main tool lies in a maximal technical simplification of the operators. As a characteristic implementation of such an idea let us recall the Bender's and Boettcher's requirement [5] of having the Hamiltonians non-Hermitian but parity-time symmetric ($\mathcal{PT}-$symmetric). Indeed, in effect, the latter restriction made the $\mathcal{PT}-$symmetric theory popular even far beyond its initial closed-system applications [35, 36, 37].

Incidentally, the widespread belief in the consistency of the $\mathcal{PT}-$symmetric theory has recently been opposed [14, 38], reconfirming the older criticism of the theory by mathematicians [39]. Fortunately, a specific strategy circumventing the criticism can be found described in the physics-oriented review paper [9]. The recommendation lies in the exclusive use of the operators of observables which are bounded. Such a recommendation will be followed in our present paper.

From the point of view of many phenomenologically oriented users, the innovative potential of the amended theory may be best illustrated by the observation that the "conventional" choice (1) of a Hamiltonian can be complemented by a qualitatively different, manifestly non-Hermitian toy model

$$H^{(2)}(\lambda) = \left[\begin{array}{cc} -1 & \lambda \\ -\lambda & 1 \end{array} \right] . \tag{8}$$

Still, the easily obtained formula $E_\pm(\lambda) = \pm\sqrt{1-\lambda^2}$ which defines eigenvalues implies that they stay real and non-degenerate whenever $\lambda \in \mathcal{D} = (-1, 1)$. In this interval of parameters, matrix (8) becomes Hermitizable. It can, therefore, play the role of a closed-system quantum Hamiltonian. We can infer that one of the main qualitative innovations is that in model (8) the distance ϱ of the EPs from the real axis of parameters λ is zero (in the conventional model of Eq. (1) we had $\varrho = 1$). This means that one can expect the emergence of multiple innovative dynamical features of the system. In particular, the observable system can get *arbitrarily close* to its "unphysical" EP-mediated phase-transition extreme.

In the light of the off-diagonal antisymmetry of model (8), the matrix is, certainly, *maximally* non-Hermitian [40]. In this sense it can be treated as an inspiration of an AHO generalization

$$H^{(AHO)}_{\text{(antisymmetric)}}(\lambda) = \left[\begin{array}{cccc} 1 & \lambda\, V_{0,1}(\lambda) & \lambda\, V_{0,2}(\lambda) & \ldots \\ -\lambda\, V_{0,1}(\lambda) & 3 & \lambda\, V_{1,2}(\lambda) & \ddots \\ -\lambda\, V_{0,2}(\lambda) & -\lambda\, V_{1,2}(\lambda) & 5 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{array} \right] . \tag{9}$$

Naturally, the practical applicability of such an over-ambitious model seems very much suppressed by the purely numerical nature of the related predictions.

3.3 Tridiagonality constraint and the assumption of PT-symmetry

We saw that even some small and manifestly non-Hermitian matrices could play the role of a standard, non-numerically tractable toy-model Hamiltonian. In [40, 41] such an observation was generalized and extended to the whole family

$$H^{(N)}_{(\text{tridiagonal})}(\lambda) = \left[\begin{array}{ccccc} 1 & b_1(\lambda) & 0 & \ldots & 0 \\ -\,b_1(\lambda) & 3 & b_2(\lambda) & \ddots & \vdots \\ 0 & -b_2(\lambda) & 5 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & b_{N-1}(\lambda) \\ 0 & \ldots & 0 & -b_{N-1}(\lambda) & 2N-1 \end{array}\right] \tag{10}$$

of the *tridiagonal* real N by N matrix candidates for a non-Hermitian but hermitizable perturbed-harmonic-oscillator-type Hamiltonian.

In the first step of the construction an inessential shift $E \to E - N$ of the origin of the energy scale was used to transform the main diagonal written in the boxed-symbol form $\boxed{\text{1,3,5,...,2N-1}}$ into its centrally symmetric version $\boxed{\text{1-N,3-N,...,N-3,N-1}}$. Such a shift renders the matrix traceless. In the second step the (still excessively large) number of the variable matrix elements was halved by a decisive simplifying assumption of the symmetry of the matrix of perturbations with respect to the second diagonal. The latter simplification was based on the popularity of $\mathcal{PT}-$symmetry [35] in a specific version imposed upon the anharmonicity (see [41] for a more extensive complementary discussion). Its implementation yielded the final form of tridiagonal AHO Hamiltonians

$$H^{(N)}_{(\text{toy})}(\lambda) = \left[\begin{array}{ccccc} 1-N & b_1(\lambda) & 0 & \ldots & 0 \\ -\,b_1(\lambda) & 3-N & \ddots & \ddots & \vdots \\ 0 & -b_2(\lambda) & \ddots & b_2(\lambda) & 0 \\ \vdots & \ddots & \ddots & N-3 & b_1(\lambda) \\ 0 & \ldots & 0 & -b_1(\lambda) & N-1 \end{array}\right]. \tag{11}$$

The main mathematical merit of such a choice of the benchmark lies in a manifestly non-numerical form of the Hermitization (see the details in [42]). In our present paper, an even more important phenomenological merit of the choice is to be seen in the availability of the extreme, singular $\lambda = \lambda^{(EPN)}$ limits of the Hamiltonians in closed forms. Indeed, the sequence of the EP limits obtained, in [40], at all N is formed by elementary matrices

$$H^{(2)}_{(\text{toy})}(\lambda^{(EP)}) = \left[\begin{array}{cc} -1 & 1 \\ -\,1 & 1 \end{array}\right], \qquad H^{(3)}_{(\text{toy})}(\lambda^{(EP)}) = \left[\begin{array}{ccc} -2 & \sqrt{2} & 0 \\ -\,\sqrt{2} & 0 & \sqrt{2} \\ 0 & -\sqrt{2} & 2 \end{array}\right], \tag{12}$$

$$H^{(4)}_{(\text{toy})}(\lambda^{(EP)}) = \left[\begin{array}{cccc} -3 & \sqrt{3} & 0 & 0 \\ -\sqrt{3} & -1 & 2 & 0 \\ 0 & -2 & 1 & \sqrt{3} \\ 0 & 0 & -\sqrt{3} & 3 \end{array} \right], \qquad H^{(5)}_{(\text{toy})}(\lambda^{(EP)}) = \left[\begin{array}{ccccc} -4 & 2 & 0 & 0 & 0 \\ -2 & -2 & \sqrt{6} & 0 & 0 \\ 0 & -\sqrt{6} & 0 & \sqrt{6} & 0 \\ 0 & 0 & -\sqrt{6} & 2 & 2 \\ 0 & 0 & 0 & -2 & 4 \end{array} \right],$$

etc.

4 Pentadiagonal Hamiltonians

In a small vicinity of any one of the latter extremes the character of dynamics is interesting, determined by the closeness of the EPN-related "catastrophe" [20, 43]. Unfortunately, a more detailed analysis of the quantum-catastrophic scenario leads to the major disappointment: in Eq. (5), all of the EPN limits (12) lead, at any N, to *the same*, non-clustering, obstinate $K = 1$ degeneracies. For the *tridiagonal* $\lambda-$dependent matrices (11), in this manner, the menu of the eligible scenarios of the EPN-related quantum phased transition is too narrow. The mathematically user-friendly tridiagonality constraint proves phenomenologically over-restrictive.

4.1 Search for anomalous degeneracies

In contrast to the "standard" exceptional points with a minimal geometric multiplicity $K = 1$, their $K > 1$ alternatives may be called "anomalous". The latter term was conjectured, in [12], as reflecting the highly plausible one-to-one correspondence between the tridiagonality of the Hamiltonian and the $K = 1$ form of its EPN-boundary loss-of-the-observability limit. In this sense the most natural candidates for an "anomalous" EPN-related dynamical regime may be Hamiltonians which are pentadiagonal. Unfortunately, the corresponding general ansatz

$$H^{(N)}_{(\text{pentadiagonal})}(\lambda) = \left[\begin{array}{cccccc} 1-N & b_1(\lambda) & c_1(\lambda) & 0 & \ldots & 0 \\ -b_1(\lambda) & 3-N & b_2(\lambda) & \ddots & \ddots & \vdots \\ -c_1(\lambda) & -b_2(\lambda) & \ddots & \ddots & c_2(\lambda) & 0 \\ 0 & \ddots & \ddots & N-5 & b_2(\lambda) & c_1(\lambda) \\ \vdots & \ddots & -c_2(\lambda) & -b_2(\lambda) & N-3 & b_1(\lambda) \\ 0 & \ldots & 0 & -c_1(\lambda) & -b_1(\lambda) & N-1 \end{array} \right] \qquad (13)$$

can only be analyzed by numerical methods. For our present purposes, a further simplification is needed. Thus, a non-numerical tractability of matrix (13) can be achieved, e.g., by an additional assumption that $b_n(\lambda) = 0$ at all n. In order to see this clearly, the resulting, simplified Hamiltonian may be partitioned,

$$H^{(N)}_{(\text{pent.special})}(\lambda)=\left[\begin{array}{c|c|c|c|c|c} 1-N & 0 & c_1(\lambda) & 0 & \ldots & 0\\ \hline 0 & 3-N & 0 & \ddots & \ddots & \vdots\\ \hline -c_1(\lambda) & 0 & \ddots & \ddots & c_2(\lambda) & 0\\ \hline 0 & \ddots & \ddots & N-5 & 0 & c_1(\lambda)\\ \hline \vdots & \ddots & -c_2(\lambda) & 0 & N-3 & 0\\ \hline 0 & \ldots & 0 & -c_1(\lambda) & 0 & N-1 \end{array}\right]. \qquad (14)$$

This reveals that the matrix is equal to a direct sum of the two decoupled tridiagonal matrices

$$H^{(N)}_{(\text{component one})}(\lambda)=\left[\begin{array}{cccc} 1-N & c_1(\lambda) & 0 & \ldots\\ -c_1(\lambda) & 5-N & c_3(\lambda) & \ddots\\ 0 & -c_3(\lambda) & 9-N & \ddots\\ \vdots & \ddots & \ddots & \ddots \end{array}\right] \qquad (15)$$

and

$$H^{(N)}_{(\text{component two})}(\lambda)=\left[\begin{array}{cccc} 3-N & c_2(\lambda) & 0 & \ldots\\ -c_2(\lambda) & 7-N & c_4(\lambda) & \ddots\\ 0 & -c_4(\lambda) & 11-N & \ddots\\ \vdots & \ddots & \ddots & \ddots \end{array}\right]. \qquad (16)$$

In detail, every matrix in question is identified, uniquely, by its main diagonal. Thus, the matrix of Eq. (14) may be assigned the boxed symbol $\boxed{\text{1-N,3-N,...,N-1}}$, etc. Also the direct-sum decomposition of the latter matrix can be abbreviated as follows,

$$\boxed{\text{1-N,3-N,...,N-1}}=\boxed{\text{1-N,5-N,9-N, ...}}\oplus\boxed{\text{3-N,7-N,11-N,...}}.$$

The last elements of the summands are not displayed because they vary with the parity of N. After the explicit specification of the parity of N we arrive at the following two conclusions.

Lemma 1 *At the even matrix dimension $N = 2J$, the decomposition of the pentadiagonal sparse-matrix model (14) into its tridiagonal AHO components (15) and (16) only supports the two standard $K = 1$ EPJ limits (12), with the two different energies $\eta_\pm = \pm 1 \neq 0$ in Eq. (4).*

Proof. The main diagonal $\boxed{\text{1-N,3-N,...,N-1}}=\boxed{\text{1-2J,3-2J,...,2J-1}}$ of matrix (14) does not contain a central zero. This means that the central interval $(-1, 1)$ is "too short". Its two elements -1 and 1 will be distributed among both of the components (15) and (16). In the resulting direct sum

$$\boxed{\text{1-N,3-N,...,N-1}}=\boxed{\text{1-2J,5-2J, ...,2J-3}}\oplus\boxed{\text{3-2J,7-2J,...,2J-1}},$$

for this reason, both of the components will be centrally asymmetric. The search for an anomalous EP with $K = 2$ fails. Even after a successful J by J realization of the two separate EPJ limits using building blocks (12), the requirement (4) will offer two different, incompatible values $\eta_\pm = \pm 1$ of the eligible limiting EP energies. The direct-sum decomposition yields the two non-anomalous $K = 1$ EPs of the same small order $J = N/2$. $\square$

Lemma 2 *At odd $N = 2J+1$, both of the tridiagonal matrices (15) and (16) admit the respective realizations (12) of their EP limits. The related energies coincide so that the direct sum (14) admits the anomalous EPN limit with geometric multiplicity two.*

Proof. We have

$$\boxed{\text{1-N,3-N,}\ldots\text{,N-1}} = \boxed{\text{-2J,2-2J,}\ldots\text{,2J}} = \boxed{\text{-2J,4-2J, }\ldots\text{ ,2J}} \oplus \boxed{\text{2-2J,6-2J,}\ldots\text{,2J-2}} \tag{17}$$

so that out of the central triplet of integers $(-2, 0, 2)$, the doublet $(-2, 2)$ remains long enough to be a component of one of the sub-boxes. For this reason, their respective dimensions $J + 1$ and J are now different. This is compensated by the central symmetry of the summands which implies the coincidence of the separate EP-related energies, $\eta_\pm = \eta = 0$. In the direct sum (14) of these matrices the respective EP(J+1) and EPJ limits degenerate to a single, shared, manifestly anomalous EPN limit. The "$K-$tuple clusterization" phenomenon (5) takes place at $K = 2$. $\square$

Our two Lemmas may be read as an empirical support of the "no-go" conjecture of Ref. [12] which attributed the long-lasting failures of the trial-and-error search for the anomalous, $K > 1$ EPN degeneracies to the underlying matrix-tridiagonality restriction. Our Lemmas confirm that the mere weaker, pentadiagonality constraint or its further band-matrix generalizations need not necessarily help too much. At the same time, the model has several methodical merits. First, it shows that any attempted classification of the Hamiltonians considered in their EPN extreme must involve the anomalous cases with $K > 1$. Second, it offers an independent support for an apparently arbitrary but unexpectedly fortunate choice of the specific, AHO-based Hamiltonian-operator (sub)matrices. Third, the pentadiagonality facilitates the decoupling of the system into two subsystems, an idea which inspires, immediately, a generalization to the $K-$component partitionings with $K > 2$. Fourth, one should emphasize that the study of the pentadiagonal models may be perceived as paving the way towards its full-matrix extensions, especially at $\lambda \approx \lambda^{(EP)}$ where there emerges a number of technical problems ranging from the numerical ill-conditioning difficulties [24] up to the complicated nature of the perturbation-approximation tractability of the stable and unitary systems when occurring near the EPN singularities [25].

Certain unusual physical phenomena might emerge involving not only the collapse of the system (via a "fall into the EPN singularity") but also, in opposite direction, the processes of an escape from the degeneracy resembling the Big Bang in cosmology [44]. In both of these directions there are illustrative examples available in the literature. *Pars pro toto* let us mention Ref. [45] where the authors showed that in a complex Bose-Hubbard model the many-bosonic system can either escape from, or fall in, the Bose-Einstein condensation singularity.

4.2 Corridors of unitarity

In an overall model-building strategy one should feel aware of the deep difference between the effective-operator studies of resonances in open systems (working with complex energies and, hence, not considered here) and the closed system models in which the spectrum of energies is assumed real. In the latter cases a corridor of a possible unitary-evolution could exist and connect the weakly-anharmonic (WA) and the strong-coupling (SC) dynamical regimes.

The partitioning of our present specific quantum Hamiltonians (14) may be expected to simplify the construction of the corridors. First of all, its perturbation specification in the WA dynamical regime (in which all of the off-diagonal elements $c_n(\lambda)$ have to remain small) would be routine. In the opposite extreme of the strongly non-Hermitian SC domain, both of the tridiagonal-matrix components (15) and (16) of the Hamiltonian become *separately* tractable as small perturbations of their respective SC EP limits (12). Thus, as long as the matrix dimensions N remain finite, an approximate construction of the bound states becomes feasible in both of the WA or SC perturbation regimes. The available perturbation approximations will determine, roughly at least, the families of the admissible matrix elements $c_n(\lambda)$ for which the spectrum remains real and discrete, i.e., for which the evolution of the quantum system in question remains unitary and, hence, stable.

In the gap between the two comparatively small WA- and SC-applicability subintervals of $\lambda \leq \lambda_{\max}^{(WA)} \ll \lambda^{(EP)}$ and $\lambda \geq \lambda_{\min}^{(SC)} \lessapprox \lambda^{(EP)}$ it will be more difficult to guarantee the reality of the spectrum. Fortunately, what we still know is that for $\lambda \in (\lambda_{\max}^{(WA)}, \lambda_{\min}^{(SC)})$ there always exists a special unitarity-compatible parametrization of matrix elements applicable to both of the *independent* tridiagonal-matrix sub-Hamiltonians (15) and (16) and constructed in [40, 41]. This implies that in the light of Lemma 2 a unitarity corridor will exist, in the anomalous $K = 2$ case based on the direct sum of these components, for all of our pentadiagonal $\mathcal{PT}-$symmeric anharmonic-oscillator Hamiltonians (14) with odd N. In particular, in the $N = 7$ exemplification of our antisymmetrically anharmonic pentadiagonal Hamiltonian (14)

$$H^{(7)}(\lambda) = \left[\begin{array}{ccccccc} 1 & 0 & \sqrt{3}g & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & \sqrt{2}g & 0 & 0 & 0 \\ -\sqrt{3}g & 0 & 5 & 0 & 2g & 0 & 0 \\ 0 & -\sqrt{2}g & 0 & 7 & 0 & \sqrt{2}g & 0 \\ 0 & 0 & -2g & 0 & 9 & 0 & \sqrt{3}g \\ 0 & 0 & 0 & -\sqrt{2}g & 0 & 11 & 0 \\ 0 & 0 & 0 & 0 & -\sqrt{3}g & 0 & 13 \end{array} \right] \tag{18}$$

(where we returned, for the sake of clarity of physics, to the initial, unshifted harmonic-oscillator energy scale), the $\lambda-$dependence of the off-diagonal matrix elements is controlled by a single function $g = g(\lambda)$. Such a simplification implies that the related Schrödinger bound-state problem

$$H^{[7]}(g)\,|\psi_n(g)\rangle = E_n(g)\,|\psi_n(g)\rangle \tag{19}$$

is solvable exactly,

$$E_0(g) = 7\,, \qquad E_{\pm 1}(g) = 7 \pm \sqrt{4-g^2}$$
$$E_{\pm 2}(g) = 7 \pm 2\,\sqrt{4-g^2}\,, \qquad E_{\pm 3}(g) = 7 \pm 3\,\sqrt{4-g^2}\,.$$

The model exemplifies the system in which there exists a corridor of unitarity which connects the self-adjoint harmonic-oscillator dynamics realized at $g = 0$ with the EP7 extreme where $g = 2$. Thus, whenever the growth of $g(\lambda)$ does not deviate too much from the linear function, the bound-state energies remain real and well separated along a path connecting the WA and SC ends of the open interval of values $g = g(\lambda) \in (0, 2)$.

4.3 SC dynamical extreme

In the WA regime with $\lambda \leq \lambda_{\max}^{(WA)}$ the optional auxiliary (and, say, monotonously increasing) function $g(\lambda)$ is to be kept small. Then, the anharmonicity will remain easily tractable by the standard Rayleigh-Schrödinger perturbation methods. Not only at our $N = 7$ but also in all of the models of a pentadiagonal direct-sum type with arbitrary odd N. Near the opposite SC boundary where $g \lessapprox 2$ the EP7 degeneracy (4) is reached. We may introduce a new small parameter $\kappa = \kappa(\lambda) \in (0, 1)$, redefine $g = \widetilde{g}(\kappa) = 2\,(1-\kappa^2)$ and consider the related ("tilded") modification of our spectral problem (19) with, naturally, the same exact eigenvalues written now in a reparameterized, SC-friendly form

$$\widetilde{E}_0(\kappa) = 7\,, \qquad \widetilde{E}_{\pm 1}(\kappa) = 7 \pm 2\,\sqrt{-\kappa^4 + 2\,\kappa^2} \sim 7 \pm 2\,\sqrt{2}\kappa + \mathcal{O}(\kappa^3)\,,$$
$$\widetilde{E}_{\pm 2}(\kappa) = 7 \pm 4\,\sqrt{-\kappa^4 + 2\,\kappa^2}\,, \qquad \widetilde{E}_{\pm 3}(\kappa) = 7 \pm 6\,\sqrt{-\kappa^4 + 2\,\kappa^2}\,.$$

With $\kappa \in (0, 1)$ these values remain all real.

In the SC EP7 limit $\kappa \to 0$ the spectrum becomes degenerate and the Hamiltonian ceases to be diagonalizable. Even after the determination of the EP value $\eta = 7$ of the degenerate energy, a modified eigenvalue problem can be considered in this limit,

$$H^{(7)}(\lambda^{(EP7)})\,Q^{[7]} = Q^{[7]}\,J^{[7]}(\eta)\,. \tag{20}$$

In the light of Lemma 2 we may immediately pick up the so called canonical representation

$$J^{[7]}(\eta) = \left[\begin{array}{cccc|ccc} \eta & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \eta & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \eta & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \eta & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \eta & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \eta & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \eta \end{array}\right] \tag{21}$$

of our limiting toy-model Hamiltonian in (20). Although such a choice is not unique (see [46] for some related mathematical comments and technicalities), its $K = 2$ direct-sum form (21) composed of the two Jordan-block matrices is by far its most popular version.

The partitioning in Eq. (21) re-emphasizes that there is no mutual coupling between the even and odd indices in our toy-model Hamiltonian (18). Also in the EP7 limit, matrix $H^{(7)}(\lambda^{(EP7)})$ may be interpreted as a direct sum of the two smaller components,

$$H^{(7)}(\lambda^{(EP7)}) = H^{[odd]} \bigoplus H^{[even]}$$

where

$$H^{[odd]} = \left[\begin{array}{cccc} 1 & 2\sqrt{3} & 0 & 0 \\ -2\sqrt{3} & 5 & 4 & 0 \\ 0 & -4 & 9 & 2\sqrt{3} \\ 0 & 0 & -2\sqrt{3} & 13 \end{array} \right], \qquad H^{[even]} = \left[\begin{array}{ccc} 3 & 2\sqrt{2} & 0 \\ -2\sqrt{2} & 7 & 2\sqrt{2} \\ 0 & -2\sqrt{2} & 11 \end{array} \right].$$

Another specific illustrative merit of our $N = 7$ model lies the availability of the explicit transition-matrix solution

$$Q^{[7]} = \left[\begin{array}{ccccccc} -48 & 24 & -6 & 1 & 0 & 0 & 0 \\ 0 & 8 & -4 & 1 & 8 & -4 & 1 \\ -48\sqrt{3} & 16\sqrt{3} & -2\sqrt{3} & 0 & 0 & 0 & 0 \\ 0 & 8\sqrt{2} & -2\sqrt{2} & 0 & 8\sqrt{2} & -2\sqrt{2} & 0 \\ -48\sqrt{3} & 8\sqrt{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 & 8 & 0 & 0 \\ -48 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

of Eq. (19). Indeed, in the constructions of the SC perturbation series the transition matrices can be treated as an optimal substitute for the non-existent complete sets of unperturbed eigenvectors. In this sense the role of an unperturbed SC basis is played by the transition matrices themselves (see, e.g., [15]).

5 Maximally anharmonic full-matrix models

The simulation of quantum dynamics near EPNs using multidiagonal matrix Hamiltonians was proposed in paper [12]. The scope of the study was restricted there, due to the apparently purely numerical nature of the problem, to the smallest matrix dimensions $N \leq 6$. The decision was a bit unfortunate because in the light of subsection 4.2 and Lemmas 1 and 2 above, the next option with $N = 7$ would have been perceivably more instructive. Still, the message delivered by Ref. [12] remained significant, showing that the search for anomalous EPN singularities with optional geometric multiplicities K should be based on a systematic study of non-tridiagonal, multi-diagonal matrix models.

On such a background we are prepared to make the next step towards the construction of a complete list of the first few AHO-based phase-transition scenarios. Our considerations will start from the entirely general real N by N matrix ansatz

$$H^{(N)}_{(\text{full})}(\lambda) = \left[\begin{array}{cccccc} 1-N & b_1(\lambda) & c_1(\lambda) & d_1(\lambda) & \ldots & \omega_1(\lambda) \\ -\,b_1(\lambda) & 3-N & b_2(\lambda) & c_2(\lambda) & \ddots & \vdots \\ -\,c_1(\lambda) & \ddots & \ddots & \ddots & \ddots & d_1(\lambda) \\ -\,d_1(\lambda) & \ddots & -b_3(\lambda) & N-5 & b_2(\lambda) & c_1(\lambda) \\ \vdots & \ddots & -c_2(\lambda) & -b_2(\lambda) & N-3 & b_1(\lambda) \\ -\,\omega_1(\lambda) & \ldots & -d_1(\lambda) & -c_1(\lambda) & -b_1(\lambda) & N-1 \end{array} \right] \qquad (22)$$

carrying, in the diagonally dominated cases at least, the interpretation of an acceptable physical Hamiltonian. In the language of reviews [6, 7, 9, 47] such a Hamiltonian generates, in the corresponding quasi-Hermitian Schrödinger picture, the standard unitary evolution of a closed and stable quantum system with the states living in an $N-$dimensional physical Hilbert space $\mathcal{H}^{(N)}$.

5.1 Classification of EPNs using generalized boxed symbols

What lies in the very center of our attention are the EPN-degeneracy requirements (4) and (5) in application to the general $\mathcal{PT}-$symmeric anharmonic-oscillator Hamiltonian (22). We will assume that the real and antisymmetric-matrix anharmonicity (i.e., in a way, a maximally non-Hermitian anharmonicity) vanishes in the unperturbed-harmonic-oscillator limit of $\lambda \to \lambda^{(HO)} = 0$.

After the model leaves the weakly anharmonic dynamical regime controlled by the diagonally dominated Hamiltonian, one has to start to fine-tune the $\lambda-$dependence of all of the off-diagonal matrix elements in order to guarantee the survival of the reality and non-degeneracy of the bound-state energy spectrum, i.e., of the stability and unitarity of the evolution in a "physical" interval of $\lambda \in (0, \lambda^{(\max)})$. This opens the theoretically as well as phenomenologically highly attractive possibility of the existence of such a set of the off-diagonal matrix elements (i.e., of the *ad hoc* $\lambda-$dependent anharmonicities) that the ultimate loss of the observability of the system (i.e., the loss of the reality and non-degeneracy of the energy spectrum at $\lambda^{(\max)}$) would have a very specific complete-degeneracy EPN form as prescribed by Eq. (4) above. This would mean that the system can reach such a strong-perturbation regime with $\lambda \approx \lambda^{(\max)} = \lambda^{EP)}$ where the system reaches its EPN degeneracy as characterized by Eq. (3).

Our project of the study of such a possibility was inspired by the recent paper [12] in which it has been emphasized (and, via a few *ad hoc* $N = 6$ examples, demonstrated) that the complete EPN degeneracy of the spectrum may be "anomalous", accompanied by the mere $K-$clustered, geometric multiplicity reflecting degeneracy of the eigenstates as described by Eq. (5). We are going to complement Ref. [12] by extending its purely numerical $K = 2$ and $K = 3$ samples of the EP6-related Hamiltonians to a universal non-numerical construction of the EPN-related models characterized by the arbitrary integer matrix dimensions $N \geq 2$ and by the arbitrary integer geometric multiplicites $K \geq 1$.

The idea of our constructions is threefold. Firstly, in a way inspired by the specific $K = 2$ pentadiagonal-matrix constructive results of Lemma 2 above, the full-matrix Hamiltonian (22) will again be characterized, without any danger of confusion, by the slightly generalized version of its identification using the description of its main diagonal via the same boxed left-right symmetric symbol

$$\mathcal{S}(N,1) = \boxed{\text{1-N,3-N,}\ldots\text{,N-3,N-1}} \tag{23}$$

(see Definition 4 in Appendix A below). Secondly, in the same spirit, we will search for a multiterm (i.e., more precisely, $K-$term) generalization of the direct-sum $K = 2$ expansion (17). Thirdly, we will emphasize that or present physics-related, AHO-based requirement of the non-numerical tractability of our toy-model EP limits is in a one-to-one correspondence with the mathematics-related requirement that all of the components of our innovative and systematic $K-$term direct-sum expansions must remain centrally symmetric.

5.2 Enhancement of clusterization $K \longrightarrow K+1$

Some of the combinatorial aspects of the direct-sum expansions are clarified in Appendix A. In a more pragmatic spirit let us return to the concept of the partitioning which proved to be useful in the pentadiagonal-matrix case above. On a way toward its $K > 2$ generalizations let us first concentrate, for introduction, on a special form of the most elementary partitioning-motivated separation of the outer rows and columns followed by their reduction to the mere two non-vanishing elements. This yields the matrix

$$H^{(N)}_{(\text{spec.partit.})}(\lambda) = \left[\begin{array}{c|ccccc|c} 1-N & 0 & 0 & \ldots & 0 & \omega_1(\lambda) \\ \hline 0 & 3-N & b_2(\lambda) & \ldots & z_2(\lambda) & 0 \\ 0 & -b_2(\lambda) & \ddots & \ddots & \vdots & 0 \\ \vdots & \vdots & \ddots & N-5 & b_2(\lambda) & \vdots \\ 0 & -z_2(\lambda) & \ldots & -b_2(\lambda) & N-3 & 0 \\ \hline -\omega_1(\lambda) & 0 & 0 & \ldots & 0 & N-1 \end{array} \right] \tag{24}$$

which can be perceived as a direct sum of the two decoupled and fully independent smaller matrices,

$$H^{(N)}_{(\text{spec.partit.})}(\lambda) = \left[(N-1) \times H^{(2)}_{(\text{toy})}(\lambda) \right] \oplus H^{(N-2)}_{(\text{full})}(\lambda)\,.$$

Such a decomposition preserves the possible AHO-related nature of both of the components. In our present abbreviated notation of Appendix A this means that the centrally symmetric main diagonal of the left-hand-side Hamiltonian (assigned the appropriate boxed symbol) can be reinterpreted as the following composition of the two other AHO-representing symbols,

$$\boxed{\text{1-N,3-N,}\ldots\text{,N-1}} = \boxed{\text{1-N,N-1}} \oplus \boxed{\text{3-N,5-N,}\ldots\text{,N-3}}\,.$$

Immediately, the trick leading to the latter direct-sum decomposition can be generalized to any centrally symmetric partitioning. At a few illustative matrix dimensions N the complete lists of these decompositions may be found listed, using a slightly simplified notation, in Appendix A.

6 Maximally anharmonic sparse-matrix models

Partitioning of the AHO-representing centrally symmetric main diagonal $\mathcal{S}(N,1)$ of Eq. (23) into $K-$plets of its shorter, centrally symmetric equidistant subsets

$$\mathcal{S}(M,L) = \boxed{\text{(1-M) L,(3-M) L,}\ldots\text{,(M-3) L,(M-1) L}} \tag{25}$$

forms our main model-building principle. In its spirit, every AHO-based Hamiltonian is to be represented as a direct sum of its AHO-based sub-Hamiltonian building blocks. At the first few dimensions N, the systematic application of this recipe is to be illustrated in what follows.

6.1 The choice of $N = 2$ and $N = 3$: no anomalous degeneracies

For our present AHO class of $\lambda-$dependent Hamiltonians (22) there exists strictly one, unique EP2 limit satisfying our restrictions at $N = 2$, namely, the matrix $H^{(2)}_{(\text{toy})}(\lambda^{(EP)})$ as displayed in Eq. (12). In the abbreviated notation using the centrally symmetric boxed symbols such a matrix is characterized by $\mathcal{S}(2,1) = \boxed{\text{-1,1}}$. The number $a(N)$ of eligible scenarios is one, $a(2) = 1$.

Similarly, at $N = 3$ we have $a(3) = 1$ and the unique limit $H^{(3)}_{(\text{toy})}(\lambda^{(EP3)})$ represented by the boxed $K = 1$ symbol $\mathcal{S}(3,1) = \boxed{\text{-2,0,2}}$ and by the matrix displayed in Eq. (12).

6.2 The simplest anomalous case with $N = 4$ and $K = 2$

Besides the trivial $K = 1$ option with symbol $\boxed{\text{-3,-1,1,3}}$, there exists strictly one other centrally symmetric possibility of decomposition

$$\boxed{\text{-3,-1,1,3}} = \boxed{\text{-1,1}} \oplus \boxed{\text{-3,3}}\,, \qquad K = 2\,.$$

In the limit $\lambda \to \lambda^{(EP4)}$ this direct sum represents the seven-diagonal but very sparse AHO matrix

$$H^{(4)}_{(K=2)}(\lambda^{(EP4)}) = \left[\begin{array}{cccc} -3 & 0 & 0 & 3 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -3 & 0 & 0 & 3 \end{array}\right]. \tag{26}$$

In the unitarity domain where $\lambda \neq \lambda^{(EP4)}$ the number of the eligible dynamical scenarios is two, $a(4) = 2$.

6.3 Two $K = 2$ options at $N = 5$

At $N = 5$ the number of scenarios is three, $a(5) = 3$. Indeed, besides the trivial case, we have the two $K = 2$ decompositions $\boxed{\text{-4,-2,0,2,4}} = \boxed{\text{-2,0,2}} \oplus \boxed{\text{-4,4}}$ and $\boxed{\text{-4,-2,0,2,4}} = \boxed{\text{-4,0,4}} \oplus \boxed{\text{-2,2}}$, representing the two respective EP5 limiting matrices, viz., the nine-diagonal

$$H^{(5)}_{(K=2,a)}(\lambda^{(EP5)}) = \left[\begin{array}{ccccc} -4 & 0 & 0 & 0 & 4 \\ 0 & -2 & \sqrt{2} & 0 & 0 \\ 0 & -\sqrt{2} & 0 & \sqrt{2} & 0 \\ 0 & 0 & -\sqrt{2} & 2 & 0 \\ -4 & 0 & 0 & 0 & 4 \end{array}\right]$$

and the pentadiagonal

$$H^{(5)}_{(K=2,b)}(\lambda^{(EP5)}) = \left[\begin{array}{ccccc} -4 & 0 & 2\sqrt{2} & 0 & 0 \\ 0 & -2 & 0 & 2 & 0 \\ -2\sqrt{2} & 0 & 0 & 0 & 2\sqrt{2} \\ 0 & -2 & 0 & 2 & 0 \\ 0 & 0 & -2\sqrt{2} & 0 & 4 \end{array}\right].$$

The latter matrix fits in the classification pattern as provided by Lemma 2 above.

6.4 The first occurrence of $K = 3$ at $N = 6$

Besides the trivial, non-degenerate EP6 limit with $K = 1$ we have to consider its anomalous descendants, viz., the unique $K = 2$ decomposition $\boxed{\text{-5,-3,-1,1,3,5}} = \boxed{\text{-3,-1,1,3}} \oplus \boxed{\text{-5,5}}$ and the unique $K = 3$ decomposition $\boxed{\text{-5,-3,-1,1,3,5}} = \boxed{\text{-1,1}} \oplus \boxed{\text{-3,3}} \oplus \boxed{\text{-5,5}}$. In both of these cases the necessary direct-sum components of $H^{(6)}_{(K=2,K=3)}(\lambda^{(EP6)})$ may be found displayed in Eq. (12). In the latter case, for example, we get

$$H^{(6)}_{(K=3)}(\lambda^{(EP6)}) = \left[\begin{array}{cccccc} -5 & 0 & 0 & 0 & 0 & 5 \\ 0 & -3 & 0 & 0 & 3 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & -3 & 0 & 0 & 3 & 0 \\ -5 & 0 & 0 & 0 & 0 & 5 \end{array}\right].$$

Summarizing, the number of scenarios is $a(6) = 3$. Incidentally, the role and consequences of small perturbations of the latter matrix have numerically been analyzed in [12].

6.5 Paradox of decrease of $a(N)$ between $N = 7$ and $N = 8$

At $N = 7$ the number of the eligible EP7 scenarios is $a(7) = 6$ because the usual trivial $K = 1$ option can be accompanied by the following quintuplet of anomalous EP7 direct sums,

$$\boxed{\text{-6,-4,-2,0,2,4,6}} = \boxed{\text{-4,-2,0,2,4}} \oplus \boxed{\text{-6,6}}\,, \qquad K = 2\,,$$

$$\boxed{\text{-6,-4,-2,0,2,4,6}} = \boxed{\text{-2,0,2}} \oplus \boxed{\text{-4,4}} \oplus \boxed{\text{-6,6}}\,, \qquad K = 3\,,$$

$$\boxed{\text{-6,-4,-2,0,2,4,6}} = \boxed{\text{-4,0,4}} \oplus \boxed{\text{-2,2}} \oplus \boxed{\text{-6,6}}\,, \qquad K = 3$$

$$\boxed{\text{-6,-4,-2,0,2,4,6}} = \boxed{\text{-4,0,4}} \oplus \boxed{\text{-6,-2,2,6}}\,, \qquad K = 2\,,$$

$$\boxed{\text{-6,-4,-2,0,2,4,6}} = \boxed{\text{-6,0,6}} \oplus \boxed{\text{-2,2}} \oplus \boxed{\text{-4,4}}\,, \qquad K = 3\,.$$

In contrast, at $N = 8$ we have $a(8) = 4$, i.e., only the triplet of the anomalous, $K > 1$ direct sums becomes available, viz.,

$$\boxed{\text{-7,-5,-3,-1,1,3,5,7}} = \boxed{\text{-5,-3,-1,1,3,5}} \oplus \boxed{\text{-7,7}}\,, \qquad K = 2\,,$$

$$\boxed{\text{-7,-5,-3,-1,1,3,5,7}} = \boxed{\text{-3,-1,1,3}} \oplus \boxed{\text{-5,5}} \oplus \boxed{\text{-7,7}}\,, \qquad K = 3\,,$$

and, finally, the first four-term direct-sum decomposition

$$\boxed{\text{-7,-5,-3,-1,1,3,5,7}} = \boxed{\text{-1,1}} \oplus \boxed{\text{-3,3}} \oplus \boxed{\text{-5,5}}\,, \oplus \boxed{\text{-7,7}}\,, \qquad K = 4$$

representing a fifteen-diagonal but very sparse matrix $H^{(8)}_{(K=4)}(\lambda^{(EP8)})$.

7 Discussion

The concept of the exceptional-point value $\lambda^{(EP)}$ of a real parameter $\lambda \in \mathbb{R}$ in a linear operator $H(\lambda)$ proved, originally, useful just in mathematics [2, 8]. Physics behind the EPs remained obscure. The situation has changed after several authors discovered that the concept admits applicability in multiple branches of quantum as well as non-quantum physics [6, 36, 37]. *Pars pro toto* let us mention that in the subdomain of quantum physics the values of $\lambda^{(EP)}$ acquired the status of instants of an experimentally realizable quantum phase transition [5, 9, 22]. The boundary-of-stability role played by the values of $\lambda^{(EP)}$ attracted, therefore, attention of experimentalists [19, 48] as well as of theoreticians [14, 49].

7.1 Unitary vs. non-unitary quantum systems

Whenever one keeps the evolution unitary, the values of $\lambda^{(EP)}$ mark the points of the loss of the observability of the system [20, 50]. In this sense, our present hierarchy of specific AHO models may be treated as certain exactly solvable quantum analogue of the Thom's typology of classical catastrophes [51], with potential applicability to closed as well as open systems.

In our present paper we were exclusively interested in the former type of applications. We showed that the dynamics of any unitary quantum system (i.e., typically, its stability with respect to small perturbations) is particularly strongly influenced by the EPs. We should only add that an extreme care must be paid to the Stone theorem [52] requiring the Hermiticity of $H(\lambda)$ in the related physical Hilbert space $\mathcal{H}$. This means that a hermitization of the Hamiltonian is needed [9]. Such a process usually involves a reconstruction of an appropriate amended inner product in the conventional but manifestly unphysical Hilbert space $\mathcal{K}$. Interested readers may find a sample of such a reconstruction of $\mathcal{H}$, say, in [53].

In the realistic models one often encounters a paradox that the construction of the correct, amended inner product may happen to be prohibitively complicated, i.e., from the pragmatic point of view, inaccessible. This is the reason why people often postpone the problem and use, temporarily, a simplified inner product. For our present, user-friendly, matrix-represented AHO Hamiltonians of closed systems with $N < \infty$ such a purely technical obstacle does not occur. The reconstruction of $\mathcal{H}$ (left to the readers) would be a routine application of linear algebra. Reclassifying the real AHO matrices which are non-Hermitian in our auxiliary space $\mathcal{K} = \mathbb{R}^N$ (that's why we write $H \neq H^\dagger$) into operators which are, by construction, Hermitian in $\mathcal{H}$ (in [47], for example, we wrote $H = H^\ddagger$).

Let us add that in the more common studies of effective non-Hermitian Hamiltonians $H_{\text{eff}}(\lambda)$ describing the manifestly non-unitary evolution of the open quantum systems the role of the physical space is transferred back from the "unfriendly" $\mathcal{H}$ to the "original" $\mathcal{K}$. This means that the difficult construction of a nontrivial inner product is not needed. Thus, it is not too surprising that in the framework of the quantum theory of open systems the use of various non-Hermitian Hamiltonians becomes increasingly popular [36]. The study and/or the search for the manifestations of the presence and structure of the open-system EPs enters, due to their simpler forms, many new theoretical as well as phenomenological territories [37, 54].

7.2 Conclusions

In our present paper we proposed one of the potentially most useful classifications of the unitary quantum processes of phase transitions mediated by the Kato's exceptional points. Emphasis has been put upon the exact solvability of the underlying, AHO-based benchmark models. One of the main merits of our constructive classification is that the exact solvability of these N by N matrix loss-of-observability benchmark models is guaranteed at an arbitrarily large matrix dimension N.

From the point of view of physics, the success of our construction appeared to be a consequence of our fortunate initial choice of the family of the phenomenological real-matrix Hamiltonians which combined the harmonic-oscillator-inspired equidistance of their diagonal matrix elements with the entirely general but extreme, maximally non-Hermitian (i.e., antisymmetric) but $\mathcal{PT}-$symmetric full-matrix form of the anharmonicities. In such a setup our search for a complete set of the EPN-supporting benchmark models which would remain non-numerical at any N was entirely straightforward. A constructive classification of the closed quantum systems living in a vicinity of their EPN-mediated phase transition has been achieved. We believe that in the nearest future the direct-sum nature of our models may be expected to facilitate the task of a guarantee of stability with respect to small perturbations at larger N and K.

The most welcome serendipitious byproduct of our construction is two-fold. First, we managed to overcome the widespread belief that in the study of the EPN-related phenomena the techniques (as sampled, say, in Ref. [12]) must necessarily remain numerical and restricted to the systems with the smallest matrix dimensions N. Second, we found that an increase of the number of diagonals in $H^{(N)}(\lambda)$ does not play any significant role. We have shown, in particular, that the next-to-tridiagonal (i.e., pentadiagonal) choice of matrices does not help too much. We may conclude that the number of diagonals is inessential, and that at N as small as six, and for the geometric multiplicity as small as $K = 3$, one needs as many as eleven diagonals to simulate the anomalous behavior of the system near its EP6 singularity.

In the language of mathematics we were forced to introduce a certain new version of the notion of partitioning. Starting from the idea of the uniqueness of a simplified representation of our specific matrices by their main diagonals we took into consideration the availability of tridiagonal building-block sub-Hamiltonians. Then we added the concept of a direct sum of these AHO-based sub-Hamiltonians forming a decomposition of the main N by N Hamiltonian. This led us to the development of a special partitioning technique which is described in Appendix A. Along these lines we arrived, at an arbitrary preselected matrix dimension N, at an exhaustive characterization and "numbering" of all of the eligible EPN-related setups. In a final step, this result opened the way to the guarantee of the completeness of the classification.

We can summarize that in the context of the study and classification of quantum systems in a small vicinity of the instant of phase transition, our key idea of decomposition of certain toy-model N by N matrix Hamiltonians into direct sums of their AHO-matrix components proved efficient and productive. It was shown to lead to an exhaustive list of multidiagonal AHO-related matrices $H^{(N)}(\lambda)$ forming an infinite family of benchmark EPN-supporting Hamiltonians. These representatives of the systems near a collapse happened to be unexpectedly user-friendly, especially due to their characteristic sparse-matrix structure.

Appendix A: Non-equivalent EPN scenarios

In the present AHO-based phenomenological models numbered by the Hamiltonian-matrix dimensions $N = (1), 2, 3, \ldots$, the counts of the eligible non-equivalent EPN-related dynamical scenarios form a sequence

$$a(N) = (0), 1, 1, 2, 3, 3, 6, 4, 11, 6, 17, 7, 32, 8, 39, 13, 40, \ldots . \tag{27}$$

The evaluation of the sequence is important and useful for at least two physical reasons. First, beyond the smallest N, it enables us to check the completeness of the EPN-related dynamical alternatives. Second, the asymptotically exponential growth of the sequence indicates that at the larger Ns, the menus of the EPN-supporting toy models will be dominated by the anomalous, multidiagonal $K > 1$ Hamiltonians.

Besides that, the properties of the sequence are of an independent mathematical interest. First of all we notice that our sequence seems composed of the two apparently simpler, monotonously increasing integer subsequences. They have to be discussed separately.

A.1. Subsequence of $a(N)$ with even $N = 2J$, $J = 1, 2, \ldots$.

The values of the subsequence

$$b(J) = a(2J) = 1, 2, 3, 4, 6, 7, 8, 13, 14, 15, 25, 26, 33, 50, \ldots \tag{28}$$

may be generated by the algorithm described in [55]. Recalling this source (available, temporarily, under the identification number A336739) let us summarize a few key mathematical features of the (sub)sequence.

Definition 3 *The quantity b(n) is the number of decompositions of B(n,1) into disjoint unions of B(j,k) where B(j,k) is the set of numbers* $\{(2i\text{-}1)(2k\text{-}1),\ 1 \le i \le j\}$.

It may be instructive to display a few examples:

B(n,1) are the sets {1}, {1,3}, {1,3,5}, {1,3,5,7}, ...,
B(n,2) are the sets {3}, {3,9}, {3,9,15}, {3,9,15,21}, ...,
B(n,3) are the sets {5}, {5,15}, {5,15,25}, {5,15,25,35}, ...,

etc. Thus, one can conclude that there are two decompositions of B(2,1) = {1,3}, viz., trivial B(2,1) and nontrivial B(1,1) + B(1,2) = {1} + {3}. Similarly, the complete list of the $a(5) = 6$ decompositions of {1,3,5,7,9} is as follows:

{{1,3,5,7,9}},
{{1,3,5,7}, {9}},
{{1,3,5}, {7}, {9}},
{{1,3}, {5}, {7}, {9}},
{{1}, {3}, {5}, {7}, {9}},
{{3,9}, {1}, {5}, {7}}.

We should add that the notation used in definition 3 of quantities B(j,k) is mathematically optimal. For the purposes of our present paper, nevertheless, it is necessary to recall the equivalence of every B(j,k) to one of the present boxed symbols. For example, in place of B(3,1) = {1,3,5} we should write $\mathcal{B}[3,1] = \boxed{\text{-5,-3,-1,1,3,5}}$, etc. Thus, definition 3 could be modified as follows.

Definition 4 *The quantity b(n) is the number of different decompositions of* $\mathcal{B}[n,1]$ *into unions of* $\mathcal{B}[j,k]$ *where* $\mathcal{B}[J,K]$ *is defined as the boxed symbol*

. $\boxed{(2K\text{-}1)(1\text{-}2J),(2K\text{-}1)(3\text{-}2J),(2K\text{-}1)(5\text{-}2J),\ \ldots\ ,(2K\text{-}1)(2J\text{-}1)}$.

A.2. Subsequence of $a(N)$ with odd $N = 2J + 1$, $J = 1, 2, \ldots$.

The values of the subsequence

$$c(J) = a(2J + 1) = 1, 3, 6, 11, 17, 32, 39, 40, 56, \ldots \tag{29}$$

may be found discussed in [56]. Using this source let us summarize a few key aspects of this sequence which carries, temporarily, the identification number A335631.

Definition 5 *The quantity c(n) is the number of decompositions of C(n,1) into disjoint unions of C(j,k) and G(q,r) where C(j,k) is the set of numbers* $\{ i\, k,\ 0 \leq i \leq j \}$ *and where G(q,r) is the set of numbers* $\{ (2\,p\text{-}1)\, r,\ 1 \leq p \leq q \}$.

In a more explicit manner let us point out that

C(n,1) are the sets {0,1}, {0,1,2}, {0,1,2,3}, {0,1,2,3,4}, ...,

C(n,2) are the sets {0,2}, {0,2,4}, {0,2,4,6}, {0,2,4,6,8}, ...,

C(n,3) are the sets {0,3}, {0,3,6}, {0,3,6,9}, {0,3,6,9,12}, ...,

etc., and that

G(n,1) are the sets {1}, {1,3}, {1,3,5}, {1,3,5,7}, ...,

G(n,2) are the sets {2}, {2,6}, {2,6,10}, {2,6,10,14}, ...,

G(n,3) are the sets {3}, {3,9}, {3,9,15}, {3,9,15,21}, ...,

etc. We can say that $a(2) = 3$ because the decompositions of C(2,1) = {0,1,2} involve not only the trivial copy C(2,1) but also the nontrivial formulae C(1,2) + G(1,1) = {0,2} + {1} and C(1,1) + G(1,2) = {0,1} + {2}. Similarly: why $a(3) = 6$? Because the decompositions of {0,1,2,3} are as follows:

{{0,1,2,3}},

{{0,1,2}, {3}},

{{0,1}, {2}, {3}},

{{0,2}, {1,3}},

{{0,2}, {1}, {3}},

{{0,3}, {1}, {2}}.

The one-to-one correspondence and the translation of this notation to our present boxed-symbol language is again obvious, fully analogous to the one described in the preceding Appendix A.1.

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