Second Order Differential Equations with Hypergeometric Solutions of Degree Three

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ABSTRACT

Let L be a second order linear homogeneous differential equation with rational function coefficients. The goal in this paper is to solve L in terms of hypergeometric function ${}_2F_1(a,b;c\,|\,f)$ where f is a rational function of degree 3.

Categories and Subject Descriptors

I.1.2 [**Symbolic and Algebraic Manipulation**]: Algorithms; G.4 [**Mathematics of Computing**]: Mathematical Software

General Terms

Algorithms

Keywords

Symbolic Computation, Differential Equations, Closed Form Solutions

1. INTRODUCTION

A linear differential equation with rational function coefficients corresponds to a differential operator $L \in \mathbb{C}(x)[\partial]$ where $\partial = \frac{d}{dx}$. For example, if $L = a_2\partial^2 + a_1\partial + a_0$ is a differential operator with $a_2, a_1, a_0 \in \mathbb{C}(x)$, then the corresponding differential equation $L(y) = 0$ is $a_2y'' + a_1y' + a_0y = 0$. We assume that L has no Liouvillian solutions, otherwise L can be solved quickly using Kovacic's algorithm [7].

Definition 1.1. *If $S(x)$ is a special function that satisfies a differential operator L_S (called a base equation) of order n, then a function y is called a linear S-expression if there exist algebraic functions $f, r, r_0, r_1, \ldots$ such that $y =$*

$$\exp(\int r\,dx)\cdot\Big(r_0S(f)+r_1S(f)'+\cdots+r_{n-1}S(f)^{(n-1)}\Big). \quad (1)$$

More generally, we say that y can be expressed in terms of S if it can be written in terms of expressions of the form (1), using field operations and integrals.

*Supported by NSF grant 1017880

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ISSAC'13, June 26–29, 2013, Boston, Massachusetts, USA.
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Higher derivatives are not needed in (1) since they are linear combinations of $S(f), S(f)', \ldots, S(f)^{(n-1)}$. If $L_S \in \mathbb{C}(x)[\partial]$ is of order n and $k = \mathbb{C}(x, r, f, r_0, r_1, \ldots) \subseteq \overline{\mathbb{C}(x)}$ then y satisfies an equation $L \in k[\partial]$ of order $\leq n$.
If $L \in \mathbb{C}(x)[\partial]$ has order 3 or 4, and S is a special function that satisfies a second order equation, then the problem of solving L in terms of S can be reduced, with an algorithm and implementation [12], to the problem of solving second order equations. This reduction of order motivates a focus on second order equations.
If y and S satisfy second order operators, then products of (1) are not needed, and the form reduces to

$$y = \exp(\int r\,dx)\cdot\Big(r_0S(f)+r_1S(f)'\Big). \quad (2)$$

Although form (2) looks technical, it is the most natural form to consider, because it is closed under the known transformations that send irreducible second order operators in $\mathbb{C}(x)[\partial]$ to second order linear operators. Given an input operator L_{inp} of order 2, finding a solution of the form (2) corresponds to finding a sequence of transformations that sends L_S to L_{inp} (or a right hand factor of L_{inp}, but we assume L_{inp} to be irreducible):

(i) Change of variables: $y(x) \mapsto y(f)$
(ii) Gauge transformation: $y \mapsto r_0y + r_1y'$
(iii) Exponential product: $y \mapsto \exp(\int r\,dx)$

The function f in (i) above is called the *pullback* function. These transformations are denoted as $\xrightarrow{f}_C$, $\xrightarrow{r_0,r_1}_G$ and $\xrightarrow{r}_E$ respectively. They send expressions in terms of S to expressions in terms of S. So any solver for finding solutions in terms of S, if it is complete, then it must be able to deal with all three transformations. In other words, it must be able to find any solution of the form (2).
The goal in this paper is the following:

Given $L_{inp} \in \mathbb{C}(x)[\partial]$, irreducible, order 2, find, if it exists, a nonzero solution of form (2) where $S(x) = {}_2F_1(a,b;c\,|\,x)$, $f, r, r_0, r_1 \in \mathbb{C}(x)$ and f has degree 3.

Given L_{inp}, our task is to find:

$$L_S \xrightarrow{f}_C M \xrightarrow{r_0,r_1}_G \xrightarrow{r}_E L_{inp}.$$

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There are algorithms [2] to find the transformations $\xrightarrow{r_0,r_1}_G$ and $\xrightarrow{r}_E$ but to apply them we first need M (or equivalently, f and L_S). Thus the crucial part is to compute f.
We compute f from the singularities of M. Since we do not yet know M, the only singularities of M that we know are those singularities of L_{inp} that can not *disappear* (turn into regular points) under transformations $\xrightarrow{r_0,r_1}_G$ and $\xrightarrow{r}_E$.

DEFINITION 1.2. *A singularity is called* non-removable *if it stays singular under any combination of* $\xrightarrow{r_0,r_1}_G$ *and* $\xrightarrow{r}_E$.

A singularity $x = p$ of L_{inp} that can become a regular point under $\xrightarrow{r_0,r_1}_G$ and/or $\xrightarrow{r}_E$ need not be a singularity of M. Such singularities (*removable* singularities) provide no information about f. They include *apparent* singularities (singularities p where all solutions are analytic at $x = p$, such singularities can disappear under $\xrightarrow{r_0,r_1}_G$). More generally, if there exist functions u, y_1, y_2 with y_1, y_2 analytic at $x = p$ such that uy_1, uy_2 is a basis of local solutions of L at $x = p$, then $x = p$ is removable (such p can be sent to an apparent singularity with $\xrightarrow{r}_E$).

1.1 Motivation

Equations with a ${}_2F_1$-type solution are common. We examined integer sequences $u(0), u(1), u(2), \ldots$ from the Online Encyclopedia of Integer Sequences (oeis.org) for which $y = \sum_n u(n)x^n \in \mathbb{Z}[x]$ is (a) convergent, and (b) holonomic, meaning that y satisfies a linear differential operator L. Among the L's obtained this way, all second order L's (including dozens that had no Liouvillian solutions) turned out to have ${}_2F_1$-type solutions. For third order operators we used order $3 \to 2$ reduction [12] to find solutions of the form $\exp(\int r dx)\Big(r_0(S(f))^2 + r_1 S(f)S(f)' + r_2(S(f)')^2\Big)$, where $S(x) = {}_2F_1(a, b; c \,|\, x)$.
The key step to find ${}_2F_1$-type solutions is to find the pullback function f, and the ${}_2F_1$-parameters a, b, c. Classifying all *rational* functions $f \in \mathbb{C}(x)$ that can occur as a pullback function for some L with d *non-removable* singular points is ongoing work, see [8] for $d = 4$ and [9] for $d = 5$ (with at least one logarithmic singularity). For a fixed d, a large table is needed to ensure that we can solve every second order L with d singularities that has a ${}_2F_1$-type solution. But the table can be greatly reduced by developing algorithms such as 2-*descent* [6]. If f is a rational function with a degree 2 decomposition, then we can apply the 2-descent algorithm to L to reduce the degree of f in half.
After trying 2-descent, the next case is to solve every L that has a ${}_2F_1$-type solution where f is a rational function of degree 3. This is useful in its own right because it solves many equations, but it also significantly reduces the tabulation work that is needed (many f's from [8],[9] have a decomposition factor of degree 2 or 3).

1.2 Hypergeometric solutions, an example

Consider the operator $L =$

$$2\left(2x^2-1\right)\left(8x^2-1\right)\partial^2 + 4x\left(24x^2-7\right)\partial + 24x^2 - 3. \quad (3)$$

L can be solved in terms of ${}_2F_1(a, b; c \,|\, f)$ where f is a rational function of degree 3. We give one such solution (a second independent solution looks similar):

$$sol_L = (1-2x\sqrt{2})^{-\frac{1}{3}}(1+x\sqrt{2})^{-\frac{1}{6}} \cdot \; {}_2F_1\Big(\frac{1}{6}, \frac{1}{3}; \frac{5}{6} \,|\, f\Big) \quad (4)$$

where $f = \frac{(2x-\sqrt{2})(4x+\sqrt{2})^2}{(2x+\sqrt{2})(4x-\sqrt{2})^2}$.
L is defined over $\mathbb{Q}$, i.e. $L \in \mathbb{Q}(x)[\partial]$ but $f \notin \mathbb{Q}(x)$; instead $f \in \mathbb{Q}(\sqrt{2}, x)$. Such a field extension can only occur when f is not unique (replacing $\sqrt{2}$ by $-\sqrt{2}$ gives another solution). The non-uniqueness of f in this example is explained by the fact that L has a symmetry $x \mapsto -x$ (The change of variables $x \mapsto -x$ produces an operator L_{-x} that equals L). The change of variables $x \mapsto \sqrt{x}$ produces an operator $L_{\sqrt{x}}$ that is still in $\mathbb{Q}(x)[\partial]$ (this is a trivial case of 2-descent).

$$L_{\sqrt{x}} = x(2x-1)(8x-1)\partial^2 + (32x^2 - 12x + \frac{1}{2})\partial + 3x - \frac{3}{8}. \quad (5)$$

Our program produces the following solution of $L_{\sqrt{x}}$:

$$sol_{L_{\sqrt{x}}} = \; {}_2F_1\Big(\frac{1}{12}, \frac{1}{4}; \frac{1}{2} \,|\, 2x\,(8x-3)^2\Big). \quad (6)$$

Applying $x \mapsto x^2$ to (6) produces another solution of L:

$$Sol_L = \; {}_2F_1\Big(\frac{1}{12}, \frac{1}{4}; \frac{1}{2} \,|\, 2x^2\,\big(8x^2-3\big)^2\Big). \quad (7)$$

The pullback function f in (7) has degree 6, but is nicer than the degree 3 function in (4). So we chose in our implementation to search only for f's which are defined over the field of constants specified in the input, and to apply 2-descent for the cases where f requires a field extension.
The following diagrams show the impact of the change of variables $x \mapsto f$.

Notation: (see section 2 for more details and definitions).
p: non-removable singularity with exponent-difference Δ_p.
$H^{a,b}_{c,x}$: hypergeometric differential operator.

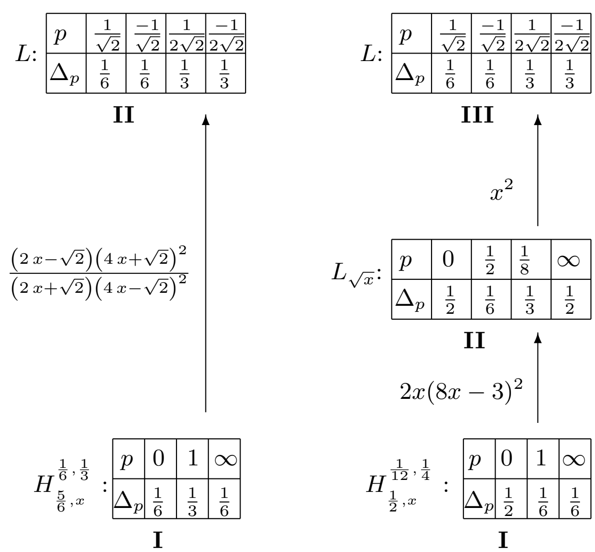

Diagram 1 **Diagram 2**

The diagrams give the singularity structures (the p's and Δ_p's, see section 2 for definitions) of $L, L_{\sqrt{x}}$ and $H^{a,b}_{c,x}$. The hypergeometric function ${}_2F_1(a, b; c \,|\, x)$ is a solution of $H^{a,b}_{c,x}$ (see Section 2.2). Choosing $(a, b, c) = (\frac{1}{6}, \frac{1}{3}, \frac{5}{6})$ makes the exponent-differences of $H^{a,b}_{c,x}$ equal to $(\frac{1}{6}, \frac{1}{3}, \frac{1}{6})$. Then the pullback function $f = \frac{(2x-\sqrt{2})(4x+\sqrt{2})^2}{(2x+\sqrt{2})(4x-\sqrt{2})^2}$ in Diagram 1 sends the singularity structure of $H^{a,b}_{c,x}$ to that of L.

Diagram 2 has two changes of variables, $x \mapsto 2x(8x-3)^2$ produces the singularity structure of $L_{\sqrt{x}}$ from a hypergeometric equation with exponent-differences $(\frac{1}{2}, \frac{1}{6}, \frac{1}{6})$. After that, $x \mapsto x^2$ produces the singularity structure of L.
Presenting (4) in the general form (2), we get $r = \frac{u'}{u} \in \mathbb{Q}(x, \sqrt{2})$ where $u = (1+2x\sqrt{2})^{-\frac{1}{3}}(1-x\sqrt{2})^{-\frac{1}{6}}$, $r_0 = 1$ and $r_1 = 0$. For (7), we also have $r_1 = 0$ (with $r = 0$, $r_0 = 1$). The case $r_1 \neq 0$ is more complicated as this involves integer shifts of the exponent-differences, see *section 2.4* for an example.

1.3 Goal of the paper

The examples in the above diagrams illustrate the effect of change of variables $x \mapsto f$ on the singularities and their exponent-differences; this paper will use that to reconstruct f whenever it has degree 3, a problem that turns out to consist of 18 cases. One of those cases (case sing5.2 in *section 4.2*) was treated previously in *section 2.6* in [4]. The goal in this paper is to cover all cases with f of degree 3. Combining this work with the 2-descent algorithm from [6] and the tables from [8, 9] leads to a solver that can find ${}_2F_1$-type solutions for many second order equations. The combined solver is very effective; it appears that closed form solutions exist for every second order L that has a non-zero convergent solution of the form $\sum_{n=0}^{\infty} u(n)x^n$ with $u(n) \in \mathbb{Z}$.

2. PRELIMINARIES AND NOTATIONS

This section gives a brief summary of prior results needed for this paper. Notations used throughout the paper:

C: a subfield of $\mathbb{C}$.
$L_{inp} \in C(x)[\partial]$: input differential operator.
$H^{a,b}_{c,x}$: Gauss hypergeometric differential operator.
$S(x) = {}_2F_1(a, b; c|x)$: hypergeometric function (a solution of $H^{a,b}_{c,x}$).
e_0, e_1, e_∞: exponent differences of $H^{a,b}_{c,x}$ at $0, 1, \infty$.
$f \in C(x)$: a rational function of degree 3.
$H^{a,b}_{c,f}$: obtained from $H^{a,b}_{c,x}$ by a change of variables $x \mapsto f$.
$S(f) = {}_2F_1(a, b; c \,|\, f)$: a solution of $H^{a,b}_{c,f}$.

2.1 Differential Operators and Singularities

A derivation $\partial = \frac{d}{dx}$ on $\mathbb{C}(x)$ produces a non-commutative ring $\mathbb{C}(x)[\partial]$. A differential operator of order n is an element $L \in \mathbb{C}(x)[\partial]$ of the form $L = \sum_{i=0}^{n} a_i \partial^i$, with $a_i \in \mathbb{C}(x)$ and $a_n \neq 0$. The solution space (set of all solutions in a *universal extension*) of L is denoted by $V(L) = \{y \,|\, L(y) = 0\}$.
After clearing denominators, we may assume $a_i \in \mathbb{C}[x]$. Then $p \in \mathbb{C}$ is called a *regular* (or *non-singular*) point when $a_n(p) \neq 0$. Otherwise it is called a *singular* point (or a *singularity*). The point $p = \infty$ is called regular if the change of variable $x \mapsto 1/x$ produces an operator $L_{1/x}$ with a regular point at $x = 0$. Given $p \in \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$, we define the local parameter as $t_p = \begin{cases} x - p & \text{if } x \neq \infty \\ \frac{1}{x} & \text{if } x = \infty. \end{cases}$

DEFINITION 2.1. *$L \in \mathbb{C}(x)[\partial]$ is called* Fuchsian *(or* regular singular*) if all its singularities are regular singularities. A singularity p is a* regular singularity *when:*
(1) (if $p \neq \infty$): $t_p^i \cdot \frac{a_{n-i}}{a_n}$ is analytic at p for $1 \leq i \leq n$.
(2) (if $p = \infty$): $L_{1/x}$ has a regular singularity at $x = 0$.
Otherwise p is called irregular singular.

This paper considers only Fuchsian operators, of order 2. For closed form solutions of non-Fuchsian equations (L having at least one irregular singularity) see [13].

THEOREM 2.2. *([10], Sections 2.1–2.4). Suppose L has order 2 and $p \in \mathbb{P}^1$. If $x = p$ is a regular singularity or a regular point of L, then there exists the following basis of $V(L)$ at $x = p$*

$y_1 = t_p^{e_1} \sum_{i=0}^{\infty} a_i t_p^i$, $a_0 \neq 0$ *and*

$y_2 = t_p^{e_2} \sum_{i=0}^{\infty} b_i t_p^i + c y_1 \log(t_p)$, $b_0 \neq 0$

where $e_1, e_2, a_i, b_i, c \in \mathbb{C}$ such that:
(i) If $e_1 = e_2$ then c must be non zero.
(ii) Conversely, if $c \neq 0$ then $e_1 - e_2$ must be in $\mathbb{Z}$.

NOTATION 2.3. *In Theorem 2.2:*

1. *If $c \neq 0$ then $x = p$ is called a* logarithmic *singularity.*
2. *e_1, e_2 are called the* exponents *of L at $x = p$ (they are computed as the roots of the* indicial equation*).*
3. *$\Delta_p(L) := \pm(e_1 - e_2)$ is the* exponent difference *at p.*
4. *If $\Delta_p(L_1) \equiv \Delta_p(L_2) \mod \mathbb{Z}$ then we say that $\Delta_p(L_1)$* matches *$\Delta_p(L_2)$.*

REMARK 2.4. *Logarithmic singularities are always non-removable (they stay logarithmic under the transformations in Definition 1.2). If $e_1 - e_2 \in \mathbb{Z}$ and $x = p$ is non logarithmic then $x = p$ is either a regular point or a removable singularity. Proofs and more details can be found in [5]. The relation with Theorem 2.2 is as follows:*
(1) $x = p$ is non-singular $\iff \{e_1, e_2\} = \{0, 1\}$ and $c = 0$.
(2) $x = p$ is non-removable $\iff c \neq 0$ or $e_1 - e_2 \notin \mathbb{Z}$.

DEFINITION 2.5. *The* singularity structure *of L is:*
$Sing(L) = \{(p, \Delta_p(L) \mod \mathbb{Z}) : p$ is non-removable$\}$.

It is often more convenient to express singularities in terms of monic irreducible polynomials.

DEFINITION 2.6. *Let C be a field of characteristic 0. places(C):= $\{f \in C[x] \,|\, f$ is monic and irreducible$\} \bigcup \{\infty\}$. The* degree *of a place p is 1 if $p = \infty$ and* $\deg(p)$ *otherwise.*

EXAMPLE 2.7. *Consider L in section 1.2.*

$$Sing(L) = \Big\{\Big(\frac{1}{\sqrt{2}}, \frac{1}{6}\Big), \Big(-\frac{1}{\sqrt{2}}, \frac{1}{6}\Big), \Big(\frac{1}{2\sqrt{2}}, \frac{1}{3}\Big), \Big(-\frac{1}{2\sqrt{2}}, \frac{1}{3}\Big)\Big\}.$$

In terms of places($\mathbb{Q}$) it is written as:

$$Sing(L) = \Big\{\Big(x^2 - \frac{1}{2}, \frac{1}{6}\Big), \Big(x^2 - \frac{1}{8}, \frac{1}{3}\Big)\Big\}.$$

2.2 Gauss Hypergeometric Equation

The Gauss hypergeometric differential equation (GHE) is:

$$x(1-x)y'' + (c - (a+b+1)x)y' - aby = 0. \quad (8)$$

It has three regular singularities, with exponents $\{0, 1-c\}$ at $x = 0$, $\{0, c-a-b\}$ at $x = 1$ and $\{a, b\}$ at $x = \infty$. The corresponding differential operator is denoted by:

$$H^{a,b}_{c,x} = x(1-x)\partial^2 + (c - (a+b+1)x)\partial - ab. \quad (9)$$

One of the solutions at $x = 0$ is ${}_2F_1(a, b; c \,|\, x)$. Computing a ${}_2F_1$-type solution of a second order L_{inp} corresponds to computing transformations from $H^{a,b}_{c,x}$ to L_{inp}.

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REMARK 2.8. *The exponent differences of $H_{c,x}^{a,b}$ can be obtained from the parameters a, b, c and vice versa: $(e_0, e_1, e_\infty) = (1-c, c-a-b, b-a)$.*

REMARK 2.9. *We assume that $H_{c,x}^{a,b}$ has no Liouvillian solutions. For such $H_{c,x}^{a,b}$, the points $0, 1, \infty$ are never non-singular or removable singularities. So if $H_{c,x}^{a,b}$ has $e_p \in \mathbb{Z}$ (with $p \in \{0, 1, \infty\}$) then p is a logarithmic singularity.*

2.3 Properties of Transformations

For second order operators, we use the notation $L_1 \longrightarrow L_2$ if L_1 can be transformed to L_2 with any combination of the three transformations from *section 1.* If $L_1 \longrightarrow L_2$ then $L_1 \xrightarrow{f}_C \xrightarrow{r_0,r_1}_G \xrightarrow{r}_E L_2$. More details can be found in [1].

REMARK 2.10.

1. *$\xrightarrow{r_0,r_1}_G$ and $\xrightarrow{r}_E$ are equivalence relations.*

2. *Δ_p remains same under $\xrightarrow{r}_E$ but may change by an integer under $\xrightarrow{r_0,r_1}_G$.*
   So if $L_1 \xrightarrow{f}_C M \xrightarrow{r_0,r_1}_G \xrightarrow{r}_E L_{inp}$ for some input L_{inp} with L_1, M unknown, then $\Delta_p(M)$ can be ($\bmod \mathbb{Z}$ and up to $\pm$) read from $\Delta_p(L_{inp})$,
   $$Sing(L_{inp}) = Sing(M).$$
   Hence L_1, f, M should be reconstructed from $Sing(L_{inp})$.

3. *If one of e_0, e_1, e_∞ is in $\frac{1}{2}+\mathbb{Z}$ then $H_{c,x}^{a,b}$ is determined, up to the equivalence relation $\xrightarrow{r_0,r_1}_G \xrightarrow{r}_E$, by the triple (e_0, e_1, e_∞) up to $\pm$ and $\bmod \mathbb{Z}$.*
   If $\{e_0, e_1, e_\infty\} \bigcap (\frac{1}{2}+\mathbb{Z}) = \emptyset$ then the triple leaves two separate cases for $H_{c,x}^{a,b}$ up to $\xrightarrow{r_0,r_1}_G \xrightarrow{r}_E$; we need to consider (e_0, e_1, e_∞) up to $\pm$ and $\bmod \mathbb{Z}$, and (e_0+1, e_1, e_∞) up to $\pm$. See Theorem 8, section 5.3 in [14] for details.

Because of the transformation $M \xrightarrow{r_0,r_1}_G \xrightarrow{r}_E L_{inp}$ in *remark 2.10* only non-removable singularities of L_{inp} provide usable data for M and f.

DEFINITION 2.11. *Two operators L_1, L_2 are called projectively equivalent (notation: $L_1 \sim_p L_2$) if $L_1 \xrightarrow{r_0,r_1}_G \xrightarrow{r}_E L_2$.*

The following lemma gives the effect of $\xrightarrow{f}_C$ on the singularities and their exponent differences: (see [4] for more details)

LEMMA 2.12. *Let e_0, e_1, e_∞ be the exponent differences of $H_{c,x}^{a,b}$ at $0, 1, \infty$. Let $H_{c,f}^{a,b}$ be the operator obtained from $H_{c,x}^{a,b}$ by applying $x \mapsto f$. Let $d = \Delta_p$ be the exponent difference of $H_{c,f}^{a,b}$ at $x = p$. Then:*

1. *If p is a root of f with multiplicity m, then $d = me_0$.*

2. *If p is a root of $1-f$ with multiplicity m, then $d = me_1$.*

3. *If p is a pole of f of order m, then $d = me_\infty$.*

2.4 An Example Involving All Three Transformations

Let $u(0) = 1,\ u(1) = 828,\ u(n+2) =$

$$\frac{4(592(n-1)^2 - 977)u(n+1) - 28^3(16n^2-9)u(n))}{(n+2)^2}. \quad (10)$$

This defines a sequence 1, 828, -121212, ... How to prove that this is an integer sequence?
Consider the following differential operator:

$$\tilde{L} = (x-37)\left(x^2+3\right)\partial^2 + \left(x^2+3\right)\partial - \frac{9}{16}(x+9). \quad (11)$$

Our implementation solves this equation. One solution is:

$$sol_{\tilde{L}} = s\left(g \cdot {}_2F_1\left(\frac{1}{12}, \frac{5}{12}; 1 \,|\, f\right) + h \cdot {}_2F_1\left(\frac{5}{12}, \frac{13}{12}; 1 \,|\, f\right)\right). \quad (12)$$

where $s = \frac{98^{\frac{1}{4}}}{126(3x-13)^{\frac{5}{4}}}$, $g = (3x+1)(3x-13)$, $h = 36x+40$ and $f = \frac{27(x-37)(x^2+3)}{(3x-13)^3}$.

One can convert between differential equations and recurrences (*see 'gfun' package in Maple*) and find:

$$sol_{\tilde{L}} = \sum_{n=0}^{\infty} u(n)\left(\frac{x-37}{2^7 \cdot 7^3}\right)^n \quad (13)$$

where $u(n)$ are given by the recurrence relation in (10).
The explicit expression (12) can be used to prove $u(n) \in \mathbb{Z}$ for $n = 0, 1, \ldots$ (it is not clear if there is a different way to prove that for this example).
The following diagram shows the effects of $\xrightarrow{f}_C$ and $\xrightarrow{r_0,r_1}_G$ on the set of non-removable singularities and their exponent differences ($\xrightarrow{r}_E$ does not affect them):

$\tilde{L}$:

p	37	$\sqrt{-3}$	$-\sqrt{-3}$	∞
Δ_p	0	1	1	$\frac{3}{2}$

III ↑ $y \mapsto r_0 y + r_1 y'$

$H_{1,f}^{\frac{1}{12},\frac{5}{12}}$:

p	37	$\sqrt{-3}$	$-\sqrt{-3}$	∞
Δ_p	0	0	0	$\frac{1}{2}$

II ↑ $f = \frac{27(x-37)(x^2+3)}{(3x-13)^3}$

$H_{1,x}^{\frac{1}{12},\frac{5}{12}}$:

p	0	1	∞
Δ_p	0	$\frac{1}{2}$	$\frac{1}{3}$

I

Diagram 3

Suppose $y(x) = \sum_{n=0}^{\infty} u(n)x^n$ is convergent with $u(n) \in \mathbb{Z}$, $(n = 0, 1, 2, \ldots)$ and satisfies a second order differential operator $L \in \mathbb{Q}(x)[\partial]$. In all known examples such $y(x)$ is either algebraic or expressible in terms of ${}_2F_1$ hypergeometric functions. Hence algorithms for finding ${}_2F_1$-type solutions are useful for integer sequences.

3. PROBLEM DISCUSSION

Starting with a Fuchsian linear differential operator L_{inp}, of order 2, and no Liouvillian solutions, we want to find a solution of such L_{inp} in the form :

$$y = e^{\int r}(r_0 S(f) + r_1 S(f)') \qquad (14)$$

where $S(x) = {}_2F_1(a, b; c \,|\, x)$, $f, r, r_0, r_1 \in \mathbb{C}(x)$ and f has degree 3. There are two key steps:

- Compute f and (e_0, e_1, e_∞) such that

  $$Sing(H^{a,b}_{c,f}) = Sing(L_{inp}). \qquad (15)$$

  (see *remark 2.8* for the relation between (e_0, e_1, e_∞) and (a, b, c))

- Compute projective equivalence $\sim_p$ between $H^{a,b}_{c,f}$ and L_{inp} which sends solutions $S(f) = {}_2F_1(a, b; c \,|\, f)$ of $H^{a,b}_{c,f}$ to solutions of L_{inp} of the form (14).

If we find f then [2] takes care of the second step. Hence the crucial part is to compute f (as well as a, b, c, or equivalently, e_0, e_1, e_∞).
Let $f = A/B$ where $A, B \in \mathbb{C}[x]$ with $gcd(A, B) = 1$. The hypergeometric operator $H^{a,b}_{c,x}$ has singularities at $x = 0, 1, \infty$. So one might expect L_{inp} to have singularities whenever $f = 0, 1$ or ∞; i.e. at the roots of A, $A - B$ and B. If all roots of $A, A - B, B$ would appear among the singularities of L_{inp}, then it would be fairly easy to reconstruct $f = A/B$. However, that is not true in general (it is true for 8 out of the 18 cases in *Tab. 1* in *Section 4.2*). For example; if f has a root p with multiplicity 2 and e_0 is a half-integer (an odd integer divided by 2), then p will be a removable singularity or a non-singular point of $H^{a,b}_{c,f}$. Such p does not appear in $Sing(L_{inp})$.

4. COMPUTING PULLBACK FROM THE SINGULARITY STRUCTURE

4.1 Relating singularities to f

Let $[[a_1, \ldots, a_i], [b_1, \ldots, b_j], [c_1, \ldots, c_k]]$ denote the *branching pattern* of f. It contains the branching orders of f above 0, 1 and ∞ respectively (So f has i distinct roots with multiplicities $a_1, \ldots, a_i$. Likewise $1 - f$ and $\frac{1}{f}$ have j resp. k distinct roots). Using *lemma 2.12* and *remark 2.4*, the singularities of $H^{a,b}_{c,f}$ are as follows:

$$P_0 = \{x : f = 0 \text{ and } (e_0 \in \mathbb{Z} \text{ or } a_l\, e_0 \notin \mathbb{Z}) \text{ for } 1 \le l \le i\}$$

$$P_1 = \{x : 1 - f = 0 \text{ and } (e_1 \in \mathbb{Z} \text{ or } b_l\, e_1 \notin \mathbb{Z}) \text{ for } 1 \le l \le j\}$$

$$P_\infty = \{x : \tfrac{1}{f} = 0 \text{ and } (e_\infty \in \mathbb{Z} \text{ or } c_l\, e_\infty \notin \mathbb{Z}) \text{ for } 1 \le l \le k\}$$

where (e_0, e_1, e_∞) are the exponent differences of $H^{a,b}_{c,x}$ at $(0, 1, \infty)$ respectively. The union of P_0, P_1 and P_∞ are the non-removable singularities of $H^{a,b}_{c,f}$ (or L_{inp}, by eq. (15)).
Since f has degree 3, L_{inp} could have at most 9 singularities. The least we could have is 2 when we choose the branching pattern of f as [[3],[1,2],[1,2]] and $(e_0, e_1, e_\infty) \equiv (\pm\frac{1}{3}, \frac{1}{2}, \frac{1}{2})$ mod $\mathbb{Z}$. But a hypergeometric equation with two exponent-differences in $\frac{1}{2} + \mathbb{Z}$ has Liouvillian solutions, so we do not treat this case here. If L_{inp} has 3 non-removable singularities, then we can transform these to $0, 1, \infty$ via a Möbius transformation and express the solution accordingly (with a rational function of degree 1). This case is already treated in [14]. So we exclude these cases (Liouvillian and 3 non-removable singularities) from our consideration.

4.2 Tabulating cases

Let d be the total number of non-removable singularities in L_{inp}. From *Section 4.1* we have $4 \le d \le 9$. The first step is to enumerate all possibilities for exponent differences e_0, e_1, e_∞ and branching patterns above $\{0, 1, \infty\}$ for each d. We express all such possibilities for degree 3 rational function f in the following table:

NOTATION 4.1.
d: number of non-removable singularities in L_{inp}.
$*$, E_1, E_2, E_3: elements of $\mathbb{C}$.
$\frac{*}{2}$: an element of $\frac{1}{2} + \mathbb{Z}$.
$\frac{*}{3}$: an element of $(\frac{1}{3} + \mathbb{Z}) \cup (\frac{2}{3} + \mathbb{Z})$.

d	Case	Exponent difference at 0, 1, ∞ resp.	Branching pattern above 0, 1, ∞ resp.
4	case4.1	$\frac{*}{2}$, $\frac{*}{3}$, E_1	[1,2], [3], [1,1,1]
	case4.2	$\neq \frac{*}{3}$, $\frac{*}{3}$, E_1	[3], [3], [1,1,1]
	case4.3	$\neq \frac{*}{2}$, $\neq \frac{*}{2}$, $\frac{*}{3}$	[1,2], [1,2], [3]
	case4.4	$\neq \frac{*}{3}$, $\neq \frac{*}{2}$, $\frac{*}{2}$	[3], [1,2], [1,2]
	Liouv	$\neq \frac{*}{2}$, $\frac{*}{2}$, $\frac{*}{2}$	[1,2], [1,2], [1,2]
5	case5.1	$\neq \frac{*}{3}$, $\neq \frac{*}{3}$, E_1	[3], [3], [1,1,1]
	Liouv	$\frac{*}{2}$, $\frac{*}{2}$, E_1	[1,2], [1,2], [1,1,1]
	case5.2	$\neq \frac{*}{2}$, $\frac{*}{3}$, E_1	[1,2], [3], [1,1,1]
	case5.3	$\frac{*}{2}$, E_1, $\neq \frac{*}{3}$	[1,2], [1,1,1], [3]
	case5.4	$\neq \frac{*}{2}$, $\neq \frac{*}{2}$, $\frac{*}{2}$	[1,2], [1,2], [1,2]
	case5.5	$\neq \frac{*}{3}$, $\neq \frac{*}{2}$, $\neq \frac{*}{2}$	[3], [1,2], [1,2]
6	case6.1	$\frac{*}{3}$, E_1, E_2	[3], [1,1,1], [1,1,1]
	case6.2	$\neq \frac{*}{2}$, $\frac{*}{2}$, E_1	[1,2], [1,2], [1,1,1]
	case6.3	$\neq \frac{*}{3}$, $\neq \frac{*}{2}$, E_1	[3], [1,2], [1,1,1]
	case6.4	$\neq \frac{*}{2}$, $\neq \frac{*}{2}$, $\neq \frac{*}{2}$	[1,2], [1,2], [1,2]
7	case7.1	$\neq \frac{*}{3}$, E_1, E_2	[3], [1,1,1], [1,1,1]
	case7.2	$\frac{*}{2}$, E_1, E_2	[1,2], [1,1,1], [1,1,1]
	case7.3	$\neq \frac{*}{2}$, $\neq \frac{*}{2}$, E_1	[1,2], [1,2], [1,1,1]
8	case8.1	$\neq \frac{*}{2}$, E_1, E_2	[1,2], [1,1,1], [1,1,1]
9	case9.1	E_1, E_2, E_3	[1,1,1], [1,1,1], [1,1,1]

Table 1: Cases for degree 3 pullback up to permutation of $0, 1, \infty$

Two cases (denoted *Liouv*) in *Tab.1* correspond to the hypergeometric equations with two singularities having a half-integer exponent difference. Such equations have Liouvillian solutions (this follows from Kovacic' algorithm and also from Theorem 8, section 5.3 in [14]). Now the main task is to compute f for the remaining 18 cases. Recall that non removable singularities of $H^{a,b}_{c,f}$ come from (form a subset of) the roots of f, $1 - f$ and poles of f. We will use the singularity structure of L_{inp} to recover f.

4.3 Treating one case

The main algorithm in *Section 5* takes as input C, L_{inp}, x where C is a field of characteristic 0, and $L_{inp} \in C(x)[\partial]$ has

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order 2 and no Liouvillian solutions. It computes $Sing(L_{inp})$ and d. Then it loops over the corresponding cases in *Tab.1*. For example; if $d = 4$ then it loops over cases 4.1 − 4.4 in *Tab.1*. Each case in *Tab.1* is a subprogram. Each of these subprograms takes C, $Sing(L_{inp})$ as input, checks if $Sing(L_{inp})$ is compatible with that particular case, and if so, returns a set of candidates for $f, (e_0, e_1, e_\infty)$ that are compatible with that particular case. We give details for only one case, namely Algorithm[5.3] (notation: casei, j is handled by Algorithm[i.j]). The other cases are treated by similar programs (details can be found in our implementation [11]).

Let $L_{inp} \in C(x)[\partial]$ be input differential operator with 5 non-removable singularities. In terms of places(C), there are 7 ways to end up with 5 points:

1. One place of degree 5 (note: a place of degree > 1 is always a monic irreducible polynomial of that degree. A place of degree 1 can be either ∞ or a monic polynomial of degree 1.)
2. Places of degrees 4, 1.
3. Places of degrees 3, 2.
4. Places of degrees 3, 1, 1.
5. Places of degrees 2, 2, 1.
6. Places of degrees 2, 1, 1, 1.
7. Places of degrees 1, 1, 1, 1, 1.

Algorithm[5.3]: Compute $f \in C(x)$ of degree 3 and exponent differences (e_0, e_1, e_∞) for $H^{a,b}_{c,x}$ corresponding to 'case5.3' in *Tab.1*.

Input: Field C and $Sing(L_{inp})$ in terms of *places*(C).

Output: A set of lists $[f, (e_0, e_1, e_\infty)]$ where $f \in C(x)$ has degree 3 and branching pattern [1,2], [1,1,1], [3] above $0, 1, \infty$ such that $Sing\left(H^{a,b}_{c,f}\right) = Sing(L_{inp})$ where (a, b, c) corresponds to (e_0, e_1, e_∞) by *remark 2.8* and (*see Tab.1*) $e_0 \in \frac{1}{2} + \mathbb{Z}$, e_1 is arbitrary, and $e_\infty \notin \pm\frac{1}{3} + \mathbb{Z}$.

1. Check if $Sing(L_{inp})$ is consistent with case 5.3 (if not, return the empty set and stop) as follows:

   The branching pattern [1,2] at $f = 0$ indicates that f has two roots $a_1, a_2 \in C \bigcup \{\infty\}$ with multiplicities 1 resp. 2. Then $x = a_1$ will have an exponent-difference $e_0 \in \frac{1}{2} + \mathbb{Z}$ but $x = a_2$ will be a regular point or a removable singularity, and so it does not appear in $Sing(L_{inp})$.

   The branching pattern [3] at $f = \infty$ indicates that f has precisely one pole $b \in C \bigcup \{\infty\}$, of order 3. Then $x = b$ will have an exponent-difference $\pm 3e_\infty$ mod $\mathbb{Z}$. In case 5.3 we have $e_\infty \notin \pm\frac{1}{3} + \mathbb{Z}$ and hence the point $x = b$ must be a non-removable singularity. Combined with $x = a_1$ we see that case 5.3 is only possible when $Sing(L_{inp})$ has at least two places of degree 1. So in the above listed 7 cases (5, 4+1, ...), we can exit Algorithm[5.3] immediately if we are not in case 4, 6, or 7.

   The branching pattern [1,1,1] at $f = 1$ indicates that $1 - f$ has three distinct roots, each of multiplicity 1. Thus there must be at least three distinct singularities that match the exponent-difference $\pm e_1$ mod $\mathbb{Z}$. If we can not find three singularities (one place of degree 3, or places of degrees 2 and 1, or three places of degree 1) whose exponent-differences match (up to $\pm$ and mod $\mathbb{Z}$) then Algorithm[5.3] stops. This condition determines e_1 (up to $\pm$ and mod $\mathbb{Z}$).

   We know from Kovacic' algorithm that if there are two $e_i \in \frac{1}{2} + \mathbb{Z}$ then $H^{a,b}_{c,x}$ will have Liouvillian solutions. Since we exclude Liouvillian cases, it follows that only e_0 is in $\frac{1}{2} + \mathbb{Z}$. We conclude that $Sing(L_{inp})$ must have either 1 or 2 singularities in $C \bigcup \{\infty\}$ with an exponent-difference in $\frac{1}{2} + \mathbb{Z}$ and that 2 such singularities can only occur when $e_\infty \in \pm\frac{1}{6} + \mathbb{Z}$. So if there are more than 2, then Algorithm[5.3] stops.

2. Set Cand $= \emptyset$ and write $f = k_1 \frac{(x-a_1)(x-a_2)^2}{(x-b)^3}$ where $a_1, a_2, b \in C \bigcup \{\infty\}$ and $k_1 \in C$. We replace any factor $x - \infty$ in f by 1 in the implementation. Compute the set of places with an exponent-difference in $\frac{1}{2} + \mathbb{Z}$. This set may only have 1 or 2 elements that must have degree 1. Now a_1 loops over this set, and e_0 is the exponent-difference at $x = a_1$.

3. Loop b over the places in $Sing(L_{inp})$ of degree 1, skipping a_1, and only considering a_1, b for which the remaining three singularities have matching exponent-differences. Let e_b be the exponent-difference at $x = b$. Now loop e_∞ over $\frac{e_b}{3}, \frac{(e_b-1)}{3}, \frac{(e_b+1)}{3}$. For e_1 one can take the exponent-difference at any of the 3 remaining singularities. The reason that there are three cases for e_∞ is because we have to determine e_∞ mod $\mathbb{Z}$. Now $3e_\infty = e_b$ but if a gauge transformation occurred, i.e. if the r_1 in the form (2) in *Section 1* is non-zero, then e_b is only known mod $\mathbb{Z}$, and this leaves in general three candidate values for e_∞ mod $\mathbb{Z}$ (it suffices to compute the e_i mod $\mathbb{Z}$, see section 5.3 in [14], summarized in Remark 2.10).

4. Among the remaining 3 singularities, let $P \in C[x]$ be the product of their places (replacing $x - \infty$ by 1 if that is among them). So P has degree 3 if ∞ is not among the 3 remaining singularities, and otherwise it has degree 2. In each loop, the a_1, b appearing in f are known, while k_1 and a_2 are unknown. Take the numerator of $1 - f$ and compute its remainder mod P. Equate the coefficients of this remainder to 0. This gives $\deg(P)$ equations for k_1, a_2. If $\deg(P) = 2$ we obtain one more equation by setting $f(\infty) = 1$ (the resulting equation is $k_1 = 1$). Then we have 3 equations for 2 unknowns k_1, a_2. Compute the solutions $k_1 \in C$ and $a_2 \in C \bigcup \{\infty\}$. If any solution exists, then add the resulting $[f, (e_0, e_1, e_\infty)]$ to the set Cand.

5. Return the set Cand (which could be empty, but could also have one or more members).

Example 4.2.
Take $C = \mathbb{Q}$. Let $Sing(L_{inp})$ in terms of places($\mathbb{Q}$) *be given by:*

$$Sing(L_{inp}) = \left\{[\infty, -\tfrac{1}{2}], [x, \tfrac{2}{7}], [x-2, \tfrac{1}{2}], [x^2 + 26\,x + 44, \tfrac{5}{7}]\right\}.$$

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Our input is the following:

$Sing(L_{inp}) = \{[1, -\frac{1}{2}], [x, \frac{2}{7}], [x-2, \frac{1}{2}], [x^2+26\,x+44, \frac{5}{7}]\}$. *Notations in the steps below come from* Algorithm[5.3]. *Write* $f(x) = k_1 \frac{(x-a_1)(x-a_2)^2}{(x-b)^3}$.

Step 1: *$Sing(L_{inp})$ satisfies the conditions for 'sing5.3';*
(1) $[1, -\frac{1}{2}]$ *and* $[x-2, \frac{1}{2}]$ *have degree 1 and both have a half-integer exponent difference.*
(2) The exponent differences in $[x, \frac{2}{7}]$ *and* $[x^2+26\,x+44, \frac{5}{7}]$ *match, after all, we are working up to* $\pm$ *and* mod $\mathbb{Z}$.

Step 2: *The candidates for* $x - a_1$ *are* 1 *and* $x-2$. *For the first case, we get* $f = k_1 \frac{(x-a_2)^2}{(x-b)^3}$ *and* $e_0 = -\frac{1}{2}$ *(note: we may equally well take* $\frac{1}{2}$*). For the second case, we get* $f = k_1 \frac{(x-2)(x-a_2)^2}{(x-b)^3}$ *and* $e_0 = \frac{1}{2}$.

Step 3: *For the first case,* $x-b$ *can only be* $x-2$ *and* $e_b = \frac{1}{2}$ *(because if we take* $x - b = x$ *then there would not remain three singularities with matching exponent-differences). Likewise, for the second case,* $x-b$ *can only be* 1 *and* $e_b = -\frac{1}{2}$.

First case: $f = k_1 \frac{(x-a_2)^2}{(x-2)^3}$ *and* $e_\infty = \frac{1}{6}$ *(note: we should consider* $e_\infty \in \{\frac{e_b}{3}, \frac{(e_b+1)}{3}, \frac{(e_b-1)}{3}\}$ *since* e_b *is determined mod* $\mathbb{Z}$*, and we have to determine* e_∞ *mod* $\mathbb{Z}$*. However,* $\frac{(e_b+1)}{3} = \frac{1}{2}$ *is discarded since there should not be two* $e_i's$ *in* $\frac{1}{2}+\mathbb{Z}$*. And* $\frac{(e_b-1)}{3} = -\frac{1}{6}$ *but an exponent-difference* $-\frac{1}{6}$ *is equivalent to an exponent-difference* $\frac{1}{6}$*.)*

Second case: $f = k_1(x-2)(x-a_2)^2$ *and* $e_\infty = -\frac{1}{6}$.

Step 4: *In both cases* $P = x \cdot (x^2+26x+44)$ *and* $e_1 = \frac{2}{7}$ *(we could equally well take* $\frac{5}{7}$*). Dividing the numerator of* $1-f$ *by* P *produces equations in* k_1 *and* a_2*. In first case the equations have a solution;* $\{k_1 = -32, a_2 = -\frac{1}{2}\}$*, and in second case they do not.*

Step 5: *The output Cand has one element, namely*
$\left\{[-32\,\frac{(x+1/2)^2}{(x-2)^3}, (-\frac{1}{2}, \frac{2}{7}, \frac{1}{6})]\right\}$.

5. MAIN ALGORITHM

We have developed the algorithms to compute f's and possible exponent differences (e_0, e_1, e_∞) for $H^{a,b}_{c,x}$ corresponding to all 18 cases as given in *Tab.1*. Now we give our main algorithm:

Let $C \subseteq \mathbb{C}$ be a field and $L_{inp} \in C(x)[\partial]$ be the input differential operator. The main algorithm first computes the singularity structure of L_{inp} in terms of *places*(C). Suppose d is the total number of non-removable singularities of L_{inp}. Now we call all algorithms corresponding to d to produce a set of candidates for $f \in C(x)$ and the exponent differences $(e_0, e_1, e_\infty) = (1-c, c-a-b, b-a)$. For each member from that list we compute $H^{a,b}_{c,x}$, $H^{a,b}_{c,f}$ and apply *projective equivalence* [2] between $H^{a,b}_{c,f}$ and L_{inp} to find (if it exists) a nonzero map from $V\left(H^{a,b}_{c,f}\right)$ to $V(L_{inp})$ which sends solutions $S(f) = {}_2F_1(a, b; c\,|\,f)$ of $H^{a,b}_{c,f}$ to solutions $e^{\int r}(r_0 S(f) + r_1 S(f)')$ of L_{inp}.

Algorithm: Solve an irreducible second order linear differential operator $L_{inp} \in C(x)[\partial]$ in terms of ${}_2F_1(a, b; c\,|\,f)$, with $f \in C(x)$ of degree 3.

Input: A field C of characteristic 0, $L_{inp} \in C(x)[\partial]$ of order 2 which has no Liouvillian solutions, and a variable x.

Output: A non zero solution $y = e^{\int r}(r_0 S(f) + r_1 S(f)')$, if it exists, such that $L_{inp}(y) = 0$, where $S(f) = {}_2F_1(a, b; c\,|\,f)$, $f, r, r_0, r_1 \in C(x)$ and f has degree 3.

Step 1: Find the singularity structure of L_{inp} in terms of *places*(C). Let d be the total number of non-removable singularities.

Step 2: Let k be the total number of cases in *Tab.1* for d. For example; if $d = 6$ then $k = 4$.
Let *Candidates* $= \bigcup$ *Algorithm[n.a]*, where $a = \{1 \ldots k\}$. That produces a set of lists $[f, (e_0, e_1, e_\infty)]$ of all possible rational function $f \in C(x)$ of degree 3 and corresponding exponent differences (e_0, e_1, e_∞) for $H^{a,b}_{c,x}$.

Step 2.1 : $H^{a,b}_{c,x} = x(1-x)\partial^2 + (c - (a+b+1)x)\partial - ab$ where a, b, c come from the relation
$(e_0, e_1, e_\infty) = (1-c, c-a-b, b-a)$.
For each element $[f, (e_0, e_1, e_\infty)]$ in *Candidates* (*Step 2*),
(i) If $\{e_0, e_1, e_\infty\} \bigcap \frac{1}{2}+\mathbb{Z} \neq \emptyset$ then Cand := $\{[f, (e_0, e_1, e_\infty)]\}$ otherwise Cand := $\{[f, (e_0, e_1, e_\infty)], [f, (e_0+1, e_1, e_\infty)]\}$ (That determines $H^{a,b}_{c,x}$ up to *projective equivalence, see remark 2.10*).
(ii) From each element in Cand above (a) compute a, b, c, (b) substitute the values of a, b, c in $H^{a,b}_{c,x}$ and (c) apply the change of variable $x \mapsto f$ on $H^{a,b}_{c,x}$. That produces a list of operators $H^{a,b}_{c,f}$.

Step 2.2 : Compute the projective equivalence [2] between each operator $H^{a,b}_{c,f}$ in *Step 2.1* and L_{inp}. If the output is zero, then go back to *Step 2.1* and take the next element from *Candidates*. Otherwise, we get a map of the form: $G = e^{\int r}(r_0 + r_1\partial)$, where $r, r_0, r_1 \in C(x)$ and $\partial = \frac{d}{dx}$.

Step 2.3: $S(f) = {}_2F_1(a, b; c\,|\,f)$ is a solution of $H^{a,b}_{c,f}$. Now compute $G(S(f))$. That gives a solution of L_{inp}.

Step 2.4: Repeat the same procedure for each element in *Candidates*. That gives us a list of solutions of L_{inp}.

Step 2.5: Choose the best solution (with shortest length) from the list (to obtain a second independent solution of L_{inp}, just use a second solution of $H^{a,b}_{c,x}$).

EXAMPLE 5.1. *Consider the operator in* Section 2.4*;*

$$L_{inp} = (x-37)\left(x^2+3\right)\partial^2 + \left(x^2+3\right)\partial - \tfrac{9}{16}(x+9).$$

Procedure to solve this equation is the following:

Step 1: *Read the file* hypergeomdeg3.txt *from the folder hypergeomdeg3 in* `www.math.fsu.edu/~vkunwar`.

Step 2: *$L_{inp} \in \mathbb{Q}(x)[\partial]$. We want the solution of L_{inp} in the base field $\mathbb{Q}$. Type hypergeomdeg3($\{\}$, L_{inp}, x). (in Maple $\{\}$ is the code for $\mathbb{Q}$)*

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Step 3: *The program first finds the singularity structure;*

$Sing(L_{inp}) = \{[1, -\frac{3}{2}], [x-37, 0], [x^2+3, 1]\}$. *(our implementation uses "1" to encode a singularity at* ∞*, and polynomials to encode finite singularities).*

Step 4: *We get* $d = 4$. *The program loops over the four subprograms corresponding to case4.1,... case4.4 to compute* f*:*

1. Algorithm[4.1] *returns the following:* $F =$

$$\left\{[f, [-\frac{3}{2}, 0, \frac{1}{3}]], [f, [-\frac{3}{2}, 1, \frac{1}{3}]], [f, [-\frac{3}{2}, 0, \frac{2}{3}]], [f, [-\frac{3}{2}, 1, \frac{2}{3}]]\right\}$$

where $f = 8\,\frac{(9\,x+10)^2}{(3\,x-13)^3}$.
Note: this set contains $\sim_p$*-duplicates, the four triples* (e_0, e_1, e_∞) *all give projectively equivalent* $H^{a,b}_{c,x}$ *so we could delete three and still find a solution (if it exists). The reason they were left in the current version of the implementation is because they may help to find a solution of smaller size. In the next version, we plan to make the code more efficient by removing* $\sim_p$*-duplicates, keeping only those for which the integer-differences between the exponent-differences of* $H^{a,b}_{c,f}$ *and* L_{inp} *are minimized (in this example, only the second element of* F *would be kept in this approach).*

2. Algorithm[4.2] *returns NULL.*

3. Algorithm[4.3] *and* Algorithm[4.4] *require at least 3 linear polynomials in* $\mathbb{Q}[x]$ *for* $Sing(L_{inp})$ *which is not the case here. So* $Sing(L_{inp})$ *does not qualify the conditions for these algorithms.*

Hence F *gives the Candidates. Note that we are in the case* $\{e_0, e_1, e_\infty\} \bigcap \frac{1}{2} + \mathbb{Z} \neq \emptyset$.

Step 5: *Taking first element* $i = [8\frac{(9x+10)^2}{(3x-13)^3}, [-\frac{3}{2}, 0, \frac{1}{3}]]$ *in Candidates and applying* Step 2.1 *and* Step 2.2 *of the above main algorithm, we get* $G = e^{\int r}(r_0 + r_1\partial)$ *with*
$e^{\int r} = \frac{(\frac{9}{10}x+1)(\frac{1}{3}x^2+1)(-\frac{1}{37}x+1)}{(\frac{1}{12}x+1)(-\frac{3}{13}x+1)^{\frac{13}{4}}}$, $r_1 = 1 + \frac{90}{19}x - \frac{27}{19}x^2$ *and*
$r_0 = \frac{3}{38}\frac{729x^4-19845x^3-251919x^2+1114345x+239772}{(x-37)(3x-13)(9x+10)}$.

Step 6: *We have* $S(f) = {}_2F_1(\frac{13}{12}, \frac{17}{12}; \frac{5}{2}\,|\,8\frac{(9x+10)^2}{(3x-13)^3})$. *Computing* $G(S(f))$ *we get* $y = e^{\int r}(r_0S(f) + r_1S(f)')$ *as a solution of* L_{inp} *where* $e^{\int r}, r_0, r_1$ *are given in* Step 5.

Step 7: *Taking second element* $i = [8\frac{(9x+10)^2}{(3x-13)^3}, [-\frac{3}{2}, 1, \frac{1}{3}]]$ *in Candidates we get another solution* y *with*
$e^{\int r} = \frac{(\frac{9}{10}x+1)}{(\frac{1}{12}x+1)(-\frac{3}{13}x+1)^{\frac{7}{4}}}$, $r_0 = \frac{(2187x^3+22284x^2-37813x+116484)}{98(13-3x)(9x+10)}$,
$r_1 = x^2+3$ *and* $S(f) = {}_2F_1(\frac{7}{12}, \frac{11}{12}; \frac{5}{2}\,|\,8\frac{(9x+10)^2}{(3x-13)^3})$.

Steps 8 and 9: *Process the third and fourth element. Each produces a solution that looks quite similar to that given in Steps 6 and 7.*

Step 10: *The solution in* Step 7 *has the shortest length. So the implementation returns that as a solution of* L_{inp}.

After minor simplification this leads to the solution given in section 2.4.

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