New Lower Bounds for the Number of Equilibria in Bimatrix Games

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April 15, 1997

Abstract. A class of nondegenerate $n \times n$ bimatrix games is presented that have asymptotically more than $2.414^n/\sqrt{n}$ Nash equilibria. These are more equilibria than the $2^n - 1$ equilibria of the game where both players have the identity matrix as payoff matrix. This refutes the Quint–Shubik conjecture that the latter number is an upper bound on the number of equilibria of nondegenerate $n \times n$ games. The first counterexample is a 6×6 game with 75 equilibria. The approach uses concepts from polytope theory, which imply a known upper bound of $2.6^n/\sqrt{n}$.

Keywords. Algorithm, bimatrix game, degenerate game, Nash equilibrium, polytopes.

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Technical Report #264, Departement Informatik, ETH Zürich.
Electronically available from:
ftp://ftp.inf.ethz.ch/pub/publications/tech-reports/

*The author thanks Nicola Galli, Gyula Karolyi, Andrew McLennan, Raimund Seidel, and Emo Welzl for stimulating discussions. This research was supported by a Heisenberg grant from the Deutsche Forschungsgemeinschaft.

1. Introduction

As a tool for game-theoretic analysis, algorithms for finding Nash equilibria have found increasing interest (recent surveys are McKelvey and McLennan, 1996, and von Stengel, 1996). When looking for all Nash equilibria of a game, it is interesting to know upper bounds on their number to terminate the search, and lower bounds to know the possible output size of the algorithm. Bounds and distributions for certain kinds of Nash equilibria are considered by Stanford (1996), McKelvey and McLennan (1997), and McLennan (1997).

We study the number of equilibria of nondegenerate $n \times n$ bimatrix games (many statements hold also for games that are not square). A trivial upper bound is 2^{2n}, the number of possible supports of mixed strategy pairs. This bound can be slightly improved to $\binom{2n}{n}$, which is asymptotically $4^n/\sqrt{\pi n}$, since in a nondegenerate game, both players use the same number of pure strategies in equilibrium, so any equilibrium support corresponds to an n-subset S of $\{1, \ldots, 2n\}$ defining the supports $S \cap \{1, \ldots, n\}$ and $\{n+1, \ldots, 2n\} - S$ of mixed strategies for player 1 and 2, respectively.

A much better upper bound is $\sqrt{27/4}^{\,n}/\sqrt{n}$, approximately $2.6^n/\sqrt{n}$. As observed by Keiding (1997), this can be derived from the Upper Bound Theorem for polytopes (McMullen, 1970). The polyhedral approach to equilibrium enumeration is due to Vorob'ev (1958), Kuhn (1961), and Mangasarian (1964), and works even for degenerate games. An elegant vertex enumeration algorithm for polytopes due to Avis and Fukuda (1992) has apparently not yet been applied to bimatrix games.

A lower bound for the number of Nash equilibria of nondegenerate $n \times n$ bimatrix games is $2^n - 1$, which holds for the "coordination game" where both player's payoffs are given by the identity matrix. Quint and Shubik (1997) conjectured this to be the upper bound as well. For $n \leq 3$ this follows from the Upper Bound Theorem. For $n = 4$ it has been shown by Keiding (1997) using Grünbaum and Sreedharan's (1967) characterization of the relevant 4-polytopes, and by McLennan and Park (1996) using the geometry of 3-space.

However, the Quint–Shubik conjecture is false in general. We show a new lower bound of about $2.414^n/\sqrt{n}$. Our construction is based on the polars of cyclic polytopes, which have a simple combinatorial definition (due to Gale, 1963) and a maximal number of vertices. The inequalities defining these polytopes are permuted in a certain way to obtain games with asymptotically $(1+\sqrt{2})^n/\sqrt{n}$ many equilibria, except for a constant factor. This is not far from the upper bound $2.6^n/\sqrt{n}$ and suggests that vertex enumeration of polytopes is indeed an efficient approach to equilibrium enumeration.

The complementarity condition for Nash equilibria has a geometric interpretation with labels for mixed strategies marking the best responses of the other player (see Shapley, 1974). This subdivision of the mixed strategy sets corresponds to the facets of a polyhedron in one dimension higher, considering the maximum of the payoff functions. This simplifies a complexity study, and is known similarly for Voronoi diagrams in computational geometry (see, for example, Mulmuley, 1994). The polyhedron with the payoff as one unbounded coordinate, in turn, corresponds to a simpler, bounded polytope where mixed strategies are not normalized. The computational equivalence of these different views is straightforward and well known. Nevertheless, a geometric interpretation, which we explain in Section 2, may also be considered helpful.

Another question of general interest may be the definition of a degenerate game. This is usually stated ad hoc and merely "similarly" to related papers (see Lemke and Howson, 1964; Shapley, 1974; van Damme, 1987, p. 52; and others). We have proved elsewhere that these notions are in fact equivalent (von Stengel, 1996), and repeat this theorem here. In Section 2, we clarify and summarize these polytope-related issues.

The lower bound construction is shown in Section 3. The first interesting case is a 6×6 game with 75 Nash equilibria, which we provide explicitly. An asymptotic expression for the bound is derived in Section 4.

2. Finding equilibria as a polytope problem

We use the following notation. Let (A, B) be a bimatrix game, where A and B are $m \times n$ matrices of payoffs to the row player 1 and the column player 2, respectively. $B^\top$ is the matrix B transposed. A vector or matrix with all components zero is denoted $\mathbf{0}$. Inequalities like $x \geq \mathbf{0}$ between two vectors hold for all components. The vector $(1, \ldots, 1)^\top$ in $\mathbb{R}^n$ is denoted $\mathbf{1}_n$. The $n \times n$ identity matrix is I_n. We always assume

$$A \text{ and } B^\top \text{ are nonnegative and have no zero column.} \tag{2.1}$$

This assumption can be made without loss of generality since a constant can be added to all payoffs without changing the game in a material way. We could simply assume that A and B are positive but want to admit examples like $A = B = I_n$ (if $m = n$) where some payoffs are zero.

Consider the two polyhedral sets

$$\begin{aligned} P_1 &= \{x \in \mathbb{R}^m \mid x \geq \mathbf{0},\ B^\top x \leq \mathbf{1}_n\,\}, \\ P_2 &= \{y \in \mathbb{R}^n \mid Ay \leq \mathbf{1}_m,\ y \geq \mathbf{0}\,\}. \end{aligned} \tag{2.2}$$

The purpose of condition (2.1) is to assure that P_1 and P_2 are bounded and therefore *polytopes*, that is, bounded intersections of halfspaces. We recall some notions from polytope theory (see Ziegler, 1995). The vectors $z_1, \ldots, z_k$ are called *affinely* *independent* iff (if and only if) the vectors $\begin{bmatrix} z_1 \\ 1 \end{bmatrix}, \ldots, \begin{bmatrix} z_k \\ 1 \end{bmatrix}$ are linearly independent. A convex set has *dimension* d if it has $d+1$, but no more, affinely independent points. A d-polytope is a polytope of dimension d. (P_1 has dimension m, P_2 has dimension n.) A *face* of a polytope P is a subset of P of the form $\{z \in P \mid cz = p_0\}$ for a row vector c and scalar p_0 where $cz \le p_0$ holds for all z in P. A *vertex* of P is the unique element of a 0-dimensional face of P. A *facet* of a d-polytope P is a face of dimension $d-1$. It corresponds to an inequality used in the definition of P which is *binding*, that is, it holds as equality, and *irredundant*, that is, it cannot be omitted without changing the polytope.

Let a_i and b_j denote the rows of A and $B^\top$,

$$A = \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix}, \qquad B^\top = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}.$$

For $x \in P_1$ and $y \in P_2$, let

$$\begin{aligned} L_1(x) &= \{i \mid x_i = 0\} \;\cup \{m+j \mid b_j x = 1\}, \\ L_2(y) &= \{i \mid a_i y = 1\} \cup \{m+j \mid y_j = 0\}. \end{aligned}$$

$L_1(x)$ and $L_2(y)$ are sets of *labels* where x or y has a label in $\{1, \ldots, m+n\}$ if the respective inequality in (2.2) is binding. Labels are useful for identifying equilibria. We call (x, y) an *equilibrium* of the polytope pair (P_1, P_2) if $x \in P_1$, $y \in P_2$, and

$$L_1(x) \cup L_2(y) = \{1, \ldots, m+n\}. \tag{2.3}$$

This is justified by the following observation.

Proposition 2.1. *Given (2.1), the mixed strategy pair $(\overline{x}, \overline{y})$ is a Nash equilibrium of the bimatrix game (A, B) iff there is an equilibrium (x, y) of the polytope pair (P_1, P_2), $(x, y) \neq (\mathbf{0}, \mathbf{0})$, and $\overline{x} = x \cdot (1/\mathbf{1}_m^\top x)$, $\overline{y} = y \cdot (1/\mathbf{1}_n^\top y)$.*

Proof. Clearly, $(\overline{x}, \overline{y})$ is a Nash equilibrium of (A, B) iff, for suitable reals u, v,

$$\mathbf{1}_m^\top \overline{x} = 1, \quad \overline{x} \ge \mathbf{0}, \qquad\quad b_j \overline{x} \le v \quad (1 \le j \le n), \tag{2.4}$$

$$a_i \overline{y} \le u \quad (1 \le i \le m), \quad \mathbf{1}_n^\top \overline{y} = 1, \quad \overline{y} \ge \mathbf{0}, \tag{2.5}$$

and

$$\begin{aligned} \overline{x}_i > 0 \quad &\Longrightarrow \quad a_i \overline{y} = u \quad (1 \le i \le m), \\ \overline{y}_j > 0 \quad &\Longrightarrow \quad b_j \overline{x} = v \quad (1 \le j \le n). \end{aligned} \tag{2.6}$$

Condition (2.6) says that only pure best responses are played with positive probability. By (2.1), the equilibrium payoffs u, v are positive. With $x = \overline{x} \cdot (1/v)$, $y = \overline{y} \cdot (1/u)$, conditions (2.4), (2.5), and (2.6) imply $x \in P_1$, $y \in P_2$, and (2.3), respectively. Conversely, any pair $(x, y) \neq (\mathbf{0}, \mathbf{0})$ in $P_1 \times P_2$ with (2.3) and

$$v = 1/\mathbf{1}_m^\top x, \quad \overline{x} = x \cdot v, \quad u = 1/\mathbf{1}_n^\top y, \quad \overline{y} = y \cdot u \tag{2.7}$$

fulfills (2.4), (2.5), and (2.6). $\square$

The vectors x and y in (2.2), which are not normalized, are converted by (2.7) to mixed strategies $\overline{x}$ and $\overline{y}$ and payoffs v, u. This transformation is common, for example for the algorithm by Lemke and Howson (1964) as described in Wilson (1992). This algorithm connects the equilibrium $(\mathbf{0}, \mathbf{0})$ of (P_1, P_2) to a Nash equilibrium along a path of points (x, y) where, say, label 1 may be missing, that is, $\{2, \ldots, m+n\} \subseteq L_1(x) \cup L_2(y)$. In nondegenerate games, all equilibria of (P_1, P_2) are separate endpoints of such Lemke–Howson paths, so their number is even, and the number of Nash equilibria of (A, B) is odd.

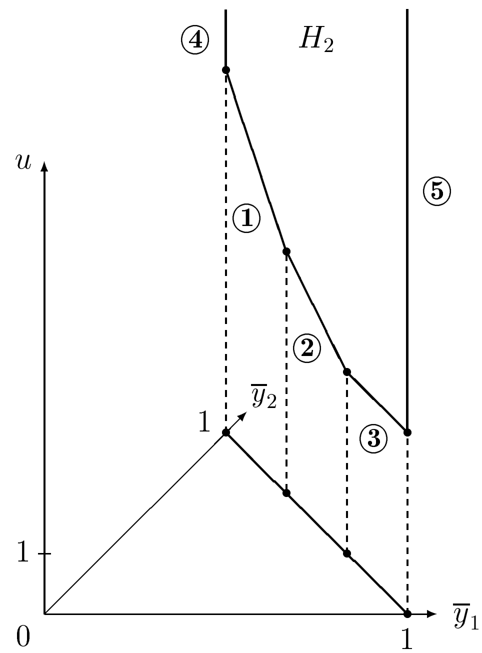

Figure 2.1. The polyhedron H_2 for the game (2.8), and its projection to the set $\{ (\overline{y}, 0) \mid (\overline{y}, u) \in H_2 \}$. The vertical scale is displayed shorter. The circled numbers label the facets of H_2 and identify pure best responses of player 1 or unplayed pure strategies of player 2.

For an enumeration of equilibria, Mangasarian (1964) considered the polyhedral sets defined by (2.4) and (2.5), namely

$$H_1 = \{ (\overline{x}, v) \mid \mathbf{1}_m^\top \overline{x} = 1,\ \overline{x} \geq \mathbf{0},\ B^\top \overline{x} \leq \mathbf{1}_n v \},$$
$$H_2 = \{ (\overline{y}, u) \mid A\overline{y} \leq \mathbf{1}_m u,\ \mathbf{1}_n^\top \overline{y} = 1,\ \overline{y} \geq \mathbf{0} \}.$$

For the game

$$A = \begin{bmatrix} 0 & 6 \\ 2 & 5 \\ 3 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 4 & 4 \end{bmatrix}, \tag{2.8}$$

the set H_2 is shown in Figure 2.1. The facets of H_2 have labels similar to the elements of $L_2(y)$ labeling the facets of P_2. For identifying equilibria, it suffices to consider these labels for the mixed strategies $\overline{y}$ of player 2, in Figure 2.1 indicated by the projection $(\overline{y}, 0)$ of $(\overline{y}, u)$, and similarly the mixed strategies of player 1. The resulting subdivision of the mixed strategy sets into best response regions can be used to visualize Nash equilibria if $m, n \leq 4$ (Shapley, 1974), and is called a Lemke–Howson diagram by Quint and Shubik (1997).

The polyhedra H_1 and H_2 are by (2.7) in one-to-one correspondence to $P_1 - \{\mathbf{0}\}$ and $P_2 - \{\mathbf{0}\}$, respectively. Figure 2.2 shows a geometric interpretation of the (nonlinear) map $(\overline{y}, u) \mapsto \overline{y} \cdot (1/u)$, which is a *projective transformation* (see Ziegler, 1995, Sect. 2.6). The points y in $P_2 - \{\mathbf{0}\}$ arise as points $(y, 1)$ on the lines connecting any $(\overline{y}, u)$ in H_2 with $(\mathbf{0}, 0)$ in $\mathbb{R}^{n+1}$.

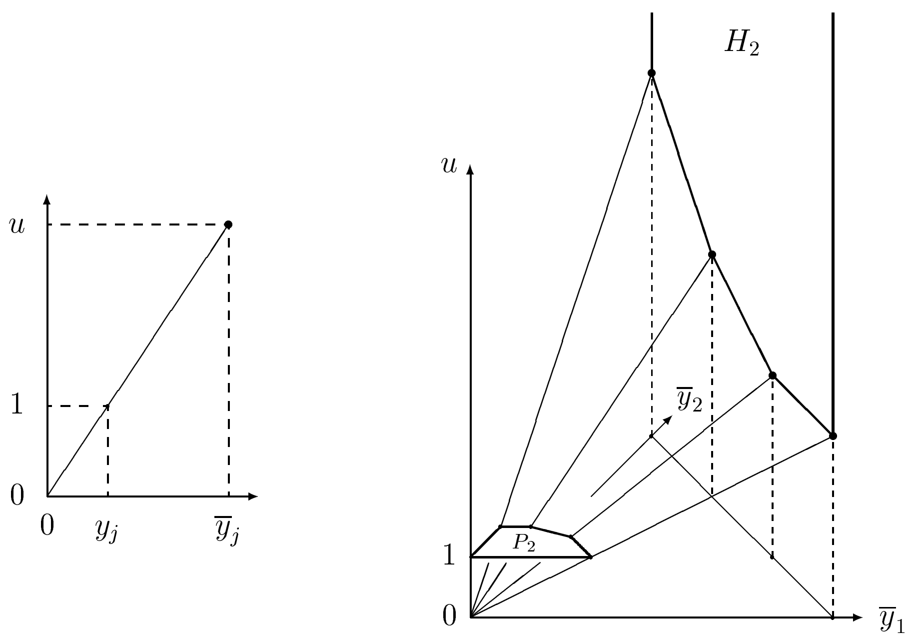

Figure 2.2. The map $H_2 \to P_2$, $(\overline{y}, u) \mapsto y = \overline{y} \cdot (1/u)$ as a projective transformation from $\mathbb{R}^{n+1}$ to the hyperplane $\{(y, 1) \mid y \in \mathbb{R}^n\}$ with projection point $(\mathbf{0}, 0)$. The left hand side shows this for a single component $\overline{y}_j$ of $\overline{y}$, where $y_j = \overline{y}_j / u$. The right hand side shows how P_2 arises in this way from H_2 in the example (2.8).

Any equilibrium of (P_1, P_2) is a convex combination of *extreme* equilibria (x, y) where x is a vertex of P_1 and y is a vertex of P_2 (Mangasarian, 1964; Winkels, 1979; Jansen, 1981). We consider only *nondegenerate* games where only pairs of vertices can be equilibria. Otherwise, the game may have infinitely many equilibria (as convex combinations of extreme equilibria). Furthermore, even the number of extreme equilibria may trivially be very large, for example if all entries of B are identical (so all vertices of P_1 except $\mathbf{0}$ have all but one label) and P_2 is a polytope with a maximum number of vertices. Nondegeneracy holds (with probability one) for a "generic" game (where each payoff is chosen independently from a continuous distribution). We use the following definition, where the *support* of a vector z is

$$\operatorname{supp}(z) = \{\, i \mid z_i \neq 0 \,\}.$$

Definition 2.2. *A bimatrix game is called nondegenerate if no mixed strategy z of a player has more than $|\operatorname{supp}(z)|$ pure best responses.*

Interpreted for the polytopes P_1 and P_2, degeneracy has two possible reasons. The first is a redundancy of the *description* of the polytope, that is, certain inequalities in (2.2) do not define facets of P_1 or P_2. For P_2, say, the inequalities $y_j \geq 0$ for $j = 1, \ldots, n$ are clearly irredundant, so every equality $y_j = 1$ defines a facet. Using linear programming duality, it can be shown that an inequality of the form $a_i y \leq 1$ is redundant for P_2 iff the pure strategy i of player 1 is weakly dominated by or payoff equivalent to a different mixed strategy $\overline{x}$ of player 1, that is, $a_i \leq \overline{x}^\top A$. If the pure strategy i is strictly dominated ($a_i < \overline{x}^\top A$ for some $\overline{x}$), then i is never played in equilibrium. Redundant inequalities $a_i y \leq 1$ of this sort can safely be omitted. However, a weakly but not strongly dominated strategy leads to a degenerate game (von Stengel, 1996, Theorem 2.8).

The second reason for degeneracy can be recognized from the polytope as a set. Assume that each inequality defines a facet. Then in a degenerate game, P_1 or P_2 has a vertex that belongs to more than d facets, where d is the dimension (m or n) of the polytope. A polytope where each vertex belongs to exactly d facets is called *simple*. In the game (2.8), P_1 is not simple because its vertex $(0, 0, {}^1\!/_4)^\top$ belongs to four facets. This game is degenerate since the pure strategy 3 of player 1 has two pure best responses. In general, one can show the following (for a proof see von Stengel, 1996).

Proposition 2.3. *Let (A, B) be a bimatrix game and let (2.1) hold. The following are equivalent:*

(a) The game is nondegenerate.

(b) The rows of $\begin{bmatrix} -I_m \\ B^\top \end{bmatrix}$ corresponding to the labels in $L_1(x)$ for any x in P_1 are linearly independent, and the corresponding condition holds for any y in P_2.

(c) P_1 and P_2 are simple polytopes, and any pure strategy of a player that is weakly dominated by or payoff equivalent to another mixed strategy is strictly dominated.

Condition (b) is used by Lemke and Howson (1964). As another equivalent condition for nondegeneracy, Shapley (1974) requests essentially for sets of labels L that the set $\{x \in P_1 \mid L \subseteq L_1(x)\}$ has dimension at most $m - |L|$, and the corresponding condition for P_2.

Finding a nondegenerate game (A, B) with a certain number of Nash equilibria can be phrased in terms of polytopes alone: Let P_1 and P_2 be simple polytopes of dimension m and n, respectively, both with $m + n$ facets labeled $1, \ldots, m + n$ in some order. A vertex has the labels of the facets it lies on. A pair (x, y) of vertices is called an equilibrium of (P_1, P_2) if x and y together have all labels $1, \ldots, m + n$. Clearly, this is the situation for P_1 and P_2 in (2.2). The only special property of these polytopes is the vertex pair $(\mathbf{0}, \mathbf{0})$, which is an equilibrium, and the directions of the facets meeting there. The latter can be achieved for any polytope by an affine transformation, which does not change the combinatorial structure (the face incidences) of the polytope. Except for one equilibrium of (P_1, P_2) that takes the role of $(\mathbf{0}, \mathbf{0})$, the simple polytopes P_1 and P_2 and their labeling can be arbitrary:

Proposition 2.4. *The following are equivalent:*

(a) *There is a nondegenerate $m \times n$ bimatrix game (A, B) with E Nash equilibria.*

(b) *There are simple polytopes P_1 and P_2 of dimension m and n, respectively, both with $m + n$ facets labeled $1, \ldots, m + n$, so that (P_1, P_2) has $E + 1$ equilibria (completely labeled vertex pairs), $E \geq 0$.*

Proof. It remains to show that (b) implies (a). Let P_1' and P_2' be simple polytopes of dimension m and n, respectively, each with $m+n$ labeled facets. By assumption, (P_1', P_2') has at least one equilibrium (x', y'). We permute the labels $1, \ldots, m + n$ (in the same way for P_1' and P_2') such that x' has labels $1, \ldots, m$ and y' has labels $m + 1, \ldots, m + n$, which does not change the equilibria of (P_1', P_2'). Let

$$P_1' = \{z \in \mathbb{R}^m \mid Cz \leq p,\ Dz \leq q\}$$

where $Cz \leq p$ represents the m inequalities for the facets $1, \ldots, m$ and $Dz \leq q$ the remaining n inequalities. For the vertex x', we have $Cx' = p$ and $Dx' < q$ since P_1' is simple. The m binding inequalities for x' are linearly independent, so C is invertible and $z \mapsto x = -Cz + p$ is an affine transformation with inverse $z = -C^{-1}(x - p)$. Let $P_1 = \{x \in \mathbb{R}^m \mid -C^{-1}(x - p) \in P_1'\}$. Then, with $r = q - DC^{-1}p$,

$$P_1 = \{x \in \mathbb{R}^m \mid -x \leq \mathbf{0},\ -DC^{-1}x \leq r\}.$$

Corresponding points of P_1 and P_1' have the same labels. Since the vertex $\mathbf{0}$ of P_1 corresponds to x' in P_1', $\mathbf{0} < r$. Thus, the jth row of $-DC^{-1}x \leq r$ can be normalized by multiplication with the scalar $1/r_j$, so we can assume $r = \mathbf{1}_n$. Then P_1 is defined as in (2.2) with the $n \times m$ transposed payoff matrix $B^\top = -DC^{-1}$. Similarly, we can find an $m \times n$ matrix A so that P_2 in (2.2) is an affine transform of P_2'. If desired, a constant can be to the entries of A and B to obtain (2.1), which does not change the combinatorial structure of P_1 and P_1' (see Figure 2.2). The game (A, B) is nondegenerate by Proposition 2.3. □

As mentioned, the Lemke–Howson paths show that the number of equilibria of a polytope pair (P_1, P_2) is even. For general polytopes, it is possible that (P_1, P_2) has no equilibria, so this case is explicitly excluded in Proposition 2.4(b).

3. Using cyclic polytopes

By Proposition 2.4, nondegenerate games with many equilibria correspond to pairs (P_1, P_2) of simple polytopes with many equilibria. Every vertex is part of at most one equilibrium, so we look for polytopes with many vertices. For the $n \times n$ game with $A = B = I_n$, both polytopes P_1 and P_2 in (2.2) are equal to the unit cube $[0,1]^n$ which has 2^n vertices and where every vertex is part of an equilibrium. The Quint–Shubik conjecture states that 2^n is the maximum number of equilibria of a polytope pair (P_1, P_2) for an $n \times n$ game.

Our construction, which refutes this conjecture, is based on the polars of cyclic polytopes, which have a maximum number of vertices. For any subset P of $\mathbb{R}^d$, its *polar* $P^\triangle$ (see Ziegler, 1995, Section 2.3) is defined by

$$P^\triangle = \{y \in \mathbb{R}^d \mid y^\top x \le 1 \text{ for all } x \in P\}.$$

Suppose P is a polytope with $\mathbf{0}$ in its interior. Then

$$P = \{x \in \mathbb{R}^d \mid c_i^\top x \le 1,\ 1 \le i \le N\} \tag{3.1}$$

for suitable d-vectors $c_1, \ldots, c_N$. Then $P^\triangle$ is the convex hull of these vectors $c_1, \ldots, c_N$, so $P^\triangle$ is a polytope. Furthermore, $P^\triangle$ has $\mathbf{0}$ in its interior, and $P^{\triangle\triangle} = P$. Suppose further that no inequality $c_i x^\top \le 1$ in (3.1) can be omitted. Then it defines a facet of P, and c_i is a vertex of $P^\triangle$. More generally, any face of P of dimension $d-k$ is defined by k binding inequalities in (3.1), and corresponds to a face of dimension $k-1$ of $P^\triangle$, given by the convex hull of the k corresponding vertices of $P^\triangle$. The polytope P is simple (no vertex of P belongs to more than d facets) iff its polar $P^\triangle$ is *simplicial* (no facet of $P^\triangle$ contains more than d vertices). We obtain a simple polytope P with N facets and V vertices as the polar $Q^\triangle$ of a simplicial polytope Q with N vertices and V facets (after possibly translating Q so that $\mathbf{0}$ is an interior point).

The *cyclic polytope* $C_d(N)$ (see Ziegler, 1995, p. 14) in dimension d with N vertices is defined as the convex hull of any N points on the *moment curve* $\{\mu(t) \mid t \in \mathbb{R}\}$ in $\mathbb{R}^d$, $\mu(t) = (t, t^2, \ldots, t^d)^\top$. Any $d+1$ points on this curve are affinely independent, so $C_d(N)$ is simplicial. The particular choice of the vertices $\mu(t_1), \ldots, \mu(t_N)$ does not affect the combinatorial structure of the polytope $C_d(N)$. Assume $t_1 < \cdots < t_N$. A set S of d vertices corresponds to a 0-1 string $s = s_1 s_2 \ldots s_N$ with $s_i = 1$ if $\mu(t_i) \in S$ and $s_i = 0$ otherwise. The hyperplane H through the points in S defines a facet of $C_d(N)$ iff the string s fulfills the *Gale* *evenness* condition (Gale, 1963), that is, it contains no substring $s_i \ldots s_j = 01\cdots 10$ with an odd number $i-j-1$ of 1's (like 01110). Otherwise, if $i-j-1$ was odd (with $s_i = s_j = 0$, $s_{i+1} = \cdots = s_{j-1} = 1$), then the two vertices $\mu(t_i)$ and $\mu(t_j)$ would be on opposite sides of H, since the moment curve changes from one side of H to the other at the points $\mu(t_i)$, $i \in S$.

We will use this representation of the facets of $C_d(N)$ by 0-1 strings. Since the 1's in these strings come in pairs (except possibly at the beginning or end), the number $\Phi(d,N)$ of these strings is determined as follows. Suppose d is odd, $d = 2l+1$. Then l pairs of 1's and $N-d$ 0's can be arranged in $\binom{l+N-d}{l}$ many ways, and the remaining 1 be put at the beginning or end, to obtain the string s, so

$$\Phi(2l+1, N) = 2\binom{N-l-1}{l}. \tag{3.2}$$

If d is even, $d = 2l$, then either s starts and ends with an even number of 1's, and is composed of l substrings 11 and $N-d$ 0's, or s is such a string with $l-1$ substrings 11 and an additional 1 put at each end. Hence,

$$\Phi(2l, N) = \binom{N-l}{l} + \binom{N-l-1}{l-1} = \frac{N}{l}\binom{N-l-1}{l-1}. \tag{3.3}$$

No d-polytope with N vertices has more facets than the cyclic polytope $C_d(N)$, according to the *Upper Bound Theorem* for polytopes (McMullen, 1970; for a self-contained proof see Mulmuley, 1994). Applied to the polars, this implies that no d-polytope with N facets has more than $\Phi(d,N)$ vertices. Hence, the polytopes P_1 and P_2 in (2.2) have at most $\Phi(m, m+n)$ and $\Phi(n, m+n)$ vertices, respectively. The bound is stricter for the polytope of smaller dimension since (3.2) and (3.3) imply $\Phi(d,N) < \Phi(d+1,N)$ if $d < N/2$. Thus, we can state the following bound on the number of equilibria, as observed by Keiding (1997):

Proposition 3.1. *A nondegenerate $m \times n$ bimatrix game, $m \leq n$, has at most $\Phi(m, m+n) - 1$ Nash equilibria.*

For $m = n$, $\Phi(n, 2n)$ grows asymptotically from n to $n+1$ by a factor $\sqrt{27/4} = 2.598...$, much faster than 2^n. We consider more precise asymptotics in Section 4.

For the rest of the paper, $m = n = d$. Let $P_1^\Delta = P_2^\Delta = C_d(2d)$. In this polar version of the equilibrium problem, both P_1^Δ and P_2^Δ have $N = 2d$ vertices which are labeled $1, \ldots, N$. Every equilibrium is a pair of facets of P_1^Δ and P_2^Δ such that the labels of the vertices incident to these facets form the set $\{1, \ldots, N\}$.

It suffices to look at the combinatorial definition of these facets. A facet of P_1^Δ corresponds to a certain 0-1 string $s = s_1 s_2 \ldots s_N$, for example $s = 01101100$ if $d = 4$, $N = 8$. These strings are *balanced*, that is, contain the same number of 0's and 1's, and fulfill the Gale Evenness condition. We can assume that the labeling of the N vertices of P_1^Δ is in the order of the positions in this string. The labeling of the vertices of P_2^Δ is given by a certain permutation ν of $1, \ldots, N$, such that s (defining a facet of P_1^Δ) is part of an equilibrium iff the complementary permuted string

$$\overline{s}_\nu := \overline{s}_{\nu(1)}\overline{s}_{\nu(2)} \ldots \overline{s}_{\nu(N)}$$

defines a facet of P_2^Δ, that is, fulfills the Gale Evenness condition, where $\overline{0} = 1$ and $\overline{1} = 0$. For example, suppose that ν is the identity permutation. Then for $s = 01101100$, $\overline{s}_\nu = 10010011$ which does not fulfill Gale Evenness, whereas $\overline{s}_\nu$ does for $s = 00011110$. For these two strings s, the opposite holds when considering the permutation

$$\nu(i) = \begin{cases} i-1 & \text{if } i \text{ is even} \\ i+1 & \text{if } i \text{ is odd,} \end{cases} \qquad 1 \le i \le N. \tag{3.3}$$

With the identity permutation ν, the two cyclic polytopes P_1^Δ and P_2^Δ do not have more than 2^d equilibria, since only the strings s that are composed of substrings 00 and 11, except at the ends, have the property that both s and $\overline{s}_\nu$ fulfill Gale Evenness. However, the permutation ν in (3.3) leads to a number of equilibria that exceeds 2^d for $d = 6$ and all $d \ge 8$.

Proposition 3.2. *Let $S(l)$ be the set of balanced 0-1 strings of length $4l$ composed of the substrings 00, 11, and 0110. Let s be any balanced 0-1 string of length $N = 2d$. Then for the permutation ν in (3.3), s and $\overline{s}_\nu$ fulfill the Gale Evenness condition iff*

(a) if $d = 2l$: $s \in S(l)$ or $s = 10s'01$ for some $s' \in S(l-1)$,

(b) if $d = 2l+1$: $s = 10s'$ or $s = s'01$ for some $s' \in S(l)$.

Proof. Clearly, any string s in (a) or (b) fulfills the Gale Evenness condition. The substrings 00, 11, and 0110 in s are complemented to 11, 00, and 1001, respectively, and permuted by ν to substrings 11, 00, and 0110, respectively, in $\overline{s}_\nu$. Similarly, an initial or terminal substring 10 or 01 is left as it is, so $\overline{s}_\nu$ also fulfills Gale Evenness. Conversely, suppose s is not of the described form. If s starts with the substring 10, remove it. Then, remove repeatedly all initial substrings 00, 11, or 0110 from s. If the remainder starts with 10, 0100, or 0101, the Gale Evenness condition fails for s. If it starts with 0111 (the only possibility left), it becomes 0100 in $\overline{s}_\nu$ so the condition fails there. ☐

Let $E(d)$ be the number of equilibria in our construction, where $P_1^\Delta = P_2^\Delta = C_d(2d)$ and the labels of P_2^Δ are permuted by ν. By Proposition 3.2, $E(d)$ is determined by the number $\sigma(l) := |S(l)|$ of balanced 00-11-0110 strings of length $4l = 2d$, namely

$$E(2l) = \sigma(l) + \sigma(l-1), \qquad E(2l+1) = 2\sigma(l). \tag{3.4}$$

If a string in $S(l)$ contains k substrings 00, $0 \le k \le l$, then it contains the same number of substrings 11 since it is balanced, and $l-k$ substrings 0110. These substrings may be arranged in any manner, with $(l+k)!/(k!\,k!\,(l-k)!)$ many possibilities. Hence,

$$\sigma(l) = \sum_{k=0}^{l} \frac{(l+k)!}{k!\,k!\,(l-k)!} = \sum_{k=0}^{l} \binom{l+k}{k}\binom{l}{k}. \tag{3.5}$$

The first values of $\sigma(l)$ are given as follows. The numbers $\tilde{\sigma}(l)$ are an asymptotic approximation that we will prove in the next section.

l	0	1	2	3	4	5	6	7	8
$\sigma(l)$	1	3	13	63	321	1683	8989	48639	265729
$\tilde{\sigma}(l)$		3.4	13.8	65.5	330.4	1722.6	9165.3	49456.6	269636.8

(3.6)

Our construction produces the first counterexample to the Quint–Shubik conjecture for $d = 6$ since $E(6) = 76 > 2^6$ (already $E(4) = 16 = 2^4$, where the equilibrium supports are quite different from the game where $A = B = I_4$). In general, $E(d) > 2^d$ for all $d \geq 8$.

A specific 6×6 bimatrix game with 75 Nash equilibria is obtained as follows. The 12 points $\mu(t)$ on the moment curve in $\mathbb{R}^6$ for $t = -6, -5, \ldots, -1$ and $t = 1, \ldots, 6$ determine the vertices of a cyclic polytope $C_6(12)$, which is translated to have $\mathbf{0}$ in its interior, here chosen to coincide with the barycenter of the vertices. The polar is defined by 12 inequalities, which for P_2 are pairwise interchanged (the first and second inequality, third and forth, and so on), according to ν in (3.3). The affine transformation in the proof of Proposition 2.4 is applied to represent P_1 and P_2 as in (2.2). Multiplying all payoffs by 1584 to obtain integers gives

$$A = \begin{bmatrix} 9504 & -660 & 19976 & -20526 & 1776 & -8976 \\ -111771 & 31680 & -130944 & 168124 & -8514 & 52764 \\ 397584 & -113850 & 451176 & -586476 & 29216 & -178761 \\ 171204 & -45936 & 208626 & -263076 & 14124 & -84436 \\ 1303104 & -453420 & 1227336 & -1718376 & 72336 & -461736 \\ 737154 & -227040 & 774576 & -1039236 & 48081 & -300036 \end{bmatrix},$$

$$B = \begin{bmatrix} 72336 & 48081 & 29216 & 14124 & 1776 & -8514 \\ -461736 & -300036 & -178761 & -84436 & -8976 & 52764 \\ 1227336 & 774576 & 451176 & 208626 & 19976 & -130944 \\ -1718376 & -1039236 & -586476 & -263076 & -20526 & 168124 \\ 1303104 & 737154 & 397584 & 171204 & 9504 & -111771 \\ -453420 & -227040 & -113850 & -45936 & -660 & 31680 \end{bmatrix}.$$

The obvious open question is if this construction produces $d \times d$ games with a maximum number of equilibria (for $d = 6$ and $d \geq 8$). The cyclic polytopes are plausible candidates because they have a maximum number of vertices. However, not all of their vertices can be part of equilibria. For $d \leq 6$, a check by computer shows that among all permutations ν, the one in (3.3) maximizes the number of equilibria. It is not the only such permutation, but all others produce very similar sets of equilibria. For $d = 6$, the second largest number of equilibria that occurs is 60. Permutations that yield no equilibria exist as well. Checking larger dimensions is difficult because of the enormous growth of the number $(2d)!$ of permutations. The case $d = 6$ was checked in 12 hours on a workstation, $d = 7$ would take 390 times longer than that (changing from 12! to 14! permutations and from 112 to 240 vertices of $C_d(2d)$). The required computing effort for a brute-force check is too large for $d > 7$. While the cyclic polytopes with their regular structure might eventually permit a proof that the above construction is optimal, other polytopes with many vertices may be very difficult to examine.

4. Asymptotics of upper and lower bounds

A nondegenerate $d \times d$ game has as most $\Phi(d, 2d) - 1$ many Nash equilibria by Proposition 3.1 and may have $E(d) - 1$ many as defined by (3.4) and (3.5). In order to compare these functions better with 2^d, we will find asymptotically equal expressions. Functions f, g are called asymptotically equal, denoted $f(n) \sim g(n)$ as $n \to \infty$, if $f(n)/g(n) \to 1$, that is, the relative error goes to zero (a very accessible introduction to asymptotics and generating functions is Graham, Knuth, and Patashnik, 1991). We apply Stirling's formula

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

to the upper bound $U(d) := \Phi(d, 2d)$ in (3.2) and (3.3), which yields

$$\begin{aligned} U(2l) &= 2\binom{3l-1}{l} = 2\frac{2l}{3l}\binom{3l}{l} \sim \frac{2}{3}\sqrt{\frac{3}{\pi l}}\left(\frac{27}{4}\right)^l, \\ U(2l+1) &= 2\binom{3l+1}{l} = 2\frac{3l+1}{2l+1}\binom{3l}{l} \sim \frac{3}{2}\sqrt{\frac{3}{\pi l}}\left(\frac{27}{4}\right)^l. \end{aligned}$$

Expressed in terms of d,

$$U(d) \sim \begin{cases} 2\sqrt{\dfrac{2}{3\pi}}\,\dfrac{\sqrt{27/4}^{\,d}}{\sqrt{d}} \approx .921\dfrac{2.5981^d}{\sqrt{d}}, & d \text{ even} \\[2ex] \sqrt{\dfrac{2}{\pi}}\,\dfrac{\sqrt{27/4}^{\,d}}{\sqrt{d}} \approx .798\dfrac{2.5981^d}{\sqrt{d}}, & d \text{ odd.} \end{cases} \tag{4.1}$$

Finding a similar asymptotic expression for $\sigma(l)$ in (3.5) is more interesting. This integer sequence has been studied before, as (3.6) looked up in Sloane and Plouffe (1995) (and its electronic server, described there) reveal. The number $\sigma(n)$ is the number of "King paths on a chessboard" (Moser, 1955), that is, the number of paths in a two-dimensional integer lattice from $(0,0)$ to (n,n) where the allowed steps are one unit right, up, or diagonal (each such step corresponding to a substring 00, 11, or 0110, respectively). According to an exercise in Comtet (1974, p. 81), $\sigma(n) = P_n(3)$ for the nth Legendre polynomial P_n defined explicitly by

$$P_n(x) = \sum_{k=0}^{n} \binom{n+k}{k} \binom{n}{k} \left(\frac{x-1}{2}\right)^k \tag{4.2}$$

(Moser and Zayachkowski, 1961) or recursively by $P_0(x) = 1$, $P_1(x) = x$ and

$$P_n(x) \;=\; x\,(2 - 1/n)\,P_{n-1}(x) \;-\; (1 - 1/n)\,P_{n-2}(x)\,. \tag{4.3}$$

The recurrence (4.3) can be verified by (4.2). For $x = 3$, $\sigma(n) = P_n(3)$, it can – with some effort – also be given a combinatorial interpretation in terms of the lattice paths with diagonal steps. Using the generating function

$$g(y) = \sum_{n\geq 0} \sigma(n)\, y^n, \tag{4.4}$$

the recurrence (4.3) for $x = 3$ is equivalent to the differential equation

$$g'(y)(1 - 6y + y^2) + g(y)(y - 3) = 0$$

which, with $g(0) = \sigma(0) = 1$, has the unique solution

$$g(y) = \frac{1}{\sqrt{1 - 6y + y^2}}\,. \tag{4.5}$$

Regarded as a function on the complex plane $\mathbb{C}$, the function g is analytic around the origin with Taylor coefficients $\sigma(n)$ as in (4.4). We use a theorem by Flajolet and Odlyzko (1990) that shows how to obtain information about these coefficients from the behavior of g at its dominant singularity (the one with smallest absolute value). For simplicity, we state this theorem with overly strong assumptions, which hold here, concerning the domain of the function; $[1,\infty)$ denotes the set of all reals z with $z \geq 1$.

Theorem 4.1. *(Flajolet and Odlyzko, 1990, Corollary 2.) Assume that $f(z)$ is analytic in $\mathbb{C} - [1,\infty)$, and that as $z \to 1$ in $\mathbb{C}$,*

$$f(z) \sim K\,(1 - z)^{\alpha}$$

where K and α are real constants, α not a positive integer. Then, as $n \to \infty$,

$$[z^n]f(z) \sim \frac{K}{\Gamma(-\alpha)} n^{-\alpha-1}. \tag{4.6}$$

In (4.6), $[z^n]f(z)$ is the Taylor coefficient of z^n in the expansion of $f(z)$, and Γ is the Gamma function, where $\Gamma(1/2) = \sqrt{\pi}$. We use Theorem 4.1 for $\alpha = -1/2$ but have to normalize the dominant singularity of $g(y)$ to one. It is given by the smaller root r of the roots r and R of the polynomial $1 - 6y + y^2$,

$$r = 3 - 2\sqrt{2}, \qquad R = 3 + 2\sqrt{2},$$

so that

$$g(y) = \Big((r-y)(R-y)\Big)^{-1/2} = \Big(r\,(R-y)(1-y/r)\Big)^{-1/2}.$$

Let $z = y/r$, $y = rz$,

$$f(z) = g(rz) = \Big(r\,(R-rz)(1-z)\Big)^{-1/2} \sim \Big(r\,(R-r)\Big)^{-1/2}(1-z)^{-1/2}$$

as $z \to 1$. Using $1/r = R = (1+\sqrt{2})^2$, (4.6) yields

$$[z^n]f(z) \sim \frac{1+\sqrt{2}}{2^{5/4}\sqrt{\pi n}}$$

as $n \to \infty$ and, since $g(y) = f(y/r)$,

$$\sigma(n) \sim \tilde{\sigma}(n) := \frac{1+\sqrt{2}}{2^{5/4}\sqrt{\pi}}\ \frac{(1+\sqrt{2})^{2n}}{\sqrt{n}}. \tag{4.7}$$

The relative error of this approximation is for $n \geq 6$ less than two percent, as (3.6) shows. A better approximation would introduce factors like $(1+c/n)$ for a constant c so that the relative error is of order $O(n^{-2})$ rather than $O(n^{-1})$ (as it is known for Stirling's formula, see Graham et al., 1991), which we have not investigated.

The asymptotic expression becomes simpler when used for the number $E(d)$ of equilibria in (3.4) since $1+\sqrt{2}$ appears in the denominator and cancels. Expressed in terms of d, as $d \to \infty$,

$$E(d) \sim \begin{cases} \sqrt{\dfrac{2\sqrt{2}}{\pi}}\ \dfrac{(1+\sqrt{2})^d}{\sqrt{d}} \approx .949\dfrac{2.414^d}{\sqrt{d}}, & d \text{ even} \\ \sqrt{\dfrac{\sqrt{2}}{\pi}}\ \dfrac{(1+\sqrt{2})^d}{\sqrt{d}} \approx .671\dfrac{2.414^d}{\sqrt{d}}, & d \text{ odd.} \end{cases} \tag{4.8}$$

As in (4.1), the numerical constants are rounded (with $\sqrt{27/4}^{\,d}$ in (4.1) rounded *up* to 2.5981^d so that the upper bound $U(d)$ is asymptotically true).

As (4.1) and (4.8) show, the number $E(d)$ of equilibria in our construction is not that far away from the upper bound $U(d)$, at least compared with the previously known lower bound 2^d. We summarize our result (where the upper bound is due to Keiding, 1997).

Theorem 4.2. *The possible number of Nash equilibria in a nondegenerate $d \times d$ bimatrix game is asymptotically, as $d \to \infty$, bounded from above by $U(d)$ in (4.1) and from below by $E(d)$ in (4.8).*

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