Exact and Asymptotic Solutions of a Divide-and-Conquer Recurrence Dividing at Half: Theory and Applications

HSIEN-KUEI HWANG, Academia Sinica
SVANTE JANSON, Uppsala University
TSUNG-HSI TSAI, Academia Sinica

Divide-and-conquer recurrences of the form

$$f(n) = f\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + f\left(\left\lceil \frac{n}{2} \right\rceil\right) + g(n) \qquad (n \geqslant 2),$$

with $g(n)$ and $f(1)$ given, appear very frequently in the analysis of computer algorithms and related areas. While most previous methods and results focus on simpler crude approximation to the solution, we show that the solution always satisfies the simple *identity*

$$f(n) = nP(\log_2 n) - Q(n)$$

under an optimum (iff) condition on $g(n)$. This form is not only an identity but also an asymptotic expansion because $Q(n)$ is of a smaller order than linearity. Explicit forms for the *continuous* periodic function P are provided. We show how our results can be easily applied to many dozens of concrete examples collected from the literature and how they can be extended in various directions. Our method of proof is surprisingly simple and elementary but leads to the strongest types of results for all examples to which our theory applies.

CCS Concepts: • **General and reference** → **Evaluation**; • **Mathematics of computing** → *Combinatoric problems*; *Enumeration*; *Differential calculus*; Combinatorial algorithms; • **Theory of computation** → *Divide and conquer*; Sorting and searching;

Additional Key Words and Phrases: Analysis of algorithms, recurrence relation, asymptotic linearity, periodic oscillation, identity, master theorems, functional equation, asymptotic approximation, uniform continuity, additivity, sensitivity analysis

ACM Reference format:
Hsien-Kuei Hwang, Svante Janson, and Tsung-Hsi Tsai. 2017. Exact and Asymptotic Solutions of a Divide-and-CONQUER Recurrence Dividing at Half: Theory and Applications. *ACM Trans. Algorithms* 13, 4, Article 47 (October 2017), 43 pages.
https://doi.org/10.1145/3127585

47

Part of this work was carried out while the first author was visiting the Department of Mathematics, Uppsala University; he thanks the Department for its hospitality and support. Part of the work of the second author was carried out during visits to the Isaac Newton Institute for Mathematical Sciences (EPSCR Grant Number EP/K032208/1) and was partially supported by a grant from the Simons Foundation and a grant from the Knut and Alice Wallenberg Foundation; he thanks these foundations for their hospitality and support.
Authors' addresses: H.-K. Hwang and T.-H. Tsai, Institute of Statistical Science, Academia Sinica, 128, Sec. 2, Academia Road, Taipei 11529, Taiwan; emails: {hkhwang, chonghi}@stat.sinica.edu.tw; S. Janson, Department of Mathematics, Uppsala University, PO Box 480, SE-751 06, Uppsala, Sweden; email: svante.janson@math.uu.se.

https://doi.org/10.1145/3127585

1 INTRODUCTION

Divide-and-conquer is one of the most widely used design paradigms in computer algorithms; it often appears in the form of subproblems of nearly the same cardinalities. Such a "principle of balancing" has long been observed to be "a basic guide to good algorithm design" (see [1, §2.7]) and has found fruitful applications in algorithmics; typical examples can be found in computer arithmetics, mergesort, sorting and merging networks, digital sums, fast Fourier transform, computational geometry algorithms, combinatorial sequences, random trees, and more. The analysis of the corresponding algorithms often leads, in its simplest form, to recurrences of the form

$$f(n) = f\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + f\left(\left\lceil \frac{n}{2} \right\rceil\right) + g(n) \qquad (n \geqslant 2) \tag{1.1}$$

for given $g(n)$ and $f(1)$; here, the function $g(n)$ is often called the *toll function*. The recurrence Equation (1.1) can also be written as

$$\begin{cases} \quad f(2n) = 2f(n) + g(2n), \\ f(2n+1) = f(n) + f(n+1) + g(2n+1) \end{cases} \qquad (n \geqslant 1). \tag{1.2}$$

For simplicity, we refer to Equation (1.1) (or Equation (1.2)) as the *BDC (Balanced Divide-and-Conquer) recurrence*. Such a recurrence also naturally arises as the solution of the recurrences with maximization or minimization, such as

$$f(n) = \min_{1 \leqslant j < n} \{f(j) + f(n-j)\} + g(n) \qquad (n \geqslant 2),$$

when $g(n)$ is convex (i.e., the second difference of $g(n)$ is nonnegative and $g(3) \geqslant g(2)$); see [37, 42, 49]).

Effective Bounds for (1.1). In most cases, one seeks crude upper or lower bounds for the solution of the BDC recurrence (Equation (1.1)); for that purpose, there are many different approaches used in the literature, three common ones being as follows.

- *Change the two-sided recurrence (Equation (1.1)) into a one-sided one*: Replace the floor function in Equation (1.1) by ceiling function or the other way round, resulting in the two recurrences

$$\begin{cases} f(n) = 2f\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + g(n), \\ f(n) = 2f\left(\left\lceil \frac{n}{2} \right\rceil\right) + g(n), \end{cases} \qquad (n \geqslant 2), \tag{1.3}$$

  which provide good lower and upper bounds to the original solution. Such one-sided recurrences are easier to solve because $\lfloor \frac{1}{2} \lfloor \frac{n}{2} \rfloor \rfloor = \lfloor \frac{n}{4} \rfloor$ and $\lceil \frac{1}{2} \lceil \frac{n}{2} \rceil \rceil = \lceil \frac{n}{4} \rceil$ for all n, so that their solutions can be readily obtained by iteration:

$$\begin{cases} f(n) = \sum_{0 \leqslant k < L_n} 2^k g\left(\left\lfloor \frac{n}{2^k} \right\rfloor\right) + 2^{L_n} f(1), \\ f(n) = \sum_{0 \leqslant k \leqslant L_{n-1}} 2^k g\left(\left\lceil \frac{n}{2^k} \right\rceil\right) + f(1) 2^{L_{n-1}+1}, \end{cases} \qquad (n \geqslant 2),$$

  respectively, where, *here and throughout this article*, $L_x := \lfloor \log_2 x \rfloor$ for $x > 0$. Then, the asymptotic behavior of $g(n)$ can be translated into that of $f(n)$ by a direct bounding argument. In particular, we have that $f(n) = O(n \log n)$ when $g(n) = O(n)$.
- *From power-of-two to general n*: Alternatively, the BDC recurrence can be solved by assuming that n is a power of two and then by iterating the resulting difference equation, giving

$$f(n) = 2f\left(\frac{n}{2}\right) + g(n) = \sum_{0 \leqslant k < L_n} 2^k g\left(\frac{n}{2^k}\right) + 2^{L_n} f\left(\frac{n}{2^{L_n}}\right), \tag{1.4}$$

Table 1. Master Theorems for Some Recurrences

Bentley et al. (9) Verma (74), Mogos (57)	$f(n) = cf(bn) + g(n)$
Wang and Fu (76)	$f(n) = c_n f(b_n) + g(n)$
Akra and Bazzi (2) Leighton (55) Kao (52), Verma (75) Schöning (68), Yap (77)	$f(x) = \sum_{1\leqslant k\leqslant r} c_k f(b_k x) + g(x)$
Roura (67)	$f(n) = \sum_{1\leqslant k\leqslant r} c_{n,k} f(b_{n,k}) + g(n)$
Drmota and Szpankowski (27)	$f(n) = \sum_{1\leqslant k\leqslant r} c_k f\left(\lfloor b_k n + \delta_k \rfloor\right)$ $+ \sum_{1\leqslant k\leqslant r} c'_k f\left(\lceil b_k n + \delta'_k \rceil\right) + g(n)$

Note: Except for the last two references, most results are of an O-type and one major proof-technique is based on iteration and induction. Here, c, c_n, $c_{n,k}$, c'_k are all positive constants, b, b_n, $b_{n,k} \in (0, 1)$ and δ_k, $\delta'_k = O(k^{1-\varepsilon})$ for some $\varepsilon > 0$.

and then the growth order of $f(n)$ may be deduced from that of $g(n)$ by induction or by monotonicity.

- *Master Theorems*: Yet another widely used approach is to apply the so-called "Master Theorems," which, for our BDC recurrence, has the form

$$f(n) = \begin{cases} O(n), & \text{if } g(n) = O(n^{1-\varepsilon}), \\ O(n\log n), & \text{if } g(n) = O(n), \\ \Theta(g(n)), & \text{if } g(n) = \Omega(n^{1+\varepsilon}) \text{ and regular varying.} \end{cases} \tag{1.5}$$

  We see particularly that linearity serves as a "watershed function" [77] separating small and large cost: very roughly, if $g(n)$ is sufficiently smaller than linear, then $f(n)$ is always linear, while if $g(n)$ is larger than linear, then $f(n)$ is of the same order as that of $g(n)$. This form was proposed by Bentley et al. [9], which is the first paper on Master Theorems and shaped much of the early development of the topic; note that special cases, such as $g(n) = O(1)$ and $g(n) = O(n)$, were discussed in the classical book by Aho et al. [1] on algorithms. On the other hand, "Master Theorems" first appeared in the book by Cormen et al. [22].

Master Theorems such as (1.5) for different recursions have been the subject of many papers; we briefly summarize the major ones in Table 1.

It is worth mentioning that recurrences of similar types, particularly the form examined by Akra and Bazzi [2] and Leighton [55], were also studied in number theory, functional equations (often referred to as "linear functional equations") and other areas; see, for example, [32, 47], [54, Ch. 6] and the references therein.

Asymptotic Linearity of Equation (1.1). Returning to the BDC recurrence (Equation (1.1)), as far as the asymptotic linearity of $f(n)$ is concerned, namely, $f(n) = O(n)$—the following conditions on $g(n)$ have been proved to be sufficient. Here and throughout this article, $\varepsilon > 0$ represents a small constant whose value may differ from one occurrence to another.

- Aho et al. [1]: $g(n) = O(1)$;
- Bentley et al. [9]: $g(n) = O(n^{1-\varepsilon})$;
- Brassard and Bratley [11, p. 77], Yap [77]: $g(n) = O(\frac{n}{(\log n)^{1+\varepsilon}})$;

- Verma [74]:

$$g(n) \geqslant 0, \frac{g(n)}{n} \text{ nonincreasing and } \sum_{k\geqslant 1} \frac{g(2^k)}{2^k} \text{ converges;} \tag{V}$$

- Akra and Bazzi [2] and Leighton [55]:

$$g(x) \geqslant 0, c_1 g(x) \leqslant g(u) \leqslant c_2 g(x) \text{ for } \frac{1}{2}x \leqslant u \leqslant x \text{ and } \sum_{1\leqslant k\leqslant n} \frac{g(k)}{k^2} = O(1). \tag{ABL}$$

Our natural motivating question was: *what is the optimum (necessary and sufficient) condition for the asymptotic linearity of $f(n)$*, namely, *under what condition(s) on $g(n)$ does $f(n)$ satisfy the estimate $f(n) = \Theta(n)$ and vice versa?* Verma addressed this question in [74] and argued that $f(n) = \Theta(n)$ iff $g(n)$ satisfies conditions (V). However, as we will see, his sufficient conditions are not necessary; for example, neither positivity nor monotonicity is needed. On the other hand, the conditions (ABL) are not necessary either because the polynomial growth condition is very strong and does not apply to sequences containing gaps (e.g., $g(n) = \mathbf{1}_{n \text{ odd}}$). Also, $g(n)$ in general may oscillate between positive and negative values.

Since the monotonicity condition in (V) and the polynomial growth condition in (ABL) are both very restrictive, we then ask if the boundedness of the two partial sums appeared in both conditions (V) and (ABL) alone are optimum? This is a very natural guess in view of the closeness of the other sufficient conditions to n that we listed above. However, the answer is still in the negative, as the following two examples show (they are not even sufficient). More precisely, that the condition $\sum_{0\leqslant k\leqslant m} \frac{g(2^k)}{2^k} = O(1)$ is insufficient for $f(n) = O(n)$ is seen by the example

$$g(n) = \begin{cases} \frac{2^\ell}{\ell}, & \text{if } n = 3\cdot 2^\ell,\ \ell \geqslant 1 \\ 0, & \text{otherwise} \end{cases} \implies \begin{cases} \sum_{0\leqslant k\leqslant m} \frac{g(2^k)}{2^k} = O(1) \\ \text{but } f(3\cdot 2^m) = \Theta(2^m \log m). \end{cases}$$

Similarly, the insufficiency of the condition $\sum_{0\leqslant k\leqslant n} \frac{g(k)}{k^2} = O(1)$ becomes obvious through the example

$$g(n) = \begin{cases} \frac{2^\ell}{\ell}, & \text{if } n = 2^\ell,\ \ell \geqslant 1; \\ 0, & \text{otherwise} \end{cases} \implies \begin{cases} \sum_{0\leqslant k\leqslant n} \frac{g(k)}{k^2} = O(1) \\ \text{but } f(2^m) = \Theta(2^m \log m). \end{cases}$$

As we will see, these conditions are, although not sufficient, very close to being optimum.

Note that partial sums of the form $\sum_{1\leqslant k\leqslant n} \frac{g(k)}{k^2}$ also appeared in other contexts, such as

- divide-and-conquer algorithms in computational geometry [21, 24, 25]
- quicksort and search trees [19, 20, 48]
- linearity of subadditive functions [41, 42],

and the partial sum $\sum_{1\leqslant k\leqslant m} \frac{g(2^k)}{2^k}$ arises in the analysis of queue-mergesort [16] and bounds for recurrences with minimization or maximization [49, 56].

Periodic Oscillations of Equation (1.1). In addition to more rough O-bounds, the exact and asymptotic aspects exhibited by the BDC recurrence lead to many interesting periodic oscillating phenomena (as will be demonstrated in this article through many concrete examples), which have been less explored so far. One of the main goals of this article is to show that the BDC recurrence (Equation (1.1)), under very general conditions on $g(n)$, has always an *exact solution* of the form

$$f(n) = F(n) + nP(\log_2 n) - Q(n) \qquad (n \geqslant 2), \tag{1.6}$$

where $F(n)$ is either 0 or larger than linear, $P(x)$ is 1-periodic, and $Q(n) = o(n)$. Furthermore, each of these functions can be readily computed or even admit a simple closed-form expression. This

Table 2. Some Examples of $g(n)$ and Their Interpolated Extensions $g(x)$.

$g(n)$	$g(x)$	$g(n)$	$g(x)$
c	c	n	x
$\mathbf{1}_{n \text{ is odd}}$	$\begin{cases} \{x\} & \text{if } \lfloor x \rfloor \text{ is even} \\ 1-\{x\} & \text{if } \lfloor x \rfloor \text{ is odd} \end{cases}$	$\mathbf{1}_{n\equiv 2 \bmod 4}$	$\begin{cases} \{x\} & \text{if } \lfloor x \rfloor \equiv 1 \bmod 4 \\ 1-\{x\} & \text{if } \lfloor x \rfloor \equiv 2 \bmod 4 \\ 0 & \text{if } \lfloor x \rfloor \equiv \{0,3\} \bmod 4 \end{cases}$
$\lfloor \log_2 n \rfloor$	$\begin{cases} \lfloor \log_2 x \rfloor + \{x\} & \text{if } \lfloor x \rfloor + 1 = 2^{\lfloor \log_2 x \rfloor + 1} \\ \lfloor \log_2 x \rfloor & \text{otherwise} \end{cases}$	$\lfloor \frac{n}{2} \rfloor$	$\begin{cases} \left\lfloor \frac{x}{2} \right\rfloor + \{x\} & \text{if } \lfloor x \rfloor \text{ is odd} \\ \left\lfloor \frac{x}{2} \right\rfloor & \text{if } \lfloor x \rfloor \text{ is even} \end{cases}$

implies that most crude or asymptotic approximations to Equation (1.1) by using uniquely ceiling or floor functions are to some extent unnecessary. We also show that approximating Equation (1.1) by Equation (1.3) will not only lose precision of approximation but also result in *discontinuous* periodic functions as opposed to *continuous* P in Equation (1.6). Thus, *the continuity of* P *represents a characteristic property of the BDC recurrence.*

Asymptotic solutions to Equation (1.1) were systematically analyzed in [33, 34] by a novel, powerful analytic approach based on Mellin-Perron integral, finite differences and Dirichlet series; see also [35, 38]. This approach was later refined in [39, 45, 46], leading to exact solutions that are also asymptotic in nature. These articles deal with more specific problems, although the approaches used are quite general. By a completely different approach, Kieffer [53] shows that

$$g(n) = O(1) \Longrightarrow f(n) = nP(\log_2 n) + o(n), \tag{1.7}$$

where $P(t)$ is a *continuous* 1-periodic function. Then, it is also natural to ask: *what is the iff-condition for the estimate on the right-hand side of Equation (1.7)*? See also [33, 34, 39, 44, 60] for more examples with explicitly computable periodic function P and more precise approximations.

Our Main Results. The key to our optimum condition of the asymptotic linearity of $f(n)$ relies on *linear interpolation*, which extends the sequence $f(n)$ to a function defined for all real $x \geqslant 0$ by

$$f(x) := f(\lfloor x \rfloor) + \{x\}(f(\lfloor x \rfloor + 1) - f(\lfloor x \rfloor)) \qquad (x \geqslant 0), \tag{1.8}$$

where $\{x\}$ denotes the fractional part of x, and $g(x)$ is defined similarly; see Table 2 for a few concrete examples of a sequence and its interpolated function. With the introduction of this relation, the recurrence (Equation (1.1)) can then be written in the more general yet much simpler form (see Lemma 1),

$$f(x) = 2f\left(\frac{x}{2}\right) + g(x) \qquad (x \geqslant 2), \tag{1.9}$$

whose solution is readily obtained by iteration as in Equation (1.4), provided that we define $g(0)$ and $g(1)$ properly; see Lemma 2 for more details.

Theorem 1 (Asymptotic linearity of $f(n)$: O-bound). *Define the sequence $f(n)$ by Equation (1.1) and $G_m(t) := \sum_{0\leqslant k\leqslant m} 2^{-k} g(2^k t)$. Then,*

$$f(n) = O(n) \quad \textit{iff} \quad G_m(t) = O(1) \textit{ for } m \geqslant 1 \textit{ and } t \in [1,2]. \tag{1.10}$$

We see that our optimum condition requires neither positivity nor monotonicity nor polynomial growth condition of $g(n)$ such as that in (ABL) but instead relies on the boundedness of a weighted partial sum of the interpolated function. Note that the results mentioned above from [1, 2, 9, 11, 55, 74, 77] yielding $f(n) = O(n)$ under various conditions all follow immediately.

It turns out that in almost all cases of interest, the O-bound can be replaced by more precise asymptotic or exact expressions under a slightly stronger condition. Recall that a sequence $\{f_n(x)\}$ of functions *converges uniformly* to a limiting function $f(x)$ for $x \in [a,b]$ if for any $\varepsilon > 0$ there exists an integer N such that $|f_n(x) - f(x)| < \varepsilon$ for all $n \geqslant N$ and for all $x \in [a,b]$. While the usual continuity is defined at a point, the uniform continuity is defined on an interval.

THEOREM 2 (ASYMPTOTIC LINEARITY OF $f(n)$: ASYMPTOTICS AND IDENTITY). *Define* $g(0) = g(1) = 0$. *Then, the following statements are equivalent.*

(i) $f(n) = nP(\log_2 n) + o(n)$ *as* $n \to \infty$ *for some continuous and 1-periodic function* P *on* $\mathbb{R}$.
(ii) $f(x) = xP(\log_2 x) + o(x)$ *as* $x \to \infty$ *for some 1-periodic function* P *on* $\mathbb{R}$.
(iii) $G_m(t) := \sum_{0\leqslant k\leqslant m} 2^{-k} g(2^k t)$ *converges uniformly to* $G(t) := \sum_{k\geqslant 0} 2^{-k} g(2^k t)$ *for* $t \in [1,2]$ *as* $m \to \infty$.

When these conditions hold, we have an identity

$$f(x) \equiv xP(\log_2 x) - Q(x) \qquad (x \geqslant 1), \tag{1.11}$$

and the closed-form expression for the 1-periodic function P *and the remainder* Q

$$P(t) := \sum_{k\in\mathbb{Z}} 2^{-k-\{t\}} g(2^{k+\{t\}}) + f(1) = \sum_{k\geqslant 0} 2^{-k-\{t\}} g(2^{k+\{t\}}) + f(1) \qquad (t \in \mathbb{R}) \tag{1.12}$$

and

$$Q(x) := G(x) - g(x) = \sum_{k\geqslant 1} 2^{-k} g(2^k x), \tag{1.13}$$

with $Q(x) = o(x)$ *as* $x \to \infty$.

Note that the continuity of P in (ii) is not part of the condition and is automatically implied if (ii) holds.

A trivial case when $g(n) \equiv c$ gives $P \equiv c + f(1)$ and $Q(n) = c$.

The following sufficient condition is stronger but in most cases easier to check.

COROLLARY 1. *If* $g(n) = O(n(\log n)^{-1-\varepsilon})$ *with* $\varepsilon > 0$, *then* $f(n) = nP(\log_2 n) - Q(n)$ *for* $n \geqslant 1$, *where* P, Q *are defined as in Theorem 2 and* $Q(n) = O(n(\log n)^{-\varepsilon})$.

We give many examples below in which $g(n)$ is known explicitly and it is possible to compute $P(t)$ and $Q(n)$ exactly by Equations (1.12) and (1.13). However, Theorem 2 and Corollary 1 are as useful in cases in which we only have an estimate of the toll function $g(n)$; in this case Equation (1.11) still yields a representation of $f(n)$ and Equations (1.12) and (1.13) can be used to derive estimates of the periodic function $P(t)$ and the error term $Q(n)$. As an example, the result Equation (1.7) by Kieffer [53] follows immediately; we obtain a stronger error term $Q(n) = O(1)$ under his condition $g(n) = O(1)$. Similarly, if $g(n) = O(n^{1-\varepsilon})$, then $Q(n) = O(n^{1-\varepsilon})$.

A common case encountered in many examples below is $g(n) = 0$ when n is even. In this case, $Q(n) = 0$ for $n \geqslant 1$ by (1.13).

COROLLARY 2. *If* $g(n) = 0$ *when* n *is even, then* $f(n) = nP(\log_2 n)$ *for* $n \geqslant 1$.

While Theorem 2 and the two corollaries are formulated in terms of a sublinear toll function $g(n)$, their use is not limited to this range. If $g(n)$ is of a higher order, then one can often normalize $f(n)$ properly so that the resulting sequence satisfies Equation (1.1) with a sublinear $g(n)$ for which our framework applies. Roughly, for a suitable $F(n)$, the sequence $f(n) - F(n)$ satisfies Equation (1.1) with a new $g(n)$ satisfying our conditions, which yields Equation (1.6). For example, if $g(n) = \lfloor \frac{n}{2} \rfloor$ (see Example 5.2(b) below), then one can write $g(n) = \frac{n}{2} - \{\frac{n}{2}\}$ and express the solution into two parts: the part corresponding to $\frac{n}{2}$ can be easily solved by iterating Equation (1.9), leading to a simple closed-form expression, and the part corresponding to $\{\frac{n}{2}\}$ is well within the range of applicability of Theorem 2. See Section 5 for details.

The key idea of linear interpolation that we used here also extends to the more general recurrence

$$f(n) = \alpha f\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + \beta f\left(\left\lceil \frac{n}{2} \right\rceil\right) + g(n) \qquad (n \geqslant 2), \tag{1.14}$$

with $f(1)$ and $g(n)$ given, but the technicalities are more involved because the interpolation function is no more linear when $\alpha \neq \beta$. This and finer properties of the periodic function P under stronger conditions will be discussed in a companion paper [50].

From a methodological point of view, it is of interest to mention that many different techniques have been developed for clarifying the asymptotics of general divide-and-conquer recurrences of the form in Equation (1.14) and their extensions. These include (i) real-analytic (including calculus, functional iteration, linear algebra, additivity, repertoire, and so on): see, for example, [2, 9, 40, 42, 65, 74, 77]; (ii) complex-analytic: [27, 33–35, 39, 45]; (iii) Tauberian theorems: [27, 37]; (iv) renewal theory: [32]; and (v) fractal geometry and iterated function system: [28, 53, 59]; These techniques show not only the wide occurrence of the recurrence Equation (1.14) but also its rich mathematical connections to other tools.

This article is structured as follows. We prove Theorem 1 and 2 in the next section. Applications to a large number of examples, mostly from analysis of algorithms and Sloane's Online Encyclopedia of Integer Sequences (OEIS) [70], will be discussed in Sections 3 to 6, grouping according as the growth order of $g(n)$ being bounded, linear, quadratic, or higher. We then consider a few variants and extensions in Section 7, such as the recurrence arising from dividing into $q \geqslant 2$ parts of nearly the same sizes

$$f(n) = \sum_{1 \leqslant k \leqslant q} f\left(\left\lfloor \frac{n+k-1}{q} \right\rfloor\right) + g(n), \tag{1.15}$$

which reduces to Equation (1.1) when $q = 2$. The final section deals briefly with the simpler cases (Equation (1.3)).

Notation. For convenience, we introduce the operator Λ as follows:

$$\Lambda[f](n) := f(n) - f\left(\left\lfloor \frac{n}{2} \right\rfloor\right) - f\left(\left\lceil \frac{n}{2} \right\rceil\right),$$

so that Equation (1.1) can be written as $\Lambda[f](n) = g(n)$ or simply as $\Lambda[f] = g$ (where $n \geqslant 2$ is tacitly understood). Let $L_x = \lfloor \log_2 x \rfloor$ when $x > 0$. The (generic) functions P, Q, G are always defined as in Theorem 2 (except in Sections 7 and 8).

2 THE RECURRENCE $\Lambda[f] = g$ AND ITS SOLUTION

We prove Theorems 1 and 2 in this section. Observe first that the recursion Equation (1.1) for $n \geqslant 2$ does not involve $f(0), g(0)$, and $g(1)$; thus, we may choose their values arbitrarily. For definiteness and for our purposes, we will later choose $f(0) = g(0) = g(1) = 0$.

From the Sequence $f(n)$ to the Continuous Function $f(x)$

Lemma 1. *If we extend $f(n)$ to $f(x)$ and $g(n)$ to $g(x)$ by the linear interpolation Equation (1.8), then $f(x)$ satisfies Equation (1.9) for $x \geqslant 2$.*

Proof. If $x = n$ is an integer, then Equation (1.9) is the same as Equation (1.1), recalling Equation (1.8). Hence, Equation (1.9) holds for integer $x = n \geqslant 2$. Moreover, both sides of Equation (1.9) are linear on each interval $[n, n+1]$; thus, since they are equal at the endpoints, they are equal for all $x \in [n, n+1]$, $n \geqslant 2$. □

A few concrete cases of g discussed below are listed in Table 2 together with their interpolated version.

Identities. By iterating the functional Equation (1.9), we obtain first the following relation.

LEMMA 2. *For any $x \geqslant 1$ and $0 \leqslant m \leqslant L_x$,*

$$f(x) = \sum_{0\leqslant k<m} 2^k g\left(2^{-k}x\right) + 2^m f\left(2^{-m}x\right). \tag{2.1}$$

Remark 1. Lemmas 1 and 2 are valid for any $f(0)$, $g(0)$, and $g(1)$ since we only claim Equation (1.9) for $x \geqslant 2$. If we choose $f(0) = f(1)$ and $g(0) = g(1) = -f(1)$, then Equation (1.1) also holds for $n = 0, 1$ and the proof above shows that Equation (1.9) holds for all $x \geqslant 0$. These choices provide a more elegant formulation that may have other uses, but, for our purposes, we find it simpler to choose $g(1) = 0$ and consider only $x \geqslant 2$ in (1.9).

From now on and throughout this section, we choose $g(0) = g(1) = 0$ so that $g(x) = 0$ for $x \in [0, 1]$. With this choice of $g(0)$ and $g(1)$, we obtain the following basic identities.

LEMMA 3. *The identities*

$$x^{-1} f(x) = \sum_{0\leqslant k\leqslant L_x} (2^{-k}x)^{-1} g(2^{-k}x) + f(1) \tag{2.2}$$

$$= \sum_{k\geqslant 0} (2^{-k}x)^{-1} g(2^{-k}x) + f(1) \tag{2.3}$$

hold for $x \geqslant 1$.

In particular, if $f(1) = 0$, then

$$x^{-1} f(x) = \sum_{k\geqslant 0} (2^{-k}x)^{-1} g(2^{-k}x) \qquad (x \geqslant 1).$$

Note also that the Master Theorems in Equation (1.5) follow immediately from Equation (2.2).

PROOF. By Equation (1.9), $f(2) = 2f(1) + g(2)$. Thus, by Equation (1.8),

$$f(x) = f(1) + (f(1) + g(2))(x-1) = f(1)x + g(2)(x-1) \qquad (1 \leqslant x \leqslant 2).$$

But, since $g(1) = 0$, we have that $g(x) = g(2)(x-1)$ for $1 \leqslant x \leqslant 2$. Thus,

$$f(x) = f(1)x + g(x) \qquad (1 \leqslant x \leqslant 2).$$

Substituting this relation into Equation (2.1) with $m = L_x$ gives, for $x \geqslant 1$,

$$\begin{aligned} f(x) &= \sum_{0\leqslant k<L_x} 2^k g\left(2^{-k}x\right) + 2^{L_x} f\left(2^{-L_x}x\right) \\ &= \sum_{0\leqslant k\leqslant L_x} 2^k g\left(2^{-k}x\right) + f(1)x, \end{aligned} \tag{2.4}$$

since $1 \leqslant 2^{-L_x}x < 2$. This proves Equation (2.2), and Equation (2.3) follows since $g(2^{-k}x) = 0$ for $k > L_x$. □

Proof of Theorem 1. Write $\theta_x := \{\log_2 x\}$, so that $x = 2^{L_x+\theta_x}$. Then, by Equation (2.2) and making the change of variables $k \mapsto L_x - k$, we see that for $x \geqslant 1$,

$$x^{-1} f(x) = \sum_{0\leqslant k\leqslant L_x} 2^{-k-\theta_x} g\left(2^{k+\theta_x}\right) + f(1) = 2^{-\theta_x} G_{L_x}(2^{\theta_x}) + f(1). \tag{2.5}$$

Thus, if $G_m(t) = O(1)$, then $f(x) = O(x)$, and vice versa. □

Proof of Theorem 2. (iii) (*uniform convergence of* $G_m(t)$) ⇒(i),(ii) (*asymptotics of* $f(n)$ *and* $f(x)$): Assume that (iii) holds. Then, we first show that the series

$$P_1(t) := \sum_{k\in\mathbb{Z}} 2^{-(k+t)} g\left(2^{k+t}\right) = \sum_{k\in\mathbb{Z}} 2^{-(k+\{t\})} g\left(2^{k+\{t\}}\right) \tag{2.6}$$

is a well-defined continuous 1-periodic function. Consider first $t \in [0, 1]$. Since $g(x) = 0$ for $0 \leqslant x \leqslant 1$, we then have that

$$P_1(t) = \sum_{k\geqslant 0} 2^{-(k+t)} g(2^{k+t}) = 2^{-t} G(2^t) \qquad (0 \leqslant t \leqslant 1), \tag{2.7}$$

where $G(2^t) = \lim_{m\to\infty} G_m(2^t)$ converges uniformly for $t \in [0, 1]$ by assumption. Hence, the first sum in Equation (2.6) converges; moreover, the uniform convergence theorem and the continuity of $g(x)$ imply that $P_1(t)$ is continuous on $[0, 1]$. Furthermore, by replacing k by $k - \lfloor t\rfloor$, we see that for every $t \in \mathbb{R}$, the two sums in Equation (2.6) are both convergent and identical; thus, P_1 is well defined and 1-periodic on $\mathbb{R}$. Consequently, P_1 and $P = P_1 + f(1)$ are a continuous 1-periodic function on $\mathbb{R}$.

To show (ii), we apply Equation (2.5) and obtain, with $\theta_x = \{\log_2 x\}$ and using Equation (2.7),

$$\begin{aligned} x^{-1} f(x) &= 2^{-\theta_x} G_{L_x}(2^{\theta_x}) + f(1) = 2^{-\theta_x} G(2^{\theta_x}) + f(1) + o(1) \\ &= P_1(\theta_x) + f(1) + o(1) = P(\theta_x) + o(1) = P(\log_2 x) + o(1), \end{aligned} \tag{2.8}$$

as $x \to \infty$. Thus, (ii) holds with the continuity of P, which, in turn, implies (i).

(i) ⇒(ii): Assume that (i) holds. We prove that

$$|x^{-1} f(\lfloor x\rfloor) - P(\log_2 x)| \to 0 \quad \text{and} \quad |x^{-1} f(\lceil x\rceil) - P(\log_2 x)| \to 0, \tag{2.9}$$

as $x \to \infty$, which will then imply (ii) since $f(x)$ linearly interpolates between $f(\lfloor x\rfloor)$ and $f(\lfloor x\rfloor + 1)$. We split the first difference into three parts:

$$\begin{aligned} |x^{-1} f(\lfloor x\rfloor) - P(\log_2 x)| \leqslant{} & |x^{-1} f(\lfloor x\rfloor) - \lfloor x\rfloor^{-1} f(\lfloor x\rfloor)| \\ & + |\lfloor x\rfloor^{-1} f(\lfloor x\rfloor) - P(\log_2 \lfloor x\rfloor)| \\ & + |P(\log_2 \lfloor x\rfloor) - P(\log_2 x)|. \end{aligned}$$

By assumption, $n^{-1} f(n)$ is bounded; thus, the first term satisfies

$$|x^{-1} - \lfloor x\rfloor^{-1}||f(\lfloor x\rfloor)| = O\left(x^{-1}\lfloor x\rfloor^{-1} f(\lfloor x\rfloor)\right) = O(x^{-1}).$$

The second term on the right-hand side tends to zero as $x \to \infty$ by assumption. Finally, since the continuity of P ensures uniform continuity and $|\log_2\lfloor x\rfloor - \log_2 x| = O(x^{-1})$, we see that the third term also converges to zero. This proves the first relation in Equation (2.9). The proof of the other convergence in Equation (2.9) is similar.

(ii) ⇒ (iii): Assume that (ii) holds. Then, for any $\varepsilon > 0$, there exists $K > 0$ such that for all $x \geqslant 2^K$,

$$|x^{-1} f(x) - P(\log_2 x)| < \varepsilon. \tag{2.10}$$

For $t \in [1, 2]$, let $x = 2^k t$, where $k > K$. Then, Equation (2.10) yields

$$|(2^k t)^{-1} f(2^k t) - P(\log_2 t)| < \varepsilon, \tag{2.11}$$

since $P(\log_2 x) = P(k + \log_2 t) = P(\log_2 t)$. By Equation (2.5), for $1 \leqslant t < 2$ and by continuity for $t = 2$,

$$(2^k t)^{-1} f(2^k t) = t^{-1} G_k(t) + f(1).$$

Thus, as $k \to \infty$, Equation (2.11) yields

$$t^{-1} G_k(t) \to P(\log_2 t) - f(1)$$

uniformly for $t \in [1, 2]$, which implies (iii).

To complete the proof of Theorem 2, we observe that if we define $Q(x) := xP(\log_2 x) - f(x)$, implying that Equation (1.11) holds, then for $x \geqslant 1$, by Equations (2.7) and (2.5), since $\log_2 x = L_x + \theta_x$ where $\theta_x = \{\log_2 x\}$,

$$\begin{aligned} Q(x) &= xP(\theta_x) - f(x) = xP_1(\theta_x) + xf(1) - f(x) = x2^{-\theta_x} G\left(2^{\theta_x}\right) - x2^{-\theta_x} G_{L_x}\left(2^{\theta_x}\right) \\ &= 2^{L_x} \sum_{k > L_x} 2^{-k} g\left(2^{k+\theta_x}\right) = \sum_{j \geqslant 1} 2^{-j} g\left(2^j x\right) = G(x) - g(x), \end{aligned}$$

showing Equation (1.13). Moreover, $Q(x) = o(x)$ as $x \to \infty$ by Equation (1.11) and (ii). □

The following sufficient condition is generally simpler to apply.

Corollary 3. *Define* $A_m := \sup_{2^m \leqslant n \leqslant 2^{m+1}} |g(n)|$. *Then,*

$$\sum_{m \geqslant 0} 2^{-m} A_m < \infty \quad \textit{implies} \quad f(x) = xP(\log_2 x) - Q(x) \quad \textit{for } x \geqslant 1,$$

where P *is continuous, 1-periodic, and is given by Equation (1.12) and* $Q(x) := \sum_{k \geqslant 1} 2^{-k} g(2^k x) = o(x)$.

Similar conditions on blockwise suprema appear in many other areas of mathematics, such as the "direct Riemann integrability" in renewal theory; see [66, §3.10].

Remark 2. If Theorem 2 applies, then necessarily $g(n) = o(n)$. In fact, (iii) implies $2^{-k} g(2^k t) \to 0$ uniformly for $t \in [1, 2]$ as $k \to \infty$, and thus $g(x)/x \to 0$ as $x \to \infty$.

Remark 3. The sum $G(t) := \sum_{k \geqslant 0} 2^{-k} g(2^k t)$ in (iii) may fail to converge absolutely. One counterexample is given by taking $g(n) := \frac{(-1)^k}{k} \min(n - 2^k, 2^{k+1} - n)$ for $n \in [2^k, 2^{k+1})$, $k \geqslant 1$. Then, $G(\frac{3}{2}) = \frac{1}{2} \sum_{k \geqslant 1} \frac{(-1)^k}{k}$ $(= -\frac{1}{2} \log 2)$.

Remark 4. Any continuous 1-periodic function $P(x)$ can occur in Theorem 2 for some $f(1)$ and $g(n)$. For example, given P, we may take $f(1) = P(0)$ and then define P_1 and G backwards by $P_1(t) := P(t) - f(1)$ and Equation (2.7), implying that $G(1) = G(2) = P_1(1) = 0$. Then, define $G_m(t)$ for $t \in [1, 2]$ by linear interpolation between the values $G_m(2^{-m} n) := G(2^{-m} n)$, $n \in [2^m, 2^{m+1}]$. There exists a $g(x)$ on $[1, \infty)$ such that $G_m(t) = \sum_{0 \leqslant k \leqslant m} 2^{-k} g(2^k t)$ for $t \in [1, 2]$ and $m \geqslant 0$. This function is linear on each interval $[n, n+1]$ and is thus given by linear interpolation of the sequence $g(n)$. Finally, note that $G_m(t) \to G(t)$ uniformly on $[1, 2]$ since $G(t)$ is continuous. See the graphic renderings of various periodic functions in Sections 3 to 7 on applications.

An Example with Nonuniform Convergence. We now show by a simple example that uniform convergence of $G_m(t)$ is needed for the continuity of P, which also reflects the difference between Theorem 1 and Theorem 2.

Define

$$f(n) = \begin{cases} 0, & \text{if } n = \left\lfloor \frac{2^k}{3} \right\rfloor \text{ or } n = \left\lceil \frac{2^k}{3} \right\rceil, k \geqslant 1; \\ n, & \text{otherwise,} \end{cases}$$

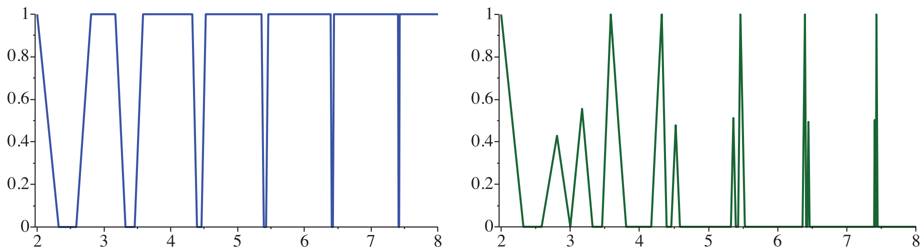


Fig. 1. The functions $\frac{f(x)}{x}$ and $\frac{g(x)}{x}$ in logarithmic scale.

and let $g(n)$ be defined by Equation (1.1). Then, $g(n) = 0$ unless $|n - \frac{2^k}{3}| \leqslant \frac{7}{3}$ for $k \geqslant 1$. Note that $\lceil \frac{2^k}{3} \rceil = \lfloor \frac{2^k}{3} \rfloor + 1$ for $k \geqslant 0$. More precisely, $g(n) \neq 0$ if and only if, writing $n_k := \lfloor \frac{2^k}{3} \rfloor$,

$$n \in \{4, 7\} \text{ or } n \in \bigcup_{k \geqslant 5 \text{ odd}} \{n_k - 1, n_k + 2, n_k + 3\} \text{ or } n \in \bigcup_{k \geqslant 6 \text{ even}} \{n_k - 2, n_k - 1, n_k + 2\}.$$

Note first that $f(n) = O(n)$ and thus $G_m(t) = O(1)$ for $t \in [1, 2]$ by Theorem 1.

Furthermore, $f(x) = x$ unless $|x - \frac{2^k}{3}| \leqslant \frac{5}{3}$ for some k. If $x \in [1, 2]$ and $x \neq \frac{4}{3}$ (or for any $x \notin \{\frac{2^k}{3}\}_{k \in \mathbb{Z}}$), then this holds for $2^m x$ for all large x, and thus $f(2^m x) \sim 2^m x$. However, $f(\frac{2^m}{3}) = 0$ for all m. Thus, we see that $(2^m x)^{-1} f(2^m x) \to P(\log_2 x)$ as $m \to \infty$, where the function

$$P(x) = \begin{cases} 0, & x + \log_2 3 \in \mathbb{Z}; \\ 1, & \text{otherwise}, \end{cases} \tag{2.12}$$

is not continuous; see Figure 1.

Moreover, it follows from Equation (2.5) that if $1 \leqslant x < 2$ and $m \geqslant 0$, thus $L_{2^m x} = m$, and then $(2^m x)^{-1} f(2^m x) = x^{-1} G_m(x)$. Consequently, $x^{-1} G_m(x) \to P(\log_2 x)$ as $m \to \infty$, and thus

$$G_m(x) \to G(x) = xP(\log_2 x) \tag{2.13}$$

for $1 \leqslant x < 2$; it is easily verified that $G_m(2) = 2G_{m+1}(1)$; thus, Equation (2.13) holds for $x = 2$ too. However, since the limit Equation (2.12) is discontinuous, the convergence is not uniform on $[1, 2]$. In fact, $g(n) = n$ for arbitrarily large n; thus, $g(n)$ is not $o(n)$; see Remark 2. In this example, $f(n) = 0$ for arbitrarily large n; thus, $n^{-1} f(n) \nrightarrow 1$, although Equation (1.12) converges for every t.

Fourier Expansions. The periodic function P can be computed, in addition to the series expansion in Equation (1.12), via its Fourier series. Although the polynomial convergence rate of the Fourier series is generally much worse than the exponential rate provided by Equation (1.12), the viewpoint from the frequency domain (rather than from the time domain) provides much information. For example, the mean value of P in the unit interval is given by the 0th Fourier coefficient, and the other coefficients yield an estimate of the magnitude of the oscillations of P.

Theorem 3 (Fourier series expansion of P). *Suppose that the equivalent conditions (i) to (iii) in Theorem 2 hold. Let*

$$\chi_k := \frac{2k\pi i}{\log 2} \qquad (k \in \mathbb{Z}), \tag{2.14}$$

and let

$$D(s) := \sum_{n \geqslant 2} g(n) \Big((n+1)^{-s} - 2n^{-s} + (n-1)^{-s} \Big), \tag{2.15}$$

which converges at least for $s \in \{\chi_k : k \in \mathbb{Z}\} \cup \{s : \Re(s) > 0\}$. *Then,* $P(t)$ *has the Fourier series expansion:*

$$P(t) \sim f(1) + \frac{D'(0)}{\log 2} + \frac{1}{\log 2}\sum_{k\neq 0}\frac{D(\chi_k)}{\chi_k(\chi_k+1)}\, e^{2k\pi i t}, \tag{2.16}$$

where

$$D'(0) := \sum_{n\geqslant 2} g(n)\Big(2\log n - \log(n+1) - \log(n-1)\Big). \tag{2.17}$$

Compare the expansions in [34]. Here, we use the symbol "∼" for the Fourier series since the series may not converge for every t (although it does in typical examples); see Remark 8.

Proof. Since $P(t)$ is 1-periodic and integrable (in fact, continuous), it has a Fourier series expansion $P(t) \sim \sum_{k\in\mathbb{Z}} \hat{P}(k)e^{2k\pi i t}$, and since $P(t) = P_1(t) + f(1)$, we have that $\hat{P}(k) = \hat{P}_1(k) + \delta_{k0} f(1)$. By Equation (2.7) and the uniform convergence of G_m to G on $[1,2]$ and noting that $2^{\chi_k} = 1$,

$$\begin{aligned}
\hat{P}_1(k) &= \int_0^1 P_1(t)e^{-2k\pi i t}\,\mathrm{d}t = \int_0^1 G(2^t)2^{-t}e^{-2k\pi i t}\,\mathrm{d}t = \frac{1}{\log 2}\int_1^2 G(v)v^{-2-\chi_k}\,\mathrm{d}v\\
&= \frac{1}{\log 2}\lim_{m\to\infty}\int_1^2 \sum_{0\leqslant j\leqslant m} 2^{-j}g(2^j v)v^{-2-\chi_k}\,\mathrm{d}v\\
&= \frac{1}{\log 2}\lim_{m\to\infty}\sum_{0\leqslant j\leqslant m}\int_{2^j}^{2^{j+1}} g(y)y^{-2-\chi_k}\,\mathrm{d}y\\
&= \frac{1}{\log 2}\lim_{m\to\infty}\int_1^{2^{m+1}} g(y)y^{-2-\chi_k}\,\mathrm{d}y.
\end{aligned} \tag{2.18}$$

Furthermore, $g(n) = o(n)$ (see Remark 2); thus, $\int_{2^m}^{2^{m+1}} |g(x)|x^{-2}\,\mathrm{d}x = o(1)$ as $m \to \infty$. Consequently, Equation (2.18) shows that

$$\hat{P}_1(k) = \frac{1}{\log 2}\int_1^\infty g(y)y^{-2-\chi_k}\,\mathrm{d}y = \frac{1}{\log 2}\int_0^\infty g(y)y^{-2-\chi_k}\,\mathrm{d}y, \tag{2.19}$$

where the integrals converge conditionally in the usual sense, namely, as $\lim_{A\to\infty}\int^A$.

Now, the linear interpolation in Equation (1.8) can be written as

$$g(x) = \sum_{n\geqslant 2} g(n)\min(x-(n-1), n+1-x)\mathbf{1}_{n-1\leqslant x\leqslant n+1}, \tag{2.20}$$

and thus for any s such that the integral $\int_1^\infty g(x)x^{-2-s}\,\mathrm{d}x$ converges conditionally, as $N \to \infty$,

$$\begin{aligned}
\int_1^\infty g(x)x^{-2-s}\,\mathrm{d}x &= \int_1^N g(x)x^{-2-s}\,\mathrm{d}x + o(1)\\
&= \sum_{2\leqslant n\leqslant N} g(n)\int_{n-1}^{n+1}\frac{\min(x-(n-1), n+1-x)}{x^{2+s}}\,\mathrm{d}x + o(1).
\end{aligned} \tag{2.21}$$

An elementary integration yields, for $s \neq 0, -1$,

$$\int_{n-1}^{n+1}\frac{\min(x-(n-1), n+1-x)}{x^{2+s}}\,\mathrm{d}x = \frac{1}{s(s+1)}\Big((n-1)^{-s} - 2n^{-s} + (n+1)^{-s}\Big) \tag{2.22}$$

and thus by Equations (2.21) and (2.15), for every s such that integral converges (at least conditionally),

$$\int_1^\infty g(x)x^{-2-s}\,\mathrm{d}x = \frac{D(s)}{s(s+1)}, \tag{2.23}$$

with the sum in Equation (2.15) converging. In particular, Equations (2.19) and (2.23) yield

$$\hat{P}_1(k) = \frac{D(\chi_k)}{\chi_k(\chi_k+1)\log 2} \qquad (k \neq 0). \tag{2.24}$$

For $k = 0$, we similarly obtain $\int_1^\infty g(x)x^{-2}\,\mathrm{d}x = D'(0)$ given by Equation (2.17) using an analogue of Equation (2.22) (or by letting $s \to 0$ in Equation (2.22)), and thus

$$\hat{P}_1(0) = \frac{D'(0)}{\log 2}. \tag{2.25}$$

This completes the proof of Theorem 3. □

Remark 5. By Equation (2.19), $\hat{P}_1(k)$ equals $1/\log 2$ times the Mellin transform $\tilde{g}(s) := \int_0^\infty g(x)x^{s-1}\,\mathrm{d}x$ evaluated at $s = -1 - \chi_k$. Since $g(x) = o(x)$ as $x \to \infty$ and $g(x) = 0$ for $x \leqslant 1$, the Mellin transform converges absolutely and is analytic at least for $\Re(s) < -1$; however, we are interested in points on the boundary of this domain. The proof above shows only that the Mellin transform converges conditionally at the points $s = -1 - \chi_k$ at which absolute convergence may not be guaranteed. There may exist other s with $\Re(s) = -1$ where the Mellin transform does not even converge conditionally; a counterexample is given by $g(n)$ in Remark 3.

Similarly, $D(s)$ converges absolutely for $\Re(s) > 0$ and is analytic there. The proof above shows that it converges at least conditionally for $s = \chi_k$, but the same counterexample shows that absolute convergence may not be guaranteed there.

On the other hand, if $\sum_{n\geqslant 1} |g(n)|n^{-2} < \infty$, then the sum $D(s)$ converges absolutely also for $\Re(s) = 0$, including $s = \chi_k$, and under the stronger assumption that $g(n) = O(n^{1-\varepsilon})$, $D(s)$ converges and is analytic for $\Re(s) > -\varepsilon$ (and similarly for the Mellin transform $\tilde{g}$).

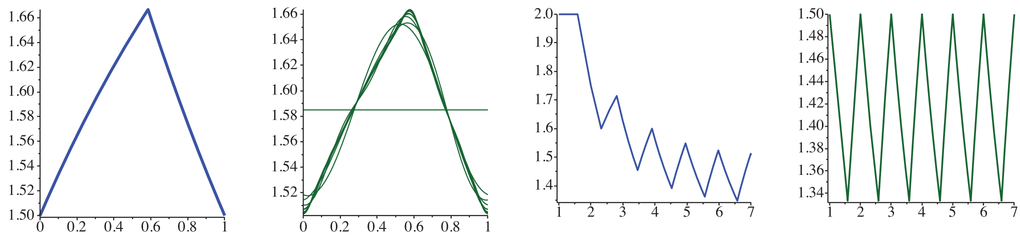
Remark 6. Since the series $D(s)$ may not be defined in an interval around 0, it may not be differentiable in the standard sense at 0. Nevertheless, the right derivative at 0 always exists and equals $D'(0)$ as defined in Equation (2.15). In fact, $\int_1^\infty g(x)x^{-2}\,\mathrm{d}x$ exists by the proof above; it follows easily by an integration by parts that $s \mapsto \int_1^\infty g(x)x^{-2-s}\,\mathrm{d}x$ is continuous for $s \geqslant 0$, and then Equation (2.23) implies that $D(s)/s \to D'(0)$ as $s \searrow 0$.

Remark 7. The series in Equation (2.15) can be rearranged as a Dirichlet series

$$D(s) = \sum_{n\geqslant 1} \big(g(n+1) - 2g(n) + g(n-1)\big)n^{-s}, \tag{2.26}$$

provided that $\Re(s)$ is so large that the latter series converges.

Remark 8. The function $P(t)$ may be any continuous 1-periodic function (see Remark 4), and thus the Fourier series in Equation (2.16) converges for almost every t by a well-known theorem of Carleson [12]. However, the Fourier series may not converge for every t, but instead converge under suitable summation techniques such as Cesàro means (or Fejér sums) [79, Theorems VIII.1.1 and III.3.4]; see [50] for a more detailed discussion of convergence of the Fourier series.



(3.4) Fourier approximation A080637 without shift A080637 with shift

Fig. 2. Left: The periodic function $P(t)$ in (3.4). Middle-left: Truncated Fourier series approximation to Equation (3.4); middle-right ($\frac{\text{A080637}(n)+1}{n}$) and right ($\frac{\text{A080637}(n-1)+1}{n}$): For $n = 2, \ldots, 128$ in logarithmic scale.

3 APPLICATIONS. I. BOUNDED $g(n)$

We apply our results derived above to examples involving the BDC recurrence in Equation (1.1) with bounded $g(n)$ in this section and to larger order $g(n)$ in the next three sections.

Example 3.1 (constant $g(n)$). The simplest case is when $g(n) \equiv c$ for some constant c. If the recurrence in Equation (1.1) holds for $n \geqslant 2$ and $f(1)$ is given, then the solution is easily seen to be

$$f(n) = (f(1) + c)n - c. \tag{3.1}$$

Many practical cases either have more complicated toll functions or start the recurrence from $n \geqslant n_0$ with $n_0 > 2$. For simplicity, we assume that $n_0 = 3$ and $g(n) = c$ for $n \geqslant 3$. The cases for which $n_0 > 3$ can be treated similarly. Note that $f(n) + c$ satisfies Equation (1.1) with $g(n) = 0$ for $n \geqslant 3$. We choose $m = L_n - 1$ in Equation (2.1), so that $2 \leqslant \frac{n}{2^m} < 4$ and for $n \geqslant 2$

$$f(n) + c = nP(\log_2 n), \tag{3.2}$$

where $P(t) = P(\{t\})$ is defined for $t \in [0, 1]$ by

$$P(t) := 2^{-1-t}(f(2^{1+t}) + c) = 2^{-1-t}\Big(\{2^{1+t}\}f(\lfloor 2^{1+t}\rfloor + 1) + (1 - \{2^{1+t}\})f(\lfloor 2^{1+t}\rfloor) + c\Big).$$

Note that $\lfloor 2^{1+t}\rfloor$ assumes either 2 or 3 for $t \in [0, 1]$. Thus, if $\log_2(1 + \frac{r}{2}) \leqslant t < \log_2(1 + \frac{r+1}{2})$ for $r = 0, 1$, then $\lfloor 2^{1+t}\rfloor = 2 + r$, and

$$P(t) = 2^{-1-t}(f(2 + r) + c) + (1 - 2^{-t} - r2^{-1-t})(f(3 + r) - f(2 + r)) \tag{3.3}$$

for $r = 0, 1$. The periodic function P thus consists of two different pieces of smooth functions (see Figure 2), and the values needed here are $\{f(2), f(3), f(4)\}$, where $f(3)$ and $f(4)$ can be computed from $f(1)$ and $f(2)$.

Example 3.2 (finding the minimum and the maximum in a set of n elements by divide-and-conquer). This is one of the classical divide-and-conquer examples described in, for example, the classic book by Aho et al. [1] on algorithms. It finds the smallest and largest elements of a file of n given elements simultaneously by splitting the input into two equal halves with sizes $\lfloor \frac{n}{2}\rfloor$ and $\lceil \frac{n}{2}\rceil$, respectively, by finding the smallest and largest in the two subfiles and then by completing the task by two additional comparisons; see [1, 43]. The number of comparisons used obviously satisfies Equation (1.1) with $g(n) = 2$ $(n \geqslant 3)$ and $f(1) = 0$ and $f(2) = 1$. Applying Equations (3.2) and (3.3), we obtain $f(n) + 2 = nP(\log_2 n)$ for $n \geqslant 2$, where $P(t) = P(\{t\})$ is defined in the unit interval by

$$P(t) = \begin{cases} 2 - 2^{-1-t} & t \in [0, \log_2 \frac{3}{2}] \\ 1 + 2^{-t} & t \in [\log_2 \frac{3}{2}, 1]. \end{cases} \tag{3.4}$$

Table 3. Twenty Sequences from OEIS Directly Expressible in Terms of $f(n)$ of Example 3.2 (for Min-Max Finding)

OEIS seq.	in terms of f	for $n \geqslant ?$	Notes
A159615$(n-1)$	$f(n)+1$	2	= A275202$(n)-1$ for $n \geqslant 2$ ("odious numbers")
A005942$(n+1)$	$2(f(n)+2)$	1	= A214214$(n)+1$ (complexity of Thue-Morse seq.)
A006165(n)	$f(n)-n+2$	1	= A066997$(n-1)$ for $n \geqslant 3$ = A078881$(n-1)$ for $n \geqslant 2$ (2nd order Josephus problem)
A053646(n)	$2f(n)-3n+4$	2	= A080776$(n-1)$ (distance to nearest power of 2)
A166079$(n+1)$	$2n-1-f(n)$	1	= A060973$(n)+1$ for $n \geqslant 1$ (phone-user arrangement problem)
A007378(n)	$3n-2-f(n)$	2	= A080645(n) for $n \geqslant 3$ (↑ seq. with $a(a(n))=2n$)
A080637$(n-1)$	$3n-3-f(n)$	2	= A079905$(n-1)$ for $n \geqslant 3$ (↑ seq. with $a(a(n))=2n+1$)
A080653$(n-2)$	$3n-4-f(n)$	3	= A079945$(n-3)+1$ for $n \geqslant 3$ = A080596$(n-3)+1$ for $n \geqslant 5$ = A080702$(n-4)+2$ for $n \geqslant 5$ = A115836$(n-1)$ for $n \geqslant 2$

Equivalently, for $n \geqslant 2$, $f(n)+2 = n + \min\{n-2^{L_n-1}, 2^{L_n}\}$; see also [43, 49]. By Equation (2.16) (or Equations (2.24) and (2.25)), we see that the average value of P equals $\widehat{P}(0) = \log_2 3 \approx 1.584$, and $\widehat{P}(k) = \frac{1-3^{-\chi_k}}{(\log 2)\chi_k(\chi_k+1)}$ $(k \neq 0)$; see Figure 2.

While the sequence $f(n)$ is not in OEIS, it is connected to many sequences there, which all satisfy Equation (1.1) (after properly shifted) with constant g. Twenty of them are listed in Table 3.

Note that the question "whether A078881 equals A006165" posed on OEIS can be directly proved, see Appendix A of this article.

On the other hand, for some of the sequences in the table, shifting is a crucial step in getting a simpler form for $g(n)$. Take, for example, $f(n) :=$ A080637(n) ($f(n)$ equals the smallest positive integer consistent with the sequence being monotonically increasing and satisfying $f(1)=2$, $f(f(n)) = 2n+1$ for $n > 1$), which in our format satisfies $f(2)=3$ and

$$g(n) = \lfloor \log_2(n+1) \rfloor - \lfloor \log_2 \tfrac{4}{3}(n+1) \rfloor \qquad (n \geqslant 3).$$

The sequence g consists of a block of 2^k 0s followed by a block of -1s of the same length for $k \geqslant 1$ and $n \geqslant 3$. If we define $\bar{f}(n) = f(n-1)+1$ for $n \geqslant 2$ with $\bar{f}(1) = 1$, then we obtain a sequence (which coincides with A007378) still satisfying the same recurrence (Equation (1.1)) but with $g(n) = 0$ for $n \geqslant 3$. We then deduce that $f(n-1) = nP(\log_2 n) - 1$, where $P(t) = P(\{t\})$ is defined by

$$P(t) = \begin{cases} 1 + 2^{-1-t} & t \in [0, \log_2 \frac{3}{2}] \\ 2 - 2^{-t} & t \in [\log_2 \frac{3}{2}, 1]; \end{cases}$$

see Figure 2 for an illustration. About half of the examples listed in the Table 3 have the same $P(t)$, for example, A079945, A080653, and A007378.

Example 3.3 (OEIS: the role of initial conditions). Consider the sequence $f(n) :=$ A080639(n), which equals the smallest integer larger than $f(n-1)$ and consistent with the condition "for $n > 1$, n is a member of the sequence if and only if $f(n)$ is even." In our format, this sequence satisfies Equation (1.1) but with a nonconstant $g(n)$ having a more complicated pattern. If we define instead $\bar{f}(n) := f(n-2) + 2$ with $\bar{f}(1) = 1$ and $\bar{f}(2) = 2$, then $\bar{f}$ satisfies Equation (1.1) with g given by

n	$\leqslant 4$	5	6	7	8	9	$\geqslant 10$
$g(n)$	0	2	3	3	3	1	0

By extending the argument used in Example 3.1, we deduce that

$$\bar{f}(n) = \begin{cases} n + 3 \cdot 2^{L_n - 3}, & \text{if } 2^{L_n} \leqslant n < \frac{9}{8} 2^{L_n}, \\ 2n - 3 \cdot 2^{L_n - 2}, & \text{if } \frac{9}{8} 2^{L_n} \leqslant n < \frac{3}{2} 2^{L_n}, \\ n + 3 \cdot 2^{L_n - 2}, & \text{if } \frac{3}{2} 2^{L_n} \leqslant n < 2^{L_n + 1}, \end{cases} \qquad (n \geqslant 5),$$

or $f(n-2) + 2 = nP(\log_2 n)$, where

$$P(t) = \begin{cases} 1 + 3 \cdot 2^{-3-t}, & \text{if } 0 \leqslant t \leqslant \log_2 \frac{9}{8}, \\ 2 - 3 \cdot 2^{-2-t}, & \text{if } \log_2 \frac{9}{8} \leqslant t < \log_2 \frac{3}{2}, \\ 1 + 3 \cdot 2^{-2-t}, & \text{if } \log_2 \frac{3}{2} \leqslant t < 1. \end{cases} \tag{3.5}$$

Other sequences with a very similar behavior include A088720, A088721, A079000, and A079253. A079000(n) = A080639($n + 1$) − 1 and A079253(n) = A080639($n + 2$) − 2.

Example 3.4 (Optimal algorithms for finding the minimum and maximum in a set of n elements). The balanced divide-and-conquer algorithm for finding the minimum and maximum in a set of n elements we mentioned above is simple but not optimal for general n (e.g., $n = 6$). A better divide-and-conquer strategy is to split the elements into two parts of sizes $2^{\lfloor \log_2 \frac{2}{3} n \rfloor}$ and $n - 2^{\lfloor \log_2 \frac{2}{3} n \rfloor}$, respectively, leading to the recurrence

$$f(n) = f\left(2^{\lfloor \log_2 \frac{2}{3} n \rfloor}\right) + f\left(n - 2^{\lfloor \log_2 \frac{2}{3} n \rfloor}\right) + 2 \qquad (n \geqslant 3),$$

with $f(1) = 0$ and $f(2) = 1$. The solution is easily seen to be (see [16, 49])

$$f(n) = \left\lceil \frac{3}{2} n \right\rceil - 2 = \frac{3}{2} n - 2 + \left\{ \frac{1}{2} n \right\} \qquad (n \geqslant 1).$$

The complexity is identical to that of the optimum algorithm proposed by Pohl [63]. It is easy to show that such an $f(n)$ also satisfies Equation (1.1) with $g(n) = 2 - \mathbf{1}_{n \equiv 2 \bmod 4}$ for $n \geqslant 2$; see also Example 3.7. On the other hand, $f(n)$ coincides with A032766($n - 1$), for which many combinatorial interpretations can be found on its OEIS webpage. Also, a huge number of OEIS sequences of the form $cn + d + h(n)$ with $h(n)$ periodic satisfy Equation (1.1) with bounded and periodic g; examples include A032766, A047335, A004523, and A047229.

Example 3.5 (Mergesort). The variance of the number of comparisons used by the top-down mergesort (see [34, 46]) satisfies Equation (1.1) with

$$g(n) = \frac{2 \lceil \frac{n}{2} \rceil^2 \left(\lceil \frac{n}{2} \rceil - 1\right)}{\left(\lceil \frac{n}{2} \rceil + 1\right)^2 \left(\lceil \frac{n}{2} \rceil + 2\right)} \qquad (n \geqslant 2). \tag{3.6}$$

Since g is bounded for all n, our theorems apply and it is easy to see that

$$f(n) = nP(\log_2 n) - Q(n) \qquad (n \geqslant 1), \tag{3.7}$$

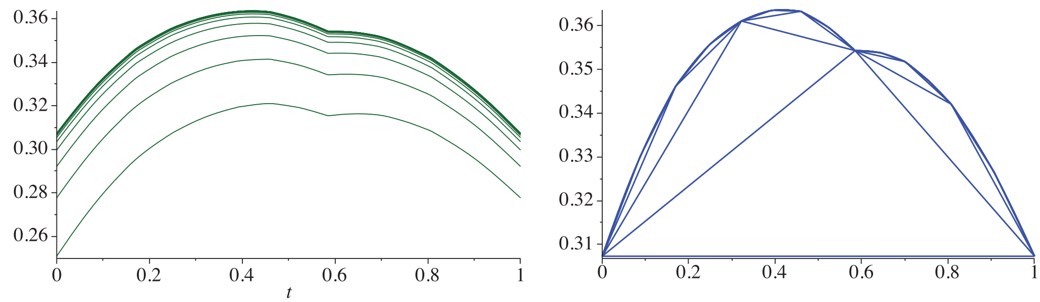


Fig. 3. The periodic function arising from the variance of mergesort as approximated by the first N terms of the series in Equation (3.8) (left) for $N = 5, \ldots, 20$ and by Equation (3.7) (right) for $n = 1$ to $n = 2048$ (plotted against $\{\log_2 n\}$).

where

$$P(t) = \sum_{k\geqslant 1} 2^{-k-\{t\}} g\left(2^{k+\{t\}}\right) \qquad (t \in \mathbb{R}), \tag{3.8}$$

and

$$Q(n) = \sum_{k\geqslant 1} 2^{-k} g(2^k n) = 2 + \sum_{k\geqslant 0} \frac{1}{2^k} \left(\frac{7}{2^k n + 1} - \frac{12}{2^k n + 2} - \frac{2}{(2^k n + 1)^2} \right).$$

Note that $g(t) = 0$ for $t \in [0, 2]$ because $g(0) = g(1) = g(2) = 0$. The identity Equation (3.7) was derived in [46] by a purely analytic approach based on second difference and Mellin-Perron integrals; the elementary proof here is more general and to some extent simpler. Also, Equation (3.8) is new.

The Fourier coefficients can be computed by applying Theorem 3. We obtain from Equations (2.24), (2.26), and (3.6) easily, with $\chi_k = \frac{2k\pi i}{\log 2}$, as usual,

$$\widehat{P}(k) = \frac{2}{(\log 2)\chi_k(1+\chi_k)} \sum_{m\geqslant 1} \frac{m(5m^2 + 10m + 1)}{(m+1)^2(m+2)^2(m+3)} \left((2m)^{-\chi_k} - (2m+1)^{-\chi_k}\right)$$

for $k \neq 0$, which is identical to that derived in [34]. Similarly, when $k = 0$, the mean value of P over the unit interval equals, using Equations (2.25) and (2.17),

$$\begin{aligned}\widehat{P}(0) &= \frac{1}{\log 2} \sum_{m\geqslant 1} \frac{2m(5m^2 + 10m + 1)}{(m+1)^2(m+2)^2(m+3)} \log \frac{2m+1}{2m} \\ &\approx 0.34549\,32539\,59979\,17006\,74766\ldots\end{aligned}$$

See Figure 3 for two different plots of $P(t)$.

Higher-order cumulants of the number of comparisons used all satisfy the same recurrence in Equation (1.1) with bounded $g(n)$ and can be treated in the same manner; see [46] for the third and the fourth orders.

Example 3.6 (lossless compression of balanced trees). The logarithm of the total number of the 2-balanced trees with n leaves (A110316 in OEIS) satisfies Equation (1.1) with $g(n) = \mathbf{1}_{n \text{ is odd}}$ for $n \geqslant 2$ and $f(1) = 0$; see [60]. We then obtain $Q(n) = 0$ by Equation (1.13). Thus, $f(n) = nP(\log_2 n)$, where

$$P(t) = \sum_{k\geqslant 1} 2^{-k-\{t\}} g\left(2^{k+\{t\}}\right), \tag{3.9}$$

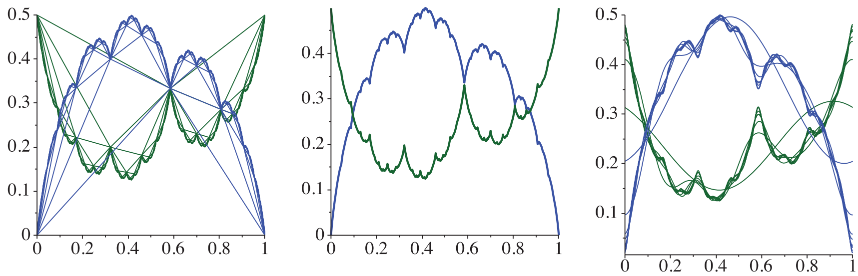


Fig. 4. The periodic fluctuations of the two sequences in Example 3.6: periodic functions successively refined by $\frac{f(n)}{n}$ (in blue) and $\frac{\bar f(n)+1}{n}$ (in green) plotted against $\{\log_2 n\}$ (left), rendered by their series expressions of the form (3.9) (middle) and their Fourier series representations (right).

with $g(x) = \{x\}$ if $\lfloor x\rfloor \geqslant 2$ is even and $g(x) = 1 - \{x\}$ if $\lfloor x\rfloor \geqslant 3$ is odd; see Figure 4. The Fourier coefficients can be computed by Equation (2.16). Note that Equation (2.15) yields

$$D(s) = \sum_{m\geqslant 1}\left((2m+2)^{-s} - 2(2m+1)^{-s} + (2m)^{-s}\right) = (2^{2-s}-2)\zeta(s) - 2^{-s} + 2, \tag{3.10}$$

(where ζ denotes Riemann's zeta function; see [78, Ch. XIII]), first for $\Re(s) > 1$, and thus by analytic extension for $\Re(s) > -1$ (where the sum converges absolutely). In particular,

$$D'(0) = -4(\log 2)\zeta(0) + 2\zeta'(0) + \log 2 = 3\log 2 + 2\zeta'(0) = 2\log 2 - \log \pi. \tag{3.11}$$

Thus, Equation (2.16) provides the Fourier series expansion for $P(t)$:

$$P(t) = 2 - \log_2 \pi + \frac{1}{\log 2}\sum_{k\neq 0}\frac{1+2\zeta(\chi_k)}{\chi_k(\chi_k+1)}\, e^{2k\pi i t} \qquad (t\in\mathbb{R}); \tag{3.12}$$

see Figure 4. In particular, the mean value of P equals

$$\widehat{P}(0) = \frac{D'(0)}{\log 2} = 2 - \log_2 \pi \approx 0.34850\,38705\ldots.$$

By the known estimate for Riemann's zeta function (see [78, p. 276])

$$|\zeta(it)| = O\left(|t|^{\frac{1}{2}+\varepsilon}\right) \qquad (|t|\geqslant 1) \tag{3.13}$$

for any $\varepsilon > 0$, we see that the Fourier series in Equation (3.12) is absolutely convergent.

A "conjugate" sequence (A003661) arises in the context of bipartite Steinhaus graphs for which the total number on $n+1$ nodes equals $2n + \bar f(n)$ (see [31]), where $\Lambda[\bar f] = \mathbf{1}_{n \text{ is even}}$ with $\bar f(n) = 0$ for $n \leqslant 3$. We then obtain $\bar f(n) + 1 = n\bar P(\log_2 n)$, where $\bar P$ has the same series expression as Equation (3.9), with g there replaced by $\bar g(x) = \{x\}$ if $\lfloor x\rfloor$ is odd and $\bar g(x) = 1 - \{x\}$ if $\lfloor x\rfloor$ is even, for $x \geqslant 3$; see Figure 4. The corresponding Fourier series is then given by

$$\bar P(t) = \log_2(3\pi) - 3 - \frac{1}{\log 2}\sum_{k\neq 0}\frac{3^{-\chi_k} + 2\zeta(\chi_k)}{\chi_k(\chi_k+1)}\, e^{2k\pi i t} \qquad (t\in\mathbb{R}).$$

On the other hand, the sequence A268289$(n-1)$ satisfies the same recurrence and the same toll function but with different initial conditions.

Example 3.7 (a sensitivity test). Motivated by Example 3.4 above and Example 5.5 below, we consider and compare the four sequences $\Lambda[f_j] = g_j$ with $f_j(0) = f_j(1) = 0$, where $g_j(n) := \mathbf{1}_{n\equiv j \bmod 4}$ for $j = 0, 1, 2, 3$. While the definitions are almost identical, their periodic behaviors differ significantly. The simplest case among these four is $f_2(n) = \lfloor \frac{n}{2}\rfloor$ for $n \geqslant 1$, the other three having

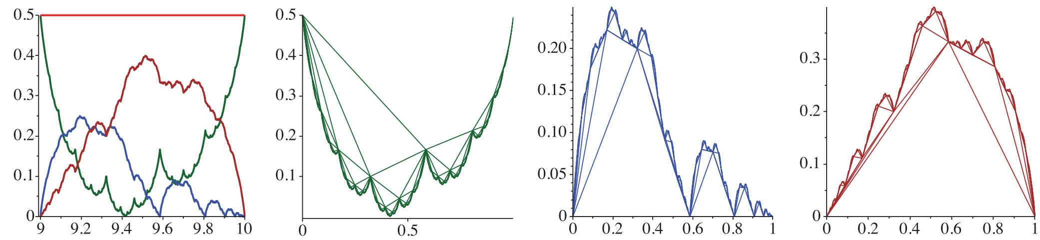


Fig. 5. Periodic fluctuations of the four cases corresponding to different $g_j(n) = \mathbf{1}_{n \equiv j \bmod 4}$ and approximations of $P_j(\log_2 n)$ by $\frac{f_j(n)-Q_j(n)}{n}$, for $n = 2, \dots, 1024$ and $j = 0, 1, 3$: P_0 in green, P_1 in blue, and P_3 in brown. P_2 is a constant.

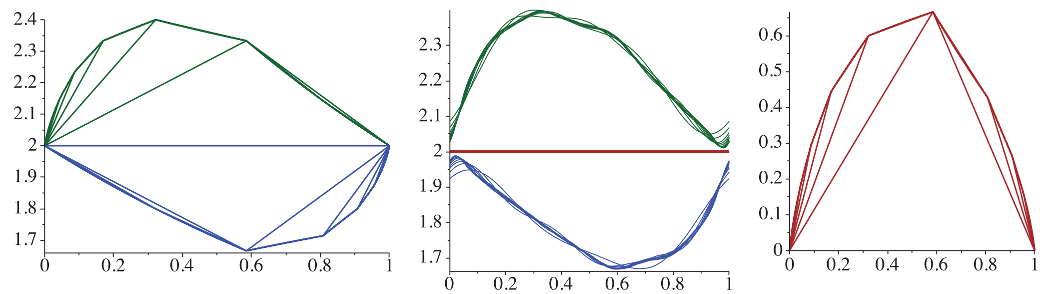


Fig. 6. The periodic functions arising in the two log-cases: $\lceil \log_2 n \rceil$ (upper part) and $\lfloor \log_2 n \rfloor$ (lower part) for $n = 2, \dots, 1024$ (left), approximated by Fourier partial sums (middle), and the difference between the two periodic functions (right).

no such explicit form. This means that $f_2(n) = nP_2(\log_2 n) - \{\frac{n}{2}\}$, where the periodic function $P_2(t) = \frac{1}{2}$ is a constant. Note also that $\sum_{0 \leqslant j \leqslant 3} g_j(n) = 1$ for all $n \geqslant 2$; thus, by Equation (3.1), $\sum_{0 \leqslant j \leqslant 3} f_j(n) = n - 1$ and $\sum_{0 \leqslant j \leqslant 3} P_j(t) = 1$. See Figure 5 for an illustration. These examples show how *a minor change in the toll function g results in rather different periodic fluctuations in P*. Such a sensitive change in fluctuations becomes invisible if one absorbs all $g_j(n)$ by $O(1)$.

4 APPLICATIONS. II. SUBLINEAR $g(n)$

We begin with logarithmic orders $g(n) = \lceil \log_2 n \rceil$ and $g(n) = \lfloor \log_2 n \rfloor$ for which we can still derive rather precise expressions for the periodic functions, and then discuss cases when $g(n) = \Theta((\log n)^d)$ with $d \geqslant 1$ and $g(n) = \Theta(n^\tau)$ with $\tau \in (0, 1)$, which arise in the analysis of computational geometry algorithms using divide-and-conquer.

Example 4.1 (heights in balanced binary trees). The sum of heights of the nodes in a certain balanced binary tree with n leaves gives a sequence (A213508 in OEIS) such that $f(n) =$ A213508$(n-1)$ satisfies Equation (1.1) with $g(n) = \lceil \log_2 n \rceil$ and $f(1) = 0$; see [14].

We now simplify $f(n)$ and prove that

$$f(n) = nP(\log_2 n) - \lceil \log_2 n \rceil - 2 \qquad (n \geqslant 1), \tag{4.1}$$

where the periodic and continuous function P has the closed form (see Figure 6)

$$P(t) = \begin{cases} 2^{1-\{t\}} + (1 - 2^{-\{t\}})\Big(2^{1-\{\log_2(2^{\{t\}}-1)\}} - \lfloor \log_2(2^{\{t\}} - 1) \rfloor\Big), & t \notin \mathbb{Z}, \\ 2, & t \in \mathbb{Z}. \end{cases} \tag{4.2}$$

This is one of the few cases beyond bounded $g(n)$ for which P admits a closed-form expression. Of course, the sublinear term $-\lceil \log_2 n\rceil - 2$ in (4.1) is nothing but $Q(n)$, but the proof for Equation (4.2) is more complicated.

To prove Equation (4.2), we start from the identity in Equation (2.4) together with Equation (1.8)

$$f(n) = \sum_{0\leqslant k\leqslant L_n} 2^k \left(g\left(\left\lfloor \frac{n}{2^k}\right\rfloor\right) + \left\{\frac{n}{2^k}\right\}\left(g\left(\left\lfloor \frac{n}{2^k}\right\rfloor + 1\right) - g\left(\left\lfloor \frac{n}{2^k}\right\rfloor\right)\right)\right).$$

Note that $g(n+1) - g(n) = 1$ only if n is a power of two, that is, if $n = 2^{L_n}$. If $n = (1b_{L_n-1}\dots b_0)_2$, and $\kappa_0 = \kappa_0(n) := L_{n-2^{L_n}}$ denotes the position of the largest k smaller than L_n such that $b_k = 1$, then $\mathbf{1}_{\lfloor \frac{n}{2^k}\rfloor = 2^{L_n-k}} = \mathbf{1}_{\kappa_0<k\leqslant L_n}$, which holds also when $n = 2^{L_n}$ if we define $L_0 := -1$ and thus in this case $\kappa_0(n) := -1$. Hence, for $0 \leqslant k \leqslant L_n$

$$\begin{cases} g\left(\left\lfloor \frac{n}{2^k}\right\rfloor\right) = L_n - k + 1 - \mathbf{1}_{\kappa_0<k\leqslant L_n}, \\ g\left(\left\lfloor \frac{n}{2^k}\right\rfloor + 1\right) - g\left(\left\lfloor \frac{n}{2^k}\right\rfloor\right) = \mathbf{1}_{\kappa_0<k\leqslant L_n}. \end{cases}$$

Thus, we get that

$$f(n) = \sum_{0\leqslant k\leqslant L_n} 2^k(L_n - k + 1) - \sum_{\kappa_0<k\leqslant L_n} 2^k\left(1 - \left\{\frac{n}{2^k}\right\}\right).$$

The first sum equals $2^{L_n+2} - L_n - 3$. For the second sum, observe that when $\kappa_0 < k \leqslant L_n$ or, equivalently, $\lfloor \frac{n}{2^k}\rfloor = 2^{L_n-k}$, then $2^k\{\frac{n}{2^k}\} = n - 2^k\lfloor\frac{n}{2^k}\rfloor = n - 2^{L_n}$. Thus,

$$\sum_{\kappa_0<k\leqslant L_n} 2^k\left(1 - \left\{\frac{n}{2^k}\right\}\right) = \sum_{\kappa_0<k\leqslant L_n} (2^k + 2^{L_n} - n) = 2^{L_n+1} - 2^{\kappa_0+1} + (2^{L_n} - n)(L_n - \kappa_0).$$

We thus obtain

$$f(n) = 2^{L_n+1} - L_n - 3 + 2^{\kappa_0+1} + (n - 2^{L_n})(L_n - \kappa_0) \qquad (n \geqslant 1).$$

In particular, when $n = 2^{L_n}$, so $2^{\kappa_0+1} = 1$ by our convention, $f(n) = 2n - L_n - 2$, which verifies Equation (4.1) with $P(L_n) = 2$.

Assume now that $n \neq 2^{L_n}$. Write $L_n = \log_2 n - \vartheta_n$, where $\vartheta_n := \{\log_2 n\} > 0$. Thus, $n - 2^{L_n} = n(1 - 2^{-\vartheta_n}) = 2^{L_n}(2^{\vartheta_n} - 1)$. Let $\vartheta'_n := \{\log_2(n - 2^{L_n})\} = \{\log_2(2^{\vartheta_n} - 1)\}$. Then,

$$\kappa_0 = L_{n-2^{L_n}} = \log_2 n + \log_2(1 - 2^{-\vartheta_n}) - \vartheta'_n = L_n + \lfloor \log_2(2^{\vartheta_n} - 1)\rfloor.$$

Thus, $2^{\kappa_0+1} = 2n(1 - 2^{-\vartheta_n})2^{-\vartheta'_n}$ and

$$\begin{aligned} \frac{f(n) + \lceil \log_2 n\rceil + 2}{n} &= \frac{2^{L_n+1} + 2^{\kappa_0+1} + (n - 2^{L_n})(L_n - \kappa_0)}{n} \\ &= 2^{1-\vartheta_n} + 2(1 - 2^{-\vartheta_n})2^{-\vartheta'_n} - (1 - 2^{-\vartheta_n})\lfloor \log_2(2^{\vartheta_n} - 1)\rfloor, \end{aligned}$$

from which we deduce Equations (4.1) and (4.2).

We then obtain the mean value of P over the unit interval

$$\widehat{P}_0 = \int_0^1 P(t)\,\mathrm{d}t = 1 + \frac{1}{2\log 2} + \int_0^\infty \frac{2^{\{v\}} + \lfloor v\rfloor}{(2^v + 1)^2}\,\mathrm{d}v.$$

For other Fourier coefficients, we can still use Equation (4.2) to simplify $\widehat{P}(k)$, but it is simpler to apply Equation (2.16) as follows (alternatively one may apply the analytic approach developed in [34, 46]). Define $\tilde{f}(n) = f(n) + \lceil \log_2 n\rceil + 2$. Then, $\tilde{f}(n)$ satisfies (1.1) with $g(n) = \delta_n$ and $\tilde{f}(1) = 2$, where $\delta_n = 1$ when $n = 2^k + 1$, $k \geqslant 1$ and $\delta_n = 0$ otherwise. Thus, we deduce the identity $f(n) + \lceil \log_2 n\rceil + 2 = nP(\log_2 n)$, where, by Equation (2.16),

$$P(t) := 2 + \frac{\tilde{D}'(0)}{\log 2} + \frac{1}{\log 2}\sum_{j\neq 0}\frac{\tilde{D}(\chi_j)}{\chi_j(\chi_j+1)}\,e^{2k\pi i t},$$

with

$$\tilde{D}(s) := \sum_{k\geqslant 1}\left(2^{-ks} - 2(2^k+1)^{-s} + (2^k+2)^{-s}\right) \qquad (\Re(s) > -2).$$

Numerically, the mean value of the periodic function equals (see Figure 6)

$$\widehat{P}_0 = 2 + \frac{\tilde{D}'(0)}{\log 2} \approx 2.25352\,40379\,34699\,65912\ldots.$$

A very similar example is A213509 (which comes from [14]): if we define $f(n) :=$ A213509$(n-1)-1$, then $\Lambda[f] = \lceil \log_2 n\rceil$ for $n \geqslant 4$. A closed-form expression of this sequence can be similarly characterized.

Example 4.2 (the case in which $g(n) = \lfloor \log_2 n\rfloor$ with $f(1) = 0$). By the same arguments used above for $\lceil \log_2 n\rceil$, we have that

$$f(n) = 2^{L_n+1} - L_n - 2 + \sum_{0\leqslant k\leqslant L_n} 2^k\left\{\frac{n}{2^k}\right\}\prod_{k\leqslant j\leqslant L_n} b_j = 2^{L_n+1} - L_n - 2 + \sum_{\kappa_1\leqslant k\leqslant L_n} 2^k\left\{\frac{n}{2^k}\right\},$$

where

$$\kappa_1 := \min\left\{k \,:\, \prod_{k\leqslant j\leqslant L_n} b_j = 1\right\} = L_{2^{L_n+1}-n-1} + 1 = \left\lceil \log_2(2^{L_n+1}-n)\right\rceil,$$

where, as above, $L_0 := -1$. Since when $\kappa_1 \leqslant k \leqslant L_n$, $2^k\{\frac{n}{2^k}\} = n - 2^{L_n+1} + 2^k$, we see that

$$f(n) + L_n + 2 = 2^{L_n+1} + n - (2^{L_n+1}-n)(L_n - \kappa_1) - 2^{\kappa_1}.$$

We thus deduce, similarly as above, the exact expression

$$f(n) + \lfloor \log_2 n\rfloor + 2 = nP(\log_2 n) \qquad (n \geqslant 1),$$

where (see Figure 6)

$$P(t) = 1 + 2^{1-\{t\}} - (2^{1-\{t\}}-1)\left(2^{1-\{\log_2(2-2^{\{t\}})\}} - \lfloor \log_2(2-2^{\{t\}})\rfloor - 1\right).$$

In particular, the mean value of P over the unit interval is given by

$$\widehat{P}(0) = \int_0^1 P(t)\,\mathrm{d}t = 1 + \frac{1}{\log 2} - \int_0^\infty \frac{2^{\{u\}} + \lfloor u\rfloor}{(2^{1+u}-1)^2}\,\mathrm{d}u.$$

The same approach as above using $\tilde{f}(n) := f(n) + \lfloor \log_2 n\rfloor + 2 = nP(\log_2 n)$, leads to, by Equation (2.16),

$$P(t) := 2 + \frac{\tilde{D}'(0)}{\log 2} + \frac{1}{\log 2}\sum_{j\neq 0}\frac{\tilde{D}(\chi_j)}{\chi_j(\chi_j+1)}\,e^{2k\pi i t},$$

where

$$\tilde{D}(s) := -\sum_{k\geqslant 2}\left((2^k-2)^{-s} - 2(2^k-1)^{-s} + 2^{-ks}\right) \qquad (\Re(s) > -2).$$

Numerically, the mean value of the periodic function equals

$$\widehat{P}_0 = 2 + \frac{\tilde{D}'(0)}{\log 2} \approx 1.79191\,68246\,62028\,52468\ldots.$$

Example 4.3 (computational geometry algorithms). Divide-and-conquer with balanced part sizes has been one of the most widely used design paradigms in computational geometry (see [64]). In

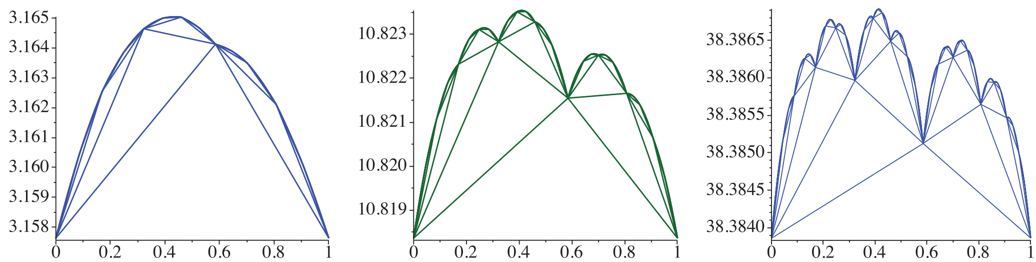


Fig. 7. The periodic functions arising in the expected cost of maxima-finding algorithms using divide-and-conquer: $d = 2$ (left), $d = 3$ (middle), and $d = 4$ (right), approximated by using Equation (4.3) for $n = 2, \ldots, 1024$ and plotted against $\{\log_2 n\}$.

terms of the average-case time complexity, such a paradigm yields simple yet efficient procedures, leading often to many $O(n)$ or $O(n(\log n)^c)$ expected time algorithms. Typical problems of this category include convex hull, maxima-finding, closest pairs, etc.; see, for example, [10, 30, 64].

Recall that the maxima of a set of points in $\mathbb{R}^d$ are the points dominated by no other points (a point dominating another if the coordinate-wise difference has no negative entry). A simple way to find the maxima of a set of points is to first split the input points into two halves, find the maxima of each half recursively, and then merge the two sets of maxima by pairwise comparisons; see [17, 33, 64] for more information on maxima and related algorithms. If we assume that the input n points are randomly chosen from the d-dimensional hypercube $[0, 1]^d$, then it is known that the expected number of maxima can be computed recursively by the recurrence

$$M_{n,d} = \frac{1}{d-1} \sum_{1 \leqslant j < d} H_n^{(d-j)} M_{n,j} \quad \text{where} \quad H_n^{(a)} := \sum_{1 \leqslant j \leqslant n} j^{-a},$$

with $M_{n,1} \equiv 1$ for $n \geqslant 1$; see [7] and the references therein. In particular, $M_{n,2} = H_n$ and $M_{n,3} = \frac{1}{2}(H_n^2 + H_n^{(2)})$. For fixed $d \geqslant 2$, $M_{n,d} = \Theta(\log^{d-1} n)$.

Let $f(n)$ be the expected number of pairwise comparisons. A naive pairwise comparison gives the toll function $g(n) = M_{\lfloor \frac{n}{2} \rfloor, d} M_{\lceil \frac{n}{2} \rceil, d}$ for $n \geqslant 2$ with $g(1) = f(1) = 0$. Note that $g(n) = \Theta(\log^{2(d-1)} n)$. Thus, we obtain an identity of the form

$$f(n) = nP(\log_2 n) - \sum_{k \geqslant 0} 2^{-k-1} M_{2^k n, d}^2, \tag{4.3}$$

where $P(t) := \sum_{k \geqslant 0} 2^{-k-\{t\}} g(2^{k+\{t\}})$ and the series converges absolutely. In particular, when $d = 2$, $M_{2^k n, d}^2 = H_{2^k n}^2$. Note that the error term provided by the series on the right-hand side is crucial in the graphic rendering of the periodic function P; see Figure 7.

From Figure 7, we see that the mean values of the periodic functions increase very fast with d; these can be reduced by using more efficient algorithms to merge the two sets of maxima; see [17, 26] for more references.

The same divide-and-conquer algorithm applies to computing the convex hull of a given set of points; see [10, 30, 64]. According to known theory, the expected number of extreme points is in different typical situations of order $(\log n)^\upsilon$ or n^τ with $\upsilon > 0$ and $\tau \in (0, 1)$; see [30, 64]. However, in most cases, we do not have an exact expression for the toll function, but we can get estimates. Suppose, for example, that $|g(n)| \leqslant An^\tau$ for some constants $\tau < 1$ and $A < \infty$. Then, Theorem 2 shows that $f(n) = nP(\log_2 n) - Q(n)$, where the error term $Q(n)$ can be estimated by $|Q(n)| \leqslant A(2^{1-\tau} - 1)^{-1} n^\tau$.

5 APPLICATIONS. III. LINEAR $g(n)$

Linear toll functions abound in algorithmics and related structures; they are often of the form $g(n) = n + \bar{g}(n)$, where $\bar{g}(n) = O(1)$. By additivity, we can separate the toll function into two parts: one with n and the other with $\bar{g}(n)$ for which we already showed how such sequences can be systematically handled.

Example 5.1 (binary entropy function, A003314). When $g(n) = n$ ($n \geqslant 2$) and $f(1) = 0$, the sequence is called the binary entropy function in OEIS (A003314). An exact solution can be obtained by taking $m = L_n$ in Equation (2.1) (so that $1 \leqslant 2^{-m} n \leqslant 2$), giving

$$f(n) = nL_n + 2n - 2^{L_n+1} \qquad (n \geqslant 1). \tag{5.1}$$

Accordingly,

$$f(n) = n\log_2 n + nP(\log_2 n),$$

where

$$P(t) = 2 - \{t\} - 2^{1-\{t\}} = \frac{3}{2} - \frac{1}{\log 2} + \frac{1}{\log 2}\sum_{k\neq 0}\frac{e^{2k\pi it}}{\chi_k(1+\chi_k)} \qquad (t \in \mathbb{R}) \tag{5.2}$$

is a continuous periodic function.

As in the bounded toll function cases, the sequence $f(n)$ is also connected to many other sequences in OEIS. In particular, $f(n) = \text{A123753}(n-1) - 1$. Some others are listed as follows.

OEIS seq.	in terms of f	for $n \geqslant ?$	notes ($a(n) = \text{Axxxxxx}(n)$)
A001855(n)	$f(n) - n + 1$	1	max # comparisons used by mergesort
A083652($n-1$)	$f(n) - n + 2$	1	sums of lengths of binary numbers
A033156(n)	$f(n) + n$	1	$a(1) = 1$ and for $n \geqslant 2$ $a(n) = n + \min_{1\leqslant k<n}\{a(k) + a(n-k)\}$
A054248(n)	$f(n) + \mathbf{1}_{n \text{ is odd}}$	1	$a(1) = 1, a(2) = 2$ and for $n \geqslant 3$ $a(n) = n + \min_{1\leqslant k<n}\{a(k) + a(n-k)\}$
A097383($n-1$)	$f(n) - \left\lfloor \frac{3}{2}n \right\rfloor + 1$	2	optimal binary search with equality
A061168($n-1$)	$f(n) - 2n + 2$	1	$\sum_{1\leqslant k\leqslant n}\lfloor\log_2 k\rfloor$

We will discuss some of these later.

Example 5.2 (Mergesort). We discussed in Example 3.5 the variance of the number of comparisons used by the top-down mergesort (see [34]). We consider here the number itself in the worst, the average, and the best cases, whose treatments are similar. In all cases, $f(1) = 0$.

(a) *Worst-case*: This has the toll function $g(n) = n - 1$, which implies that $g(x) = x - 1$ for $x \geqslant 1$. This yields, for example, by Equation (2.1) or Equation (2.4), the exact solution $f(n) = nL_n + n - 2^{L_n+1} + 1$, which can be written as

$$f(n) = n\log_2 n + nP(\log_2 n) + 1, \tag{5.3}$$

where

$$P(t) = 1 - \{t\} - 2^{1-\{t\}} = \frac{1}{2} - \frac{1}{\log 2} + \frac{1}{\log 2}\sum_{k\neq 0}\frac{e^{2k\pi it}}{\chi_k(1+\chi_k)} \qquad (t \in \mathbb{R}). \tag{5.4}$$

This sequence is A001855 in OEIS and also enumerates a few other objects, such as the number of switches in an AS-Waksman network [8] and (shifted by 1) n times the expected total number of probes for a successful binary search.

Note that compared to Example 5.1, $g(n)$ differs by 1 and thus the sequence $f(n)$ differs by $n-1$ from A003314 there; see (3.1). The sequence $f(n)$ here can also be expressed in terms of other OEIS sequences, as in Example 5.1. In addition to those mentioned above, A001855 is also connected to A097384 (shifted by 1), which satisfies (1.1) with $f(1) = f(2) = 0$ and $g(n) = n-1$ for $n \geqslant 3$. Thus, it differs from A001855 by A060973 mentioned in Example 3.2.

(b) *Best case*: The minimum number of comparisons used by merging two sorted subfiles of sizes $\lfloor \frac{n}{2} \rfloor$ and $\lceil \frac{n}{2} \rceil$ equals $\lfloor \frac{n}{2} \rfloor$. Hence, the minimum number of comparisons used by top-down mergesort satisfies Equation (1.1) with $g(n) = \lfloor \frac{n}{2} \rfloor$. The sequence $f(n)$ with this toll function (A000788 in OEIS, shifted by 1) occurs in a large number of different contexts, such as optimal search, quicksort, hypercube graphs, game theory, random generation, binary trees, and sorting networks,; see [18, 49, 72] for more information and references. The most notable connection is that $f(n)$ counts the total number of 1s in the binary expansions of the first n nonnegative integers, for which there is a rich literature; see the survey paper [18].

In addition, the sequence here A000788$(n) = f(n+1)$ equals essentially A078903 (differing by n) and A076178 (twice that of A078903). Other connected sequences include A163095$(n) = f(n+1)^2$, A059015$(n) =$ A083652$(n) - f(n+1)$, and A122247$(n) = n(n+1) - f(n+1)$ (see also Example 6.2).

Write $g(n) = \lfloor \frac{n}{2} \rfloor = \frac{n-1}{2} + \bar{g}(n)$, where $\bar{g}(n) = \frac{1}{2} - \{\frac{n}{2}\} = \frac{1}{2}\mathbf{1}_{n \text{ is even}}$ for $n \geqslant 2$. Recall that we treated the case with essentially the same toll function in Example 3.6 but with different initial conditions. The sequence $\bar{f}(n)$ satisfying $\Lambda[\bar{f}] = 2\bar{g}$ and $\bar{f}(1) = 0$ equals A268289$(n-1)$ in OEIS. This says that the minimum number of comparisons used to sort n elements by top-down mergesort equals half the maximum number plus a roughly linear term.

Applying Equation (1.13) to $\bar{g}$ yields $\bar{Q}(n) = \frac{1}{2}$ for $n \geqslant 1$. We then deduce from Theorem 2 and Equation (5.3) that $f(n) = \frac{1}{2} n \log_2 n + nP(\log_2 n)$, where P is the Trollope–Delange function (see [5, 23]):

$$P(t) = \frac{1}{2} - \frac{1}{2}\{t\} - 2^{-\{t\}} + \sum_{k \geqslant 0} 2^{-k-\{t\}} \bar{g}\left(2^{k+\{t\}}\right), \tag{5.5}$$

where $\bar{g}(x) = \frac{1}{2}(1 - \{x\})$ if $\lfloor x \rfloor$ is even and $\bar{g}(x) = \frac{1}{2}\{x\}$ if $\lfloor x \rfloor$ is odd. The function defined by the infinite series is often referred to as the Takagi function; see the recent survey paper [3] for more information. Furthermore, we also get the Fourier series expansion

$$P(t) = \frac{\log_2 \pi}{2} - \frac{1}{4} - \frac{1}{2\log 2} - \frac{1}{\log 2} \sum_{k \neq 0} \frac{\zeta(\chi_k)}{\chi_k(\chi_k + 1)} e^{2k\pi i t}, \tag{5.6}$$

where the Fourier series is absolutely convergent by Equation (3.13). See Figure 8.

Similarly, the total number of zeros in the binary expansions of $1, 2, \ldots, n-1$ satisfies Equation (1.1) with $g(n) = \lceil \frac{n}{2} \rceil - 1$ and $f(1) = 0$. This time, $g(n) = \frac{n-1}{2} - \bar{g}(n)$, with the same $\bar{g}(n) = \frac{1}{2}\mathbf{1}_{n \text{ is even}}$ as above. We then get $f(n) = \frac{n}{2}\log_2 n + nP(\log_2 n) + 1$, where $P = P_{(5.4)} - P_{(5.6)}$. This yields the two sequences A181132 and A059015 (differing by 1; both shifted by 1).

(c) *Average case*: if $f(n)$ is the average number of comparisons, then Equation (1.1) holds with $g(n) = n - \frac{\lfloor \frac{n}{2} \rfloor}{\lceil \frac{n}{2} \rceil + 1} - \frac{\lceil \frac{n}{2} \rceil}{\lfloor \frac{n}{2} \rfloor + 1}$ for $n \geqslant 2$. It suffices to consider the toll function

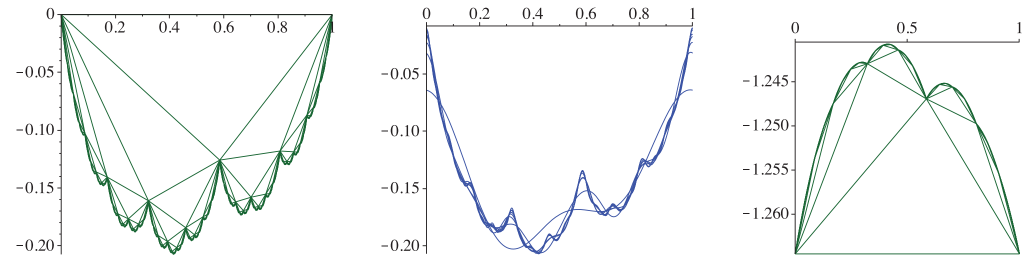


Fig. 8. The periodic functions arising in the best case (left and middle) and average-case (right) of mergesort: $P(t)$ in the best case approximated by $\frac{f(n)}{n} - \frac{1}{2}\log_2 n$ (left) and approximated by truncated Fourier series Equation (5.6) (middle); $P(t)$ in the average case approximated by Equation (5.8) (right).

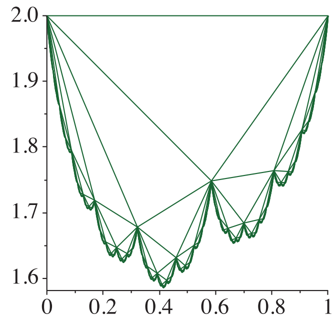


Fig. 9. $P(\log_2 n)$ (A067699).

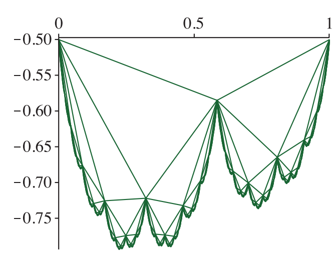


Fig. 10. $P(\log_2 n)$ (A220001).

$$\bar{g}(n) := 1 - \frac{\lfloor \frac{n}{2} \rfloor}{\lceil \frac{n}{2} \rceil + 1} - \frac{\lceil \frac{n}{2} \rceil}{\lfloor \frac{n}{2} \rfloor + 1} = -1 + \frac{2}{\lceil \frac{n}{2} \rceil + 1} \qquad (n \geqslant 1), \tag{5.7}$$

since the difference $n-1$ corresponds to the worst case, whose solution is given in (a) above. By Theorem 2, we see that, denoting by $\bar{f}(n)$, the sequence satisfying Equation (1.1) with $g(n) = \bar{g}(n)$, $\bar{f}(n) = n\bar{P}(\log_2 n) - \bar{Q}(n)$, where $\bar{P}(t) := \sum_{k\in\mathbb{Z}} 2^{-k-\{t\}}\bar{g}(2^{k+\{t\}})$ and $\bar{Q}(n) := -1 + \sum_{k\geqslant 0} \frac{1}{2^k(2^k n+1)}$. Adding this result and the cost in the worst case, we obtain the expected cost of top-down mergesort

$$f(n) = n\log_2 n + nP(\log_2 n) - Q(n), \tag{5.8}$$

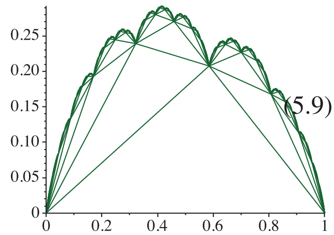


Fig. 11. $P(\log_2 n)$ (A173318).

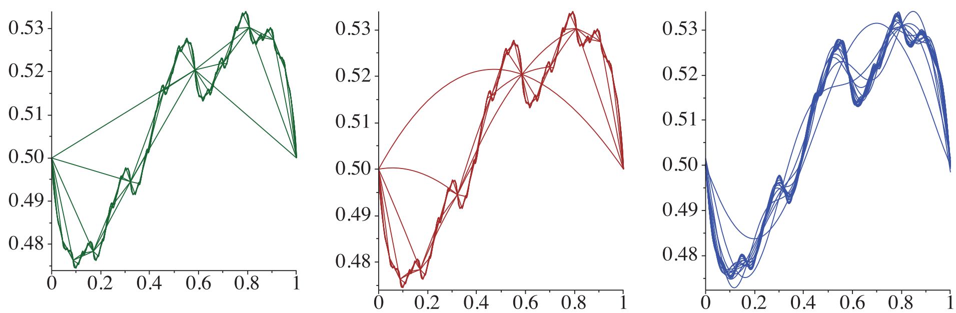


Fig. 12. The periodic function P approximated by Equation (5.14) (left), by Equation (5.15) (middle), and by its Fourier series Equation (5.16) (right), respectively.

where $Q(n) = -1 + \bar{Q}(n)$, which is consistent with the result in [34, 45]. Here, the periodic function equals, using Equation (5.4),

$$P(t) = 1 - \{t\} - 2^{1-\{t\}} + \sum_{k\in\mathbb{Z}} 2^{-k-\{t\}} \bar{g}\left(2^{k+\{t\}}\right),$$

where $\bar{g}(x)$ is extended from $\bar{g}(n)$ by linear interpolation. The Fourier coefficients have the form, using, for example, Equations (2.16) and (2.26),

$$\begin{aligned}\widehat{P}(0) &= \frac{1}{2} - \frac{1}{\log 2} - \frac{2}{\log 2} \sum_{m\geqslant 1} \frac{\log(2m+1) - \log(2m)}{(m+1)(m+2)} \\ &\approx -1.24815\,20420\,99653\,84890\,29565\,64329,\end{aligned}$$

and for $k \neq 0$

$$\widehat{P}(k) = \frac{1}{\chi_k(\chi_k+1)\log 2}\left(1 - 2\sum_{m\geqslant 1} \frac{m^{-\chi_k} - \left(m+\frac{1}{2}\right)^{-\chi_k}}{(m+1)(m+2)}\right).$$

Example 5.3 (Quicksort). The minimum number of comparisons used by the standard quicksort (see [69, pp. 106–116]) satisfies

$$a(n) = a\left(\left\lfloor \frac{n-1}{2} \right\rfloor\right) + a\left(\left\lceil \frac{n-1}{2} \right\rceil\right) + n - 1 \qquad (n \geqslant 2),$$

with $a(0) = a(1) = 0$. Write $f(n) = a(n-1)$. Then, $\Lambda[f] = n - 2$ for $n \geqslant 2$ with $f(1) = 0$. Since $g(n) = n - 2$ differs by 2 from Example 5.1 and by 1 from (a) above, it follows that $f(n) = \text{A003314}(n) - 2n + 2 = \text{A001855}(n) - n + 1$. The sequence $a(n)$ is A061168, which equals $\sum_{1\leqslant k<n} \lfloor \log_2 k \rfloor$. Another closely related sequence is A097384, mentioned in Example 5.2(a), which satisfies the same recurrence of $a(n)$ but with the toll function $n-1$ there replaced by n.

From Equation (2.1) (see also Equation (5.1)), we obtain $f(n) = nL_n - 2^{L_n+1} + 2$ for $n \geqslant 1$. It follows that $f(n) = n\log_2 n + nP(\log_2 n) + 2$, where $P(t) = -\{t\} - 2^{1-\{t\}}$; see Equations (5.3) and (5.4).

Another sequence with the same $g(n)$ but with a nonzero initial condition $f(1) = 1$ is A083652 (sum of the lengths of binary numbers), which equals $f(n) + n$.

In general, the cost used by quicksort in the best case satisfies the same recurrence but with the toll function of the form $cn + d$ (see [69, pp. 106–116]), which can be manipulated in the same manner.

There is yet another sequence connected to the best case of quicksort: A067699, which is the number of comparisons made in a version of quicksort for an array of size n with n identical elements. In our format, it satisfies Equation (1.1) with $g(n) = 2\lceil\frac{n+1}{2}\rceil = n + 2 - \mathbf{1}_{n \text{ is odd}}$. This time we obtain, for example, by combining Example 5.2 (a) and Example 3.6, $f(n) = n\log_2 n + nP(\log_2 n) - 2$ for $n \geqslant 1$, where P is given by

$$P(t) = 4 - \{t\} - 2^{1-\{t\}} - \sum_{k\geqslant 1} 2^{-k-\{t\}}\bar{g}\left(2^{k+\{t\}}\right),$$

where, for $x \geqslant 2$, $\bar{g}(x) = \begin{cases} \{x\}, & \text{if } \lfloor x\rfloor \text{ is even,} \\ 1-\{x\}, & \text{if } \lfloor x\rfloor \text{ is odd.}\end{cases}$

Example 5.4 (interconnecting networks). A Benes network is designed to realize any permutation. The number of switches $f(n)$ used by a class of networks called AS-Benes networks satisfies Equation (1.1) with $g(n) = 2\lfloor\frac{n}{2}\rfloor$ for $n \geqslant 3$ with $f(1) = 0$ and $f(2) = 1$; see [15]. See also [8] for more information. This sequence is A220001 in OEIS.

The sequence $f(n)$ is essentially twice the minimum number of comparisons used by mergesort (see Example 5.2(b)); the difference lies at the initial condition $f(2) = 1$. Thus, we denote the sequence A000788$(n-1)$ in Example 5.2(b) by $f_0(n)$ and consider the difference $\tilde{f}(n) := 2f_0(n) - f(n)$, which satisfies Equation (1.1) with $\tilde{g}(n) = 0$ $(n \geqslant 3)$, $\tilde{f}(1) = 0$, and $\tilde{g}(2) = \tilde{f}(2) = 1$. This difference sequence is indeed A060973(n) = A007378$(n) - n$ (see Example 3.2), and we obtain $\tilde{f}(n) = n\tilde{P}(\log_2 n)$, where (see (3.4))

$$\tilde{P}(\{t\}) = \begin{cases} 2^{-1-\{t\}}, & \text{if } \{t\} \in [0, \log_2 3 - 1], \\ 1 - 2^{-\{t\}}, & \text{if } \{t\} \in [\log_2 3 - 1, 1].\end{cases}$$

Thus, we obtain, using also Equation (5.5), $f(n) = n\log_2 n + nP(\log_2 n)$, where

$$P(t) = -\{t\} + \sum_{k\geqslant 0} 2^{-k-\{t\}}\bar{g}\left(2^{k+\{t\}}\right) - \begin{cases} 5\cdot 2^{-1-\{t\}} - 1, & \text{if } \{t\} \in [0, \log_2 3 - 1] \\ 2^{-\{t\}}, & \text{if } \{t\} \in [\log_2 3 - 1, 1],\end{cases}$$

where $\bar{g}(x) = 1 - \{x\}$ if $\lfloor x\rfloor$ is even and $\bar{g}(x) = \{x\}$ if $\lfloor x\rfloor$ is odd. The corresponding Fourier series is given by

$$P(t) = \log_2(3\pi) - \frac{1}{\log 2} - \frac{5}{2} + \frac{1}{\log 2}\sum_{k\neq 0}\frac{1 - 3^{-\chi_k} - 2\zeta(\chi_k)}{\chi_k(\chi_k+1)}e^{2k\pi i t},$$

which is absolutely convergent by Equation (3.13).

Example 5.5 (number of ones in Gray code representation). The Gray code representation of integers has the characteristic feature that the codes for any two neighboring integers differ in exactly one digit; such a coding scheme and its underlying concept are useful in many applications. As discussed in Example 5.2(b), the cost used in the best case of mergesort is identical to the total number of 1s in the binary expansions of the first n nonnegative integers. Enumerating the same quantity for the (binary reflected) Gray code of the first n nonnegative integers yields the same recurrence Equation (1.1) with the toll function (= A004524$(n+1)$) $g(n) = \lfloor \frac{n+1}{4} \rfloor + \lfloor \frac{n+2}{4} \rfloor$ for $n \geqslant 1$. This gives rise to sequence A173318$(n-1)$ of OEIS. There are several different ways to decompose $g(n)$ into linear and bounded terms to describe the periodic fluctuations of $f(n)$. We consider the decomposition $g(n) = \frac{n-1}{2} + \bar{g}(n)$, where $\bar{g}(n) = \frac{5}{4} - \{\frac{n+1}{4}\} - \{\frac{n+2}{4}\} = \frac{1}{2} - \frac{1}{2}\mathbf{1}_{n\equiv 1 \bmod 4} + \frac{1}{2}\mathbf{1}_{n\equiv 3 \bmod 4}$. Then, Equation (1.13) yields $\bar{Q}(n) = \frac{1}{2}$ and thus $\bar{f}(n) = n\bar{P}(\log_2 n) - \frac{1}{2}$, where $\bar{P}(t) = \sum_{k\in\mathbb{Z}} 2^{-k-\{t\}}\bar{g}(2^{k+\{t\}})$ with

$$\bar{g}(x) = \begin{cases} \frac{1}{2}(1-\{x\}), & \text{if } \lfloor x \rfloor \equiv 0 \bmod 4;\\ \frac{1}{2}\{x\}, & \text{if } \lfloor x \rfloor \equiv 1 \bmod 4;\\ \frac{1}{2}(1+\{x\}), & \text{if } \lfloor x \rfloor \equiv 2 \bmod 4;\\ 1-\frac{1}{2}\{x\}, & \text{if } \lfloor x \rfloor \equiv 3 \bmod 4. \end{cases}$$

for $x \geqslant 1$ and $\bar{g}(x) := 0$ for $x \in [0,1]$.

We then obtain from Equation (2.15) or Equation (2.26)

$$\begin{aligned} \bar{D}(s) &= \sum_{m\geqslant 0}\Big((4m+1)^{-s} - (4m+3)^{-s}\Big) - \frac{1}{2} \\ &= 2\cdot 4^{-s}\zeta\Big(s, \frac{1}{4}\Big) - (1-2^{-s})\zeta(s) - \frac{1}{2} \end{aligned} \tag{5.9}$$

for $\Re(s) > -1$, where $\zeta(s,v)$ denotes Hurwitz zeta function defined for $\Re(s) > 1$ by $\zeta(s,v) := \sum_{j\geqslant 0}(j+v)^{-s}$ ($v \in (0,1]$). Note that $\bar{D}$ is also expressible in terms of Dirichlet's L-function.

We thus obtain, again using also Example 5.2(a), $f(n) = \frac{1}{2}n\log_2 n + nP(\log_2 n)$, where

$$P(t) = \frac{1}{2}(1-\{t\} - 2^{1-\{t\}}) + \sum_{k\in\mathbb{Z}} 2^{-k-t}\bar{g}\Big(2^{k+t}\Big), \tag{5.10}$$

with the Fourier series expansion, using Equations (5.4) and (5.9),

$$P(t) = c_0 + \frac{2}{\log 2}\sum_{k\neq 0}\frac{\zeta\left(\chi_k, \frac{1}{4}\right)}{\chi_k(\chi_k+1)}e^{2k\pi i t}.$$

Here, $c_0 := -\frac{5}{4} - \frac{1}{2\log 2} - \log_2 \pi + 2\log_2 \Gamma\left(\frac{1}{4}\right)$. This rederives the results in [36] (the better converging series expansion Equation (5.10) being new). For more examples of a similar type, see [35] and [51].

A different decomposition is to start with the difference $\bar{g}(n) := \lfloor \frac{n+1}{4} \rfloor + \lfloor \frac{n+2}{4} \rfloor - \lfloor \frac{n}{2} \rfloor = \mathbf{1}_{n\equiv 3 \bmod 4}$ and then consider $\Lambda[\bar{f}] = \bar{g}$; see Examples 5.2(b) and 3.7 and the discussions there.

Example 5.6 (recurrences with minimization). The sequence A003314 (referred to as the binary entropy function in OEIS) that we examined in Example 5.1 is the solution of the following recurrence:

$$a(n) = n + \min_{1\leqslant k<n}\{a(k) + a(n-k)\} \tag{5.11}$$

for $n \geqslant 2$ with $a(1) = 0$; see [13, 58].

If we change the initial condition to $a(1) = 1$, then we get A033156 (also discussed in Example 5.1). Further changing the initial conditions to be $a(1) = 1$ and $a(2) = 2$ gives the sequence A054248, which is identical to the sequence $f(n)$ satisfying $\Lambda[f] = g$ with $g(n) = n - 2\cdot\mathbf{1}_{n\equiv 2 \bmod 4}$ and $f(1) = 1$. A proof by induction of this is given in Appendix B. Note that, for this sequence, the minimum in Equation (5.11) is attained at $k = 2\lfloor \frac{n+2}{4} \rfloor$ in contrast to the two sequences A003314 and A033156.

Then, we deduce (see (5.1) and $f_2(n)$ in Example 3.7) the closed-form solution

$$f(n) = n(L_n + 2) - 2^{L_n+1} + \mathbf{1}_{n \text{ odd}} \qquad (n \geqslant 1),$$

implying that $f(n) = n \log_2 n + nP(\log_2 n) + \mathbf{1}_{n \text{ is odd}}$, where $P(t) = 2 - \{t\} - 2^{1-\{t\}}$, as in (5.2).

On the other hand, a minor variant of Equation (5.11) has the form

$$a(n) = n + \min_{1 \leqslant k < n} \{a(k) + a(n-1-k)\} \qquad (n \geqslant 2),$$

with $a(0) = 0$. If $a(1) = 1$, then the shifted sequence $a(n-1)$ coincides with A001855 (studied in Example 5.2(a)), while if $a(1) = 0$, then the resulting sequence equals A097383. Now, the optimal choice of k is $k = 2\lfloor \frac{n+3}{4} \rfloor - 1$. By an argument similar to the proof of Lemma 5 in Appendix B, we can show by induction that the shifted sequence $a(n-1)$ satisfies the recurrence $\Lambda[f] = g$ with $g(n) = n - 1 - \mathbf{1}_{n \equiv 2 \bmod 4}$ with $f(1) = 0$. The solution is easily seen to be, for example, by combining Examples 5.2(a) and 3.7, $f(n) = n(L_n + 1) - 2^{L_n+1} - \lfloor \frac{n}{2} \rfloor + 1$ for $n \geqslant 1$, so that $f(n) = n \log_2 n + nP(\log_2 n) + 1 + \{\frac{n}{2}\}$, where $P(t) = \frac{1}{2} - \{t\} - 2^{1-\{t\}}$.

These examples show the sensitivity of recurrences with minimization under the change of initial conditions and simple shift.

Example 5.7 (Lebesgue constants of the Walsh system). This represents an example from harmonic analysis for which the periodic oscillations are rather different in look. The Lebesgue constants of the Walsh system (defined via binary coding) satisfy the recurrence (see [44])

$$\lambda(n) = \frac{1}{2}\lambda\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + \frac{1}{2}\lambda\left(\left\lceil \frac{n}{2} \right\rceil\right) + \frac{1}{2}\mathbf{1}_{n \text{ is odd}} \qquad (n \geqslant 2), \tag{5.12}$$

with $\lambda(0) := 0$ and $\lambda(1) = 1$. Then, the partial sum $f(n) := \sum_{k<n} \lambda(k) + \frac{1}{2}\lambda(n)$ satisfies the recurrence Equation (1.1) with

$$g(n) = \frac{1}{2}\left\lfloor \frac{n}{2} \right\rfloor + \frac{1}{2}\left(\lambda(n) - \lambda\left(\left\lceil \frac{n}{2} \right\rceil\right)\right).$$

We then split the toll function into two parts: $g_0(n) := \frac{1}{4}n$, which, by Example 5.1, yields $f_0(n) = \frac{1}{4}n \log_2 n + nP_0(\log_2 n)$ with $P_0(t) = \frac{1}{2} - \frac{\{t\}}{4} - 2^{-1-\{t\}}$, and

$$\bar{g}(n) = -\frac{1}{2}\left\{\frac{n}{2}\right\} + \frac{1}{2}\left(\lambda(n) - \lambda\left(\left\lceil \frac{n}{2} \right\rceil\right)\right) \qquad (n \geqslant 2).$$

Observe first that Equation (5.12) implies that $\bar{g}(2n) = 0$ and $\bar{g}(2n+1) = -\frac{1}{4}\Delta\lambda(n)$ for $n \geqslant 1$, where $\Delta\lambda(n) := \lambda(n+1) - \lambda(n)$, and that $\Delta\lambda(n)$ satisfies the recurrence

$$\Delta\lambda(n) = \frac{1}{2}\Delta\lambda\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + \frac{(-1)^n}{2} \qquad (n \geqslant 0). \tag{5.13}$$

Note that $\bar{Q}(n) = 0$ by Equation (1.13).

We then deduce that, using $\bar{f}(1) = f(1) = \frac{1}{2}$,

$$f(n) = \frac{1}{4}n \log_2 n + nP(\log_2 n), \tag{5.14}$$

where

$$P(t) = 1 - \frac{\{t\}}{4} - 2^{-1-\{t\}} + \sum_{k \geqslant 1} 2^{-k-\{t\}}\bar{g}(2^{k+\{t\}}). \tag{5.15}$$

This periodic function has a very different shape when compared with most others appearing in this article, which is also visible from the corresponding Fourier series already given in [44] by a completely analytic approach:

$$P(t) = -\frac{5}{24} - \frac{3\zeta'(-1)}{\log 2} + \frac{3}{\log 2} \sum_{k\neq 0} \frac{\zeta(-1+\chi_k)}{\chi_k(\chi_k^2-1)} e^{2k\pi i t}. \tag{5.16}$$

For more examples of Equation (1.1) with orders of the form $n(\log n)^m$, see [5, 35, 39].

6 APPLICATIONS. IV. QUADRATIC AND HIGHER ORDER $g(n)$

Fewer interesting examples were found for the recurrence Equation (1.1) with higher-order toll function $g(n)$, although many sequences in OEIS are of the form $cn^2 + dn + e$, which also satisfy Equation (1.1) with quadratic g.

Example 6.1 (Polynomials of the form n^m). The sequence A001105 in OEIS $f(n) = 2n^2$ satisfies Equation (1.1) with $g(n) = n^2 - \mathbf{1}_{n \text{ is odd}}$. This simple example is interesting because it is a nontrivial example without periodic fluctuation terms. In some sense, the fluctuation is transferred from $f(n)$ to $g(n)$. More generally, given any constant x_0, we can construct $f(n)$ containing no periodic oscillations as follows (with the right $f(1)$):

$$g(n) = \begin{cases} n^2 + x_0, & \text{if } n \text{ is even} \\ n^2 + x_0 - 1, & \text{if } n \text{ is odd} \end{cases} \implies f(n) = 2n^2 - x_0.$$

Similarly, A002378 (Oblong numbers) in which $f(n) = n(n+1)$ satisfies (with shift by 1) Equation (1.1) with $g(n) = \lfloor \frac{n^2}{2} \rfloor$. See also A046092, A000217, A005563, A001844, A161680, . . .(and many others).

It is also easy to extend such an idea of constructing $g(n)$ such that $f(n) = cn^m$ (again nonoscillating) for $m \geqslant 3$. For example, assuming that $f(1) = 1$,

$$g(n) = \begin{cases} \frac{3}{4}n^3, & \text{if } n \text{ is even} \\ \frac{3}{4}(n^3 - n), & \text{if } n \text{ is odd} \end{cases} \implies f(n) = n^3.$$

This implies that $f(n) = n^3$ (A000578) satisfies Equation (1.1) with $g(n) = \frac{3}{4}n^3 - \frac{3}{4}n\mathbf{1}_{n \text{ is odd}}$. Similarly, many other numbers (such as A000292, tetrahedral numbers) connected to the cubes also satisfies Equation (1.1) with (roughly) polynomial toll functions.

More generally, we have that $f(n) = n^m$ for $m \geqslant 2$ if $f(1) = 1$ and

$$g(n) = \begin{cases} (1-2^{1-m})n^m, & \text{if } n \text{ is even} \\ (1-2^{1-m})n^m - 2^{1-m} \sum_{1\leqslant j \leqslant \lfloor \frac{m}{2} \rfloor} \binom{m}{2j} n^{m-2j}, & \text{if } n \text{ is odd} \end{cases}.$$

Example 6.2 (A122247). The sequence A122247 consists of the partial sums of A005187, where the latter is defined as $2n - \nu(n)$, with $\nu(n)$ denoting the number of 1s in the binary expansion of n. By summing k from 1 to n, we obtain

$$\text{A122247}(n) = \sum_{1\leqslant k\leqslant n} (2k - \nu(k)) = n(n+1) - \text{A000788}(n). \tag{6.1}$$

It follows (see Examples 6.1 and 5.2(b)) that the shifted sequence $f(n) := \text{A122247}(n-1)$ satisfies the recurrence Equation (1.1) with $g(n) = \frac{n(n-1)}{2}$, the triangular numbers (A000217).

To solve this recurrence, we can use Equation (6.1) and the results in Example 5.2(b) for the best case of mergesort. We thus obtain

$$f(n) = n^2 - \frac{1}{2} n \log_2 n - n(1 + P_{(5.5)}(\log_2 n)). \tag{6.2}$$

Three other sequences in OEIS are closely connected to $f(n)$ and satisfy (after properly shifted) Equation (1.1) with quadratic $g(n)$:

- A077071$(n-1) = 2f(n)$,
- A122248$(n-1) = f(n) - \frac{1}{2}n^2 + \frac{3}{2}n - 1$, and
- A174605$(n-1) = f(n) - \frac{1}{2}n^2 + \frac{1}{2}n$.

In particular, such a connection and Equation (6.2) lead to

$$\text{A077071}(n-1) = 2n^2 - n\log_2 n - 2n(1 + P_{(5.5)}(\log_2 n)), \qquad (n \geqslant 2),$$

which clarifies and improves the statement in OEIS for A077071 "it seems that $f(n) = 2n^2 + O(n^{\frac{3}{2}})$." Note that shifting n to $n-1$ again plays an important role in getting a simpler g and the corresponding solution. On the other hand, A077071 is also connected to A067699 (discussed in Example 5.3).

7 VARIATIONS AND EXTENSIONS

The algorithmic and combinatorial literature abounds with a huge number of recurrences of multifarious forms; we discuss some interesting variants and extensions of the recurrences that we have discussed so far.

7.1 The recurrence $f(n) = -f(\lfloor \frac{n}{2} \rfloor) - f(\lceil \frac{n}{2} \rceil) + g(n)$

Most of our arguments apply well to the more general recurrence

$$f(n) = \alpha f\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + \alpha f\left(\left\lceil \frac{n}{2} \right\rceil\right) + g(n) \qquad (n \geqslant 2), \tag{7.1}$$

although our theorems do not. The essential fact is that Lemma 1 extends to this case, using the same linear interpolation to real arguments and with Equation (1.9) replaced by

$$f(x) = 2\alpha f\left(\frac{x}{2}\right) + g(x).$$

For simplicity, we here only discuss briefly the case $\alpha = -1$, whose behavior seems less anticipated; see the companion paper [50] for general results and many examples with $\alpha = 2$.

Example 7.1 (A005536: an example with a 2-periodic function). This is a "von Koch" sequence generated by the first Feigenbaum symbolic sequence A035263; it is also the sequence of partial sums of A065359; see [6]. The shifted sequence $f(n) = \text{A005536}(n-1)$ satisfies the recurrence Equation (7.1) with $\alpha = -1$ and $g(n) = \lfloor \frac{n}{2} \rfloor$ and $f(1) = 0$. By decomposing $g(n)$ into, say, $\frac{n-1}{2}$ and $\frac{1}{2} - \{\frac{n}{2}\}$, and by applying the same arguments used above for $\Lambda[f] = g$, we obtain $f(n) = nP(\log_2 n)$, where

$$P(t) = \frac{1}{4} + \frac{(-1)^{\lfloor t \rfloor}}{2}\left(\frac{1}{2} - \frac{2^{1-\{t\}}}{3}\right) + (-1)^{\lfloor t \rfloor}\sum_{j \geqslant 0}(-1)^j 2^{-j-\{t\}}\bar{g}\left(2^{j+\{t\}}\right),$$

where, for $x \geqslant 1$, $\bar{g}(x) = \frac{1}{2}(1 - \{x\})$ if $\lfloor x \rfloor$ is even and $\bar{g}(x) = \frac{1}{2}\{x\}$ if $\lfloor x \rfloor$ is odd. Note that $P(0) = P(2) = 0$ and, because of the occurrences of $(-1)^{\lfloor t \rfloor}$, *$P(t)$ is 2-periodic.* Also, it is continuous; see Figure 13. The Fourier series expansion can also be computed by the arguments used above: with $\chi'_k := \frac{(2k+1)\pi i}{\log 2}$,

$$P(t) = \frac{1}{4} + \frac{3}{\log 2}\sum_{k \in \mathbb{Z}} \frac{\zeta(\chi'_k)}{\chi'_k(\chi'_k + 1)} e^{(2k+1)\pi i t} \qquad (t \in \mathbb{R}).$$

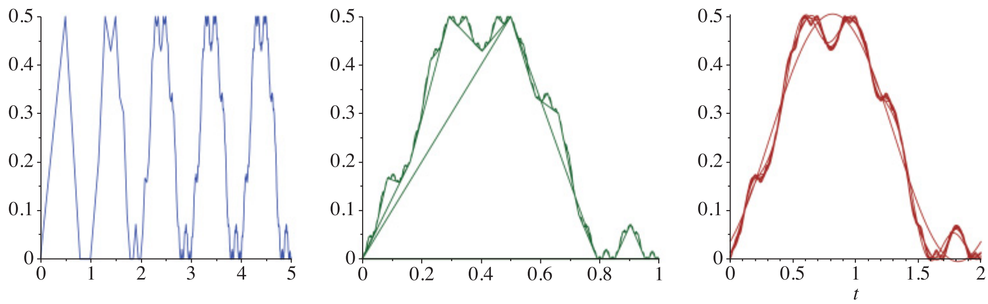


Fig. 13. Periodic fluctuations of $\frac{f(n)}{n}$ when plotted against $\log_4 n$: For n from 1 to 1024 (left), normalized in the unit interval (middle), which approximates P, and approximation of P by its Fourier series expansions using $1, 3, \dots, 21$ terms (right).

A closely related sequence is $A087733(n-1)$, which is given by $\tilde{f}(n) = \sum_{1 \leqslant k < n} (-1)^{v_2(k)}(n-k)$, where $v_2(n)$ denotes the largest power of two dividing n. This sequence satisfies Equation (7.1) with $\alpha = -1$, $\tilde{f}(1) = 0$, and $\tilde{g}(n) = \frac{n^2}{4} - \frac{1}{2}\{\frac{n}{2}\}$. Let $\bar{f}(n) = \frac{3}{2}(\tilde{f}(n) - \frac{n(n-1)}{6})$. Then, $\bar{f}(n) = f(n)$ for $n \geqslant 1$.

7.2 From Binary to q-ary

One of the most natural extensions of our study is to recurrences of the form (resulting, e.g., from dividing into $q \geqslant 2$ subproblems in the divide-and-conquer algorithm)

$$f(n) = \sum_{0 \leqslant j < q} f\left(\left\lfloor \frac{n+j}{q} \right\rfloor\right) + g(n) \qquad (n \geqslant q), \tag{7.2}$$

with $f(1), \dots, f(q-1)$ given. Alternatively, Equation (7.2) can be rewritten as

$$f(n) = q\left(1 - \left\{\frac{n}{q}\right\}\right) f\left(\left\lfloor \frac{n}{q} \right\rfloor\right) + q\left\{\frac{n}{q}\right\} f\left(\left\lceil \frac{n}{q} \right\rceil\right) + g(n).$$

We can apply the same linear interpolation techniques used in the binary case Equation (1.1) and then obtain a closed-form solution, which turns out to be useful for characterizing the corresponding asymptotic behaviors and periodic fluctuations. We thus define $f(x)$ and $g(x)$ for real x by Equation (1.8). Then, the recurrence Equation (7.2) implies that

$$f(x) = qf\left(\tfrac{x}{q}\right) + g(x) \qquad (x \geqslant q).$$

We then get the closed-form solution

$$f(x) = \sum_{0 \leqslant j < m} q^j g\left(\frac{x}{q^j}\right) + q^m f\left(\frac{x}{q^m}\right) \qquad (0 \leqslant m \leqslant \log_q x; x \geqslant 2). \tag{7.3}$$

Instead of formulating a more general theorem, we content ourselves with the discussion of two examples.

Example 7.2 (Lossless compression of balanced trees). The sequence $f(n)$ (see [60]) satisfies Equation (7.2) with $g(n) = \log_2 \binom{q}{n \bmod q}$ and $f(n) = 0$ for $n < q$; see Example 3.6, in which the case $q = 2$ was treated. We then deduce from Equation (7.3) that $f(n) = nP(\log_q n)$, where

$$P(t) := \sum_{k \in \mathbb{Z}} q^{-k-\{t\}} g\left(q^{k+\{t\}}\right) = \sum_{k \geqslant 1} q^{-k-\{t\}} g\left(q^{k+\{t\}}\right),$$

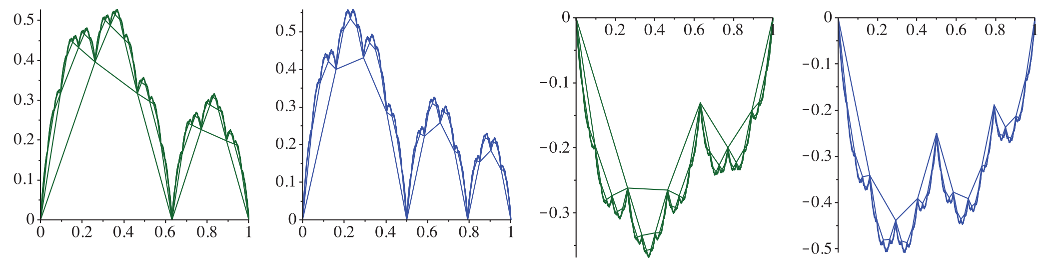


Fig. 14. Periodic fluctuations arising from the recurrence Equation (7.2) with $g(n) = \log_2 \binom{q}{n \bmod q}$ for $q = 3$ (left) and $q = 4$ (middle-left), and with g given by Equation (7.4) for $q = 3$ (middle-right) and $q = 4$ (right).

with $g(x) = 0$ for $0 \leqslant x \leqslant q$ and $g(x) = \{x\}g(\lfloor x \rfloor + 1) + (1 - \{x\})g(\lfloor x \rfloor)$ for $x \geqslant q$; see Figure 14 for a plot of P when $q = 3$ and $q = 4$. This provides an effective means of computing P; see the fractal approach in [60]. The corresponding Fourier coefficients are also easily computed (similarly to the binary case) as follows. By using $g(q\ell + j) = \bar{g}(j) := \log_2 \binom{q}{j}$ for $0 \leqslant j < q$ and $\ell \geqslant 1$, and noting that $\bar{g}(0) = 0$,

$$\begin{aligned} D(s) = D_q(s) &:= \sum_{k \geqslant q} g(k)\Big((k+1)^{-s} - 2k^{-s} + (k-1)^{-s}\Big) \\ &= \sum_{1 \leqslant j < q} \bar{g}(j) \sum_{\ell \geqslant 1} \left((q\ell + j - 1)^{-s} - 2(q\ell + j)^{-s} + (q\ell + j + 1)^{-s}\right). \end{aligned}$$

Let $h_j(s) := \sum_{\ell \geqslant 1}(q\ell + j)^{-s}$. Then, by partial summation, using $\bar{g}(0) = \bar{g}(q) = 0$,

$$\sum_{1 \leqslant j < q} \bar{g}(j)\Delta^2 h_{j-1}(s) = \bar{g}(1)h_0(s) + \sum_{1 \leqslant j < q} h_j(s)\Delta^2 \bar{g}(j-1) + \bar{g}(q-1)h_q(s).$$

Now, $h_0(s) = q^{-s}\zeta(s)$, $h_j(s) = q^{-s}\zeta(s, \frac{j}{q}) - j^{-s}$ for $1 \leqslant j < q$, and $h_q(s) = q^{-s}(\zeta(s) - 1)$. Also, $\bar{g}(j) = \bar{g}(q - j)$. Thus,

$$D_q(s) = \bar{g}(1)q^{-s}(2\zeta(s) - 1) + \sum_{1 \leqslant j < q} \Delta^2 \bar{g}(j-1)\left(q^{-s}\zeta\left(s, \frac{j}{q}\right) - j^{-s}\right).$$

In particular, we obtain, as already seen in the binary case in Equation (3.10),

$$\begin{aligned} D_2(s) &= 2 - 2^{-s} - 2(1 - 2^{1-s})\zeta(s), \\ D_3(s) &= (\log_2 3)(1 + 2^{-s} - 3^{-s}) - (\log_2 3)(1 - 3^{1-s})\zeta(s) \\ D_4(s) &= 3 - \log_2 3 - (1 - \log_2 3)2^{1-s} + (3 - \log_2 3)3^{-s} - 2 \cdot 4^{-s} \\ &\quad - \Big(3 - \log_2 3 - (5 - 3\log_2 3)2^{-s} - 2(1 + \log_2 3)4^{-s}\Big)\zeta(s). \end{aligned}$$

The Fourier series expansion is then given by, defining $\chi_k^{(q)} := \frac{2k\pi i}{\log q}$,

$$\frac{D_q'(0)}{\log q} + \frac{1}{\log q} \sum_{k \neq 0} \frac{D_q(\chi_k^{(q)})}{\chi_k^{(q)}(\chi_k^{(q)} + 1)} e^{2k\pi i t} \qquad (t \in \mathbb{R}).$$

This answers a question in [60]. Note that this result can also be derived by the analytic approach in [34] and that the series is absolutely convergent by an estimate similar to Equation (3.13) for

$|\zeta(it,c)|$; see [78, p. 276]. In particular, the mean value can be simplified as follows. Since $\zeta(0,t) = \frac{1}{2} - t$ and $\zeta'(0,t) = \log\Gamma(t) - \frac{1}{2}\log 2\pi$, we obtain, using $\bar{g}(1) = \log_2 q$,

$$\mu_q := \frac{D_q'(0)}{\log q} = \log_2 \frac{2q\Gamma\left(\frac{2}{q}\right)}{\Gamma\left(\frac{1}{q}\right)^2} + \sum_{2\leqslant j<q} \bar{g}(j)\log_q \frac{\Gamma\left(\frac{j-1}{q}\right)\Gamma\left(\frac{j+1}{q}\right)(j^2-1)}{\Gamma\left(\frac{j}{q}\right)^2 j^2}.$$

For small q, this gives

$$\begin{aligned}
\mu_2 &= 2 - \log_2 \pi,\\
\mu_3 &= \tfrac{5}{2}\log_2 3 - 2 - \log_2 \pi,\\
\mu_4 &= \tfrac{17}{4} - \log_2 \pi - \tfrac{9}{4}\log_2 3 + \tfrac{1}{2}(\log_2 3)^2,\\
\mu_5 &= -\tfrac{7}{2} - \log_2 \pi - \log_5 3 + \log_5(\sqrt{5}+1) + \tfrac{9}{4}\log_2 5 + \tfrac{1}{2}\log_2(\sqrt{5}-1).
\end{aligned}$$

A large number of concrete examples satisfying Equation (7.2), possibly after a shift by ±1 or ±2, can be found on OEIS; for example, A003605, A006166, A073849, A080722, A080723, A080724, A080726, A080727, A081134 for $q = 3$, and A073850, A080678, A275974 for $q = 4$. See also [14] for other examples connected to balanced trees.

Example 7.3 (Partial sum of the sum-of-digits function). The second example is the sum-of-digits function in the q-ary expansion for which $f(n) = \sum_{k<n} \nu_q(k)$, where $\nu_q(n) = \sum_{0\leqslant j\leqslant \lfloor \log_q n\rfloor} c_j$ when $n = \sum_{0\leqslant j\leqslant\lfloor\log_q n\rfloor} c_j q^j$ with $c_j \in \{0,\dots,q-1\}$. Such partial sums have been well studied in the literature and one finds the following correspondence of $f(n)$ in OEIS:

q	OEIS	q	OEIS	q	OEIS
2	A000788	5	A231668	8	A231680
3	A094345	6	A231672	9	A231684
4	A231664	7	A231676	10	A037123

Now, by the obvious recurrence $\nu_q(qk+j) = \nu_q(k) + j$ for $0 \leqslant j < q$, we get that

$$f(n) = \sum_{0\leqslant j<q}\sum_{k<\lfloor\frac{n+j}{q}\rfloor} \nu_q(qk+q-1-j) = \sum_{0\leqslant j<q} f\left(\left\lfloor\frac{n+j}{q}\right\rfloor\right) + g(n)$$

where, writing $n = qm + \ell$, $0 \leqslant \ell < q$,

$$g(n) = \sum_{0\leqslant j<q}(q-1-j)\left\lfloor\frac{n+j}{q}\right\rfloor = \frac{1}{2}(q-1)n - \frac{1}{2}\ell(q-\ell). \tag{7.4}$$

Then we deduce, again by Equation (7.3), Delange's closed-form expression $f(n) = \frac{q-1}{2}n\log_q n + nP(\log_q n)$ (see Figure 14), where P is a continuous and 1-periodic function; see [5, 23] for more information.

7.3 Sensitivity

The solutions to divide-and-conquer recurrences are often very sensitive to minor changes, particularly if one aims at exact solutions. This is probably one reason that some common sequences have many variants in OEIS. Nevertheless, the asymptotic aspect is generally more robust.

Some of the variants can be readily approached by our theory by either a simple shift of the parameter (as in many examples above) or a change of variables. We also discussed briefly the sensitivity of examples involving minimization in Section 3. We consider two more examples here.

Example 7.4 (A perturbed recurrence). We considered in Example 5.2(b) the recurrence $\Lambda[f] = g$ arising from the analysis of the best case of mergesort, where $g(n) = \lfloor \frac{n}{2} \rfloor$, which has the standard form $f(n) = \frac{n}{2}\log_2 n + nP(\log_2 n)$. Motivated from a heuristic for finding the minimum weighted Euclidean matching (see [45, 73]), it is of interest to compare $f(n)$ with the sequence $\tilde{f}(n)$ satisfying the perturbed recurrence

$$\tilde{f}(n) = \begin{cases} \tilde{f}\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + \tilde{f}\left(\left\lceil \frac{n}{2} \right\rceil\right) + \left\lfloor \frac{n}{2} \right\rfloor, & \text{if } n \not\equiv 0 \bmod 4, n \geqslant 2 \\ \tilde{f}\left(\frac{n}{2} - 1\right) + \tilde{f}\left(\frac{n}{2} + 1\right) + \frac{n}{2}, & \text{if } n \equiv 0 \bmod 4, n \geqslant 4, \end{cases} \tag{7.5}$$

with $\tilde{f}(0) = \tilde{f}(1) = 0$. This sequence (not in OEIS) starts with

$$\{\tilde{f}(n)\}_{n\geqslant 1} = \{0, 1, 2, 4, 5, 7, 9, 11, 13, 15, 17, 20, 22, 25, 27, 30, \dots\}.$$

We show that such a simple perturbation at multiples of four results not only in a lower cost ($\tilde{f}(n) \leqslant f(n)$ for all n) but also with a more smooth periodic function.

Consider the difference $\bar{f}(n) := \tilde{f}(n+1) - \tilde{f}(n-1)$, which satisfies the recurrence

$$\begin{cases} \bar{f}(2n) = \bar{f}(n) + 1, & (n \geqslant 1) \\ \bar{f}(4n+1) = \bar{f}(n) + 2, & (n \geqslant 1) \\ \bar{f}(4n+3) = \bar{f}(n+1) + 2, & (n \geqslant 0) \end{cases}$$

which leads to the closed-form solution $\bar{f}(n) = \lfloor \log_2(3n) \rfloor$ for $n \geqslant 1$. We then deduce that

$$\tilde{f}(n) = \sum_{1\leqslant j\leqslant \lfloor \frac{n}{2} \rfloor} \lfloor \log_2(3(n+1-2j)) \rfloor \qquad (n \geqslant 1).$$

It follows that

$$\tilde{f}(n) = \frac{n}{2}\log_2 n + nP(\log_2 n) + \begin{cases} \frac{1}{2} - \frac{(-1)^{\lfloor \log_2(3n) \rfloor}}{6}, & \text{if } n \text{ is even;} \\ \frac{1}{4} + \frac{(-1)^{\lfloor \log_2(3n) \rfloor}}{12}, & \text{if } n \text{ is odd,} \end{cases}$$

where $P(t) = \frac{1}{2}\log_2 3 - \frac{1}{2}\{t + \log_2 3\} - 2^{-\{t+\log_2 3\}}$; see Figure 15. This simplifies largely the expression in [45]. To show that $\tilde{f}(n) \leqslant f(n)$, it suffices to observe that their difference $d(n) := f(n) - \tilde{f}(n)$ satisfies $\Lambda[d](n) = 0$ when $n \not\equiv 0 \bmod 4$, and $d(4n) = 2d(2n) + \frac{1+(-1)^{\lfloor \log_2(3n) \rfloor}}{2}$ for $n \geqslant 1$.

See [45] for another example of the same type

$$f(n) = \begin{cases} \alpha f\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + \beta f\left(\left\lceil \frac{n}{2} \right\rceil\right) + \frac{1}{2}(1 + (-1)^n)c, & \text{if } n \not\equiv 0 \bmod 4 \\ \alpha f\left(\frac{n}{2} - 1\right) + \beta f\left(\frac{n}{2} + 1\right) + c, & \text{if } n \equiv 0 \bmod 4 \end{cases}$$

for $n \geqslant 2$ with $f(0) = f(1) = 0$, where $\alpha = \beta = \frac{1}{\sqrt{2}}$ and $c = \sqrt{3}$.

Example 7.5 (Two recurrences from the analysis of a "dichopile algorithm"). The following two recurrences were taken from [61, p. 45] and [62]:

$$f_1(n) = f_1\left(\left\lfloor \frac{n}{2} \right\rfloor - 1\right) + f_1\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + \left\lceil \frac{n}{2} \right\rceil,$$
$$f_2(n) = f_2\left(\left\lfloor \frac{n}{2} \right\rfloor - 1\right) + f_2\left(\left\lceil \frac{n}{2} \right\rceil\right) + \left\lfloor \frac{n}{2} \right\rfloor,$$

with the initial conditions $f_j(0) = 0$ and $f_j(1) = 1$ for $j = 1, 2$. The second sequence was also recently studied in [29]. In addition to their algorithmic connection, these two recurrences serve as additional concrete examples for illustrating the sensitivity of divide-and-conquer recurrences when compared particularly with $\Lambda[f](n) = \lfloor \frac{n}{2} \rfloor$ or (7.5).

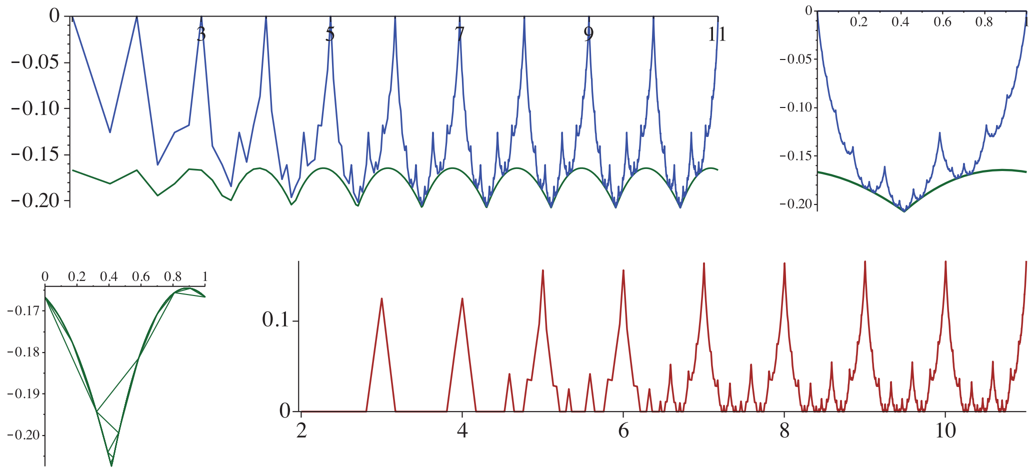


Fig. 15. The periodic functions arising from the two sequences $\frac{f(n)}{n} - \frac{1}{2}\log_2 n$ (blue) and $\frac{\tilde{f}(n)-\varepsilon(n)}{n} - \frac{1}{2}\log_2 n$ (green), where $\varepsilon(n) := \frac{1}{2} - \frac{(-1)^{\lfloor \log_2 3n \rfloor}}{6}$ if n is even and $\varepsilon(n) := \frac{1}{4} + \frac{(-1)^{\lfloor \log_2 3n \rfloor}}{12}$ if n is odd. The latter is more smooth than the former. Lower left: $\frac{\tilde{f}(n)-\varepsilon(n)}{n} - \frac{1}{2}\log_2 n$ plotted against $\{\log_2 n\}$; lower right: $\frac{f(n)-\tilde{f}(n)}{n}$.

For both sequences, we can prove that they have the asymptotic form $f_j(n) = \frac{n}{2}\log_2 n + nP_j(\log_2 n) + O(\log n)$, with different periodic functions:

$$P_1(t) = \frac{1}{2}P_{(5.4)}(t) = \frac{1}{2}\left(1 - \{t\} - 2^{1-\{t\}}\right) \quad \text{and} \quad P_2(t) = \frac{1}{3}(P_{(5.4)}(t) + P_{(5.5)}(t)).$$

Briefly, for f_1, we consider the difference $f_1(n) - f_1(n-1) - f_1(n-2) + f_1(n-3)$ and for f_2 the difference $f_2(n+1) - f_2(n) - f_2(n-2) + f_2(n-3)$, and then sum these differences back to get expressions for f_1 and f_2, respectively.

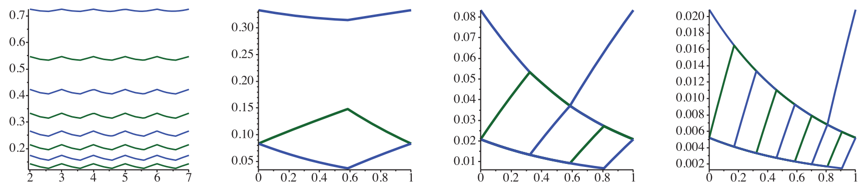
7.4 Asymptotic Robustness of Equation (1.1)

The large number of examples that we discussed show that the recurrence $\Lambda[f] = g$ can be solved in full if g is known explicitly. How to quantify the total cost of $f(n)$ when an expression of $g(n)$ is only available through regression or numerical procedures? More precisely, if $g(n)$ can somehow be approximated to within an error of order n^{-c}, where $c \geqslant 0$, then what is the maximal error made at the level of total cost $f(n)$? Thus, assume that $\Lambda[f_c] = g_c$, where $g_c(n) = n^{-c}$ for $n \geqslant 2$ with $f_c(0) = f_c(1) = 0$. Then, Theorem 2 yields

$$f_c(n) = nP_c(\log_2 n) - \frac{n^{-c}}{2^{c+1}-1} \qquad (n \geqslant 2),$$

where $P_c(t) = P(\{t\}) = \sum_{k\geqslant 1} 2^{-k-t}g(2^{k+t}) + g(2)(1-2^{-t})$ for $t \in \mathbb{R}$. A plot of $P_c(t)$ with $c = 0, \frac{1}{4}, \ldots, 2$ is given in Figure 16(i), in which we see that P_c gets smaller for increasing c.

On the other hand, if we fix $g(n) = n^{-1}$ and change the initial conditions so that $\Lambda[f^{[m]}] = n^{-1}$ for $n \geqslant m$ and $f^{[m]}(n) = 0$ for $n < m$, then we get $f^{[m]}(n) + \frac{1}{3}n^{-1} = nP^{[m]}(\log_2 n)$ for $n \geqslant m$, where $P^{[m]}$ has smaller amplitude for increasing m; see Figure 16 for an illustration.



(i) *(ii)* *(iii)* *(iv)*

Fig. 16. The periodic functions $P_c(t)$ (i) with $c = \frac{1}{4}l$ for $l = 0, 1, \ldots, 8$ (from top to bottom) and $n = 4, \ldots, 128$; $P^{[m]}$ with $m = 2, 3, 4$ (ii) (from top to bottom), $m = 4, \ldots, 8$ (iii), and $m = 8, \ldots, 16$ (iv).

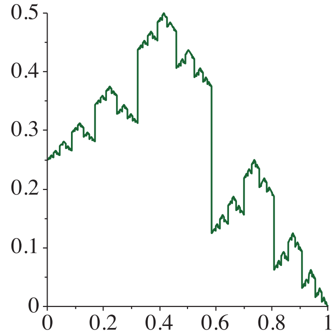


Fig. 17. $P(\log_2 n)$ (A0388554).

8 THE ONE-SIDED RECURRENCES (1.3)

We complete our study by discussing briefly the two cases in (1.3). Such cases arise more frequently than the recurrence (1.2) we analyzed above in the analysis of divide-and-conquer algorithms, mainly because cruder bounds are simpler to analyze and still useful in many practical situations.

8.1 Only Floor Function

We consider first the recurrence

$$f(n) = 2f\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + g(n) \qquad (n \geqslant 2), \tag{8.1}$$

with $f(1)$ given. Observe that when a satisfies $a(n) = 2a(\lfloor \frac{n}{2} \rfloor) + b(n)$, then the partial sum of a, say, $f(n) := \sum_{k<n} a(k)$ satisfies Equation (7.1) with $\alpha = 2$, where g denotes the partial sum of b. Thus, we expect that the corresponding periodic functions arising from Equation (8.1) will be less smooth in nature.

Our arguments used above for Equation (1.1) also apply to Equation (8.1). In particular, the extension of $f(n)$ from a sequence to all positive reals is now simply

$$f(x) = f(\lfloor x \rfloor) \qquad (x \geqslant 0);$$

in such a case, $f(x)$ is discontinuous (except in trivial cases). The solution to Equation (8.1) is easily seen to be

$$f(n) = \sum_{0 \leqslant k < m} 2^k g\left(\left\lfloor \frac{n}{2^k} \right\rfloor\right) + 2^m f\left(\left\lfloor \frac{n}{2^m} \right\rfloor\right)$$

for any $0 \leqslant m \leqslant L_n$. Taking $m = L_n$ gives

$$f(n) = \sum_{0 \leqslant k < L_n} 2^k g\left(\left\lfloor \frac{n}{2^k} \right\rfloor\right) + f(1)2^{L_n} \qquad (n \geqslant 1).$$

THEOREM 4. *Assume that f satisfies Equation (8.1) with $f(1)$ given. Define $g(1) = 0$. Then, the following statements are equivalent.*

(i) $f(n) = nP(\log_2 n) + o(n)$ as $n \to \infty$ for some 1-periodic function P on $\mathbb{R}$ satisfying

$$|P(\log_2 x) - P(\log_2 \lfloor x \rfloor)| \to 0 \qquad (x \to \infty). \tag{8.2}$$

(ii) $f(x) = xP(\log_2 x) + o(x)$ as $x \to \infty$ for some 1-periodic function P on $\mathbb{R}$.
(iii) The function $G(t) := \sum_{k \geqslant 0} 2^{-k} g(\lfloor 2^k t \rfloor)$ converges uniformly for $t \in [1, 2]$.

When these conditions hold, the periodic function P is given by

$$P(t) := 2^{-\{t\}}\left(G(2^{\{t\}}) + f(1)\right).$$

In typical cases, P is discontinuous. *Moreover, we have the exact formula $f(n) = nP(\log_2 n) - Q(n)$, where*

$$Q(n) := G(n) - g(n) = \sum_{k \geqslant 1} 2^{-k} g\left(\lfloor 2^k n \rfloor\right).$$

The proof is similar to that of Theorem 2 and is omitted here.

8.2 Only Ceiling Function

We now consider

$$f(n) = 2f\left(\left\lceil \frac{n}{2} \right\rceil\right) + g(n) \qquad (n \geqslant 2), \tag{8.3}$$

with $f(1)$ given. In this case, the extension of f is simply $f(x) = f(\lceil x \rceil)$. Again, $f(x)$ is discontinuous and we have the solution

$$f(n) = \sum_{0 \leqslant k \leqslant L_{n-1}} 2^k g\left(\left\lceil \frac{n}{2^k} \right\rceil\right) + f(1)2^{L_{n-1}+1} \qquad (n \geqslant 2).$$

Define $\{t^-\}$ as the left-continuous version of $\{t\}$, that is, $\{t^-\} = 1$ when $t \in \mathbb{Z}$, and $\{t^-\} = \{t\}$ otherwise. This can also be defined as $\{t^-\} := 1 - \{-t\}$.

THEOREM 5. *Assume that f satisfies Equation (8.3) with $f(1)$ given. Define $g(1) = 0$. Then, the following statements are equivalent.*

(i) $f(n) = nP(\log_2 n) + o(n)$ as $n \to \infty$ for some 1-periodic function P on $\mathbb{R}$ satisfying

$$|P(\log_2 x) - P(\log_2 \lceil x \rceil)| \to 0 \qquad (x \to \infty). \tag{8.4}$$

(ii) $f(x) = xP(\log_2 x) + o(x)$ as $x \to \infty$ for some 1-periodic function P on $\mathbb{R}$.
(iii) The function $G(t) := \sum_{k \geqslant 0} 2^{-k} g(\lceil 2^k t \rceil)$ converges uniformly for $t \in [1, 2]$.

When these conditions hold, the periodic function P is given by

$$P(t) := 2^{-\{t^-\}}\left(G(2^{\{t^-\}}) + 2f(1)\right).$$

In typical cases, P is discontinuous. *Moreover, we have the exact formula $f(n) = nP(\log_2 n) - Q(n)$, where*

$$Q(n) := G(n) - g(n) = \sum_{k \geqslant 1} 2^{-k} g\left(\lceil 2^k n \rceil\right).$$

A large number of examples of the types in Equations (8.1) and (8.3) can be found in OEIS and worked out as above; see [71] or [4, 5] for many such recursive sequences. We content ourselves with the following example from OEIS.

Example 8.1 (A038554, "the derivative of n"). In this sequence, $f(n)$ is obtained by XOR-ing each binary digit with the next one; equivalently, $f(n)$ is the XOR of n and its right-shift, with the first bit dropped. This sequence satisfies Equation (8.1) with

$$g(n) := \frac{1-(-1)^{\lceil n/2 \rceil}}{2} \qquad (n \geqslant 2),$$

and $f(1) = 0$. We easily see from Theorem 4 that $f(n) = nP(\log_2 n) - \frac{1}{2}\mathbf{1}_{n \text{ is odd}}$, where $P(t) := \sum_{k\geqslant 1} 2^{-k-\{t\}} g(\lfloor 2^{k+\{t\}} \rfloor)$ is a discontinuous function.

Sequence A003188, the value of the Gray code regarded as a binary number, is another sequence satisfying the recurrence Equation (8.1) with the same $g(n)$, but now $f(1) = 1$. Hence, this sequence differs from A038554 by 2^{L_n} and $P(t)$ differs by $2^{-\{t\}}$.

APPENDIX

A PROOF OF A078881 = A006165 (SHIFTED BY 1)

Table 3, with twenty sequences from OEIS, includes the sequence A078881(n). It is in OEIS noted that this equals A006165$(n+1)$ for $n \leqslant 1023$, and it is asked whether this holds for all n. For completeness, we prove here that this is true:

$$\text{A078881}(n) = \text{A006165}(n+1) \qquad (n \geqslant 1). \tag{A.1}$$

This also implies that A078881(n) = A066997(n) for $n \geqslant 2$.

We prove the following exact expression for A078881(n), which implies Equation (A.1) by comparison with formulas for A006561 in OEIS.

Lemma 4. *Let $f(n)$ = A078881(n) denote the largest size of a subset S of $\{1, 2, \ldots, n\}$ with the property*

$$i \neq j \in S \Longrightarrow (i \text{ XOR } j) \notin S, \tag{A.2}$$

where XOR is the bitwise exclusive-OR operator. Then,

$$f(n) = 2^{L_n - 1} + \min\{n - 2^{L_n} + 1, 2^{L_n - 1}\} \qquad (n \geqslant 1). \tag{A.3}$$

Proof. The method of proof consists of three steps: We first show that the expression in Equation (A.3) is a lower bound by explicitly constructing a set S of this size. We then prove two different upper bounds, corresponding to the two terms in the minimum in Equation (A.3).

Step 1: Lower bound by construction. Let the subset S be composed of two nonoverlapping parts:

(1) $A_n := \{k : k \in [2^{L_n-1}, 2^{L_n} - 1]\}$. Then, $|A_n| = 2^{L_n-1}$ and each $k \in A_n$ has the binary expansion $(01x\cdots x)_2$.
(2) $B_n := \{k : k \in [2^{L_n}, \min\{n, 2^{L_n} + 2^{L_n-1} - 1\}]\}$. Then, $|B_n| = \min\{n - 2^{L_n} + 1, 2^{L_n-1}\}$ and each $k \in B_n$ has the binary expansion $(10x\cdots x)_2$.

Then, we have Equation (A.2) for $S := A_n \cup B_n$ by checking the following properties:

- if $i, j \in A_n$, then $(i \text{ XOR } j) = (00x\cdots x)_2 \notin S$;
- if $i, j \in B_n$, then $(i \text{ XOR } j) = (00x\cdots x)_2 \notin S$; and
- if $i \in A_n$ and $j \in B_n$, then $(i \text{ XOR } j) = (11x\cdots x)_2 \notin S$.

Consequently,

$$f(n) \geqslant |S| = 2^{L_n-1} + \min\{n - 2^{L_n} + 1, 2^{L_n-1}\}. \tag{A.4}$$

Step 2: The first upper bound. Assume that $S = \{s_1, \ldots, s_k\} \subseteq \{1, 2, \ldots, n\}$ with the property Equation (A.2). Define $T := \{s_1 \text{ XOR } s_j : 2 \leqslant j \leqslant k\}$. From the property Equation (A.2), $S \cap T = \emptyset$. Note that $(s_1 \text{ XOR } s_i) \leqslant 2^{L_n+1} - 1$ for all $2 \leqslant i \leqslant k$ and $(s_1 \text{ XOR } s_i) \neq (s_1 \text{ XOR } s_j)$ if $s_i \neq s_j$. Thus, $|T| = |S| - 1$ and

$$|S| + |T| \leqslant 2^{L_n+1} - 1.$$

Thus, $|S| \leqslant 2^{L_n}$. Consequently,

$$f(n) \leqslant 2^{L_n}. \tag{A.5}$$

Step 3: The second upper bound. Assume again that $S \subseteq \{1, 2, \ldots, n\}$ with the property Equation (A.2). Consider the restriction $Q = S \cap \{k : 1 \leqslant k \leqslant 2^{L_n} - 1\}$. The set Q inherits the property Equation (A.2) from S; thus, by Equation (A.5), $|Q| \leqslant 2^{L_n-1}$. Thus,

$$|S| = |S - Q| + |Q| \leqslant n - 2^{L_n} + 1 + 2^{L_n-1} = n - 2^{L_n-1} + 1.$$

Consequently,

$$f(n) \leqslant n - 2^{L_n-1} + 1. \tag{A.6}$$

Combining Equations (A.5) and (A.6), we obtain

$$f(n) \leqslant 2^{L_n-1} + \min\{n - 2^{L_n} + 1, 2^{L_n-1}\},$$

which together with Equation (A.4) shows Equation (A.3). □

B OPTIMALITY OF A RECURRENCE WITH MINIMIZATION

We prove the first claim in Example 5.6, which, for ease of reference, is formulated as a lemma. The second claim has a similar proof, which is omitted here.

Lemma 5. *The sequence defined recursively by*

$$a(n) = n + \min_{1\leqslant k<n} \{a(k) + a(n-k)\} \qquad (n \geqslant 3), \tag{B.1}$$

with $a(1) = 1$ and $a(2) = 2$, satisfies the recurrence $\Lambda[f] = g$ with $f(1) = 1$ and $g(n) = n - 2 \cdot \mathbf{1}_{n\equiv 2 \bmod 4}$ for $n \geqslant 2$. Moreover, the minimum is reached at $k = \lfloor \frac{n}{2} \rfloor$ except for $n \equiv 2 \bmod 4$, for which the minimum is attained at $k = \lfloor \frac{n}{2} \rfloor \pm 1$.

Proof. We begin with the exact expression for $f(n)$, which is of the form (see Examples 5.1 and 3.7)

$$f(n) = n(L_n + 2) - 2^{L_n+1} + \mathbf{1}_{n \text{ is odd}} \qquad (n \geqslant 1). \tag{B.2}$$

We prove that $a(n) = f(n)$ for $n \geqslant 1$ by induction. The initial cases $f(1)$ and $f(2)$ are easy to check. Assume that $n \geqslant 3$ and $a(m) = f(m)$ for $1 \leqslant m < n$. By the definition of g, we now prove that

$$\min_{1\leqslant k<n} \{f(k) + f(n-k)\} = \begin{cases} f(\lfloor \frac{n}{2} \rfloor) + f(\lceil \frac{n}{2} \rceil) - 2, & \text{if } n \equiv 2 \bmod 4; \\ f(\lfloor \frac{n}{2} \rfloor) + f(\lceil \frac{n}{2} \rceil), & \text{otherwise.} \end{cases} \tag{B.3}$$

For that purpose, let $h(n) := n(L_n + 2) - 2^{L_n+1}$. It is easily verified that (also when $n + 1$ is a power of 2)

$$h(n+1) - h(n) = L_n + 2. \tag{B.4}$$

Hence, h is a convex function (second difference being nonnegative) for $n \geqslant 1$. This implies, by convexity, that

$$\min_{1 \leqslant k < n} \{h(k) + h(n-k)\} = h\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + h\left(\left\lceil \frac{n}{2} \right\rceil\right) \qquad (n \geqslant 2).$$

The difference between f and h is the error term $\mathbf{1}_{n \text{ is odd}}$ in Equation (B.2). This extra term may change the location of the minimum in the right-hand side of Equation (B.1).

- If n is odd, then exactly one of k and $n-k$ is odd; thus, the sum of the two error terms is always 1 for $1 \leqslant k < n$.
- If n is a multiple of 4, then both $\lfloor \frac{n}{2} \rfloor$ and $\lceil \frac{n}{2} \rceil$ are even. Thus, no extra error is produced.
- If $n \equiv 2 \bmod 4$—say, $n = 4m+2$, then there are three cases:
  - If $k = n-k = 2m+1$, then the two errors sum to 2.
  - If $k = 2m$ and $n-k = 2m+2$, then the errors sum to 0. Furthermore, Equation (B.4) implies that

    $$h(2m) + h(2m+2) = 2h(2m+1); \tag{B.5}$$

    thus, $f(2m) + f(2m+2) = 2f(2m+1) - 2$.
  - If $k < 2m$, then, by the convexity of h, we also have that

    $$f(k) + f(n-k) \geqslant h(k) + h(n-k) \geqslant h(2m) + h(2m+2) = f(2m) + f(2m+2).$$

    Thus, the minimum is reached at $k = 2m$.

In all three cases, Equation (B.3) follows. Thus, using the induction hypothesis, $a(n) = f(n)$. This completes the proof of Lemma 5. □

ACKNOWLEDGMENTS

We thank Michael Fuchs for constructive suggestions, and many participants of AofA 2016 (Kraków, Poland) for useful feedbacks and suggestions. We also thank the referees for helpful suggestions.

REFERENCES

[1] A. V. Aho, J. E. Hopcroft and J. D. Ullman. 1975. *The Design and Analysis of Computer Algorithms*. Second printing. Addison-Wesley Publishing Co., Reading, MA.
[2] M. Akra and L. Bazzi. 1998. On the solution of linear recurrence equations, *Comput. Optim. Appl.* 10, 195–210.
[3] P. C. Allaart and K. Kawamura. 2012. The Takagi function: a survey. *Real Anal. Exchange* 37, 1–54.
[4] J.-P. Allouche and J. Shallit. 1992. The ring of k-regular sequences, *Theoret. Comput. Sci.* 98, 163–197.
[5] J.-P. Allouche and J. Shallit. 2003. *Automatic Sequences. Theory, Applications, Generalizations*, Cambridge University Press, Cambridge UK.
[6] J.-P. Allouche and J. Shallit. 2003. The ring of k-regular sequences. II. Words, *Theoret. Comput. Sci.* 307, 3–29.
[7] Z.-D. Bai, L. Devroye, H.-K. Hwang and T.-H. Tsai. 2005. Maxima in hypercubes. *Random Structures Algorithms* 27, 290–309.
[8] B. Beauquier and E. Darrot. 2002. On arbitrary size Waksman networks and their vulnerability. *Parallel Process. Lett.* 12, 287–296.
[9] J. L. Bentley, D. Haken, and J. B. Saxe. 1980. A general method for solving divide-and-conquer recurrences. *ACM SIGACT News* 12.3, 36–44.
[10] J. L. Bentley and M. I. Shamos. 1978. Divide and conquer for linear expected time. *Information Processing Lett.* **7**, 87–91.
[11] G. Brassard and P. Bratley. 1988. *Algorithmics. Theory and Practice*. Prentice Hall, Inc., Englewood Cliffs, NJ.
[12] L. Carleson. 1966. On convergence and growth of partial sums of Fourier series. *Acta Math.* 116, 135–157.
[13] L. Carlitz. 1971. A sorting function. *Duke Math. J.* 38, 561–568.
[14] S.-H. Cha. 2012. On integer sequences derived from balanced k-ary trees. In *Proceedings of American Conference on Applied Mathematics*, pp. 377–381.
[15] C. Chang and R. Melhem. 1997. Arbitrary size Benes networks. *Parallel Process. Lett.* 7, 279–284.

[16] W.-M. Chen, H.-K. Hwang, G.-H. Chen. 1999. The cost distribution of queue-mergesort, optimal mergesorts, and power-of-2 rules. *J. Algorithms* 30, 423–448.

[17] W.-M. Chen, H.-K. Hwang and T.-H. Tsai. 2012. Maxima-finding algorithms for multidimensional samples: a two-phase approach. *Comput. Geom.* 45, 33–53.

[18] L. H. Y. Chen, H.-K. Hwang and V. Zacharovas. 2014. Distribution of the sum-of-digits function of random integers: a survey. *Probab. Surv.* 11, 177–236.

[19] H.-H. Chern and H.-K. Hwang. 2001. Phase changes in random m-ary search trees and generalized quicksort, *Random Structures Algorithms* 19, 316–358.

[20] H.-H. Chern, M. Fuchs, and H.-K. Hwang. 2007. Phase changes in random point quadtrees. *ACM Trans. Algorithms* 3, Art. 12, 51 pp.

[21] K. L. Clarkson and P. W. Shor. 1989. Applications of random sampling in computational geometry, II. *Discrete Comput. Geom.* 4, 387–421.

[22] T. Cormen, C. Leiserson, and R. Rivest. 1990. *Introduction to Algorithms.* McGraw-Hill, New York, NY.

[23] H. Delange. 1975. Sur la fonction sommatoire de la fonction "somme des chiffres." *Enseignement Math.* 21, 31–47.

[24] L. Devroye. 1983. Moment inequalities for random variables in computational geometry. 1983. *Computing* 30 111–119.

[25] L. Devroye. 1994. On random Cartesian trees. *Random Structures Algorithms* 5, 305–327.

[26] L. Devroye. 1999. A note on the expected time for finding maxima by list algorithms. *Algorithmica* 23, 97–108.

[27] M. Drmota and W. Szpankowski. 2013. A master theorem for discrete divide and conquer recurrences. *J. ACM* 60, Art. 16, 49 pp.

[28] S. Dube. 1995. Using fractal geometry for solving divide-and-conquer recurrences. *J. Austral. Math. Soc. Ser. B* 37, 145–171.

[29] P. Dumas. 2014. Asymptotic expansions for linear homogeneous divide-and-conquer recurrences: algebraic and analytic approaches collated. *Theoret. Comput. Sci.* 548, 25–53.

[30] R. A. Dwyer. 1988. On the convex hull of random points in a polytope. *J. Appl. Probab.* 25, 688–699.

[31] W. M. Dymàček and T. Whaley. 1995. Generating strings for bipartite Steinhaus graphs. *Discrete Math.* 141, 95–107.

[32] P. Erdős, A. Hildebrand, A. Odlyzko, P. Pudaite and B. Reznick. 1987. The asymptotic behavior of a family of sequences, *Pacific J. Math.* 126, 227–241.

[33] P. Flajolet and M. Golin. 1993. Exact asymptotics of divide-and-conquer recurrences. In *Automata, Languages and Programming* (Lund, 1993), 137–149, *Lecture Notes in Computer Sciences*, Vol. 700, Springer, Berlin.

[34] P. Flajolet and M. Golin. 1994. Mellin transforms and asymptotics. The mergesort recurrence, *Acta Inform.* 31, 673–696.

[35] P. Flajolet, P. Grabner, P. Kirschenhofer, H. Prodinger, and R. F. Tichy. 1994. Mellin transforms and asymptotics: digital sums. *Theoret. Comput. Sci.* 123, 291–314.

[36] P. Flajolet and L. Ramshaw. 1980. A note on Gray code and odd-even merge. *SIAM J. Comput.* 142–158.

[37] M. L. Fredman and D. E. Knuth. 1974. Recurrence relations based on minimization. *J. Math. Anal. Appl.* 48, 534–559.

[38] P. J. Grabner. 1993. Completely q-multiplicative functions: the Mellin transform approach. *Acta Arith.* 65, 85–96.

[39] P. J. Grabner and H.-K. Hwang. 2005. Digital sums and divide-and-conquer recurrences: Fourier expansions and absolute convergence. *Constr. Approx.* 21, 149–179.

[40] D. H. Greene and D. E. Knuth. 2008. Reprint of the third (1990) edition. Birkhäuser, Boston, MA.

[41] J. M. Hammersley. 1962. Generalization of the fundamental theorem on sub-additive functions. *Proc. Cambridge Philos. Soc.* 58, 235–238.

[42] J. M. Hammersley and G. R. Grimmett. 1974. Maximal solutions of the generalized subadditive inequality. In *Stochastic Geometry (A Tribute to the Memory of Rollo Davidson)*, 270–284, Wiley, London.

[43] E. Horowitz and S. Sahni. 1978. *Fundamentals of Computer Algorithms*, Computer Science Press, Rockville, MD, 1978.

[44] H.-K. Hwang. 1996. Asymptotic expansion for the Lebesgue constants of the Walsh system. *J. Comput. Appl. Math.* 71, 237–243.

[45] H.-K. Hwang. 1998. Asymptotics of divide-and-conquer recurrences: Batcher's sorting algorithm and a minimum Euclidean matching heuristic. *Algorithmica* 22, 529–546.

[46] H.-K. Hwang. 1998. Asymptotic expansions of the mergesort recurrences, *Acta Inform.* 35, 911–919.

[47] H.-K. Hwang and S. Janson. 2011. A central limit theorem for random ordered factorizations of integers. *Electron. J. Probab.* 16, 347–361.

[48] H.-K. Hwang and R. Neininger. 2002. Phase change of limit laws in the quicksort recurrence under varying toll functions. *SIAM J. Comput.* 31, 1687–1722.

[49] H.-K. Hwang and T.-H. Tsai. 2003. An asymptotic theory for recurrence relations based on minimization and maximization. *Theoret. Comput. Sci.* 290, 1475–1501.

[50] H.-K. Hwang, S. Janson, and T.-H. Tsai. 2017. Exact and asymptotic solutions of the recurrence $f(n) = \alpha f(\lfloor \frac{n}{2} \rfloor) + \beta f(\lceil \frac{n}{2} \rceil) + g(n)$: theory and applications. In preparation (2017).

[51] Y. Kamiya and L. Murata. 2012. Relations among arithmetical functions, automatic sequences, and sum of digits functions induced by certain Gray codes. *J. Théor. Nombres Bordeaux* 24, 307–337.
[52] M.-Y. Kao. 1997. Multiple-size divide-and-conquer recurrences. *SIGACT News*, 28, 67–69.
[53] J. Kieffer. 2012. Asymptotics of divide-and-conquer recurrences via iterated function systems. In *23rd International Meeting on Probabilistic, Combinatorial, and Asymptotic Methods in the Analysis of Algorithms (AofA'12)*, Montreal, Canada. *Discrete Mathematics and Theoretical Computer Science*, AQ, pp. 55–66, 2012.
[54] M. Kuczma, B. Choczewski and R. Ger. 1990. *Iterative Functional Equations.* Cambridge University Press, Cambridge, UK.
[55] T. Leighton. 1996. Notes on better master theorems for divide-and-conquer recurrences. MIT Lecture Notes, Cambridge, MA.
[56] Z. Li and E. M. Reingold. 1989. Solution of a divide-and-conquer maximin recurrence. *SIAM J. Comput.* 18, 1188–1200.
[57] A.-H. Mogos. 2013. Three variants of the master theorem. 2013. In *the 19th International Conference on Control Systems and Computer Science*, pp. 162–166, 2013.
[58] R. Morris. 1969. Some theorems on sorting. *SIAM J. Appl. Math.* 17, , 1–6.
[59] S. Y. Oh. 2012. *Information Theory of Random Trees Induced by Stochastic Grammars.* Ph.D. Dissertation. University of Minnesota, Minneapolis, MN.
[60] S. Y. Oh and J. C. Kieffer. 2010. Fractal compression rate curves in lossless compression of balanced trees. In *IEEE International Symposium on Information Theory Proceedings (ISIT'10)*, pp. 106–110.
[61] J. Oudinet. 2010. *Approches combinatoires pour le test statistique à grande échelle.* Ph.D. Thesis, Université de Paris-Sud, Paris.
[62] J. Oudinet, A. Denise and M.-C. Gaudel. 2013. A new dichotomic algorithm for the uniform random generation of words in regular languages. *Theoret. Comput. Sci.* 502, 165–176.
[63] I. Pohl. 1972. A sorting problem and its complexity. *Comm. ACM* 15, 462–464.
[64] F. P. Preparata and M. I. Shamos. 1985. *Computational Geometry. An Introduction.* Springer-Verlag, New York, NY.
[65] P. W. Purdom and C. A. Brown. 1985. *The Analysis of Algorithms.* Holt, Rinehart & Winston, New York, NY.
[66] S. Resnick. 1992. *Adventures in Stochastic Processes.* Birkhäuser Boston, Inc., Boston, MA.
[67] S. Roura. 2001. Improved master theorems for divide-and-conquer recurrences. *J. ACM* 48, 170–205.
[68] U. Schöning. Mastering the Master Theorem. 2000. *Bulletin of the EATCS* 71, 165–166.
[69] R. Sedgewick. 1980. *Quicksort.* Garland, New York, NY.
[70] N. J. A. Sloane et al. 2017. *The On-Line Encyclopedia of Integer Sequences.* Retrieved August 13, 2017 from https://oeis.org/somedcgf.html (Version 2004-01-1).
[71] R. Stephan. 2004. Some divide-and-conquer sequences with (relatively) simple ordinary generating functions.
[72] K. B. Stolarsky. 1977. Power and exponential sums of digital sums related to binomial coefficient parity. *SIAM J. Appl. Math.* 32, 717–730.
[73] K. J. Supowit and E. M. Reingold. 1983. Divide and conquer heuristics for minimum weighted Euclidean matching. *SIAM J. Comput.* 12, 118–143.
[74] R. M. Verma. 1994. A general method and a master theorem for divide-and-conquer recurrences with applications. *J. Algorithms* 16, 67–79.
[75] R. M. Verma. 1997. General techniques for analyzing recursive algorithms with applications. *SIAM J. Comput.* 26, 568–581.
[76] X. Wang and Q. Fu. 1996. A frame for general divide-and-conquer recurrences, *Inform. Process. Lett.* 59, 45–51.
[77] C. Yap. 2011. A real elementary approach to the master recurrence and generalizations. In *Theory and Applications of Models of Computation*, 14–26, LNCS, Vol. 6648, Springer, Berlin.
[78] E. T. Whittaker and G. N. Watson. 1996. *A Course of Modern Analysis*, reprint of the fourth (1927) edition. Cambridge University Press, Cambridge, UK.
[79] A. Zygmund. 2002. *Trigonometric Series*, third ed., Cambridge University Press, Cambridge, UK.

Received December 2016; accepted July 2017