On multi-avoidance of right angled numbered polyomino patterns

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Abstract

Recently, Kitaev, Mansour and Vella introduced numbered polyomino patterns that generalize the concept of pattern avoidance from permutations and words to numbered polyominoes. We study simultaneous avoidance of two or more right angled numbered polyomino patterns, which are 0-1 labellings of the essentially unique convex two-dimensional polyomino shape with 3 tiles. It turns out that this study gives relations to several combinatorial structures.

Keywords: numbered polyomino pattern, avoidance, matrix, permutation, hypercube, spanning tree, nonattacking kings.

1 Introduction

In [5], the authors generalized the concept of *pattern avoidance* (see [4]) from permutations and words to *numbered polyominoes*. In particular, they considered avoidance of *binary right angled polyomino patterns*, which are 0-1 labellings of the essentially unique convex two-dimensional polyomino shape with 3 tiles.

As in [7] (resp. [1], [2], [3]), where the authors deal with multi-avoidance of *classical* (resp. *generalized*) 3-patterns (see [4] for definitions), it is natural to study avoidance of two or more right angled polyomino patterns. It turns out that this study gives relations to several combinatorial structures (see Section 3).

The paper is organized as follows. In Section 2 we give all necessarily definitions. In Section 3, we show the interest to study the multi-avoidance of right angled polyomino patterns by giving relations to other combinatorial objects, such as certain permutations, hypercubes, placing of nonattacking kings on certain boards, spanning trees and others. In Section 4 (resp. 5, 6) we find the number of $m \times n$ matrices that avoid any combination of two (resp. three, four) right angled polyomino patterns. The case of avoidance of five or more of polyomino patterns (out of seven) is discussed in Section 7.

2 Preliminaries

We follow [5] to define our patterns. However, we refer to [5] and the references therein for more details and examples.

A *polyomino* is a finite subset of $\mathbb{Z}^2$. The elements of a polyomino are called *tiles*. Given an element $p \in \mathbb{Z}^2$, we denote by x_p and y_p the first and second coordinates of p respectively. A *column* (resp. *row*) of a polyomino P is a maximal set of tiles of P all having the same first (resp. second) coordinate.

Now let G be the graph with $\mathbb{Z}^2$ as vertex set, and with p, q adjacent if and only if $|x_p - x_q| + |y_p - y_q| = 1$. Then G is a self-dual planar graph and a polyomino can be thought of equivalently as a set of vertices of G or a set of faces of a square tessellation of the plane, which is an embedding of G. The latter interpretation gives the intuition behind the choice of the term "polyomino", in analogy with the word "domino".

Given two polyominoes P_1, P_2, a *polyomino isomorphism* is a bijection from P_1 to P_2 such that, for every $p, q \in P_1$, $x_p < x_q \Leftrightarrow x_{\phi(p)} < x_{\phi(q)}$ and $y_p < y_q \Leftrightarrow y_{\phi(p)} < y_{\phi(q)}$. The *width* (resp. *height*) of a polyomino P is the maximum over all pairs $\{p, q\} \subseteq P$ of $|x_p - x_q|$ (resp. $|y_p - y_q|$). The *reduction* of P is the polyomino which minimizes the width and the height among all polyominoes isomorphic to P in which all tiles have only non-negative coordinates. A *polyomino shape* (or simply a *shape*) is a polyomino which is its own reduction. If the reduction of a polyomino P is a certain shape C, we shall also say that P *has* the shape C. We shall denote shapes by a geometric depiction of the relative positions of the tiles.

Given a non-negative integer n, $[n]$ denotes the set of non-negative integers less than or equal to n; a set of this form is called an *interval*. A *numbering* ϕ of a set T is a function from T into the set of integers. If the range A of ϕ is finite, there exists a unique order-preserving bijection ψ from A onto the interval of cardinality $|A|$. The *reduction* of ϕ is the numbering

$\phi \circ \psi$, and a numbering is *reduced* if it is its own reduction. Also, for any integer $k \geq |A|$, ϕ is called a k-numbering. We shall extend our notation for shapes to numbered shapes in the obvious way.

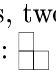
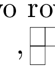
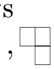
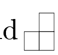
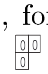
Given a polyomino P, a subset $Q \subseteq P$ is a *subpolyomino* of P. A numbered polyomino is a polyomino equipped with a numbering. If ϕ is a numbering of P, the subpolyomino Q inherits the numbering $\phi|_Q$. Given a polyomino Q' with a numbering $\phi_{Q'}$, Q is an *occurrence* of Q' in P if there exists a polyomino isomorphism μ from Q to Q' such that the numberings $\phi|_Q$ and $\mu \circ \phi_{Q'}$ have the same reduction; if the two numberings are actually the same, then the occurrence is *literal*.

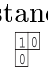
If there are no occurrences of Q' in P, P is said to *avoid* Q'.

A *numbered polyomino pattern* (or simply a *pattern*) is a polyomino shape equipped with a reduced numbering. We are concerned with occurrences of patterns in numbered shapes. Given a positive integer k, a shape C and a pattern P, $S_C^{(k)}$ denotes the set of k-numberings of C such that the corresponding numbered polyomino avoids the pattern P (the pattern P is understood and not explicitly specified in the notation).

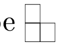
In this paper, we shall assume that $k = 2$ and only examine avoidance of polyomino patterns in (binary) matrices. Moreover, we shall assume that the matrix C has m rows and n columns, and denote $|S_C^{(2)}|$ by $M_{m,n}$.

Remark 1. The operations of *complementation* (replacing i with $k-i$) and reflection about any one of the four axes of symmetry of the square lattice (the vertical, horizontal and diagonal lines through the origin) are all involutions on the set of numbered polyominoes which preserve occurrences, in the sense that if χ is one of the above operations, and P, Q are numbered polyominoes, then P occurs in Q if and only if $\chi(P)$ occurs in $\chi(Q)$. Clearly, the same is true if χ is any composition of these operations. Note that reflecting a matrix about the line $y = -x$ and reducing the shape corresponds to taking the transpose of the matrix. As in classical permutation avoidance, these operations are often useful in reducing the enumeration of pattern-avoiding polyominoes to a smaller number of cases (patterns).

A polyomino is *right angled* if it contains precisely three tiles, two rows and two columns. There are four different right angled shapes:  ,  ,  and  , each of which can be numbered in 7 different ways (the numeration with all 1s is not in the reduced form, and therefore does not give us a pattern). However, in this paper, we shall consider simultaneous avoidance of patterns having the same shape. Thus, for instance, we are interested in simultaneous avoidance of the patterns 

and 

, but *not* in that of,

say, the patterns, one of which has the shape  , whereas the other one has the shape  . Furthermore, the operations of reflection mentioned in Remark 1 allow us to consider only the patterns

$$p_1 = \begin{smallmatrix}0&0\\0&\end{smallmatrix}\,,\ p_2 = \begin{smallmatrix}0&0\\1&\end{smallmatrix}\,,\ p_3 = \begin{smallmatrix}0&1\\0&\end{smallmatrix}\,,\ p_4 = \begin{smallmatrix}1&0\\0&\end{smallmatrix}\,,\ p_5 = \begin{smallmatrix}1&1\\0&\end{smallmatrix}\,,\ p_6 = \begin{smallmatrix}1&0\\1&\end{smallmatrix}\,,\ p_7 = \begin{smallmatrix}0&1\\1&\end{smallmatrix}\,.$$

Also, if C denotes the operation of complementation mentioned in Remark 1, T denotes the operation of transposition (see Remark 1 again), and CT denotes the composition of C and T, which is obviously commutative, then one can use Table 1 to reduce the number of cases to consider.

a pattern p	p_1	p_2	p_3	p_4	p_5	p_6	p_7
$C(p)$	p_1	p_5	p_6	p_7	p_2	p_3	p_4
$T(p)$	p_1	p_3	p_2	p_4	p_6	p_5	p_7
$CT(p)$	p_1	p_6	p_5	p_7	p_3	p_2	p_4

Table 1: Complementation, transposition, and their composition

Thus, if we determined the number $M_{m,n}$ of $m \times n$ binary matrices that simultaneously avoid, for instance, the patterns p_2, p_4 and p_5, then the number of $m \times n$ matrices that simultaneously avoid the patterns p_2, p_5 and p_7 is the same, that is $M_{m,n}$, due to the operation of complementation. Moreover, if in the the expression for that $M_{m,n}$ we switch m and n, we get the number of permutations that simultaneously avoid the patterns p_3, p_4 and p_6, as well as that avoiding the patterns p_3, p_6 and p_7, due to the operation of transposition and that of composition of complementation and transposition.

So, using Table 1, one can divide all the possibilities into equivalence classes, and we need to consider a representative from each class. We indicate the equivalence classes and their representatives in the corresponding tables of Sections 4, 5, 6 and 7. If we give any information for an equivalence class, this information concerns to the representative from this class. Since obviously $M_{1,n} = 2^n$ and $M_{m,1} = 2^m$, in many cases we give the results only for $m, n \geq 2$. Moreover, in some cases as a solution to a problem, we provide a recursion or/and the *bivariate generating function* $M(x, y)$ defined as $M(x, y) = \sum_{m,n \geq 0} M_{m,n} x^m y^n$. Thus, the variable x is responsible for the number of rows, whereas y for the number of columns. To get $M_{m,n}$ from $M(x, y)$ one can use the standard mathematical packages such as Maple.

3 Relations to other combinatorial objects

We show the interest to study multi-avoidance of right angled numbered polyomino patterns by giving connections to other combinatorial objects.

3.1 Permutations with two sequences

Let σ be a permutation on $[n]$. A *sequence* of length ℓ (≥ 2) of σ is a maximal interval of integers $[i, i+\ell-1] = \{i, i+1, \dots, i+\ell-1\}$ on which σ is monotonic.

We are concerned about the permutations with exactly two sequences. If $n = 2$, there are no such permutations, if $n = 3$, these permutations are 132, 231, 213, and 312. Indeed, for instance, in the first permutation, the sequences are 13 and 32, whereas in the last one, 31 and 12. There are 12 such permutations in the case $n = 4$, which are

1243, 1342, 1432, 2134, 2341, 2431, 3124, 3214, 3421, 4123, 4213, 4312.

One can show, in general, that there are $2^n - 4$ n-permutations with two sequences.

Proposition 2. *For $n \geq 1$, there is a bijection between the $(n+2)$-permutations with two sequences and the $2 \times n$ matrices that simultaneously avoid the patterns $p_2 = \begin{array}{|c|c|}\hline 0 & 0 \\ \hline 1 & \\ \cline{1-1} \end{array}$ and $p_5 = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 0 & \\ \cline{1-1} \end{array}$.*

Proof. Clearly, the permutations with two sequences can be divided into two groups. The first group contains permutations having increasing interval followed by decreasing one, and the second group contains all other permutations. A bijection between these groups is given by taking the complementation, that is by replacing the letter i by the letter $n+i+1$ for $(n+2)$-permutations. Thus, the groups are equally large. Moreover, the intervals in each permutation from the first group share the letter $(n+2)$, and thus such permutations can be specified by dividing the set $\{1, 2, \dots, n+1\}$ into two nonempty subsets.

It is easy to see that a matrix A avoids p_2 and p_5 if and only if the complementation of A, that is $C(A)$, does it. Using this property, we will divide the set of all matrices avoiding p_2 and p_5 into two equally large groups. Then we will construct a bijection $\mathcal{F}$ between the first $(n+2)$-permutations group and the first $2 \times n$ matrices group. A bijection between the second groups will be given by the composition $C \circ \mathcal{F} \circ C$. Thus we will get the desirable bijection.

Suppose A is a $2 \times n$ matrix that avoids p_2 and p_5. Then either A entirely consists of the columns $x = (11)^T$ and $y = (00)^T$ or A has at most one column $u = (10)^T$ and at most one column $v = (01)^T$. If A has both u and v then one of them must be the right-most column of A (any column placed to the right of both u and v would lead to an occurrence of a prohibition). Also, if there are columns in A to the right of u (resp. v) then these columns, except maybe for the last one, must be y (resp. x), and the last column can be v (resp. u).

The first $2 \times n$ matrix group consists of the following four subgroups, where the symbol "?" is used to indicate that corresponding column is either x or y, and, say, y^i denotes concatenation of i columns y.

1) $?^{n-1}y$;

2) $?^{n-1}u$;

3) $?^i u y^{n-i-2} y$, $0 \le i \le n-2$;

4) $?^i u y^{n-i-2} v$, $0 \le i \le n-2$.

This is easy to check using the considerations above, that matrices from the subgroups 1)–4) are exactly half of all matrices avoiding p_2 and p_5. All the other matrices can be obtained by the operation of complementation.

We now describe the bijection $\mathcal{F}$ that makes a correspondence between subgroup 3) (resp. 4)) and permutations from the first permutation group having 1 and 2 to the left (resp. right) of $(n+2)$. Also, to subgroup 1) (resp. 2)) there corresponds the permutations having 1 to the left (resp. right) and 2 to the right (resp. left) of $(n+2)$.

To deal with subgroups 1) or 2), we list all the $(n-1)$ elements $3, 4, \ldots, n+1$ to be placed to the right or to the left of $(n+2)$, which defines a permutation uniquely in our case. If we place an element to the right (resp. left), we assign the column x (resp. y) to it. By concatenating these assigned columns in the increasing order of the corresponding elements, and adjoining either y or u to the end, we obtain a matrix from subgroup 1) or 2). This operation is obviously a bijection.

Let us consider subgroup 3) (subgroup 4) can be considered in the same way).

As above, we list the elements $3, 4, \ldots, n+1$ and assign to them the columns x and y according to whether the elements are placed to the right of $(n+2)$ or to the left of it respectively. The fact that we have a nonempty word to the right of $(n+2)$ ensures that we have at least one column x. We concatenate the assigned columns in the same way as above, then we change the rightmost x to u, adjoin the column y from right, and by this we get a matrix from subgroup 3). This is clear how to convert this injective operation, and thus we get a bijection.

The proposition is proved. $\square$

3.2 Avoiding the patterns p_2 and p_6

Recall that $p_2 = \begin{array}{|c|c|}\hline 0 & 0 \\ \hline 1 & \\ \hline\end{array}$ and $p_6 = \begin{array}{|c|c|}\hline 1 & 0 \\ \hline 1 & \\ \hline\end{array}$.

Proposition 3. *There is a bijection between the edges in an $(n+1)$-dimensional hypercube and $2 \times n$ matrices avoiding p_2 and p_6.*

Proof. Any edge in an $(n+1)$-dimensional hypercube can be specified by the coordinate i, $1 \leq i \leq n+1$, in which 0 has been changed to 1 or vice versa in the endpoints of the edge, and a binary n-tuple, which gives values of the other coordinates.

Suppose $A = (a_{i,j})$ is a $2 \times n$ matrix that avoids p_2 and p_6.

If $a_{2,i} = 0$ for all $i = 1, 2, \ldots, n$, there are no restrictions for the first row of A, and clearly there is a one-to-one correspondence between such matrices and the edges specified by changing the $(n+1)$-st coordinate of their endpoints.

Suppose now that i is the minimum index such that $a_{2,i} = 1$, $1 \leq i \leq n$. It is easy to see, that in order to avoid p_2 and p_6, we must have $a_{1,j} = 1$ for $j = i+1, \ldots, n$, and there are no other restrictions for the elements of A. The edges that correspond to such matrices are those having change in the i-th coordinate of their endpoints.

The described correspondence is obviously a bijection. $\square$

The following proposition is related to [9] by Wilf.

Proposition 4. *There is a bijection between the ways to place n nonattacking kings on a $2 \times 2n$ chessboard for $n \geq 1$ and $2 \times n$ matrices avoiding p_2 and p_6.*

Proof. For every nonattacking placement of kings, the chessboard is naturally divided into n 2×2 cells, each containing exactly one king. We say that a cell is of type 1 (resp. 2, 3, 4) if the king sits in its NW (resp. NE, SE, SW) corner. The arrangement of kings is then completely specified by an n-word over the alphabet $\{1, 2, 3, 4\}$ that satisfies certain adjacency conditions (AC), namely that none of the following two letter words is permitted: 21, 24, 31, and 34.

The structure of $2 \times n$ matrices avoiding p_2 and p_6 is described in the proof of proposition 3. Clearly, in order to construct a bijection, we can assume that the i-th column of our matrices corresponds to the i-th cell of the chessboard with a king placed, that is to one of the letters 1,2,3, or 4.

If $a_{2,i} = 0$ for all $i = 1, 2, \ldots, n$ then the column $(00)^T$ corresponds to 1, whereas $(10)^T$ corresponds to 4. Clearly, for any such matrices we do not violate the AC, and of course this is a one-to-one correspondence.

Suppose now that i is the minimum index such that $a_{2,i} = 1$, $1 \leq i \leq n$. Thus, the columns preceding the i-th column are $(00)^T$ or $(10)^T$, and they correspond to 1 or 4 respectively as above. For the other $n - i + 1$ columns, we assume that the column $(11)^T$ corresponds to 2, $(10)^T$ and $(01)^T$ (only column i is possibly $(01)^T$) correspond to 3. Clearly we satisfy the AC.

Conversely, given a word satisfying the AC. We read it from left to right replacing each 1 by $(00)^T$ and each 4 by $(10)^T$. First time we meet 2 or 3, we replace it by $(11)^T$ or $(01)^T$ respectively. Finally, we replace 2 and 3 by $(11)^T$ and $(10)^T$ respectively.

The proposition is proved. □

Proposition 5. *There is a bijection between the number of* $2 \times (n+1)$ *0-1 matrices containing* $n+2$ *1s and having no zero row or column and* $2 \times n$ *matrices avoiding* p_2 *and* p_6.

Proof. We describe a correspondence between the matrices of the first and the second types.

The $2 \times (n+1)$ matrices under consideration have exactly one column $(11)^T$ and all other columns are either $(10)^T$ or $(01)^T$. In our correspondence between the matrices of two types, the position of $(11)^T$ determines the column of a $2 \times n$ matrix A in which we first time meet 1 in the second row reading from left to right. If this position is $i = n + 1$, the second row of A consists only of 0s (for a possible structure of A see the proof of proposition 3), in which case $(10)^T$ corresponds to itself, whereas $(01)^T$ corresponds to $(00)^T$. Otherwise, that is if $i < n+1$, $(11)^T$ is followed either by $(10)^T$ or by $(01)^T$. In the former case the i-th column of A is $(11)^T$, whereas in the last case it is $(01)^T$. Thus we glue two columns into one column. For the other columns j, $1 \leq j \leq n+1$, $j \neq i, i+1$, if $j < i$, then we proceed as in the case $i = n+1$. Otherwise, column j determines $(j-1)$-st column of A as follows. The column $(10)^T$ corresponds to itself, and $(01)^T$ corresponds to $(11)^T$.

It is easy to see that the correspondence is a bijection. □

Proposition 6. *There is a bijection between the number of spanning trees of the complete bipartite graph $K_{2,n+1}$ and $2 \times n$ matrices avoiding p_2 and p_6.*

Proof. The statement follows from proposition 5 after observing, that any spanning tree of $K_{2,n+1}$ can be coded by a $2 \times (n+1)$ matrix having exactly one column $(11)^T$ and other columns either $(10)^T$ or $(01)^T$. Indeed, suppose $K_{2,n+1} = A \cup B$, where $A = \{x, y\}$ and $B = \{1, 2, \ldots, n+1\}$, and $i \in B$ is the only node connected to both x and y. We assign $(11)^T$ to i, and for any other node $j \in B$, we assign $(10)^T$ (resp. $(01)^T$) to j if j is connected to x (resp. y). By concatenating these columns in the natural order, we get the matrix representing a spanning tree. □

Our final illustration for this subsection is probably most interesting. It is related to [6] by Robertson (for a survey on generalizations of this paper see [4, Section 2.2]). However, we leave this example without proof, and we believe this could be a good student project to solve problem 7. See, e.g., [4] for definition of a pattern in permutations, and for the concept of pattern avoidance.

Problem 7. *Find a bijection between 132-avoiding permutations of $[n+3]$ containing exactly one 123 pattern and $2 \times n$ matrices that avoid p_2 and p_6.*

3.3 Other relations

In this subsection we list some of the sequences appearing in [8] when dealing with multi-avoidance of right angled numbered polyomino patterns. These references could be considered as basis for studying the relations, which might lead to formulation of new interesting problems in this direction. We observe that most of the objects appearing in A001787 are considered in Subsection 3.2 with respect to their connection to avoidance of the patterns p_2 and p_6.

4 Multi-avoidance of two right angled polyomino patterns

There are 8 equivalence classes for two restrictions, which are presented in Table 3.

According to [5, Proposition 2], when one of the patterns to avoid is p_1, we have $M_{m,n} = 0$ for $m, n \geq 3$ with possible exception $m = n = 3$. This

restrictions	number of rows	sequence number
p_2, p_6	$m = 2$	A001787
p_2, p_7	$m = 2$	A001394
p_1, p_2, p_6	$m = 2$	A008574
p_2, p_3, p_7	$m = 3$	A005803
p_2, p_4, p_5	$m = 2$	A033484
p_2, p_4, p_7	$m = 2$	A079859
p_2, p_3, p_4, p_5	$m = 3$	A016933
p_2, p_3, p_4, p_5	$m = 4$	A017341
p_2, p_3, p_5, p_7	$m = 4$	A022144

Table 2: Sequences from [8] related to the multi-avoidance of right angled numbered polyomino patterns.

class	representative	other members of the class
$\mathcal{A}_1$	$\{p_1, p_3\}$	$C : \{p_1, p_6\}$, $T : \{p_1, p_2\}$, $CT : \{p_1, p_5\}$
$\mathcal{A}_2$	$\{p_1, p_4\}$	$C : \{p_1, p_7\}$
$\mathcal{A}_3$	$\{p_2, p_3\}$	$C : \{p_5, p_6\}$
$\mathcal{A}_4$	$\{p_3, p_4\}$	$C : \{p_6, p_7\}$, $T : \{p_2, p_4\}$, $CT : \{p_5, p_7\}$
$\mathcal{A}_5$	$\{p_2, p_5\}$	$T : \{p_3, p_6\}$
$\mathcal{A}_6$	$\{p_2, p_6\}$	$C : \{p_3, p_5\}$
$\mathcal{A}_7$	$\{p_2, p_7\}$	$C : \{p_4, p_5\}$, $T : \{p_3, p_7\}$, $CT : \{p_4, p_6\}$
$\mathcal{A}_8$	$\{p_4, p_7\}$	

Table 3: The equivalence classes for two restrictions.

is easy to check that $M_{3,3} = 6$ for $\mathcal{A}_1$, and $M_{3,3} = 4$ for $\mathcal{A}_2$. Moreover, the following proposition is true.

Proposition 8. *We have*

- $M_{m,2} = 4m + 2$ *and* $M_{2,n} = 3 \cdot 2^n - 2$ *for* $\mathcal{A}_1$, $m \geq 2$ *and* $n \geq 1$;
- $M_{2,n} = M_{n,2} = 6n - 2$ *for* $\mathcal{A}_2$ *and* $n \geq 1$.

Proof. We observe that $M_{2,n} = M_{n,2}$ for $\mathcal{A}_2$ since the transposition of the prohibited patterns gives the same patterns.

We consider only the case $n = 2$ and the class $\mathcal{A}_1$. All other statements from the proposition, can be proved similarly.

Suppose $A = (a_{i,j})$ is an $m \times 2$ matrix that avoids the patterns from $\mathcal{A}_1$, and $m \geq 3$.

If $a_{1,1} = 0$, the remain $m-1$ elements in the first column must be 0, since otherwise, independently of the value of $a_{1,2}$, we would get p_1 or p_3. In order to avoid p_1, we must have $a_{i,2} = 0$, $1 < i < m$, and there are no restrictions for the elements $a_{1,2}$ and $a_{m,2}$. So, we get 4 good matrices in this case.

If $a_{1,1} = 1$ and $a_{1,2} = 0$, the first row does not affect the rest of A, which gives $M_{m-1,2}$ good matrices.

Finally, if $a_{1,1} = 1$ and $a_{1,2} = 1$, the first column must consists of 0s in order to avoid p_1, and since $m \geq 3$, any choice of $a_{2,2}$ leads to a prohibition (either p_1 or p_3). So, there are no good matrices in this case.

So, $M_{m,2} = M_{m-1,2} + 4$ for $m \geq 3$, and that is easy to see that $M_{2,2} = 10$. This proves the statement. □

To prove Propositions 11 and 18 below concerning the classes $\mathcal{A}_3$ and $\mathcal{B}_9$, we need Lemma 10, which in turn uses Lemma 9.

Lemma 9. *Let $R_{m,n}$ denote the number of $m \times n$ binary matrices that avoid the patterns $h = \boxed{0}\boxed{1}$ and $v = \begin{array}{|c|}\hline 0 \\ \hline 1 \\ \hline \end{array}$ simultaneously. We assume that $R_{m,0} = R_{0,n} = 1$ for $m, n \geq 0$. Also, $R(x,y)$ is the bivariate generating function for the numbers $R_{m,n}$. Then*

$$R(x) = \frac{1}{1-x-y}.$$

Proof. Suppose $A = (a_{i,j})$ is an $m \times n$ matrix that avoids the patterns h and v.

If $a_{1,1} = 0$, clearly the first row and the first column must consist of 0s in order to avoid h and v, which in turn courses that all entries of A must be 0.

In the first row, all 1s must precede all 0s if any. Thus, assuming that the first i elements in the first row are 1s, $1 \leq i \leq n$, we have that $a_{j,k} = 0$ for all $1 \leq j \leq m$ and $(i+1) \leq k \leq n$, since otherwise we have an occurrence of v. The first row, as well as the submatrix consisting of 0s, do not affect the rest of A, and we have $R_{m-1,i}$ good matrices in this case. Thus, for $m, n \geq 1$,

$$R(m,n) = 1 + \sum_{i=1}^{n} R_{m-1,i} = \sum_{i=0}^{n} R_{m-1,i},$$

which using the technique of manipulations with generating functions from, for instance, the proof of Lemma 10, gives the desired. □

Lemma 10. *Let $N_{m,n}$ denote the number of $m \times n$ binary matrices that avoid the patterns $h = \begin{array}{|c|c|}\hline 0 & 1 \\ \hline \end{array}$ and $p_2 = \begin{array}{|c|c|}\hline 0 & 0 \\ \hline 1 & \\ \cline{1-1} \end{array}$ simultaneously. We assume that $N_{m,0} = N_{0,n} = 1$ for $m, n \geq 0$. Also, $N(x,y)$ is the bivariate generating function for the numbers $N_{m,n}$. Then*

$$N(x,y) = \frac{3y - 2y^2 - xy^2 - 1}{(2y-1)(x+y-1)(y-1)}$$

Proof. Suppose $A = (a_{i,j})$ is an $m \times n$ matrix that avoids the patterns h and p_2.

If the first row is $(11\dots11)$ or $(11\dots10)$, this row does not affect the rest of A, and we get $N_{m-1,n}$ good matrices in this case.

Otherwise, since h is prohibited, the first row consists of i 1s followed by 0s, where $1 \leq i \leq n-2$. Since p_2 is prohibited, the columns from $(i+1)$st to $(n-1)$ must consist of 0s, and because we have at least one such column, in order to avoid h, the last column must also consist of 0s. The remain submatrix B of A formed by first i columns without the first row, must avoid h, p_2, but also $v = \begin{array}{|c|}\hline 0 \\ \hline 1 \\ \hline \end{array}$ (otherwise we have an occurrence of p_2 with appropriate element from the last $m-i$ columns). Since if a matrix avoid v it avoids p_2, we may assume that B avoids two patterns, h and v, and thus according Lemma 9, there are $R_{m-1,i}$ good matrices in this case. We observe, that is was essential to define $R_{m,0} = R_{0,n} = 1$ in order to count all possibilities to form A. We now have

$$N_{m,n} = 2N_{m-1,n} + \sum_{i=0}^{n-2} R_{m-1,i}$$

for $m, n \geq 1$. Multiplying both parts of the equality by x^n, summing over all $n \geq 0$, and assuming that $N_m(x) = \sum_{n\geq 0} N_{m,n}$, we have

$$-1 + N_m(x) = -2 + 2N_{m-1}(x) + \frac{1}{1-x}R_{m-1}(x) - xR_{m-1}(x) - R_{m-1}(x),$$

and thus

$$N_m(x) = 2N_{m-1}(x) + \left(\frac{1}{1-x} - x - 1\right) R_{m-1}(x) - 1 =$$

$$2^m N_0(x) + \sum_{i=0}^{m-1} \left(\left(\frac{1}{1-x} - x - 1 \right) R_i(x) - 1 \right) 2^{m-i-1}.$$

We now multiply both parts of the equality by y^m, sum over all $m \geq 0$, and take into account that $N_0(x) = 1/(1-x)$ to get

$$N(x,y) = \frac{1}{(1-x)(1-2y)} + \frac{y}{1-2y}\left(\frac{1}{1-x-y}\left(\frac{1}{1-x} - x - 1\right) - \frac{1}{1-y}\right),$$

which gives the desirable after simplification. □

Proposition 11. *For the class $\mathcal{A}_3$, we have $M_{m,n} = M_{n,m}$, $M_{m,1} = M_{1,m} = 2^m$, and for $m, n \geq 2$,*

$$M_{m,n} = 2 + M_{m-1,n-1} + 2M_{m,n-1} + \sum_{i=1}^{m-2}(2N_{i,n-1} + P_{m,n,i}),$$

where $P_{m,n,0} = M_{m-1,n-1}$, $P_{m,2,i} = P_{2,m,i} = 2^{m-1}$, and for $m, n \geq 3$,

$$P_{m,n,i} = 2^i M_{m-i-1,n-1} + \sum_{k=1}^{\min(i,n-2)} \binom{i}{k}\binom{n-2}{k} P_{m-1,n-1,i-1}.$$

Proof. Suppose $A = (a_{i,j})$ is an $m \times n$ matrix that avoids the patterns from $\mathcal{A}_3$, and $m, n \geq 2$.

Suppose $a(1,1) = 0$. If $a(1,2) = 1$ then all other elements in the first column must be 1 (p_3 is prohibited), which courses that all other elements from the first row must be 1 (p_2 is prohibited). Now the first column and the first row do not affect the rest of A, and we have $M_{m-1,n-1}$ good matrices in this case. If $a(1,2) = 0$ then all other elements in the first column must be 0 (p_2 is prohibited), which gives that all other elements from the first row must be 0 (p_3 is prohibited). This is easy to see now that in order to avoid p_2 and p_3, all other elements from A but $a_{m,n}$ must be 0. So we choose $a_{m,n}$ in two ways which gives two good matrices.

Suppose now that $a(1,1) = 1$, and the first column is either $(11\dots11)^T$ or $(11\dots10)^T$. Such column does not affect the rest of A and we have $2M_{m,n-1}$ good matrices in this case.

We observe that in the remain cases having $a(1,1) = 1$, we have either

a) exactly one 0 in the first column in position $(i+1)$, or

b) the first column is $(\underbrace{11\dots1}_{i}\underbrace{00\dots0}_{m-i})^T$, where $1 \leq i \leq m-2$.

In case b), to avoid p_3, the $(i+1)$st row must consist of 0s, which gives that all elements below this row, except possibly $a_{m,n}$, must be 0 (the same considerations as above when $a(1,1) = a(1,2) = 0$). Clearly, the remain elements of A form a $i \times (n-1)$ matrix that avoids $\begin{array}{|c|c|}\hline 0 & 1 \\ \hline\end{array}$, p_2 and p_3, or just $\begin{array}{|c|c|}\hline 0 & 1 \\ \hline\end{array}$ and p_2, since once we avoid $\begin{array}{|c|c|}\hline 0 & 1 \\ \hline\end{array}$, we avoid p_3 as well. The number of such matrices $N_{i,n-1}$ was discussing in Lemma 10, and could be obtained by expanding $N(x, y)$. So, in case b) we have $2N_{i,n-1}$ good matrices.

In case a), to avoid p_2, the $(i+1)$st row must consist of 1s. Let $P_{m,n,i}$ denote the number of such matrices, that is the matrices avoiding p_2 and p_3, having $a_{i,1} = 0$, and all other elements in the first column and row i are 1 for $1 \leq i \leq m-2$. We consider the matrix B formed by intersection of columns $2, 3, \ldots, (n-1)$ and the first i rows. B is possibly empty. If all entries of B are 1s, we can choose the entries of A to the right of B in 2^i ways, and there are no restrictions, except for avoiding p_2 and p_3, for the remain elements of A, which form a $(m-i-1) \times (n-1)$ matrix. Thus, the number of good matrices in this case is $2^i M_{m-i-1,n-1}$.

Suppose now that B has at least one 0, and thus we have $n, m \geq 3$. One can see that in order to avoid p_2 and p_3, each column, as well as each row of B, must contain at most one 0. Thus, the maximum number of 0s is given by $\min(i, n-2)$. Moreover, once we place a 0 in B, the entries of A staying in the same row or column with this 0, must be 1, which does not affect the rest of A, and thus gives $P_{m-1,n-1,i-1}$ good matrices. Finally, we observe that if we want to place k 0s in B, we can do that in $\binom{i}{k}\binom{n-2}{k}$ ways by choosing first rows and then columns from the 0s. The initial conditions for $P_{m,n,i}$ are easy to get.

We sum the results in a) and b) over i from 1 to $m-2$, to get the truth the statement.

Initial values for $M_{m,n}$ can be found in Table 4.

	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
$m=1$	1	4	8	16	32
$m=2$	4	12	30	70	158
$m=3$	8	30	90	244	626
$m=4$	16	70	244	760	2214

Table 4: Initial values for $M_{m,n}$ and the class $\mathcal{A}_3$.

□

Proposition 12. *The bivariate generating function for the classes $\mathcal{A}_4$, $\mathcal{A}_6$,* *and* $\mathcal{A}_8$ *is*

$$M(x,y) = \frac{2xy}{1-2(x+y-xy)}.$$

Proof. We prove the statement for the class $\mathcal{A}_4$, all other classes can be considered in the same way.

Suppose $A = (a_{i,j})$ is an $m \times n$ matrix that avoids the patterns from $\mathcal{A}_4$.

If the first column consists of 1s, it does not affect the rest of A, and thus we have $M_{m,n-1}$ good matrices in this case. Otherwise, suppose that $a_{m-i+1,1} = 0$ and $a_{j,1} = 1$ for $m-i+2 \leq j \leq m$ and $1 \leq i \leq m$. That is the down most 0 from the first column is $a_{m-i+1,1}$. We can now choose the remain elements from the first column in 2^{m-i} ways, and our choice of $A_{j,1}$, $1 \leq j \leq m-i$, uniquely determines the j-th row of A, since we need to avoid p_3 and p_4. Thus, each of the fist $m-i$ rows of A consists of either only 1s or only 0s, which, with the first column, do not affect the rest of A, and therefore give $M_{i,m-1}$ good matrices.

We have

$$M_{m,n} = M_{m,n-1} + \sum_{i=1}^{m} M_{i,n-1} 2^{m-i}.$$

Multiplying both parts of the equality above by x^n, summing over all $n \geq 0$, assuming that $M_{m,0} = 0$ and denoting $B_m(x) = \sum_{n\geq 0} M_{m,n}x^n$, we have

$$B_m(x) = xB_{m-1}(x) + x\sum_{i=1}^{m} B_i(x)2^{m-i}x^i,$$

and thus

$$B_m(x) = \frac{x}{1-x}\sum_{i=1}^{m} B_i(x)2^{m-i}x^i.$$

Now considering $B_m(x) - 2B_{m-1}(x)$, we get

$$(1-x)(B_m(x) - 2B_{m-1}(x)) = xB_m(x),$$

which gives

$$B_m(x) = \frac{2-2x}{1-2x}B_{m-1}(x) = \left(\frac{2-2x}{1-2x}\right)^{m-1} B_1(x).$$

Clearly, $B_1(x) = \sum_{n\geq 0} 2^n x^n - 1 = \dfrac{2x}{1-2x}$, so

$$B_m(x) = \frac{2x}{1-2x}\left(\frac{2-2x}{1-2x}\right)^{m-1} = \frac{x}{1-x}\left(\frac{2-2x}{1-2x}\right)^{m}.$$

Finally, taking into account that $B_0(x)$ must be 0 rather than $x/(1-x)$ according to the formula above, we have that $M(x,y$ is given by

$$\sum_{m\geq 0} B_m(x)y^m = \frac{x}{1-x}\sum_{m\geq 0}\left(\frac{2-2x}{1-2x}\cdot y\right)^m - \frac{x}{1-x} = \frac{2xy}{1-2(x+y-xy)}.$$

□

Proposition 13. *We have that* $M_{m,2} = (m+1)2^m$ *and*

$$M_{m,n} = (m+3)2^{m+n-2} - 2^n - 2^{m+1} + 4$$

for $\mathcal{A}_5$*,* $m \geq 1$ *and* $n \geq 3$*.*

Proof. Suppose $A = (a_{i,j})$ is an $m \times n$ matrix that avoids $\mathcal{A}_5$.

We first consider the case $n = 2$. The number of good matrices having $a_{m,1} = 1$ is equal to that having $a_{m,1} = 0$, since taking the complement of p_2 and p_5, we get p_2 and p_5.

Assume that $a_{m,1} = 1$. If the other elements from the first column are 1s, the first column does not affect the second column, and we have 2^m good matrices in this case. If we have 0 in the first column, we assume that the down most 0 is in position $m-i$, where $1 \leq i \leq m-1$. Clearly, we can choose the elements above this 0 in 2^{m-i-1} ways and each such choice uniquely determines the elements $a_{j,2}$, $1 \leq j \leq m-i$, because of the prohibitions. We now can choose the remain elements in the second column in 2^i ways, to get $2^m + \sum_{i=1}^{m-1} 2^{m-i-1}2^i = (m+1)2^{m-1}$ good matrices in the case $a_{m,1} = 1$. According to the discussion above, the case $a_{m,1} = 0$ gives us the same number of good matrices, which proves the statement.

Suppose that $n \geq 3$. Like above, we can assume that $a_{m,1} = 1$ and then multiply the obtained result in this case by 2. If the first column consists of only 1s, this column does not affect the rest of A, and we get $M_{m,n-1}$ good omatrices in this case. Otherwise, assume that the down most 0 from the first column is in the $(m-i)$-th row, $1 \leq i \leq m-1$. In order to avoid p_2, all other elements from the $(m-i)$-th row must be 1. This courses that $a_{k,j} = 1$ for $m-i+1 \leq k \leq m$ and $2 \leq j \leq n-1$, since we need to avoid p_5. Moreover, we cannot have 1s above the down most 0 in the first column. Indeed, if we have 1 above the down most 0, the row corresponding to this 1 must be $(100\ldots0)$ in order to avoid p_5, but this would lead to an occurrence of p_2 (since the $(m-i)$-th row is $(011\ldots1)$, it is below, and $n \geq 3$).

So, the remain elements in the first row are 0s, which uniquely fills all the elements to the right of these 0s (they must be 1s in order to avoid p_2).

Finally, we can choose the remain elements in the last column in 2^i ways, which gives $\sum_{i=1}^{m-1} 2^i = 2^m - 2$ good matrices.

Summarizing all the cases, we have that $M_{m,n} = 2(M_{m,n-1} + 2^m - 2)$, which gives the desired, since $M_{m,1} = 2^m$. □

Proposition 14. *We have that* $M_{m,n} = (m+1)^{n-1}2^m$ *for* $\mathcal{A}_7$ *and* $m, n \geq 1$.

Proof. Suppose $A = (a_{i,j})$ is an $m \times n$ matrix that avoids $\mathcal{A}_7$.

Clearly, if $n \geq 2$ then in the first column, 0 cannot be above 1, since p_2 and p_7 are prohibited and we cannot then choose all elements for the second column. There are no extra restrictions for A, thus we choose the number of 0s in the first column in $m + 1$ ways (the places for them will be determined) and multiply it by $M_{m,n-1}$ to get $M_{m,n} = (m+1)M_{m,n-1}$. Since $M_{m,1} = 2^m$, we get the desired. □

5 Multi-avoidance of three right angled polyomino patterns

There are 12 equivalence classes for three restrictions, which are presented in Table 5.

The statements in Proposition 15 concerning the avoidance of the pattern p_1 and two more patterns, are easy to prove, and we prove only one of them to demonstrate our approach for such statements.

Proposition 15. *We have*

- $M_{m,n} = 0$ *for* $\mathcal{B}_1$–$\mathcal{B}_6$, $m, n \geq 3$, *with two exceptions:* $M_{3,3} = 4$ *for* $\mathcal{B}_4$, *and* $M_{3,3} = 2$ *for* $\mathcal{B}_5$;
- $M_{2,n} = M_{n,2} = 2n + 4$ *for* $\mathcal{B}_1$ *and* $n \geq 2$;
- $M_{2,n} = 6$ *and* $M_{m,2} = 4m$ *for* $\mathcal{B}_2$, $m \geq 2$, *and* $n \geq 3$;
- $M_{2,n} = 0$ *and* $M_{m,2} = 2^{m+1}$ *for* $\mathcal{B}_3$, $m \geq 2$, *and* $n \geq 3$;
- $M_{2,n} = M_{n,2} = 4n$ *for* $\mathcal{B}_4$ *and* $n \geq 2$;
- $M_{2,n} = 2n + 4$ *and* $M_{m,2} = 4m$ *for* $\mathcal{B}_5$ *and* $m, n \geq 2$;
- $M_{2,n} = M_{n,2} = 8$ *for* $\mathcal{B}_6$ *and* $n \geq 2$.

class	representative	other members of the class
$\mathcal{B}_1$	$\{p_1, p_2, p_3\}$	$C: \{p_1, p_5, p_6\}$
$\mathcal{B}_2$	$\{p_1, p_2, p_4\}$	$C: \{p_1, p_5, p_7\}$, $T: \{p_1, p_3, p_4\}$, $CT: \{p_1, p_6, p_7\}$
$\mathcal{B}_3$	$\{p_1, p_2, p_5\}$	$T: \{p_1, p_3, p_6\}$
$\mathcal{B}_4$	$\{p_1, p_2, p_6\}$	$C: \{p_1, p_3, p_5\}$
$\mathcal{B}_5$	$\{p_1, p_2, p_7\}$	$C: \{p_1, p_4, p_5\}$, $T: \{p_1, p_3, p_7\}$, $CT: \{p_1, p_4, p_6\}$
$\mathcal{B}_6$	$\{p_1, p_4, p_7\}$	
$\mathcal{B}_7$	$\{p_2, p_3, p_4\}$	$C: \{p_5, p_6, p_7\}$
$\mathcal{B}_8$	$\{p_2, p_3, p_5\}$	$C: \{p_2, p_5, p_6\}$, $T: \{p_2, p_3, p_6\}$, $CT: \{p_3, p_5, p_6\}$
$\mathcal{B}_9$	$\{p_2, p_3, p_7\}$	$C: \{p_4, p_5, p_6\}$
$\mathcal{B}_{10}$	$\{p_2, p_4, p_5\}$	$C: \{p_2, p_5, p_7\}$, $T: \{p_3, p_4, p_6\}$, $CT: \{p_3, p_6, p_7\}$
$\mathcal{B}_{11}$	$\{p_2, p_4, p_6\}$	$C: \{p_3, p_5, p_7\}$, $T: \{p_3, p_4, p_5\}$, $CT: \{p_2, p_6, p_7\}$
$\mathcal{B}_{12}$	$\{p_2, p_4, p_7\}$	$C: \{p_4, p_5, p_7\}$, $T: \{p_3, p_4, p_7\}$, $CT: \{p_4, p_6, p_7\}$

Table 5: The equivalence classes for three restrictions.

Proof. Let us prove that $M_{m,2} = 4m$ for $\mathcal{B}_2$ and $m \geq 2$.

Suppose $A = (a_{i,j})$ is an $m \times 2$ matrix that avoids $\mathcal{B}_2$.

Suppose $a_{m,1} = 0$. Then if we assume that there are at least two 1s in the first column, say in positions $1 \leq i_1 < i_2 < m$, then we cannot fill the position $a_{i_1,2}$ since the patterns p_1 and p_4 are prohibited. So, there is at most one 1 in the first column, which gives m possibilities. Moreover, all elements in the second column, but $a_{m,2}$ are uniquely determined because of the patterns p_1 and p_4. So, we have $2m$ possibilities in this case.

If $a_{m,1} = 1$ then in order to avoid the prohibitions, all 0s must be above all 1s, and again all elements in the second column, but $a_{m,2}$ are uniquely determined. Which gives $2m$ possibilities, and proves the statement. □

Proposition 16. *For $\mathcal{B}_7$, $M_{m,n} = M_{m,n-1} + M_{m-1,n} + 2$ for $m, n \geq 2$ and $M_{1,n} = M_{n,1} = 2^n$.*

Proof. Suppose $A = (a_{i,j})$ is an $m \times n$ matrix that avoids $\mathcal{B}_7$, and $m, n \geq 2$.

If $a_{1,1} = 0$ and $a_{1,2} = 0$ then the first column must consist of 0s (A avoids p_2), and thus the first row consists of 0s (A avoids p_3). Now all the elements but $a_{m,n}$ must be 0, since A avoids p_2 and p_3. So we have two good matrices in this case.

If $a_{1,1} = 0$ and $a_{1,2} = 1$ then the first column must consist of 1s (A avoids p_3), and thus the first row consists of 1s (A avoids p_2). Now the first row and the first column do not affect the rest of A, and thus we have $M_{m-1,n-1}$ good matrices in this case.

If $a_{1,1} = 1$ and the first column does not contain 0s, we have $M_{m,n-1}$ good matrices, since the first column does not affect the rest of A in this case.

If $a_{1,1} = 1$ and the first column has at least one 0, the first row must consist of only 1s, since p_4 is prohibited. To count the number of matrices in this case we can take the number of matrices having only 1s in the first row without additional restrictions (there are $M_{m-1,n}$ such matrices), and subtract the number of matrices having only 1s in the first row and the first column. We get $M_{m-1,n} - M_{m-1,n-1}$ good matrices in this case.

Finally, we sum the numbers from the four cases to get the desired. □

Proposition 17. *For $\mathcal{B}_8$, $M_{m,n} = (n+2)2^{m-1} + 2m(n-1) - 2$ for $m, n \geq 2$ and $M_{1,n} = M_{n,1} = 2^n$.*

Proof. Suppose $A = (a_{i,j})$ is an $m \times n$ matrix that avoids $\mathcal{B}_8$, and $m, n \geq 2$.

Let us first find $M_{m,2}$. If $a_{1,1} = 0$ and $a_{1,2} = 1$ then clearly the first column must consist of 1s and there are no restrictions for the second column. Thus we get 2^{m-1} good matrices in this case. If $a_{1,1} = 0$ and $a_{1,2} = 0$ then the first column must consist of 0s, which leads that the second column, maybe except the element $a_{m,2}$ must consist of 0s, so we get 2 good matrices in this case.

If $a_{1,1} = 1$ and $a_{1,2} = 0$ there are no restrictions for the rest of A – there are $M_{m-1,2}$ matrices in this case. If $a_{1,1} = 1$ and $a_{1,2} = 1$, since p_5 is prohibited, the first column consists of only 1s and we can choose any other element in two ways. We have 2^{m-1} good matrices in this case.

One can now see that $M_{m,2} = M_{m-1,2} + 2^m + 2 = 2^{m+1} + 2m - 2$ since $M_{1,2} = 4$.

Let us find a recursion for $M_{m,n}$ where $m \geq 2$ and $n \geq 3$.

If $a_{1,1} = 0$ then the same arguments as for the case $n = 2$ work. Thus we get $2^{m-1} + 2$ good matrices in this case.

If $a_{1,1} = 1$, then either the first row consists only of 1s, in which cases we have $M_{m,n-1}$ good matrices, or there is at least one 0 there. This 0 courses that the first row must consists of 0s (p_5 is prohibited), which using $n \geq 3$ gives that $a_{i,j} = 0$ for $1 < i \leq n-1$ and $1 < j \leq m$ (p_2 is prohibited). Since p_3 is prohibited, all the elements from the last column, but $a_{m,n}$ must be 0. Moreover, one can see that in the first column, all 0s must be below 1s, since otherwise A contains the pattern p_2. Now, we choose the number of 0s in the first column in $m-1$ ways (this number is between 1 and $m-1$), then choose the element $a_{m,n}$ in two ways, which gives $2(m-1)$ good matrices in this case.

So, $M_{m,n} = M_{m,n-1} + 2^{m-1} + 2m$ with $M_{m,2}$ given above. This proves the statement. $\square$

Proposition 18. *For $\mathcal{B}_9$, we have*

$$M(x,y) = \frac{2xy(y - xy + x - 1)}{(2x-1)(2y-1)(x+y-1)}.$$

Proof. Suppose $A = (a_{i,j})$ is an $m \times n$ matrix that avoids $\mathcal{B}_9$, and $m, n \geq 2$.

If $a_{m,1} = 0$ or $a_{m,1} = 1$ and all other elements from the first column are 1s, the first column does not effect the rest of A, and thus we have $2M_{m,n-1}$ possibilites here. Also, in order to avoid p_2 and p_7, all 1s in the first column must be above all 0s. Thus, we can assume that in the first column i 0s are below $m - i$ 1s, where $2 \leq i \leq m$.

Clearly, in order to avoid p_3, the rows $m-i+1$, $m-i+2, \ldots, m-1$ must consist only of 0s. Which, in turn, courses that each element in the last row, except possibly $a_{m,n}$ must be 0 (A avoids p_2).

Now, the $(m-i) \times (n-1)$ matrix B that is formed by all but the first column and all but the last i rows must avoid the patterns from $\mathcal{B}_9$ but also the pattern $h = \boxed{0}\boxed{1}$ (an occurrence of h in B, with the corresponding 0s in the last i rows of A will form the pattern p_3). Moreover, if a matrix avoids h, it also avoids p_3 and p_7, and thus the restrictions for B, we need to control, are p_2 and h. We can now use Lemma 10 to get that there are $N_{m-i,n-1}$ choices for B, which we must sum over i and multiply by 2, the number of ways to choose $a_{m,n}$. We observe, that it was necessarily to define $N_{0,n} = N_{m,0} = 1$ for all $m, n \geq 0$, in order to count all the possibilities to construct A.

We have, that for $m, n \geq 2$,

$$M_{m,n} = 2M_{m,n-1} + 2\sum_{i=2}^{m} N_{m-i,n-1},$$

with $M_{m,1} = 2^m$ and $M_{1,n} = 2^n$.

Suppose $M_m(x) = \sum_{n\geq 0} M_{m,n}x^m$. Using manipulations similar to that as, for instance, in Lemma 10, one can get

$$M_m(x) = \frac{2x}{1-2x}\left(2^{m-1} - m + 1 + \sum_{i=0}^{m-2} N_i(x)\right).$$

Multiplying both parts of this equality by y^m, summing over $m \geq 0$, taking into account that $M_0(x) = 0$, and simplifying the result, one can get the desired. □

Proposition 19. *For $\mathcal{B}_{10}$, we have $M_{m,n} = 2^{m+n} - 2^n - 2^m + 2$ for $m, n \geq 1$.*

Proof. Suppose $A = (a_{i,j})$ is an $m \times n$ matrix that avoids $\mathcal{B}_{10}$, and $m, n \geq 2$.

This is easy to see that if the first column of A consists of either only 0s or only 1s, the rest of A does not depend on the first column. Thus, in this cases we have $2M_{m,n-1}$ good matrices.

Suppose the first column contains at least one 0 and at least one 1. Clearly, all 0s in the first column must be above all 1s, since p_4 and p_5 are prohibited. But now all the elements to the right of the 0s must be 1s (p_2 is prohibited), which leads that all the remain elements, but that in the last column must be equal 1 (p_5 is prohibited), and there are no additional restrictions for A. So, if the number of 1s in the first column is i, the number of good matrices in this case is given by $\sum_{i=1}^{m-1} 2^i = 2^m - 2$.

Finally, $M_{m,n} = 2M_{m,n-1} + (2^m - 2)$, and we are done since $M_{m,1} = 2^m$. □

Proposition 20. *Suppose $M_1(x, y)$ and $M_2(x, y)$ are the bivariate generating functions for the class $\mathcal{B}_{11}$ and $\mathcal{B}_{12}$ respectively. Then,*

$$M_1(x, y) = M_2(y, x) = \frac{2xy}{(1-2x)(1-\frac{2-x}{1-x}y)}.$$

Proof. We consider only the class $\mathcal{B}_{11}$, since one can study $\mathcal{B}_{12}$ in the same way changing rows to columns and vice versa in the considerations. One then get the same recursion as for $\mathcal{B}_{11}$ after switching m and n.

Suppose $A = (a_{i,j})$ is an $m \times n$ matrix that avoids $\mathcal{B}_{11}$, and $m, n \geq 2$.

If $a_{1,1} = 1$ then the first row consists of 1s, since otherwise we cannot choose $a_{2,1}$ (p_4 and p_6 are prohibited). The first row now does not affect the rest of A, and we have $M_{m-1,n}$ good matrices in this case.

If $a_{1,1} = 0$ and all other elements from the first row are 1s, the first row does not affect the rest of A, and we also have $M_{m-1,n}$ good matrices in this case.

Suppose $a_{1,1} = 0$ and the first row contains at least one more 0. Clearly, all 0s must proceed all 1s in the first row, since otherwise there is an element in the second row, that we cannot choose (p_4 and p_6 are prohibited). Moreover, all the enteries below all but the rightmost 0 must be 0 (p_2 is prohibited). If the number of in the first row 0s is $i+1$, $1 \leq i \leq n-1$ then the first i columns, as well as the first row, do not affect the rest of A and could be removed, which gives $\sum_{i=1}^{n-1} M_{m-1,n-i}$ good matrices.

We now have

$$M_{m,n} = \sum_{i=0}^{n-1} M_{m-1,n-i} + M_{m-1,n}.$$

Multiplying both parts of the equality above by x^n, summing over all $n \geq 0$, assuming that $M_{m,0} = 0$ and denoting $B_m(x) = \sum_{n\geq 0} M_{m,n}x^n$, we have

$$B_m(x) = \sum_{n\geq 0}\sum_{i=0}^{n-1} M_{m-1,n-i}x^{n-i}x^i + B_{m-1}(x),$$

and thus

$$B_m(x) = B_{m-1}(x)\frac{1}{1-x} + B_{m-1}(x) = \frac{2-x}{1-x}B_{m-1}(x) = \left(\frac{2-x}{1-x}\right)^{m-1} B_1(x).$$

Clearly, $B_1(x) = \sum_{n\geq 0} 2^n x^n - 1 = \frac{2x}{1-2x}$, which gives that $M(x,y)$ is given by

$$\sum_{m\geq 0} B_m(x)y^m = \frac{2xy}{1-2x}\sum_{m\geq 1}\left(\frac{2-x}{1-x}\right)^{m-1} y^{m-1} = \frac{2xy}{(1-2x)(1-\frac{2-x}{1-x}y)}.$$

□

6 Multi-avoidance of four right angled polyomino patterns

There are 12 equivalence classes for four restrictions, which are presented in Table 6.

In the cases, when one of the patterns to avoid is p_1, it is not difficult to see that $M_{m,n} = 0$ for $m, n \geq 3$. Moreover, one can consider the cases $m = 2$ or $n = 2$ to get the truth of the following proposition.

class	representative	other members of the class
$\mathcal{C}_1$	$\{p_1, p_2, p_3, p_4\}$	$C: \{p_1, p_5, p_6, p_7\}$
$\mathcal{C}_2$	$\{p_1, p_2, p_3, p_5\}$	$C: \{p_1, p_2, p_5, p_6\}$, $T: \{p_1, p_2, p_3, p_6\}$, $CT: \{p_1, p_3, p_5, p_6\}$
$\mathcal{C}_3$	$\{p_1, p_2, p_3, p_7\}$	$C: \{p_1, p_4, p_5, p_6\}$
$\mathcal{C}_4$	$\{p_1, p_2, p_4, p_5\}$	$C: \{p_1, p_2, p_5, p_7\}$, $T: \{p_1, p_3, p_4, p_6\}$, $CT: \{p_1, p_3, p_6, p_7\}$
$\mathcal{C}_5$	$\{p_1, p_2, p_4, p_6\}$	$C: \{p_1, p_3, p_5, p_7\}$, $T: \{p_1, p_3, p_4, p_5\}$, $CT: \{p_1, p_2, p_6, p_7\}$
$\mathcal{C}_6$	$\{p_1, p_2, p_4, p_7\}$	$C: \{p_1, p_4, p_5, p_7\}$, $T: \{p_1, p_3, p_4, p_7\}$, $CT: \{p_1, p_4, p_6, p_7\}$
$\mathcal{C}_7$	$\{p_2, p_3, p_4, p_5\}$	$C: \{p_2, p_5, p_6, p_7\}$, $T: \{p_2, p_3, p_4, p_6\}$, $CT: \{p_3, p_5, p_6, p_7\}$
$\mathcal{C}_8$	$\{p_2, p_3, p_4, p_7\}$	$C: \{p_4, p_5, p_6, p_7\}$
$\mathcal{C}_9$	$\{p_2, p_3, p_5, p_6\}$	
$\mathcal{C}_{10}$	$\{p_2, p_3, p_5, p_7\}$	$C: \{p_2, p_4, p_5, p_6\}$, $T: \{p_2, p_3, p_6, p_7\}$, $CT: \{p_3, p_4, p_5, p_6\}$
$\mathcal{C}_{11}$	$\{p_2, p_4, p_5, p_7\}$	$T: \{p_3, p_4, p_6, p_7\}$
$\mathcal{C}_{12}$	$\{p_2, p_4, p_6, p_7\}$	$C: \{p_3, p_4, p_5, p_7\}$

Table 6: The equivalence classes for four restrictions.

Proposition 21. *We have*

- $M_{2,2} = 6$ *and* $M_{m,2} = M_{2,n} = 4$ *for* $\mathcal{C}_1$*, and* $m, n \geq 3$*;*
- $M_{m,2} = 2(m+1)$ *and* $M_{2,n} = 0$ *for* $\mathcal{C}_2$ *and* $\mathcal{C}_4$*, and* $m \geq 1$*,* $n \geq 3$*;*
- $M_{m,2} = M_{2,n} = 6$ *for* $\mathcal{C}_3$*, and* $m, n \geq 2$*;*
- $M_{m,2} = 2(m+1)$ *and* $M_{2,n} = 6$ *for* $\mathcal{C}_5$*, and* $m, n \geq 2$*;*
- $M_{m,2} = 6$ *and* $M_{2,n} = 4$ *for* $\mathcal{C}_6$*, and* $m \geq 2$*,* $n \geq 3$*.*

For example, to prove, in Proposition 21, that $M_{m,2} = 2(m+1)$ for $\mathcal{C}_2$ and $m \geq 1$, we proceed as follows.

Suppose $A = (a_{i,j})$ is an $m \times 2$ matrix that avoids $\mathcal{C}_2$ and $a_{1,1} = 0$. Then assuming that $m \geq 2$ and that in the first column we have an extra 0, we get a contradiction with an attempt to choose $a_{1,2}$ (p_1 and p_3 are prohibited). So, the first column consists of 1s, and $a_{1,2} = 1$ since p_2 is prohibited. Moreover, all other elements but $a_{m,2}$ must be 0 in order to avoid p_1, which gives us two matrices avoiding $\mathcal{C}_2$ after choosing $a_{m,2}$.

If $a_{1,1} = 1$ then $a_{1,2} = 0$ assuming $m \geq 2$, since otherwise we have an occurrence of p_1 or p_5. But in this case, the first row does not affect the rest of the matrix, which gives $M_{m-1,2}$ good matrices.

So, $M_{m,2} = 2 + M_{m-1,2}$ and $M_{1,2} = 4$. This recursion proves the statement.

Proposition 22. *We have*

$M_{m,n} = (n+1)2^{m-1} + 2(n-1)$ *for* $\mathcal{C}_7$, $m \geq 2$ *and* $n \geq 1$*;*

$M_{m,n} = 8$ *for* $\mathcal{C}_9$ *and* $m, n \geq 2$*;*

$M_{m,n} = 2^{m+n-1}$ *for* $\mathcal{C}_{11}$ *and* $\mathcal{C}_{12}$*, and* $m, n \geq 1$.

Proof. Suppose an $m \times n$ matrix $A = (a_{i,j})$ avoids the patterns from a class under consideration.

For the first statement, suppose $n \geq 2$. If $a_{1,1} = 1$ then $a_{i,1}$, for $2 \leq i \leq m$, must be 1, since elsewise, we cannot fill the first row (p_4 and p_5 are prohibited). Since the first row now does not affect the rest of A, we have $M_{m,n-1}$ good matrices in this case. If $a_{1,1} = 0$ then either $a_{2,1} = 0$ or $a_{2,1} = 1$. In the first subcase the first row must consists of 0s, since p_3 is prohibited, which leads that the first column consists of 0s, since p_2 is prohibited. In the second subcase, since p_2 is prohibited, the first row consists of 1s, which leads that the first column consists of 1s, since p_3 is prohibited. Since p_5 is prohibited, all columns but the last must consist of 1s. We have no additional restrictions, and the remain elements in the last column can be chosen in 2^{m-1} ways. Thus,

$$M_{m,n} = M_{m,n-1} + 2^{m-1} + 2,$$

which proves the statement since $M_{m,1} = 2^m$.

To prove the second statement, we observe that the element $a_{1,2}$ determines uniquely all the elements but $a_{1,1}$ in the first column and first row (these elements will be equal to $a_{1,2}$). This, in turn, determines all the elements but $a_{m,n}$ for $m, n > 2$ which must be also equal to $a_{1,2}$ in order to avoid all restrictions. So, we choose each of $a_{1,1}$, $a_{1,1}$ and $a_{m,n}$ in two ways, which gives 8 matrices for any $m, n \geq 2$.

In the third statement, we consider only the class $\mathcal{C}_{11}$ ($\mathcal{C}_{12}$ can be considered similarly). This is easy to see, that if $n \geq 2$ and $a_{1,1} = 0$ then in order to avoid p_2 and p_7, the first column must consists of 0s, which does not affect the rest of A, and we have $M_{m,n-1}$ good matrices in this case. If $a_{1,1} = 1$ then to avoid p_4 and p_5, the first column must consists of 1s, which also does not affect the rest of A, and we have the same number of good matrices. Thus, $M_{m,n} = 2M_{m,n-1}$, and we are done since $M_{m,1} = 2^m$. □

Proposition 23. *We have*

- $M_{m,n} = 2^m + 2^n + 2(nm - n - m)$ *for* $\mathcal{C}_8$*, and* $m, n \geq 1$*;*
- $M_{m,n} = 2^m + 2m(n-1)$ *for* $\mathcal{C}_{10}$*, and* $m, n \geq 1$*.*

Proof. Suppose an $m \times n$ matrix $A = (a_{i,j})$ avoids a class under consideration.

Let us prove the first statement. Clearly, if $n \geq 2$, we cannot have any 0 above any 1, since in this case we cannot fill the position next to 0 in the same row (p_2 and p_7 are prohibited). Moreover, if the first row consists of only 1s, we have no restrictions for the rest of A, which gives $M_{m,n-1}$ good matrices.

Suppose that $a_{m,1} = 0$ and all the other elements in the first column are 1s. Since p_4 is prohibited, all the elements, but in the last row, must be 1. Now we have now restrictions for the remain elements in the last row, and we can choose them in 2^{n-1} ways.

So, we need only to consider the case when $a_{1,1} = a_{2,1} = \cdots = a_{k,1} = 1$ for $0 \leq k \leq m-2$, and the other elements from the first column are 0s. Clearly, because of the restrictions, $a_{i,j} = 1$ for $1 \leq i \leq k$ and $2 \leq j \leq n$, and $a_{i,j} = 0$ for $k < i \leq m-1$ and $2 \leq j \leq n$. Moreover, we get at least one row consisting of only 0s, which leads to all the elements from the last row, but $a_{m,n}$ must be 0 (p_2 is prohibited). Thus, in this subcase we have 2 good matrices, which after variation of k gives $2(m-1)$ good matrices.

We now have that

$$M_{m,n} = M_{m,n-1} + 2^{n-1} + 2(m-1)$$

with $M_{m,1} = 2^m$, which gives the desired.

To prove the second statement, we use exactly the same considerations to get the recursion $M_{m,n} = M_{m,n-1} + 2m$, with $M_{m,1} = 2^m$, which gives the result.

□

7 Multi-avoidance of five or more of right angled polyomino patterns

If the number of restrictions is seven, then obviously $M_{m,n} = 0$, for $m, n \geq 2$. If we prohibit six patterns then it is easy to show the truth of the following proposition.

Proposition 24. *Suppose p is the only allowed pattern. Then*

- $M_{m,n} = 2$, *if* $p = p_1$ *and* $m, n \geq 2$;
- $M_{m,n} = 0$, *if* $p = p_4$ *or* $p = p_7$ *and* $m, n \geq 2$;
- $M_{2,n} = 2$ *and* $M_{m,n} = 0$, *if* $p = p_2$ *or* $p = p_5$, *and* $n \geq 2$ *and* $m \geq 3$;
- $M_{m,2} = 2$ *and* $M_{m,n} = 0$, *if* $p = p_3$ *or* $p = p_6$, *and* $n \geq 3$ *and* $m \geq 2$.

In the case of five restrictions, we have 8 equivalence classes, which we list in Table 7.

class	representative	other members of the class
$\mathcal{D}_1$	$\{p_1, p_2, p_3, p_4, p_5\}$	$C : \{p_1, p_2, p_5, p_6, p_7\}$, $T : \{p_1, p_2, p_3, p_4, p_6\}$, $CT : \{p_1, p_3, p_5, p_6, p_7\}$
$\mathcal{D}_2$	$\{p_1, p_2, p_3, p_4, p_7\}$	$C : \{p_1, p_4, p_5, p_6, p_7\}$
$\mathcal{D}_3$	$\{p_1, p_2, p_3, p_5, p_6\}$	
$\mathcal{D}_4$	$\{p_1, p_2, p_3, p_5, p_7\}$	$C : \{p_1, p_2, p_4, p_5, p_6\}$, $T : \{p_1, p_2, p_3, p_6, p_7\}$, $CT : \{p_1, p_3, p_4, p_5, p_6\}$
$\mathcal{D}_5$	$\{p_1, p_2, p_4, p_5, p_7\}$	$T : \{p_1, p_3, p_4, p_6, p_7\}$
$\mathcal{D}_6$	$\{p_1, p_2, p_4, p_6, p_7\}$	$C : \{p_1, p_3, p_4, p_5, p_7\}$
$\mathcal{D}_7$	$\{p_2, p_3, p_4, p_5, p_6\}$	$C : \{p_2, p_3, p_5, p_6, p_7\}$
$\mathcal{D}_8$	$\{p_2, p_3, p_4, p_5, p_7\}$	$C : \{p_2, p_4, p_5, p_6, p_7\}$, $T : \{p_2, p_3, p_4, p_6, p_7\}$, $CT : \{p_3, p_4, p_5, p_6, p_7\}$

Table 7: The equivalence classes for five restrictions.

The following statement is easy to check.

Proposition 25. *We have*

- $M_{m,2} = 4$ *for* $\mathcal{D}_1$, $\mathcal{D}_4$, $\mathcal{D}_5$, *and* $m \geq 2$;
- $M_{2,2} = 4$ *and* $M_{m,2} = M_{2,n} = 2$ *for* $\mathcal{D}_2$, *and* $m, n \geq 3$;

- $M_{2,2} = 4$ *for* $\mathcal{D}_3$*;*
- $M_{m,2} = M_{2,n} = 4$ *for* $\mathcal{D}_6$*, and* $m, n \geq 2$*;*
- $M_{m,n} = 0$ *for all non-mentioned above cases of* $m, n \geq 2$ *and the classes* $\mathcal{D}_1$*–*$\mathcal{D}_6$*.*

To illustrate how the results from Proposition 25 can be obtained, we observe that, for instance, $M_{2,n} = 4$ for $\mathcal{D}_6$ and $n \geq 2$, is given by the following considerations. A matrix that avoids $\mathcal{D}_6$ has either $(0,0)^T$ or $(1,0)^T$ as the first column, $(1,1)^T$ or $(1,0)^T$ as the last column, and a number of columns $(1,0)^T$ between them, which gives four possible matrices for each n, since otherwise we have an occurrence of prohibition.

Proposition 26. *We have*

- $M_{m,n} = 6$ *for* $\mathcal{D}_7$*, and* $m, n \geq 2$*;*
- $M_{m,n} = 2(n-1) + 2^m$ *for* $\mathcal{D}_8$*,* $m \geq 2$ *and* $n \geq 1$*.*

Proof. Suppose $A = (a_{i,j})$ is an $m \times n$ matrix without prohibitions.

To prove the first statement, we consider the element $a_{1,1}$.

If $a_{1,1} = 0$ then is is easy to see that either $a_{i,1} = a_{1,j} = 0$ or $a_{i,1} = a_{1,j} = 1$ for all $2 \leq i \leq m$ and $2 \leq j \leq n$. In the former case, all but the element $a_{m,n}$ must be 0, whereas in the second case, all but $a_{m,n}$ must be 1. If $a_{1,1} = 1$, all but $a_{m,n}$ must be 1. Now, choosing $a_{m,n}$ to be equal 0 or 1 gives us six matrices avoiding $\mathcal{D}_7$.

To prove the second statement, we also consider $a_{1,1}$.

Suppose $a_{1,1} = 0$. If $a_{i,1} = 1$ for some i, $2 \leq i \leq m$, then we get a prohibition p_2 or p_7 when we fill the first row. Thus the first column consists of 0s, which leads to the first row consists of 0s, since otherwise we have an occurrence of p_3. Now, clearly, in order to avoid p_2, all elements but $a_{m,n}$ must be 0, which gives two matrices avoiding $\mathcal{D}_8$ in this case.

Suppose $a_{1,1} = 1$. If $a_{i,1} = 0$ for some i, $2 \leq i \leq m$, then we get a prohibition p_4 or p_5 when we fill the first row. Thus the first column consists of 1s, which does not affect the rest of the matrix A. So, we have $M_{m,n-1}$ good matrices in this case.

So, $M_{m,n} = M_{m,n-1} + 2$, which with the condition $M_{m,1} = 2^m$ gives the desired. □

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