The Combinatorics of Mancala-Type Games: Ayo, Tchoukaillon, and $1/\pi$

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Abstract

Certain endgame considerations in the two-player Nigerian Mancala-type game Ayo can be identified with the problem of finding winning positions in the solitaire game Tchoukaillon. The periodicity of the pit occupancies in s stone winning positions is determined. Given n pits, the number of stones in a winning position is found to be asymptotically bounded by n^2/π.

1 Introduction

Around the world, literally hundreds of different kinds of mancala games have been observed [6, 12, 14] certain dating back to the Empire Age of ancient Egypt. Their common features involve cup-shaped depression called *pits* filled with seeds or stones. Players take turns harvesting stones by moving the around the board according to various rules. In this paper, we will study two variants: The two-player game Ayo played by the Yoruba of western Nigeria, and the solitaire game Tchoukaillon created by Deledicq and Popova [5, p. 180] as a variant of the game Tchouka played in central Europe.

We will study certain common unbalanced Ayo endgame positions which we call *determined* and show how they are related to the positions in Tchoukaillon from which a win is possible. For all s, there is a unique such position with

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s stones. Of course, certain positions are not realizable on a finite board with a fixed number of pits. However, we show that the number of stones in such a position on a board with $2n$ pits (resp. n in Tchoukaillon) is bounded by approximately n^2/π. Furthermore, we study the actual distribution of stones into pits, and discover a periodicity in the contents of the first k pits (with respect to the total number of stones) of $\text{lcm}(1, 2, \ldots, k+2)$.

2 Ayo

The game *Ayoyayo* or simply *Ayo* is played on a wooden block 20 inches long, 8 inches wide, and 2 inches thick. Two rows of six pits each about 3 inches in diameter are carved in the board. The playing pieces are either dried palm nuts, or more commonly, the stones of the shrub *caesalpina crista*. The rules [13] are as follows:

Set up. 48 stones are used. Initially, 4 are placed in each of the 12 pits. (We will generalize the game somewhat allowing boards with $2n$ pits and an arbitrary placements of stones.)

Players. Two players alternate making moves. Each player's side of the board has n pits.

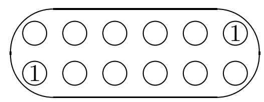
Objective. The object of the game is to capture the most stones.

Movement. To move, a player chooses a non-empty pit from his or her side of the board, and removes all of its stones. The stones are redistributed (*sown*), one per pit, among the pits in a counterclockwise direction beginning with the pit after the chosen pit.

Odu. A pit which contains $2n$ or more stones is said to be an *Odu* [13]. If the chosen pit is an *Odu*, the redistribution proceeds as usual except that the initial pit is skipped on each circuit of the board. (None of the positions we shall consider will contain an *Odu*.)

Capture. If the last pit sown by a player is on the opponent's side of the board and contains (after having been sown) two or three stones, then the stones in this pit are captured. Also captured are stones in the consecutively preceding pits which meet these conditions.

End of Game. At each turn, a player must, if possible, move in such a way that his or her opponent has a legal move. If, on some move, a player cannot move in such a way to give his or her opponent a legal move, the game is over and the player is awarded all remaining stones. If there are so few stones on the board that neither player can ever capture, but both players will always have a legal move, the game is over and each player is awarded the stones on his or her own side of the board. For example, if the position is



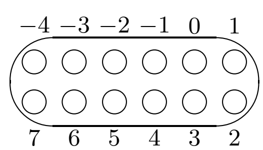
no further captures are possible, but each player can always move to give the opponent a legal move. In this case, each player is awarded a single stone.

The game opens rapidly with both players showing dexterity and skill by the speed of their movements. However, playing the game well requires remembering the number of stones in each of the twelve pits, as well as planning several moves in advance. Thus, the opening game is both interesting to watch and difficult to learn.

The endgame is less exciting, but easier to analyze. The latter stages of the game tend to be dominated by one player. She can move in such a way that her opponent has at all times only one legal move. After this sequence of moves, only a few stones usually remain, and no further captures are possible.

We shall analyze a specific type of endgame on a generalized Ayo board with $2n$ pits. For reasons which will become apparent when we examine Tchoukaillon, the pits will be numbered clockwise $-n+2, -n+1, \ldots, -1, 0, 1, \ldots, n, n+1$. (See Figure 1.) The two players will be denoted S (for South) and N (for North).

Figure 1: The standard Ayo board numbering



S makes her plays from pits numbered from $n+1$ down to 2, while N makes his plays from pits numbered from 1 down to $-n+2$. Play proceeds from higher numbered pits to lower numbered ones (and from pit $-n+2$ to pit $n+1$).

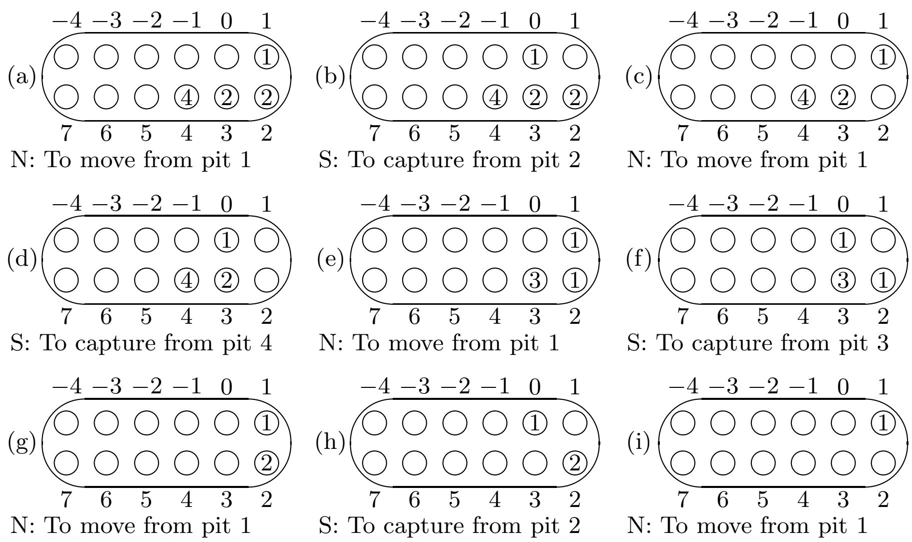
The endgame positions we shall study are those which satisfy the following definition.

Definition 1 *A* determined position *is an arrangement of stones on a generalized Ayo board where it is possible for S to move such that*

- *S captures at every turn,*
- *there is no move from an Odu,*

- *after every turn, N has only one stone on his side of the board, and*
- *all stones are captured by S except one which is awarded to N.*

Figure 2: The Determined Position with Nine stones



Note: All captures are to pit 0.

Figure 2 shows a determined position and the subsequent play between the two players on a board with 12 pits. Initially, there are nine stones and it is N's turn. Eight stones are captured by S and one is awarded to N. It is possible to show that a determined position on a 12 pit Ayo board has at most 21 stones.

It is a simple matter to establish the contents of the pits on N's side of the board in a determined position. The study of the contents of the pits on S's side of the board will be more rewarding.

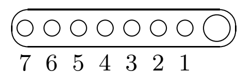
Lemma 2 *The stone on N's side of a determined Ayo position must be in pit 1 if N is to move, and in pit 0 if S is to move.*

Proof: If S is to move, she must capture and leave only one stone. Thus, the stone captured must lie in N's second pit (pit 0). Hence, before N's move, the stone must have been in pit 1.□

3 Tchoukaillon

The game Tchouka is a Russian game [16, p. 99] [15, p. 42] of possible Paleosiberian or Eskimo origin. It has seemed to have disappeared several decades ago [5, p. 99, 180]. The game is played with a number of small pits, dug in the sand, each initially containing a certain number of stones and an additional empty pit called the *Rouma*, *Cala* or *Roumba*. (See figure 3, although note that

Figure 3: A Tchouka or Tchoukaillon board



sometimes the pits are arranged in a circle.) The objective of the game is to put the stones in the Roumba. As a solitaire game, one may sow the stones from any pit, distributing them one at a time in the succeeding pits in the direction of the Roumba. If necessary, the sowing continues with the pit opposite the Roumba. There are then three possibilities:

- The last stone falls in the Roumba. The player then continues by sowing another pit of his choice.
- The last stone falls in an occupied pit (other than the Roumba). This pit must then be immediately sown.
- The last stone falls in an empty pit (other than the Roumba). The player has lost and the game is over.

When played as a two-player game, each player continues until he is forced to place a last stone in an empty pit. At that point, his opponent moves. Each player attempts to place more stones in the Roumba than does his adversary.

The solitaire game Tchoukaillon was invented [5, p. 180] as a variant of Tchouka. In this game, no wrap-around moves are allowed, and the last stone must land in the Roumba. Thus, pit i may be harvested if and only if contains exactly i stones.

For the purposes of this paper, we will not discuss any alternative or secondary objectives in the case that this is impossible. We will thus say that the game is *won* if all stones are pocketed. A position from which a win is possible is said to be a *winnable* position.

There is a close relationship between Tchoukaillon and Ayo. Suppose two people play Ayo on a board of size $2n$ while we focus our attention on pits $1, 2 \dots n+1$ disregarding which player makes a given move. We would then have the impression that we are watching a game of Tchoukaillon with pits numbered $1,\ 2,\ \dots n+1$.

Theorem 3 *There is a one-to-one correspondence between determined positions in Ayo and winnable positions in Tchoukaillon. To find the Tchoukaillon position corresponding to a determined Ayo position, ignore pits* $0, -1, -2, \dots, -n+2$.

Proof: Let D be an Ayo position corresponding to the Tchoukaillon position W. Suppose D is determined.

- If pit 1 has a stone, then by lemma 2, player N must move the stone from pit 1 to pit 0. The corresponding move in Tchoukaillon is legal: remove a stone from pit 1.
- If pit 1 is empty, then by lemma 2, player S must capture the stone in pit 0. She does so by harvesting some pit i containing exactly i stones, placing stones in pits 1, 2, $\ldots i-1$, and capturing the stone in pit 0. The corresponding move in Tchoukaillon is legal: harvest some pit i containing i stones, placing stones in pits 1, 2, 3, $\ldots$, $i-1$, and pocketing the remaining stone in the roumba.

Similarly, the legal moves in W correspond to determined moves in D. The objective in Tchoukaillon is to empty the board. At the end of determined Ayo play, the board is empty except for the pit 0. Thus, W is winning if and only if D is determined.□

Given the equivalence between these two notions, we will now concentrate on Tchoukaillon with the understanding that the results found will be equally valid for the game of Ayo.

There is a simple strategy by which any feasible win can be forced.

Proposition 4 ([7, 5]) *If a win is possible from a given Tchoukaillon position, the unique winning move must be to harvest the smallest harvestable pit.*

Proof: Suppose pits i and j are both harvestable (i.e., they contain i and j stones, respectively) and $i < j$. If pit j is harvested, then pit i will contain $i+1$ stones. It would then be "overfull," and could no longer be harvested. Further play could only increase the occupancy of pit i.□

This strategy can be used [7, 5] to enumerate a large number of winning positions. In fact, the strategy can be applied backwards. That is to say, given a winning position, one can obtain a winning position with one more stone in the following manner: Let $i \geq 1$, be the least number such that pit i is empty. Place i stones in this pit, and remove one stone from all previous pits. (This can be done since by definition they are non-empty.) Applying the winning strategy involves removing the i stones and sowing back 1 stone into all the pits from which it was removed. We thus obtain an explicit bijection between winning positions with s stones and those with $s+1$. Since there is but one position (winning to be sure) with no stones, we have the following result.

Theorem 5 *For all $s \geq 0$, there is exactly one winning position involving a total of s stones.*□

The winning positions with $s \leq 24$ are enumerated in figure 7. Note in particular that at most 21 stones may appear in a determined position on a standard Ayo board. Further calculations were done up to $s = 21,286,434$ using a simple SML program.

Figure 4: Winning Positions with up to 24 stones

Stones s	Pit 1	Pit 2	Pit 3	Pit 4	Pit 5	Pit 6	Pit 7	Pit 8	Harvest h_s
0									
1	**1**								1
2		**2**							2
3	**1**	2							1
4		1	**3**						3
5	**1**	1	3						1
6			2	**4**					4
7	**1**		2	4					1
8		**2**	2	4					2
9	**1**	2	2	4					1
10		1	1	3	**5**				5
11	**1**	1	1	3	5				1
12				2	4	**6**			6
13	**1**			2	4	6			1
14		**2**		2	4	6			2
15	**1**	2		2	4	6			1
16		1	**3**	2	4	6			3
17	**1**	1	3	2	4	6			1
18			2	1	3	5	**7**		7
19	**1**		2	1	3	5	7		1
20		**2**	2	1	3	5	7		2
21	**1**	2	2	1	3	5	7		1
22		1	1		2	4	6	**8**	8
23	**1**	1	1		2	4	6	8	1
24				**4**	2	4	6	8	1

```
fun ayo carry nil = [carry]
|   ayo carry (0::xs) = carry::xs
|   ayo carry (x::xs) = (x-1)::(ayo (carry+1) xs);
```

4 Periodicity

The columns of Figure 4 exhibit a certain periodicity. That is to say, the number of stones in the first pit depends not on s but seemingly on s modulo 2. The contents of the first two pits are periodic of period 6 and those of the first three of period 12. An extended table suggests surprisingly that the contents of the

Figure 5: Period of the contents of the first i pits

i	1	2	3	4	5	6	7	8	9	10	11
period	2	6	12	60	60	420	840	2520	2520	27720	27720

first four pits and the contents of the first five pits have the same periodicity, as do the contents of the first eight pits and the contents of the first nine. (See Figure 5.)

This periodicity can be established by an analysis of sequences of numbers determined by the winning positions. Consider the unique winning position with s stones. Let $p_{i,s}$ be the number of stones initially in pit i. Clearly, $p_{i,s} \leq i$. Let $m_{i,s}$ be the number of times pit i must be harvested in order to win and $b_{i,s}$ be the number of moves of the winning strategy which result in a stone being added to pit i. By convention, $b_{0,s} = s$. Taking the Roumba to be pit 0, we thus have $b_{0,s} = s$. Obviously, for $i \geq 1$,

$$p_{i,s} \quad = \quad im_{i,s} - b_{i,s}. \tag{1}$$

There is a one-to-one correspondence between the moves which add a stone to pit i and the moves which harvest some pit j for $j > i$. Hence $b_{i,s} = \sum_{j>s} m_{j,s}$. It follows that

$$b_{i,s} \quad = \quad m_{i+1,s} + b_{i+1,s}. \tag{2}$$

Proposition 6 *For all i, the sequence of i-tuples*

$$((p_{1,s}; p_{2,s}; \cdots; p_{i,s}))_{s \geq 0}$$

is periodic of period $\mathrm{lcm}(1, 2, 3, \ldots, i+1)$.

Proof: We show, by induction on i, that not only is the sequence of i-tuples periodic of period $t = \mathrm{lcm}(1, 2, \ldots, i+1)$, but also t is the smallest positive number such that

$$p_{j,t} = 0, \quad j = 1, 2, \ldots, i.$$

The result is trivial for $i = 1$. Assume, by induction, that the result holds for all values less than or equal to some $i \geq 1$. Let $t = \text{lcm}(1, 2, \ldots, i+1)$. The inductive hypotheses imply $p_{j,kt} = 0$, for $j = 1, 2, \ldots, i$ and $k = 1, 2, \ldots$. Thus, equations 1 and 2 imply

$$jm_{j,kt} = b_{j,kt} = m_{j+1,kt} + b_{j+1,kt} = (j+2)m_{j+1,kt}, \quad j = 1, 2, \ldots, i-1.$$

Combining these results we get

$$2m_{1,kt} = i(i+1)m_{i,kt}.$$

Since every other move is an addition to pit 1, $2m_{1,kt} = kt$. Therefore,

$$\begin{aligned} p_{i+1,kt} &= (i+1)m_{i+1,kt} - b_{i+1,kt} \\ &\equiv (i+1)(m_{i+1,kt} + b_{i+1,kt}) \qquad \bmod i+2 \\ &= (i+1)b_{i,kt} \\ &= i(i+1)m_{i,kt} \\ &= 2m_{1,kt} \\ &= kt \end{aligned}$$

The smallest positive value of k such that

$$kt \equiv 0 \mod i+2$$

is $k = (i+2)/\gcd(t, i+2)$. Setting $q = kt$, we have

$$\begin{aligned} q &= ((i+2)t)/(\gcd(t, i+2)) &&= \text{lcm}(t, i+2) \\ &= \text{lcm}((\text{lcm}(1, 2, \ldots, i), i+2) &&= \text{lcm}(1, 2, \ldots, i+2). \end{aligned}$$

Thus, we have shown

$$p_{j,q} = 0, \quad i = 1, 2, \ldots i+1$$

and q is the smallest positive multiple of t for which this is true. By the inductive assumption, we can thus deduce that q is the smallest positive integer for which this is true. In particular, the sequence of $i+1$-tuples

$$((p_{1,s}; p_{2,s}; \cdots; p_{i+1,s}))_{s \geq 0}$$

does not have period less than q.

Now, the contents of the first $i+1$ pits in the winning position with q stones are the same as the contents of these pits in the winning position with no stones. Suppose for some s that the contents of the first $i+1$ pits is the same in the unique winning position with s stones as in the winning position with $s+q$ stones. The winning positions with $s+1$ and $s+q+1$ stones, respectively, are obtained from the corresponding positions with one fewer stones by either adding stones to the same pit in both cases or by adding stones to two different pits, each having index larger than $i+1$. In either event, the effect upon the first $i+1$ pits is the same and $p_{j,s+1} = p_{j,s+q+1}$, for $j = 1, 2, \ldots, i+1$. We are able to conclude that $p_{i+1,s} = p_{i+1,s+q}$ for all $s \geq 0$. Therefore, the sequence of $i+1$-tuples has period q. $\square$

5 Asymptotics

There is a winning position for every number of stones given an unlimited number of pits. However, as in Ayo, if there is a finite number of pits n, then not all winning positions are realizable. In particular, those rows in figure 4 of length larger than n can not be realized.

Let $s(n)$ denote the smallest number of stones which actually requires the nth pit to win. (Obviously any greater number of stones will require at least n pits.) We will derive the asymptotic formula $s(n) \sim n^2/\pi$ from several interesting

Figure 6: The minimum number of stones to require n pits is well approximated by n^2/π

observations arising from an examination of the sequences $p_{i,s}$ and $m_{i,s}$.

Lemma 7 *Given the above notation,* $p_{i,s} - p_{i-1,s} = (i-1)(m_{i,s} - m_{i-1,s}) + 2m_{i,s}$.

Proof: Equations 1 and 2.□

Lemma 8 *The sequence* $(m_{i,s})_{i=1}^{\infty}$ *is non-increasing.*

Proof: From Lemma 7

$$p_{i,s} - p_{i-1,s} = (i+1)(m_{i,s} - m_{i-1,s}) + 2m_{i-1,s}.$$

Since $p_{i,s} - p_{i-1,s} \leq i$, we have $m_{i,s} - m_{i-1,s} \leq 0$ and $m_{i,s} \leq m_{i-1,s}$, as needed.□

Theorem 9 *As* n *increases,* $s(n)$ *grows as* $n^2/\pi + O(n)$.

Proof: Let s be fixed. Define $f(M)$ to be the least i such that $M = m_{i,s}$. By lemma 8, $m_{i,s} = M$ if and only if $i \in I_M$ where $I_M = \{f(M), f(M)+1, \ldots, f(M-1)-1\}$. By lemma 7, $p_{i,s} - p_{i-1,s} = 2M$ for $i, i+1 \in I_M$. Thus, the sequence $S_M = (p_{i,s})_{i \in I_M}$ is a finite arithmetic sequence with common difference $2M$.

Now,

$$p_{f(M),s} - p_{f(M)-1,s} = (f(M)-1)(M - m_{f(M)-1,s}) + 2M.$$

Figure 7: Winning position pit occupancies $p_{i,1925280}$

Since $p_{f(M)-1,s} \leq f(M)-1$ and $M - m_{f(M)-1,s} \leq -1$, we see that $p_{f(M),s}$, the leading term of S_M, satisfies

$$0 \leq p_{f(M),s} \leq 2M.$$

A similar argument shows $p_{f(M-1)-1,s}$, the final term of S_M, satisfies

$$f(M-1) - 2M + 1 \leq p_{f(M-1)-1,s} \leq f(M-1) - 1.$$

To compute the total number of terms $f(M-1) - f(M)$ in S_M we compute the difference of the leading and final terms, divide by the common difference, and add one.

$$\begin{aligned} f(M-1) - f(M) &= (p_{f(M-1)-1,s} - p_{f(M)})/2M + 1 \\ &= f(M-1)/2M + k_1 \end{aligned}$$

where $|k_1| \leq 3$. Hence,

$$f(M) = \frac{2M-1}{2M} f(M-1) + k_2$$

where $|k_2| \leq 3$, or explicitly

$$\begin{aligned} f(M) &= \frac{1 \times 3 \times 5 \times \cdots (2M-1)}{2 \times 4 \times 6 \times \cdots \times 2M} n + k_3 M \\ &= \frac{\frac{1}{2}\frac{3}{2}\cdots\frac{2M-1}{M}}{M!} n + k_3 M \\ &= \frac{\Gamma(M+\frac{1}{2})n}{M!\sqrt{\pi}} + k_3 M \end{aligned}$$

where $|k_3| \leq 3$, since $\Gamma(1/2) = \sqrt{\pi}$ taking $n + 1 = M(0)$.

We must compute $s(n) = \sum_{i=1}^{n} p_{i,s}$. Thus, we are led to the sum of each sequence I_M. The number of terms has been already computed to be $f(M-1)/2M + k_1$, where $|k_1| \leq 3$. Furthermore, the average term is

$$(p_{f(M-1)-1,s} + p_{f(M),s})/2 = f(M-1)/2 + k_4 M$$

where $|k_4| \leq 1$. Multiplying, we find the sum of I_M to be $f(M-1)^2/4M + O(f(M-1))$. We are thus led to calculate

$$s(n) \sim \sum_{M=1}^{\infty} \frac{\Gamma(M+1/2)^2 n^2}{4\pi M!(M-1)!} = \frac{n^2}{4\pi}\, {}_2F_1\left(\begin{matrix}1/2, 1/2\\ 2\end{matrix}; 1\right).$$

The result follows from Gauss's summation formula [9], since the hypergeometric function ${}_2F_1(1/2, 1/2; 2; 1)$ is equal to 4. □

6 Sieves

By slightly changing the way we think about the game, it is possible to generate the numbers $s(n)$ in a different way. Consider a Tchoukaillon board with an infinite number of pits, indexed 1,2,3, ... beginning with the first pit after the Roumba. By the remarks following Proposition 4, we can play the game backwards. The first move is to put one stone in pit 1. The k-th move is to first determine the empty pit with lowest index, say pit j, then add j stones to pit j and remove one stone from each of pits 1, 2, ..., $j-1$.

Let h_s be the pit to which stones are added on the s-th move. (See Figure 4.) Clearly $h_s = 1$ if and only if s is odd. Of the remaining terms of $(h_s)_{s\geq 1}$ not yet assigned, the first, h_2, and every third term thereafter equals 2. In general, if all terms with $h_2 \leq j$ have been assigned, the first unassigned term and every $(j+2)$-nd term thereafter has the value of $j+1$.

In the notation of Theorem 9, $s(n)$ is the first term of the sequence $(h_s)_{s\geq 1}$ which is equal to n. The sequence $(h_{s(n)})_{n\geq 1}$ can be generated by a generalized "sieve of Eratosthenes" [4, **?**]. (See Figure 8.)

Figure 8: Sieve from Tchoukaillon

```
L := list of integers, in order, beginning with 1
n := 1
repeat forever
h_n := first unslashed number
slash out 1st and every n + 1st unslashed number
n := n + 1
```

$$\begin{aligned} a_n^{(0)} &= n \\ a_n^{(i+1)} &= a^{(i)}_{\lceil (i+1)n/i \rceil} \\ s(i) &= b_1^{(i)} \end{aligned}$$

for $n, i \geq 0$ where by $\lceil x \rceil$ we denote the smallest integer greater than or equal to x.

Theorem 9 thus proves a conjecture of Erdös and Jabotinsky [8, p. 121]. They had proven that $s(n) = n^2/\pi + O(n^{4/3})$ and conjectured that $s(n) = n^2/\pi + O(n)$ based on numerical evidence.

For comparison, consider the Sieve of Eratosthenes. It is remarkable that the slight difference between these algorithms changes the output from a sequence whose nth term grows like n^2/π to one whose nth term, by the Prime Number Theorem, grows like $n(\log n)$.

Figure 9: Sieve of Eratosthenes

```
L := list of integers, in order, beginning with 2
n := 1
repeat forever
p_n := first unslashed number
slash out 1st and every p_nth unslashed number
n := n + 1
```

7 Remarks

A Tchoukaillon pit i can hold up to i stones without overfilling. Thus, a n pit position can hold up to a total of $i(i+1)/2$ stones, and the winning positions have a occupancy rate of $2/\pi + O(1/n) \sim 63.66\%$.

Similarly, since pit i has $i+1$ different possible occupancies in a winning Tchoukaillon position, there are $(i+1)!$ concievable combinations of pit occupancies for the first i pits. However, by Proposition 6, we see that of these only $\text{lcm}(1, 2, 3, \ldots, i+1)$ actually occur in winning Tchoukaillon positions.

We would like to express our surprise and satisfaction that these easily stated problems turned out to have solutions involving higher mathematics (even hypergeometric series). In particular, it was remarkable seeing the constant π appear in a combinatorial problem.

In this paper, we only studied determined positions in Ayo and positions in Tchoukaillon from which a total win is possible. It may be interested to also study strategies designed to maximize the number of stones captured.

Several researchers have studied chip-firing games played with a certain number of stones on the nodes of a directed graph [2]. A node may be fired if it contains as many stones as its out-degree. One stone is sent to each of its neighbors. Such games are very interesting from a mathematical point-of-view, since they have surprisingly many invariants despite the wealth of choices play seemingly offers. It might be of interest to relate the theory of chip-firing games to Mancala games.

The first author studied Ayo while he was teaching at the University of Ibadan, Nigeria from 1975 to 1978. The second author was introduced to Tchoukaillon at the French national congress of the Association MATh en JEANS [7] by a group of junior high school students from Collège l'Ardillière de Nézant (Saint Brice sous Forêt, France) and Collège Pierre de Ronsard (Montmorency, France). We thank Paul Campbell for putting us into contact and encouraging us to write this paper.

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