

# Series expansions of the percolation probability for directed square and honeycomb lattices.

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## Abstract

We have derived long series expansions of the percolation probability for site and bond percolation on directed square and honeycomb lattices. For the square bond problem we have extended the series from 41 terms to 54, for the square site problem from 16 terms to 37, and for the honeycomb bond problem from 13 terms to 36. Analysis of the series clearly shows that the critical exponent  $\beta$  is the same for all the problems confirming expectations of universality. For the critical probability and exponent we find in the square bond case,  $q_c = 0.3552994 \pm 0.0000010$ ,  $\beta = 0.27643 \pm 0.00010$ , in the square site case  $q_c = 0.294515 \pm 0.000005$ ,  $\beta = 0.2763 \pm 0.0003$ , and in the honeycomb bond case  $q_c = 0.177143 \pm 0.000002$ ,  $\beta = 0.2763 \pm 0.0002$ . In addition we have obtained accurate estimates for the critical amplitudes. In all cases we find that the leading correction to scaling term is analytic, i.e., the confluent exponent  $\Delta = 1$ .

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# 1 Introduction

Directed percolation (DP) was originally introduced by Broadbent and Hammersley (1957) as a model of fluid-flow through a random medium and has since been associated with a wide variety of physical processes. In *static* interpretations, the preferred direction is a spatial direction, and DP could represent the percolation of fluid through porous rock with a certain fraction of the channels blocked (De’Bell and Essam 1983b), crack propagation (Kertész and Viscek 1980) or electric current in a diluted diode network (Redner and Brown 1981). In *dynamical* interpretations, the preferred direction is time, and DP is modelled by a stochastic cellular automaton (Kinzel 1985) in which all lattice sites evolve simultaneously and the main interpretation is as an epidemic without immunisation (Harris 1974, Liggett 1985). The behaviour of these models is generally controlled by a single parameter  $p$ , which could be the probability that a channel is open or the infection probability depending on one’s favourite interpretation.

When  $p$  is smaller than a critical value  $p_c$ , the fluid does not percolate through the rock (the epidemic dies out). Let  $P(p)$  be the probability that the wetted region percolates infinitely far from the source (the ultimate survival probability in epidemic language) then one expects:

$$P(p) \propto (p - p_c)^\beta, \quad p \rightarrow p_c^+. \quad (1)$$

DP type transitions are also encountered in many other situations, perhaps most prominently in Reggeon field theory (Grassberger and Sundemeyer 1978, Cardy and Sugar 1980), chemical reactions (Schlögl 1972, Grassberger and de la Torre 1979), in numerous models for heterogeneous catalysis and surface reactions (Ziff *et al.* 1986, Köhler and ben-Avraham 1991, Zhuo *et al.* 1993, Jensen 1994), self-organized criticality (Obukhov 1990 and Paczuski *et al.* 1994) and even galactic evolution (Schulman and Seiden 1982). This short and far from complete list clearly demonstrates that directed percolation is a problem which emerges in a diverse set of physical problems and therefore deserves a great deal of attention.

In this paper we discuss series expansions for the percolation probability on directed square and honeycomb lattices. The earliest series expansion for the square bond problem was the eight terms calculated by Blease (1977). A great improvement was due to Baxter and Guttmann (1988) who extended this series to 41 terms. For the honeycomb bond problem Onody (1990) obtained a 13 term series and for the square site problem the longest series of 16 terms is due to Onody and Neves (1992) improving the previous record of 10 terms held by De’Bell and Essam (1983a). Using the finite-lattice method pioneered in this context by Baxter and Guttmann (1988) we have extended these series to 54 terms for the square bond problem, 37 terms for the

square site problem and 36 terms for the honeycomb bond problem. The percolation probability for the honeycomb site problem is related very simply to that of the square site problem,  $P^{HC}(p) = P^{SQ}(p^2)$  (Dhar *et al.* 1982, Essam and De’Bell 1982). Note also that bond percolation on the honeycomb lattice may be viewed as site-bond percolation on the square lattice (Essam and De’Bell 1982). In passing, we note that long series have been obtained for the moments of the pair connectedness for the site and bond problems on square and triangular lattices (Essam *et al.* 1986, 1988).

## 2 The finite-lattice method

We wish to calculate the series expansion of the percolation probability on square and honeycomb lattices oriented as in Figure 1. We shall consider both site and bond percolation on these lattices. In site (bond) percolation each site (bond) is independently present with probability  $p$  and absent with probability  $q = 1 - p$ . Two sites are connected if one can find a path passing through occupied sites (bonds) only, while always following the allowed directions. For an infinite system, when  $q$  is less than a critical value  $q_c$ , there is an infinite cluster spanning the lattice. The order parameter of the system is the percolation probability  $P(q)$ , i.e., the probability that a given site belongs to the infinite cluster. Note that a path passing through a given site can only lead to the sites shown in Figure 1 below the origin O. This naturally leads one to consider a finite-lattice approximation to  $P(q)$ , namely the probability  $P_N(q)$  that the origin is connected to at least one site in the  $N$ 'th row.  $P_N(q)$  is a polynomial in  $q$  with integer coefficients and a maximal order determined by the total number of sites (bonds) that may be present on the finite lattice.

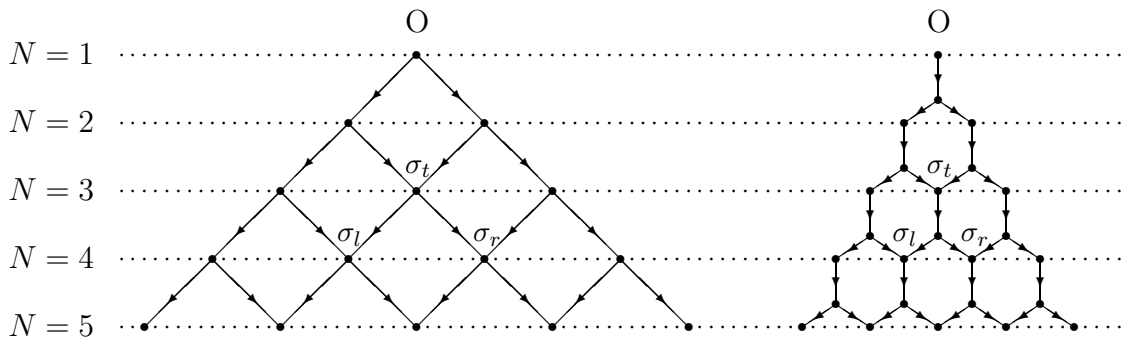


Figure 1: The directed square and honeycomb lattices with orientation given by the arrows. The rows are labelled as according to the text.

It has been proved (Bousquet-Mélou 1995), for all the problems considered in this paper, that the polynomials  $P_N(q)$  have a formal limit in the algebra of formal power series in the variable  $q$ , and therefore  $P(q) = \lim_{N \rightarrow \infty} P_N(q)$ . In all cases one finds that  $P_N(q)$  converges to  $P(q)$  in such a way that the first  $N$  (or  $N - 1$  depending on the specific problem) terms of the polynomials  $P_N(q)$  coincide with those of  $P(q)$ .

## 2.1 Specification of the models

In order to calculate  $P_N(q)$  we associate a state  $\sigma_j$  with each site, such that  $\sigma_j = 1$  if site  $j$  is connected to the  $N$ 'th row and  $\sigma_j = -1$  otherwise. We shall often write  $+/-$  for simplicity. Note that a site can be in state  $-1$  even though, in the case of site percolation, it is itself occupied, or, in the case of bond percolation, bonds emanating from the site are present. Let  $l, r$  denote the sites below  $t$  as in Figure 1. We then define the triangle weight function  $W(\sigma_t | \sigma_l, \sigma_r)$  as the probability that the top site  $t$  of the triangle is in state  $\sigma_t$ , given that the lower sites  $l$  to the left and  $r$  to the right are in states  $\sigma_l$  and  $\sigma_r$ , respectively. One can then prove (Bidaux and Forgacs 1984, Baxter and Guttmann 1988) that

$$P_N(q) = \sum_{\{\sigma\}} \prod_t W(\sigma_t | \sigma_l, \sigma_r), \quad (2)$$

where the product is over all sites  $j$  of the lattice above the  $N$ 'th row. The sum is over all values  $\pm 1$  of each  $\sigma_t$ , other than the topmost spin  $\sigma_1$  which always takes the value  $+1$ . The spins in the  $N$ 'th row are fixed to be  $+1$ . In short  $P_N(q)$  is calculated as the sum over all possible configurations of the probability of each individual configuration.

The weights  $W$  are listed in Table I. Obviously,  $W(- | \sigma_l, \sigma_r) = 1 - W(+ | \sigma_l, \sigma_r)$ . The remaining weights are easily calculated by considering the various possible arrangements of states and bonds.  $W(+ | -, -) = 0$  because the top site is connected to the  $N$ 'th row if and only if at least one of the neighbours is connected. Let us next look at the remaining square bond weights.  $W(+ | +, +) = 1 - q^2$  because the only bond configuration *not* allowed is both bonds absent which has probability  $q^2$ . Finally,  $W(+ | +, -) = W(+ | -, +) = 1 - q$  because the bond connecting the two  $+$  states has to be present, which happens with probability  $p = 1 - q$ , and the other bond can be either present or absent. For the honeycomb bond problem we find that  $W^{HC}(+ | \sigma_l, \sigma_r) = (1 - q)W^{SQ}(+ | \sigma_l, \sigma_r)$  because if the top state is  $+1$  the vertical bond has to be present. Note that one can think of the honeycomb bond problem as site-bond percolation on the directed square lattice where both sites and bonds are present with equal probability (Essam and De'Bell 1982). For the square site problem the weights are a little simpler since a site can be in state  $+1$  only if it is present and  $W$  picks up only the probability of the top state, therefore  $W(+ | -, -) = 0$  as before

Problem	$W(+ +, +)$	$W(+ +, -) = W(+ - , +)$	$W(+ - , -)$
SQ-bond	$1 - q^2$	$1 - q$	0
HC-bond	$(1 - q)(1 - q^2)$	$(1 - q)^2$	0
SQ-site	$1 - q$	$1 - q$	0
HC-site	$(1 - q)^2$	$(1 - q)^2$	0

Table I: The triangle weight functions for the various directed percolation problems. Generally one has  $W(-|\sigma_l, \sigma_r) = 1 - W(+|\sigma_l, \sigma_r)$ .

and all the other weights with a +1 top state are equal to  $1 - q$ . The honeycomb site weights are derived from the square site weights in the same manner as for the bond case. Note that it is customary to assume in site percolation problems that the origin is present with probability 1.

For the square and honeycomb site problems we therefore find:

$$P_N(q) = \sum_{\{\sigma\}} \prod_t W(\sigma_t | \sigma_l, \sigma_r) = \sum_{\{\sigma\}} W_O(0 | \sigma_2, \sigma_3) \prod_t W(\sigma_t | \sigma_l, \sigma_r)$$

The weights  $W(\sigma_t | \sigma_l, \sigma_r)$  are those of Table I and  $W_O$  is the weight of the top-most triangle. It is clear from Table I that for the *site* problem  $W^{HC}(q) = W^{SQ}(2q - q^2)$ . Since the ‘top’ weights are 1 for the square site problem and  $1 - q$  for the honeycomb site problem we find that

$$P_N^{HC}(q) = (1 - q)P_N^{SQ}(1 - (1 - q)^2), \quad (3)$$

which is essentially the relation mentioned in the Introduction, derived from the work of Dhar *et al.* (1982) by Essam and De’Bell (1982).

## 2.2 Series expansion algorithm

For small  $N$  it is quite easy to calculate  $P_N(q)$  by hand, but for larger  $N$  one obviously has to resort to computer algorithms. The algorithms are basically implementations of a transfer matrix method. From Eq. (2) we see that the evaluation of  $P_N(q)$  involves only local ‘interactions’ since the weights involve only three neighbouring sites. The sum over all configurations can therefore be performed by moving a boundary line through the lattice. At any given stage this line cuts through a number of, say  $m$ , lattice sites thus leading to a total of  $2^m$  possible configurations along this line. Any

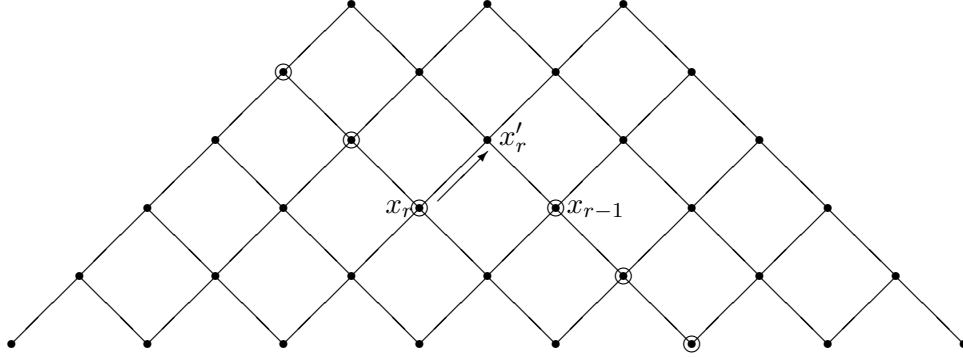


Figure 2: Part of the directed square lattice with the present boundary indicated by open circles. All weights to the left of this boundary have been summed up. The weight of the triangle given by  $(x_r, x'_r, x_{r-1})$  is picked up by moving the boundary from  $x_r$  to  $x'_r$  and updating the associated polynomials according to Eq. 4.

configuration along the line is trivially represented as a binary number by letting the  $r'$ th bit of the number equal  $(\sigma_r + 1)/2$ . For each configuration along the boundary line one maintains a (truncated) polynomial which equals the sum of the product of weights over all possible states on the side of the boundary already traversed. The boundary is moved through the lattice one site at a time. In Figure 2 we show how the boundary is moved in order to pick up the weight associated with a given triangle at position  $r$  along the boundary line. Let  $S0 = (x_1, \dots, x_{r-1}, 0, x_{r+1}, \dots, x_m)$ , be the configuration of sites along the boundary with 0 at position  $r$  and similarly  $S1 = (x_1, \dots, x_{r-1}, 1, x_{r+1}, \dots, x_m)$  the configuration with 1 at position  $r$ . Then in moving the  $r'$ th site from the bottom left to the top of the triangle we see that the polynomials associated with these configurations are updated as

$$P(S0) = W(0|0, x_{r-1})P(S0) + W(0|1, x_{r-1})P(S1), \tag{4}$$

$$P(S1) = W(1|0, x_{r-1})P(S0) + W(1|1, x_{r-1})P(S1).$$

The calculation of  $P_N(q)$  by this method is limited by memory, since one needs storage for  $2^{N-1}$  boundary configurations. To alleviate this problem one can introduce a cut into the lattice, fix the states on this cut, evaluate the lattice sum  $P_N^C(q)$  for each configuration  $C$  of the cut, and finally get  $P_N(q) = \sum_C P_N^C(q)$  as the sum over all configurations of the cut. By placing the cut appropriately the growth in memory

requirements can be reduced to  $2^{N/2}$ . Obviously the finite-lattice calculation for different configurations of the cut are independent of one another and these algorithms are therefore perfectly suited to take full advantage of modern massively parallel computers. In the following section we give a few more details of the algorithms we have used.

### 2.2.1 The bond problem algorithm

A very efficient algorithm was devised by Baxter for the square bond problem (Baxter and Guttmann 1988). All we had to do for the present work was basically to parallelize the algorithm in order to fully utilize the Intel Paragon at Melbourne University. The algorithm is based on an ingenious transformation of the square bond problem onto a honeycomb lattice. This is done by noting that the square bond weights can be written as

$$W(\sigma_t|\sigma_l, \sigma_r) = \sum_{\sigma_m=\pm 1} f(\sigma_t, \sigma_m)g(\sigma_l, \sigma_m)g(\sigma_r, \sigma_m) \quad (5)$$

where

$$\begin{aligned} f(+, +) &= -1, & f(+, -) &= f(-, +) = 1, & f(-, -) &= 0 \\ g(+, +) &= q, & g(+, -) &= g(-, +) = g(-, -) = 1. \end{aligned} \quad (6)$$

This means that if we replace each upwards pointing triangle in Figure 1 by a three-pointed star, arriving at the honeycomb lattice of Figure 3, then  $P_N(q)$  can be calculated from this lattice by assigning weights  $f(\sigma_i, \sigma_j)$  to vertical edges  $(i, j)$  and  $g(\sigma_i, \sigma_j)$  to non-vertical edges.

A cut of length  $L$  is introduced along the line RS in Figure 3 and the transfer matrix technique is used to build up the lattice to the left of RSO, starting from XR and working upwards to OS. The lattice is symmetrical around the central axis RSO and one can therefore obtain the lattice sum for the whole lattice by forming the sum of the squares of each boundary line polynomial. After this operation, the whole lattice is summed except for the edges on the center-line. So finally one has to multiply the (squared) boundary line polynomials by the weights of these edges. This is where the great advantage of the transformation becomes clear. Because  $f(-, -) = 0$  we need never consider configurations of the cut (or parts of the boundary line in the vertical position) which have any  $(-, -)$  edge. This basically means that the number of configurations of the cut which contribute to  $P_N(q)$  grow only like  $3^{L/2}$  rather than the usually expected  $2^L$ . The transformation thus provides us with an exponentially

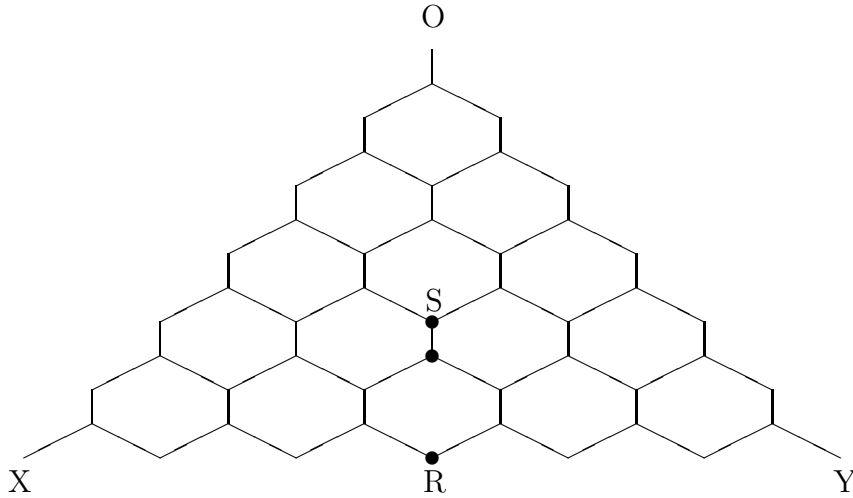


Figure 3: The transformed lattice used in the square bond algorithm. The sites marked with the full circles on the cut-line SR are fixed.

faster algorithm. Likewise, as parts of the boundary line enter the vertical position no  $(-, -)$  edges need to be considered which leads to a significant reduction in the length of the cut for a given amount of memory. The memory requirement for the algorithm is governed by the maximal extent of the boundary line, which is at XR, and hence grows like  $2^{N/2-1}$ . With this algorithm we calculated  $P_N(q)$  for  $N \leq 39$ . Since the integer coefficients occurring in the series expansion become very large the calculation was performed using modular arithmetic (see, for example, Knuth 1969). Each run, using a different modulus, took approximately 24 hours using 50 nodes on an Intel Paragon.

Virtually the same algorithm can be used for the honeycomb bond or site-bond square problem except that the  $f$  weights have to be replaced by

$$f(+, +) = -(1 - q), \quad f(+, -) = f(-, +) = 1 - q, \quad f(-, -) = q. \quad (7)$$

Since  $f(-, -)$  no longer equals 0, obviously the great advance of the original transformation vanishes and the number of configurations of the cut grow like  $2^L$ . For this reason we had to stop calculating  $P_N(q)$  at  $N = 33$ , where each modulus required about 32 hours of CPU time using 50 nodes.



### 2.2.2 The site problem algorithm

For the site problem the growth in memory can be limited to  $2^{N/2-1}$  by introducing a cut across the lattice at row  $N/2$ . The upper part of the lattice is built up first by the transfer matrix technique, yielding a partial lattice sum  $P_U^C$  and then the lower part  $P_L^C$  is done. The total lattice sum for a given cut  $P_N^C$  is simply the product of these, i.e.,  $P_N^C = P_L^C P_U^C$ . Again  $P_N(q)$  is the sum over all configurations of the cut. It might seem that the number of cuts grow as  $2^{N/2}$ . Substantial simplifications can however be obtained. Note first of all that there is symmetry around the central vertical line which basically reduces the number of cut-configurations by a factor of 2. A more subtle means of reducing the number of cuts is obtained as follows: Since all triangle weights with at least one + on the bottom are the same, it follows that for any two configurations  $C$  and  $C'$  which can be turned into one another by changing any number of +'s to -'s *without* adding or removing any (--) sequences,  $P_U^C = P_U^{C'}$ . This means, for example, that for any cut  $C$  without (--) occurrences,  $P_U^C$  equals the partial sum of the all +'s cut. It is possible to use this property to perform the lower lattice sum simultaneously for many cuts. As an example consider cuts starting with ++ and -+ but otherwise the same. The upper part is the same and the lower part is also the same except for the weight of the left-most triangle on the cut. By considering the various possibilities when moving the boundary line across this point, one can easily see that the two configurations can be summed simultaneously, i.e., the -+ cut can be made as part of the ++ cut. We calculated  $P_N(q)$  for  $N \leq 32$  which took about 48 hours for each modulus with  $N = 32$  using 50 nodes.

We also calculated the series expansion for the honeycomb site problem up to  $P_{32}(q)$ . Although we know the exact relation between the two site problems, Eq. (3), this calculation provides us with an extra check of the algorithm and the extrapolation formulas we shall discuss presently.

## 3 Extrapolation of the series

As mentioned  $P_N(q)$  will generally agree with the series for  $P(q)$  up to some order determined by  $N$ . For the square bond problem the coefficients of  $P_N(q) = \sum_{m \geq 0} a_{N,m} q^m$  agree with those of  $P(q) = \sum_{m \geq 0} a_m q^m$  to order  $N$ . Baxter and Guttmann (1988) found that the series for  $P(q)$  can be extended considerably by determining the correction terms to  $P_N(q)$ . Let us look at

$$P_N - P_{N+1} = q^N \sum_{r \geq 1} q^r d_{N,r} \quad (8)$$

then we shall call  $d_{N,r} = a_{N,N+r} - a_{N+1,N+r}$  the  $r$ 'th correction term. Obviously if one can find formulas for  $d_{N,r}$  for all  $r \leq K$  then one can use the series coefficients of  $P_N(q)$  to extend the series for  $P(q)$  to order  $N + K$  since

$$a_{N+k} = a_{N,N+k} - \sum_{m=1}^k d_{N+k-m,m} \quad (9)$$

for all  $k \leq K$ . That this method can be very efficient was clearly demonstrated by Baxter and Guttmann, who identified the first twelve correction terms, and used  $P_{29}(q)$  to extend the series for  $P(q)$  to 41 terms. To really appreciate this advance one should bear in mind that the time it takes to calculate  $P_N(q)$  grows exponentially with  $N$ , so a direct calculation correct to the same order would have taken years rather than days. In the following we will give details of the correction terms for the various cases.

### 3.1 The square bond case

The first correction term for the square bond case is given by the Catalan numbers

$$d_{N,1} = c_N = (2N!)/N!(N+1)! \quad (10)$$

a result which was proved (Bousquet-Mélou 1995) by noting that the correction term arise from compact bond animals of directed height  $N$  and perimeter  $N + 1$ . The second correction term

$$d_{N,2} = 2c_N - c_{N+1} \quad (11)$$

was also calculated exactly recently (Bousquet-Mélou 1995). As noted by Baxter and Guttmann (1988) the higher-order correction terms  $d_{N,r}$  can be expressed as rational functions of the Catalan numbers. We have found that  $d_{N,r}$  always can be written in the form

$$d_{N,r} = \sum_{k=1}^{\lfloor (r-1)/2 \rfloor} A_{r,k} \binom{N-m}{k} c_{N-m} + \sum_{j=1}^{2r-4} B_{r,j} c_{N-r+2+j} \quad (12)$$

where  $m = \max(0, r - 4 - 2k)$ . These formulas hold for all available  $N$ , provided that only Catalan numbers  $c_m$  with  $m \geq 0$  are involved. As noted by Baxter and Guttmann it is also true for  $m = -1$  provided one 'defines'  $c_{-1} = -1$  (there was a misprint at this point in the original article). Thus the extrapolation formulas are true for  $N \geq r - 4$ . For  $r \leq 15$  the coefficients  $A_{r,k}$  and  $B_{r,j}$  are either integers or fractions

		r												
j	3	4	5	6	7	8	9	10	11	12	13	14	15	
1	-2	2	2	6	28	168	1188	9438	81796	758472	7423832	75951512	806254512	
2	2	-5	13	14	40	172	940	6022	43224	337852	2821008	24820084	227759640	
3		5	$-29\frac{1}{2}$	58	76	230	916	4358	22626	116670	490220	-39210	-43047748	
4		-2	-2	-73	218	-23	16720	-238382	5439166	$-54821200\frac{3}{5}$	$1211505650\frac{4}{5}$	$-8785667878\frac{4}{5}$	$228722521818\frac{2}{5}$	
5			12	24	-360	1185	-2285	66470	-1329734	$13844727\frac{2}{5}$	$-302226767\frac{1}{5}$	$2199183639\frac{1}{5}$	$-57173632471\frac{3}{5}$	
6			-2	-6	174	-1619	3737	20838	-289156	3246796	-69778186	503746552	-13371140646	
7				0	-128	889	-7985	23543	-67332	$1144926\frac{1}{5}$	$-20852206\frac{3}{5}$	$165626416\frac{3}{5}$	$-4004317118\frac{4}{5}$	
8				0	82	-242	4320	-35091	56180	355092	-4943452	44277656	-999749068	
9					-15	-15	-1334	23956	-177605	$406391\frac{4}{5}$	$-1134497\frac{2}{5}$	$12747113\frac{2}{5}$	$-251625122\frac{1}{5}$	
10					0	18	-104	-8722	124506	$-793053\frac{3}{5}$	$1126018\frac{4}{5}$	$3369765\frac{1}{5}$	$-63360349\frac{3}{5}$	
11						-1	243	2820	-56441	675899	-3956056	7537547	-18025630	
12						0	-46	-1204	18304	-329150	3495696	-18307058	28433480	
13							-2	367	-5702	115282	-1913164	18478108	-90869764	
14							0	-44	1636	-25924	715656	-10857182	95785220	
15								2	-250	954	-200132	4463999	-61558076	
16								0	8	1036	42000	-1380082	27374696	
17									-2	-125	-6857	363300	-9209555	
18									0	-12	1064	-95182	2442068	
19										2	-163	21166	-518177	
20										0	12	-2554	86634	
21											-2	116	-11594	
22											0	-12	1316	
23												2	-140	
24												0	12	
25													-2	
26													0	
m	3	4	5	6	7	8	9	10	11	12	13	14	15	
1	-2	3	$-9\frac{1}{2}$	13	-90	-745	31152	-485988	10837718	$-109826658\frac{1}{5}$	$2422283961\frac{3}{5}$	$-17572113989\frac{3}{5}$	$457481237152\frac{4}{5}$	
2			4	-2	15	-5	51	3566	-57572	$854505\frac{1}{5}$	$-13937224\frac{4}{5}$	$130944690\frac{4}{5}$	$-2410518810\frac{2}{5}$	
3					-8	4	-24	-19	24	$-9414\frac{2}{5}$	$101155\frac{1}{5}$	$-1480387\frac{1}{5}$	$19377361\frac{3}{5}$	
4							16	0	40	$81\frac{1}{5}$	$-136\frac{4}{5}$	$21604\frac{4}{5}$	$-168086\frac{2}{5}$	
5									-32	-16	-72	-232	216	
6											64	64	144	
7													-128	

Table II: The coefficients  $A_{r,m}$  and  $B_{r,j}$  in the extrapolation formulas for the square bond problem.

$n$	$a_n$	$n$	$a_n$
0	1	28	-16161597987
1	0	29	-43448897414
2	-1	30	-117083094891
3	-2	31	-315709399172
4	-4	32	-853195535637
5	-8	33	-2306601710190
6	-17	34	-6249350665825
7	-38	35	-16933569745596
8	-88	36	-45982825444918
9	-210	37	-124847185166968
10	-511	38	-339715065397631
11	-1264	39	-923984791735474
12	-3165	40	-2518902151116767
13	-8006	41	-6861776192406434
14	-20426	42	-18738381486019497
15	-52472	43	-51115047622373452
16	-135682	44	-139811976659987636
17	-352562	45	-381836043069041990
18	-920924	46	-1046008104766969784
19	-2414272	47	-2859625985546910846
20	-6356565	48	-7845284416715093642
21	-16782444	49	-21465842456693034778
22	-44470757	50	-58976491160296065655
23	-118090648	51	-161476439366532026854
24	-314580062	52	-444296183371760430967
25	-839379548	53	-1217055970699512453538
26	-2245969278	54	-3353766967706302949866
27	-6017177104		

Table III: The coefficients  $a_n$  in the series expansion of  $P(q) = \sum_{n \geq 0} a_n q^n$  for directed bond percolation on the square lattice.

with small (2 or 5) denominators. Note that there are various relations between the Catalan numbers so there are infinitely many ways of writing (12). For several of the correction formulas the general form adopted in this paper is slightly different from that of Baxter and Guttmann (1988) who tried wherever possible to choose a form involving only integers. The trade-off for having a general expression for the correction terms is that more rational fractions become involved. However with the proliferation of powerful mathematical packages such as MAPLE and MATHEMATICA this trade-off is well worth while. In Table II we have listed the coefficients  $A_{r,j}$  and  $B_{r,j}$  for  $r \leq 15$ . Using these extrapolation formulas and the series for  $P_{39}(q)$  we have extended the series for  $P(q)$  to the 54 terms given in Table III.

### 3.2 The square site case

Inspired by the success of the extrapolation procedure for the square bond problem one might hope for similar success for other problems. And indeed one can find several of the correction terms for the square site problem, although the success is less spectacular as one is restricted to the first six correction terms. The first correction term  $d_{N,1}$  was identified by Onody and Neves (1992) and since computed exactly by Bousquet-Mélou (1995)

$$d_{N,1} = \frac{(3N)!}{N!(2N+1)!}. \quad (13)$$

This expression for the correction term was identified by Onody and Neves as the number of ways of inserting  $n-4$  sheets through a ball having  $n$  vertices on its surface such that pairs of sheets meet only on surface curves joining vertices! While this is true, a more useful and pertinent interpretation can be given. Viennot (1994) has pointed out that this is just the expression for the number of ternary trees of  $n$  vertices, which in turn is isomorphic to the number of diagonally convex directed animals (Delest and Fédou 1989). It is the identification between these animals and the first correction term that has been proved by Bousquet-Mélou. She also proved our formula for the second correction term.

As in the square bond case we can express higher correction terms as a function of  $d_{N,1}$ . Again, there are infinitely many ways of expressing the formulas for the correction terms, one of which is

$$d_N^r = \sum_{i=2}^{r-1} C_{r,i} \binom{N}{i} d_{N,1} + \sum_{j=1}^{2r-1} (NB_{r,j} + A_{r,j}) d_{K,1} \quad (14)$$

where  $K = N - r + j$  for  $r \leq 4$  and  $K = N - r - 1 + j$  for  $r \geq 5$ . These formulas are correct up to  $r = 6$ , whenever  $N \geq r$ . The coefficients are listed in Table IV. These formulas allowed us to extend the series for  $P(q)$  by an additional six terms to a total 37 terms listed in Table V.

### 3.3 The honeycomb bond case

For bond percolation on the directed honeycomb lattice Bousquet-Mélou (1995) proved that the generating function  $f = \sum_{N \geq 1} d_{N,1} t^{N-1}$  of the first correction term  $d_{N,1}$  is characterized by the algebraic equation

$r/j$	$A_{r,j}$					$B_{r,j}$					$C_{r,j}$			
	2	3	4	5	6	2	3	4	5	6	3	4	5	6
1	-6	-31	-353	-1050	-87304	9	18	126	225	15372	$3\frac{13}{36}$	$8\frac{1}{81}$	$22\frac{1037831}{1679616}$	$56\frac{32726159}{90699264}$
2	$8\frac{2}{3}$	11	-199	$-5458\frac{1}{3}$	-71645	$46\frac{1}{3}$	$-12\frac{2}{3}$	$120\frac{2}{3}$	$1388\frac{2}{3}$	$13474\frac{2}{3}$		$-6\frac{35}{216}$	$-12\frac{10969}{15552}$	$-36\frac{2734421}{5038848}$
3	-12	$-73\frac{3}{4}$	-123	$-2965\frac{2}{3}$	-85165	$-7\frac{1}{3}$	$-277\frac{5}{12}$	$150\frac{7}{27}$	1113	$23201\frac{1}{3}$			$11\frac{385}{1296}$	$19\frac{30595}{46656}$
4		$23\frac{3}{4}$	$249\frac{1}{2}$	$-530\frac{74}{81}$	$-7647\frac{61}{81}$		$-8\frac{11}{12}$	$948\frac{17}{108}$	$267\frac{2}{9}$	$2194\frac{5}{27}$				$-20\frac{5531}{7776}$
5		$18\frac{1}{3}$	904	$293\frac{2014}{2187}$	$-5605\frac{1120}{2187}$		$7\frac{1}{3}$	$862\frac{1}{4}$	$-404\frac{112}{243}$	$3084\frac{371}{729}$				
6			$-396\frac{1}{6}$	$-499\frac{193}{256}$	$134\frac{11888}{59049}$			$-149\frac{5}{12}$	$-1524\frac{353035}{559872}$	$-888\frac{17965}{19683}$				
7			0	$-8771\frac{183}{256}$	$-1225\frac{169}{1536}$			0	$-7324\frac{35}{256}$	$-5593\frac{6619423}{30233088}$				
8				$1525\frac{47}{96}$	$39744\frac{107}{512}$				$362\frac{175}{192}$	$31048\frac{1321}{1536}$				
9				392	$20245\frac{19}{192}$				112	$10870\frac{37}{48}$				
10					$-8251\frac{1}{192}$					$-2273\frac{107}{192}$				
11					0					0				

Table IV: The coefficients  $A_{r,j}$ ,  $B_{r,j}$  and  $C_{r,j}$  in the extrapolation formulas for the square site problem.

$n$	$a_n$	$n$	$a_n$
0	1	19	-92459524
1	0	20	-298142956
2	-1	21	-922424269
3	-3	22	-3098690837
4	-8	23	-9042937179
5	-21	24	-34187149573
6	-56	25	-79544646085
7	-154	26	-439149878359
8	-434	27	-313237196088
9	-1252	28	-7786443675714
10	-3675	29	16637473844344
11	-10954	30	-207593240544002
12	-33044	31	973714665769453
13	-100676	32	-7311741153076579
14	-309569	33	43345744201832502
15	-957424	34	-292472879532946388
16	-2987846	35	1867850225746155582
17	-9330274	36	-12389925641797917900
18	-29522921	37	81441868912809214904

Table V: The coefficients  $a_n$  in the series expansion of  $P(q) = \sum_{n \geq 0} a_n q^n$  for directed site percolation on the square lattice.

$$f = 1 + tf + \frac{t}{4}((7+t)f^2 + f^3). \quad (15)$$

The higher-order correction terms are given by the formulas

$$d_N^r = \sum_{k=1}^r \left[ D_{r,k} \binom{N}{3} + C_{r,k} \binom{N}{2} \right] d_{N-2r+2+k,1} + \sum_{j=1}^{2r} (NB_{r,j} + A_{r,j}) d_{N-2r+2+j,1} \quad (16)$$

which we find to be correct for  $r \leq 4$  and  $N \geq 2r - 1$ . The coefficients are listed in Table VI apart from the  $D$ 's since the only non-zero ones are  $D_{4,1} = -157281/5$  and  $D_{4,2} = 1744273/5$ . The final 36 terms series for  $P(q)$  is given in Table VII.

r/j	$A_{r,j}$			$B_{r,j}$			$C_{r,j}$		
	2	3	4	2	3	4	2	3	4
1	$-8\frac{3}{20}$	$10954\frac{1}{2}$	$-2773464\frac{741}{1000}$	$14\frac{1}{4}$	$-5696\frac{41}{50}$	$814389\frac{147}{1000}$	$-12\frac{13}{15}$	$1753\frac{9}{25}$	$-86965\frac{91}{125}$
2	$-67\frac{13}{20}$	$87495\frac{21}{50}$	$-24330909\frac{122}{125}$	$54\frac{2}{5}$	$-48502\frac{51}{100}$	$7805538\frac{1733}{2000}$	$-141\frac{8}{15}$	$19295\frac{4}{25}$	$-1118660\frac{159}{500}$
3	$-3\frac{3}{5}$	$1831\frac{14}{25}$	$415040\frac{911}{1000}$	$4\frac{9}{20}$	$-292\frac{1}{20}$	$-424945\frac{137}{2000}$	$12\frac{13}{15}$	$-1663\frac{19}{25}$	$224372\frac{37}{250}$
4		$593\frac{3}{10}$	$-276663\frac{5}{8}$		$-616\frac{9}{20}$	$98675\frac{69}{80}$			
5		$66\frac{13}{20}$	$-56787\frac{1}{5}$		$23\frac{1}{2}$	$31375\frac{77}{80}$			
6		$7\frac{4}{5}$	$-8377\frac{23}{40}$		$5\frac{1}{5}$	$9476\frac{1}{2}$			
7			$-646\frac{1}{4}$			$871\frac{19}{40}$			
8			$-315\frac{3}{5}$			$-187\frac{9}{10}$			

Table VI: The coefficients  $A_{r,j}$ ,  $B_{r,j}$  and  $C_{r,j}$  in the extrapolation formulas for the honeycomb bond problem.

$n$	$a_n$	$n$	$a_n$
0	1	19	-1103369168956
1	-1	20	-5771541600014
2	-4	21	-31153472926184
3	-12	22	-160153702442390
4	-45	23	-907425183546587
5	-188	24	-4317291410619157
6	-835	25	-28433248376749141
7	-3849	26	-99125481158184567
8	-18242	27	-1076035285073833314
9	-88265	28	-238091850291444337
10	-434295	29	-58631611223043405378
11	-2165198	30	279283045229982597450
12	-10915089	31	-4730770444199592196256
13	-55534781	32	40182669640102878093220
14	-284708699	33	-480633574529182764438221
15	-1470350760	34	4852667371105928333619923
16	-7628363273	35	-53829647651783620888423836
17	-39878267745	36	574209696129704803372604206
18	-208458228964	37	

Table VII: The coefficients  $a_n$  in the series expansion of  $P(q) = \sum_{n \geq 0} a_n q^n$  for directed bond percolation on the honeycomb lattice.



## 4 Analysis of the series

We expect that the series for the percolation probability behaves like

$$P(q) \sim A(1 - q/q_c)^\beta [1 + a_\Delta(1 - q/q_c)^\Delta + \dots], \quad (17)$$

where  $A$  is the critical amplitude,  $\Delta$  the leading confluent exponent and the  $\dots$  represents higher order correction terms. By universality we expect  $\beta$  to be the same for all the percolation problems studied in this paper and we will argue that the dominant correction term is analytic, i.e.,  $\Delta = 1$ .

In the following sections we present the results of our analysis of the series which include accurate estimates for the critical parameters  $q_c$ ,  $\beta$ ,  $A$  and  $\Delta$ . For the most part the best results are obtained using Dlog Padé (or in some cases just ordinary Padé) approximants. A comprehensive review of these and other techniques for series analysis may be found in Guttmann (1989).

### 4.1 $q_c$ and $\beta$

In Table VIII we show the Dlog Padé approximants to the percolation probability series for bond percolation on the directed square lattice. The defective approximants, those for which there is a spurious singularity on the positive real axis closer to the origin than the physical critical point, are marked with an asterisk. The overwhelming majority of the approximants cluster around the values  $q_c = 0.3552994$  and  $\beta = 0.27643$ . As always in this type of analysis it is very difficult to accurately judge the true errors of the estimates of the critical parameters, however we venture to say that the critical parameters lie in the ranges,  $q_c = 0.3552994(10)$  and  $\beta = 0.27643(10)$ , where the figures in parenthesis indicate the estimated error on the last digits. The other remarkable feature of Table VIII is that surprisingly many of the high-order approximants are defective.

The results of the analysis of the series for the square site problem are listed in Table IX. In this case there is a marked upward drift in the estimates for both  $q_c$  and  $\beta$  and the estimates do not settle down to definite values. It does however seem likely that the true critical parameters lie within the estimates:  $q_c = 0.294515(5)$  and  $\beta = 0.2763(3)$ .

The analysis of the series for the honeycomb bond problem yields the results in Table X. Again we see an upward drift in the estimates for both  $q_c$  and  $\beta$  though the estimates are somewhat more stable than in the previous case. It seems likely that the true critical parameters lie within the estimates:  $q_c = 0.177143(2)$  and  $\beta = 0.2763(2)$ .

Finally we analysed the series for the honeycomb site problem, with the results tabu-

N	[N-1,N]		[N,N]		[N+1,N]	
	$q_c$	$\beta$	$q_c$	$\beta$	$q_c$	$\beta$
11	0.3553000	0.27645	0.3553030	0.27653	0.3553023	0.27651
12	0.3553016	0.27649	0.3553011	0.27648	0.3552997	0.27644
13	0.3553028*	0.27652*	0.3553004	0.27646	0.3553000	0.27645
14	0.3552994	0.27643	0.3552972	0.27634	0.3552995	0.27643
15	0.3552991	0.27642	0.3552994	0.27643	0.3552994	0.27643
16	0.3552994	0.27643	0.3552994	0.27643	0.3552994	0.27643
17	0.3552994	0.27643	0.3552994	0.27643	0.3552997*	0.27644*
18	0.3552994	0.27643	0.3552992	0.27642	0.3552983	0.27632
19	0.3553002*	0.27643*	0.3552991	0.27641	0.3552996*	0.27644*
20	0.3552994	0.27643	0.3552994	0.27643	0.3552994	0.27643
21	0.3552994	0.27643	0.3552994	0.27643	0.3552994*	0.27643*
22	0.3552994*	0.27643*	0.3552994*	0.27643*	0.3552994*	0.27643*
23	0.3552994*	0.27643*	0.3552994	0.27643	0.3552994	0.27643
24	0.3552993*	0.27643*	0.3552993*	0.27643*	0.3552995	0.27644
25	0.3552993*	0.27643*	0.3552997	0.27645	0.3552995*	0.27644*
26	0.3552991*	0.27643*	0.3552990*	0.27643*	0.3552986*	0.27647*
27	0.3552993*	0.27643*				

Table VIII: Dlog Padé approximants to the percolation series for directed bond percolation on the square lattice.

N	[N-1,N]		[N,N]		[N+1,N]	
	$q_c$	$\beta$	$q_c$	$\beta$	$q_c$	$\beta$
5	0.2939337	0.26881	0.2943291	0.27266	0.2942979	0.27228
6	0.2942670	0.27190	0.2943175	0.27252	0.2942699*	0.27199*
7	0.2944168	0.27393	0.2944521	0.27453	0.2944777	0.27502
8	0.2945135	0.27585	0.2944742	0.27495	0.2944794*	0.27505*
9	0.2944599	0.27465	0.2944720	0.27490	0.2944739	0.27494
10	0.2944753	0.27498	0.2944656*	0.27478*	0.2944942	0.27546
11	0.2945228	0.27655	0.2945156	0.27623	0.2945020	0.27571
12	0.2945246*	0.27662*	0.2945060	0.27586	0.2945058	0.27585
13	0.2945058	0.27585	0.2945061*	0.27586*	0.2945047	0.27581
14	0.2945051	0.27578	0.2945051	0.27582	*	*
15	0.2945056	0.27584	0.2945047*	0.27581*	0.2945032*	0.27576*
16	0.2945069	0.27589	0.2945096	0.27602	0.2945089	0.27598
17	0.2945090	0.27599	0.2945095	0.27601	0.2945113	0.27612
18	0.2945134	0.27625	0.2945111	0.27611		

Table IX: Dlog Padé approximants to the percolation series for directed site percolation on the square lattice.

N	[N-1,N]		[N,N]		[N+1,N]	
	$q_c$	$\beta$	$q_c$	$\beta$	$q_c$	$\beta$
5	0.1770229	0.27331	0.1770722	0.27420	0.1771131	0.27507
6	0.1771195	0.27523	0.1770967	0.27469	0.1771067	0.27493
7	0.1771087	0.27498	0.1771161	0.27517	0.1771270	0.27552
8	0.1771320	0.27572	0.1770209*	0.27662*	0.1771414	0.27612
9	0.1771480	0.27647	0.1771294	0.27559	0.1771369	0.27591
10	0.1771391	0.27601	0.1771352	0.27584	0.1771356	0.27585
11	0.1771357	0.27586	0.1771344*	0.27580*	0.1771399	0.27609
12	0.1771412	0.27619	0.1771381	0.27598	0.1771395	0.27606
13	0.1771402	0.27612	0.1771411	0.27618	0.1771403	0.27612
14	0.1771406	0.27614	0.1771404	0.27641	0.1771403*	0.27612*
15	0.1771405	0.27613	0.1771408	0.27616	0.1771429	0.27636
16	0.1771390*	0.27605*	0.1771415	0.27622	0.1771419	0.27625
17	0.1771422	0.27629	0.1771418	0.27624	0.1771418	0.27624
18	0.1771418	0.27624				

Table X: Dlog Padé approximants to the percolation series for directed bond percolation on the honeycomb lattice.

N	[N-1,N]		[N,N]		[N+1,N]	
	$q_c$	$\beta$	$q_c$	$\beta$	$q_c$	$\beta$
5	0.1598159	0.27017	0.1599573	0.27265	0.1599491	0.27249
6	0.1599269	0.27203	0.1599516	0.27254	0.1599487*	0.27248*
7	0.1600181	0.27416	0.1600409	0.27485	0.1600545	0.27532
8	0.1600656	0.27577	0.1600476	0.27507	0.1599682	0.27195
9	0.1600378	0.27473	0.1600457	0.27501	0.1600452	0.27499
10	0.1600453	0.27499	0.1600456	0.27501	0.1600555	0.27543
11	0.1600280*	0.27462*	0.1600711	0.27640	0.1600597	0.27565
12	0.1600498	0.27515	0.1600630	0.27585	0.1600630	0.27585
13	0.1600630	0.27585	0.1600630	0.27585	0.1600622	0.27580
14	0.1600620	0.27579	0.1600625	0.27582	0.1600636	0.27589
15	0.1600630	0.27585	0.1600622*	0.27580*	0.1600391*	0.27665*
16	0.1600641	0.27593	0.1600656	0.27606	0.1600647	0.27597
17	0.1600650	0.27600	0.1600655	0.27604	0.1600662	0.27611
18	0.1600688	0.27642	0.1600662	0.27611		

Table XI: Dlog Padé approximants to the percolation series for directed site percolation on the honeycomb lattice.

lated in Table XI. As in the square site case there is a very pronounced upward drift in the estimates for both  $q_c$  and  $\beta$ . It seems likely that the true critical parameters lie within the estimates:  $q_c = 0.160067(5)$  and  $\beta = 0.2763(4)$ . We note that the expected relation between the values of  $q_c$  for the square site and honeycomb site problems,  $q_c^{SQ} = 2q_c^{HC} - (q_c^{HC})^2$ , clearly is fulfilled by the estimates. This inspires some confidence in the appropriateness of our extrapolation method in general and our error estimates in particular.

## 4.2 The critical amplitudes

From the leading critical behaviour,  $P(q) \sim A(1 - q/q_c)^\beta$ , it follows that  $(q_c - q)P^{-1/\beta}|_{q=q_c} \sim A^{-1/\beta}q_c$ . So by forming the series for  $G(q) = (q_c - q)P^{-1/\beta}$  we can estimate the critical amplitude  $A$  from Padé approximants to  $G$  evaluated at  $q_c$ . The procedure works well but requires knowledge of both  $q_c$  and  $\beta$ . For the square bond series we know both  $q_c$  and  $\beta$  very accurately, and we estimated  $A$  using values of  $q_c$  between 0.355299 and 0.3553 and values of  $\beta$  ranging from 0.2764 to 0.2765. For each  $(q_c, \beta)$  pair we calculate  $A$  as the average over all  $[N + K, N]$  Padé approximants with  $K = 0, \pm 1$  and  $2N + K \geq 45$ . The spread among the approximants is minimal for  $q_c = 0.3552994$ ,  $\beta = 0.27643$  where  $A = 1.3291475(2)$ . Allowing for values of  $q_c$  and  $\beta$  within the full range we get  $A = 1.3292(5)$ .

For the square site series we used values of  $q_c$  from 0.29451 to 0.29452 and  $\beta$  from 0.2761 to 0.2765 averaging over Padé approximants with  $2N + K \geq 27$ . In this case the spread is minimal for  $q_c = 0.294515$ ,  $\beta = 0.2763$  with  $A = 1.425164(5)$ . Again allowing for a wider choice of critical parameters we estimate that  $A = 1.425(1)$ .

For the honeycomb bond series we restricted  $q_c$  to lie between 0.177138 and 0.177148 and  $\beta$  between 0.2761 to 0.2765 using all approximants with  $2N + K \geq 26$ . The minimal spread occurs at  $q_c = 0.177143$ ,  $\beta = 0.27635$  where  $A = 1.10607(2)$ . A wider choice for  $q_c$  and  $\beta$  leads to the estimate  $A = 1.106(1)$ .

Finally in the honeycomb site case we used values of  $q_c$  in the range 0.160065 to 0.160075 and  $\beta$  from 0.2761 to 0.2765 using all approximants with  $2N + K \geq 27$ . The minimal spread occurs when  $q_c = 0.160069$ ,  $\beta = 0.2764$  where  $A = 1.16779(2)$ . With the wider choice of critical parameters we estimate that  $A = 1.167(1)$ . The exact relation, Eq. (3), between the square and honeycomb site problems means that there is a simple relation between the amplitudes in the two cases. First note that,  $A^H(1 - q/q_{c,H})^\beta \sim P^H(q) = (1 - q)P^S(2q - q^2) \sim (1 - q)(1 - (2q - q^2)/q_{c,S})^\beta$ . Since,  $q_{c,S} = 2q_{c,H} - q_{c,H}^2$ , we find that,  $(1 - (2q - q^2)/q_{c,S})^\beta = [(q_{c,H} - q)(2 - q_{c,H} - q)/q_{c,S}]^\beta$ , and therefore:  $A^H = (1 - q_{c,H})[(2 - 2q_{c,H})q_{c,H}/q_{c,S}]^\beta A^S = (1 - q_{c,H})(1 - q_{c,H}^2/q_{c,S})^\beta A^S$ . Insertions of the various critical parameters shows that this relation is indeed satisfied by our amplitude estimates.

L	SQ bond	SQ site	HC bond	HC site
1	1.29661	1.41614	1.11520	1.16740
2	1.31234	1.39775	1.12002	1.16579
3	1.31114	1.39989	1.12001	1.16607
4	1.31218	1.37739	1.11952	1.16564
5	1.31098	1.39359	1.11750	1.16546
6	1.31006	1.39001	1.11808	1.16521
7	1.32566	1.39889	1.11856	1.16486
8	1.30916	1.39582	1.11929	1.16537
9	1.31322	1.39162	1.11929	1.16534
10	1.31122	1.39449	1.11780	1.16508
11	1.31195	1.40570	1.12056	1.16578
12	1.31228	1.40306	1.12435	1.16462

Table XII: Critical amplitudes  $A$  for the four percolation problems obtained by using the method of Liu and Fisher. The estimates were calculated by averaging over various inhomogeneous differential approximants of order  $L$ .

A second method, proposed by Liu and Fisher (1989), for calculating critical amplitudes starts by assuming the functional form  $P(q) \sim A(q)(1 - q/q_c)^\beta + B(q)$ . One then transforms this function into  $g(q) = (1 - q/q_c)^{-\beta}P(q) \sim A(q) + B(q)(1 - q/q_c)^{-\beta}$ . The required amplitude is now the *background* term in  $g(q)$ , which can be obtained from inhomogeneous differential approximants (Guttmann (1989) p89). In Table XII we have listed the estimates obtained by averaging over various first order differential approximants using at least 40 terms of the series for the square bond case and at least 25 terms in the other cases. The critical parameters  $q_c$  and  $\beta$ , used in the transformation of the series, were the central values of the estimates from the previous section. This method generally yields slightly lower estimates for the amplitudes and the spread among the approximants is much larger than in the first method.

### 4.3 The confluent exponent

We studied the series using two different methods in order to estimate the value of the confluent exponent. In the first method, due to Baker and Hunter (1973), one transforms the function,  $P(q) = \sum_{i=1}^n A_i(1 - q/q_c)^{-\lambda_i} = \sum_{n=0}^{\infty} a_n q^n$ , into an auxiliary function with simple poles at  $1/\lambda_i$ . We first make the change of variable  $q = q_c(1 - e^{-\zeta})$  and find, after multiplying the coefficient of  $\zeta^k$  by  $k!$ , the auxiliary function

$$\mathcal{F}(\zeta) = \sum_{i=1}^N \sum_{k=0}^{\infty} A_i (\lambda_i \zeta)^k = \sum_{i=1}^N \frac{A_i}{1 - \lambda_i \zeta}, \quad (18)$$

which has poles at  $\zeta = 1/\lambda_i$  with residue  $-A_i/\lambda_i$ . The great advantage of this method (when it works) is that one obtains simultaneous estimates for many critical parameters, namely,  $\beta$ ,  $\Delta$ , and the critical amplitude, while there is only one parameter,  $q_c$  in the transformation. In Figure 4 we have plotted, respectively,  $\beta$  and  $\Delta$  as a function of the transformation parameter  $q_c$  for various  $[N \pm K, N]$  Padé approximants, with  $N \geq 25$ . For each ‘guess’ for  $q_c$  we performed the Baker-Hunter transformation and located the numerically largest and next-largest poles, which are the estimates for the reciprocals of  $-\beta$  and  $-(\beta + \Delta)$ , respectively. The majority of the approximants have a very narrow crossing region close to  $q_c = 0.3552996(3)$ , with  $\beta = 0.27645(3)$  and  $\Delta = 1.000(5)$ . In Table XIII we have listed the estimates for  $\beta$ ,  $\Delta$  and the corresponding critical amplitudes obtained from the Baker-Hunter transformed series with  $q_c = 0.3552996$ . The results strongly suggest that the leading correction to scaling term is analytic. Furthermore we note that the estimates for the critical amplitudes fully agree with those obtained from the first method used in the previous section.

N	M	$\beta$	A	$\Delta$	$A \times a_{\Delta}$
22	23	0.27645	1.32925	1.00097	1.03202
23	23	0.27646	1.32930	1.00013	1.03029
24	23	0.27863	1.32369	0.98439	1.01224
23	24	0.27645	1.32925	1.00090	1.03181
24	24	0.27647	1.32931	0.99994	1.02993
25	24	0.27549	1.33100	1.01375	1.05322
24	25	0.27645	1.32926	1.00078	1.03149
25	25	0.27648	1.32935	0.99922	1.02857
26	25	0.27589	1.33038	1.00698	1.04048
25	26	0.27645	1.32926	1.00064	1.03114
26	26	0.27649	1.32936	0.99906	1.02826
27	26	0.27617	1.32992	1.00305	1.03410
26	27	0.27645	1.32928	1.00037	1.03052
27	27	0.27649	1.32936	0.99911	1.02836

Table XIII: Estimates for the critical exponent  $\beta$ , critical amplitude  $A$ , confluent exponent  $\Delta$ , and confluent amplitude  $A \times a_{\Delta}$ , obtained from  $[N, M]$  Padé approximants to the Baker-Hunter transformed square bond series with  $q_c = 0.3552996$ .

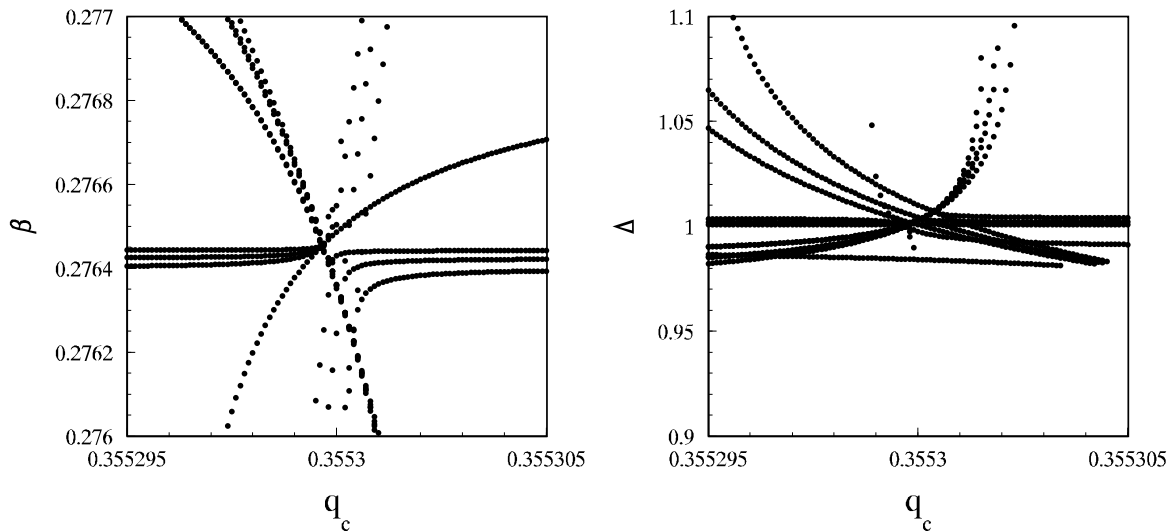


Figure 4: The critical exponent  $\beta$  and confluent exponent  $\Delta$  as a function of the parameter  $q_c$  in the Baker-Hunter transformation for the square bond series.

In the second method, due to Adler *et al.* (1981), one studies Dlog Padé approximants to the function  $G(q) = \beta P(q) + (q_c - q)dP(q)/dq$ . The logarithmic derivative to  $G(q)$  has a pole at  $q_c$  with residue  $\beta + \Delta$ . We evaluate the Dlog Padé approximants for a range of guesses for  $q_c$  and  $\beta$ . For each such guess we thus find an estimate for  $\Delta$ ; for the correct value of  $q_c$  and  $\beta$  one would expect to see a convergence region in  $(q_c, \beta, \Delta)$ -space. In practice we always froze either  $q_c$  or  $\beta$  and examined  $\Delta$  as a function of the other parameter. Figure 5 shows, respectively,  $\Delta$  as a function of  $q_c$  with  $\beta = 0.27643$  and  $\Delta$  as a function of  $\beta$  with  $q_c = 0.3552994$ . This analysis clearly support  $\Delta \simeq 1$ , and thus that there is no sign of any non-analytic corrections to scaling.

For the square site series the results from the Baker-Hunter transformation is less convincing, as there is no value of  $q_c$  at which the various approximants cross. If we look closely at the approximants evaluated at  $q_c = 0.294515$  we find, generally speaking, that only the  $[N - 1, N]$  approximants yield estimates of  $\beta$  close to the expected value with corresponding estimates for  $\Delta$  consistent with an analytic correction. The method of Adler *et al.* confirms that  $\Delta \simeq 1$ .

In the honeycomb bond case several of the approximants to the Baker-Hunter transformed series has a crossing for  $q_c = 0.177144(1)$ ,  $\beta = 0.2767(1)$  and  $\Delta = 0.89(2)$ , though it should be noted that the scatter is quite large. When we analyse the series using the second method we find that, for  $q_c$  and  $\beta$  close to the central values from the Dlog Padé analysis, a value of 1 for  $\Delta$  is fully compatible with the results.

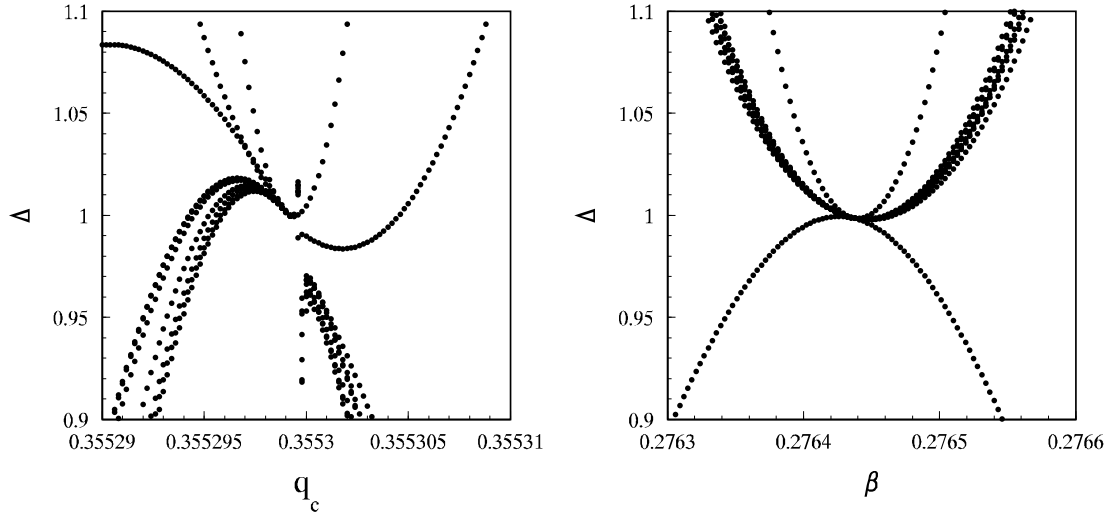


Figure 5: The confluent exponent  $\Delta$  as a function of, respectively, the parameter  $q_c$  (with  $\beta = 0.27643$ ) and the parameter  $\beta$  (with  $q_c = 0.3552994$ ) using the method of Adler *et al.* (1981).

## 5 Conclusion

In this paper we have presented extended series for the percolation probability for site and bond percolation on the square and honeycomb lattices. The analysis of the series leads to improved estimates for the percolation threshold (particularly for the honeycomb bond problem) and the order parameter exponent  $\beta$ . To summarise we estimate that

$$\begin{aligned}
 q_c &= 0.3552994(10), & \beta &= 0.27643(10) & A &= 1.3292(5) & \text{square bond problem,} \\
 q_c &= 0.294515(5), & \beta &= 0.2763(3) & A &= 1.425(1) & \text{square site problem,} \\
 q_c &= 0.177143(2), & \beta &= 0.2763(2) & A &= 1.106(1) & \text{honeycomb bond problem,} \\
 q_c &= 0.160067(5), & \beta &= 0.2763(4) & A &= 1.167(1) & \text{honeycomb site problem.}
 \end{aligned}$$

The estimates for  $q_c = 1 - p_c$  for the square bond and site problem are in excellent agreement with those obtained by Essam *et al.* (1986, 1988),  $q_c = 0.355303(6)$  and  $q_c = 0.29451(1)$ , respectively. The estimates for  $\beta$  clearly show, as one would expect, that all the models studied in this paper belong to the same universality class. The value of  $\beta$  does not suggest any simple fraction. Indeed, around the central value for  $\beta$  (square bond) we find only four fractions with denominators less than 1500. They are:  $\frac{34}{123} = 0.276422\dots$ ,  $\frac{387}{1400} = 0.276429\dots$ ,  $\frac{217}{785} = 0.276433\dots$  and  $\frac{183}{662} = 0.276435\dots$ . None of these are remotely compelling, and leave open the question as to why this



apparently simple problem has such an ugly exponent. This does seem to be a frequent characteristic of directed problems, as evidenced by the recent study of the longitudinal size exponent of square lattice directed animals (Conway and Guttmann 1994) in which it was found that  $\nu_{\parallel} = 0.81722(5)$ , a result which suggests no simple rational fraction. Finally we note that none of the series show any evidence of non-analytic confluent correction terms. This provides a hint that the model might be exactly solvable.

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r/j	$A_{r,j}$					$B_{r,j}$					
	2	3	4	5	6	2	3	4	5	6	3
1	-6	-31	-353	-1050	-87304	9	18	126	225	15372	$3\frac{13}{36}$
2	$8\frac{2}{3}$	11	-199	$-5458\frac{1}{3}$	-71645	$46\frac{1}{3}$	$-12\frac{2}{3}$	$120\frac{2}{3}$	$1388\frac{2}{3}$	$13474\frac{2}{3}$	-6
3	-12	$-73\frac{3}{4}$	-123	$-2965\frac{2}{3}$	-85165	$-7\frac{1}{3}$	$-277\frac{5}{12}$	$150\frac{7}{27}$	1113	$23201\frac{1}{3}$	
4		$23\frac{3}{4}$	$249\frac{1}{2}$	$-530\frac{74}{81}$	$-7647\frac{61}{81}$		$-8\frac{11}{12}$	$948\frac{17}{108}$	$267\frac{2}{9}$	$2194\frac{5}{27}$	
5		$18\frac{1}{3}$	904	$293\frac{2014}{2187}$	$-5605\frac{1120}{2187}$		$7\frac{1}{3}$	$862\frac{1}{4}$	$-404\frac{112}{243}$	$3084\frac{371}{729}$	
6			$-396\frac{1}{6}$	$-499\frac{193}{256}$	$134\frac{11888}{59049}$			$-149\frac{5}{12}$	$-1524\frac{353035}{559872}$	$-888\frac{17965}{19683}$	
7			0	$-8771\frac{183}{256}$	$-1225\frac{169}{1536}$			0	$-7324\frac{35}{256}$	$-5593\frac{6619423}{30233088}$	
8				$1525\frac{47}{96}$	$39744\frac{107}{512}$				$362\frac{175}{192}$	$31048\frac{1321}{1536}$	
9				392	$20245\frac{19}{192}$				112	$10870\frac{37}{48}$	
10					$-8251\frac{1}{192}$					$-2273\frac{107}{192}$	
11					0					0	