Series expansions of the percolation probability on the directed triangular lattice.

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Abstract

We have derived long series expansions of the percolation probability for site, bond and site-bond percolation on the directed triangular lattice. For the bond problem we have extended the series from order 12 to 51 and for the site problem from order 12 to 35. For the site-bond problem, which has not been studied before, we have derived the series to order 32. Our estimates of the critical exponent β are in full agreement with results for similar problems on the square lattice, confirming expectations of universality. For the critical probability and exponent we find in the site case: $q_c = 0.4043528 \pm 0.0000010$ and $\beta = 0.27645 \pm 0.00010$; in the bond case: $q_c = 0.52198 \pm 0.00001$ and $\beta = 0.2769 \pm 0.0010$; and in the site-bond case: $q_c = 0.264173 \pm 0.000003$ and $\beta = 0.2766 \pm 0.0003$. In addition we have obtained accurate estimates for the critical amplitudes. In all cases we find that the leading correction to scaling term is analytic, i.e., the confluent exponent $\Delta = 1$.

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1 Introduction

In an earlier paper (Jensen and Guttmann 1995) we reported on the derivation and analysis of long series for the percolation probability of site and bond percolation on the directed square and hexagonal lattices. In this paper we extend this work to site, bond and site-bond percolation on the directed triangular lattice. We refer to our earlier paper for a more general introduction to directed percolation and its role in the modelling of physical systems. In directed *site* percolation each site is either present (with probability p) or absent (with probability $q = 1 - p$) independent of all other sites on the lattice. Similarly for *bond* percolation each bond is absent or present independently of other bonds. Finally in *site-bond* percolation both sites and bonds may be absent or present with equal probability, but again with no dependency on any other sites or bonds. Two sites in the various models are connected if one can find a path, respecting the directions indicated in Figure 1, through occupied sites, bonds or sites *and* bonds, respectively, from one to the other. When p is smaller than a critical value p_c all clusters of connected sites remain finite, while for $p \geq p_c$ there is an infinite cluster spanning the lattice in the preferred direction. The order parameter of the system is the percolation probability $P(p)$ that a given site belongs to the infinite cluster. This quantitiy is strictly zero when $p < p_c$ and changes continuously at p_c. For $p > p_c$ the behaviour of $P(p)$ in the vicinity of p_c may be described by a critical exponent β,

$$P(p) \propto (p - p_c)^{\beta}, \quad p \to p_c^+. \tag{1}$$

The bond problem was originally studied by Blease (1977) who calculated a series to 12th order. For the site problem De'Bell and Essam (1983) derived the series to 12th order. The site-bond problem has, at least to our knowledge, never been studied before. Our main motivation for doing so in this paper is to obtain further independent estimates of the critical exponent β. Using the finite-lattice method and the extrapolation technique of Baxter and Guttmann (1988) we have extended the series for the bond problem to order 51, for the site problem to order 35 and derived the series for the site-bond problem to order 32. The site and bond problems have also been studied by Essam *et al.* (1986,1988), who derived series expansions for moments of the pair connectedness.

2 The finite-lattice method

We wish to derive a series expansion for the percolation probability on the directed triangular lattice oriented as in Figure 1. In this figure we have numbered the various levels or rows of the lattice according to which sites can be reached by a path of minimum length $N-1$ starting at the origin O. In other words all sites in the Nth row can be reached in $N-1$ steps but not in $N-2$ steps. Note that a path going through

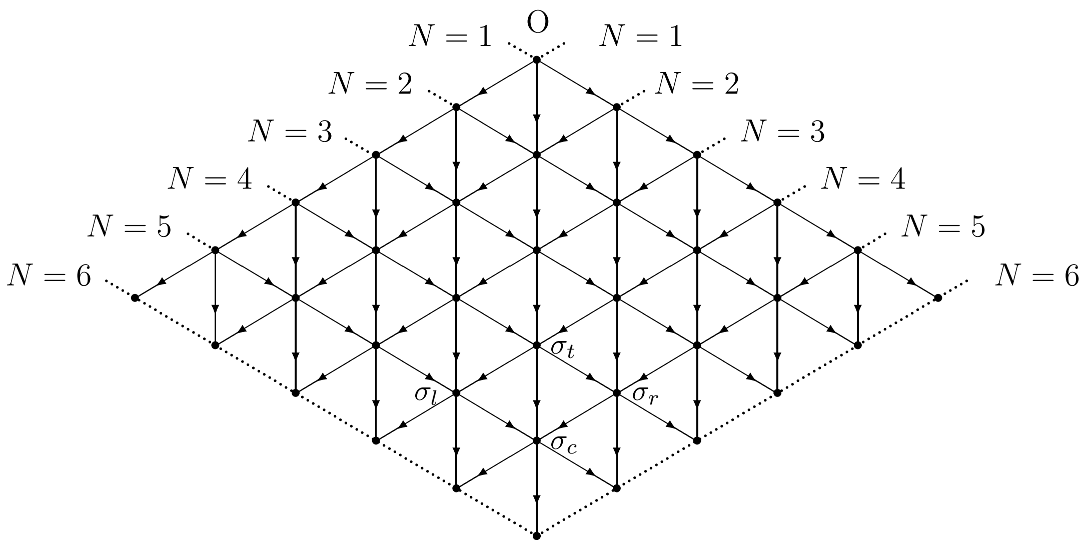

Figure 1: The directed triangular lattice with orientation given by the arrows. The rows are labelled according to the text.

a given site can only reach the part of the lattice shown in Figure 1 below the origin O. This suggests that one should look at the following finite-lattice approximation to $P(q)$, namely the probability $P_N(q)$ that the origin is connected to at least one site in the Nth row. Since we are in the high density region we have chosen to use the expansion parameter q rather than p. $P_N(q)$ is a polynomial with integer coefficients and a maximal order determined by the total number of sites and/or bonds on the finite lattice.

By the method used for the square lattice problems (Bousquet-Mélou 1995) one can prove, *mutatis mutandis*, that the polynomials $P_N(q)$ converge to $P(q)$. Indeed we may consider $P(q) = \lim_{N\to\infty} P_N(q)$ to be a more precise definition of the percolation probability. More importantly, however, from a series expansion point of view, for the site and site-bond problems the first $N+1$ coefficients of the polynomials $P_N(q)$ are identical to those of $P(q)$. In the case of bond percolation the agreement extends through the first $2N+1$ coefficients.

2.1 Specification of the models

To calculate the finite-lattice percolation probability $P_N(q)$ we associate a state σ_j with each site, such that $\sigma_j = 1$ if site j is connected to the Nth row and $\sigma_j = -1$ otherwise. We shall often write $+/-$ for simplicity. Let l, c and r denote the sites connected to a site t from the row above, as in Figure 1. We then define the weight function $W(\sigma_t|\sigma_l, \sigma_c, \sigma_r)$ as the probability that the top site t is in state σ_t, given that the lower sites l, c and r are in states σ_l, σ_c and σ_r, respectively. As for the square lattice (Bidaux and Forgacs 1984, Baxter and Guttmann 1988) we then have

$$P_N(q) = \sum_{\{\sigma\}} \prod_t W(\sigma_t|\sigma_l, \sigma_c, \sigma_r), \qquad (2)$$

where the product is over all sites t of the lattice above the Nth row. The sum is over all values ± 1 of each σ_t, other than the topmost spin σ_1 which always takes the value $+1$. The spins in the Nth row are fixed at $+1$, and $P_N(q)$ is calculated as the sum over all possible configurations of the probability of each individual configuration.

The weight functions W are calculated as follows. Obviously, $W(-|\sigma_l, \sigma_c, \sigma_r) = 1 - W(+|\sigma_l, \sigma_c, \sigma_r)$. The remaining weights are easily calculated by considering the possible arrangements of states and sites and/or bonds. $W(+|-,-,-) = 0$ because the top site is connected to the Nth row if and only if at least one of its neighbours below is connected to the Nth row. All the remaining weights for the *site* problem equal $1 - q$ because the top-site has to be occupied in order to be connected to the Nth row. Let us next look at the remaining *bond* weights. $W^B(+|+,+,+) = 1 - q^3$ because the only bond configuration *not* allowed is all three bonds absent, which has probability q^3. $W^B(+|+,+,-) = W^B(+|+,-,+) = W^B(+|-,+,+) = 1 - q^2$ because the bond to the $-$ state can be either present or absent (probability 1) while among the remaining bonds only the configuration with both bonds absent (probability q^2) is forbidden. Finally, $W^B(+|+,-,-) = W^B(+|-,+,-) = W^B(+|-,-,+) = 1 - q$ because the bond to the $+$ state has to be present, which happens with probability $p = 1 - q$, while the other bonds can be either present or absent. For the *site-bond* problem we find that $W^{SB}(+|\sigma_l, \sigma_c, \sigma_r) = (1 - q)W^B(+|\sigma_l, \sigma_c, \sigma_r)$ because if the top state is $+1$ the top site has to be present.

2.2 Series expansion algorithm

Computer algorithms for the calculation of $P_N(q)$ are readily found. These are basically implementations of the transfer matrix technique. The general features of these algorithms were described in our earlier paper (Jensen and Guttmann 1995), to which we refer for further details. The sum over configurations is performed by moving a boundary line through the lattice. For each configuration along the boundary line one maintains a (truncated) polynomial which equals the sum of the product of weights over all possible states on the side of the boundary already traversed. The boundary is moved through the lattice one site at a time. The calculation of $P_N(q)$ by this method is limited by memory, since one needs storage for 2^N boundary configurations. However, as was the case with the square lattice, this problem can be circumvented by introducing a cut into the lattice. For each fixed configuration of states on this cut one evaluates the lattice sum $P_N^C(q)$ and gets $P_N(q) = \sum_C P_N^C(q)$ as the sum over all configurations of the cut. By placing the cut appropriately, the growth in memory requirements can be reduced to $2^{N/2}$.

In Figure 2 we show the triangular lattice with a cut marked by filled circles. In the algorithm the cut is used as a pivot line by the boundary line which traverse the lattice.

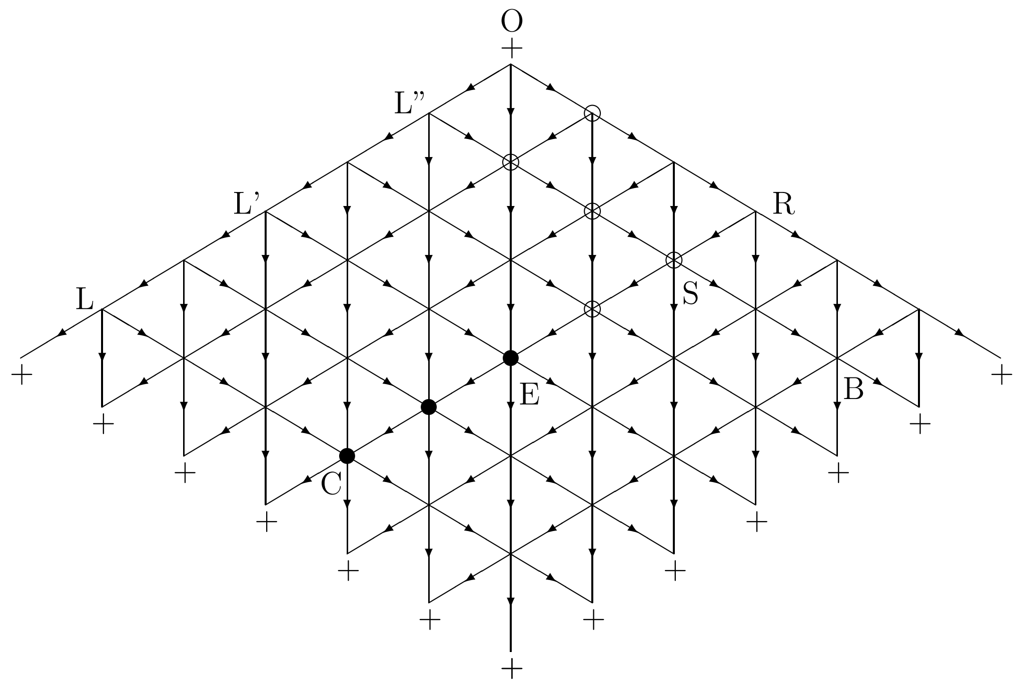

Figure 2: The directed triangular lattice with orientation given by the arrows. The sites with fixed states along the pivot line are marked by filled circles. The open circles mark one particular position of the boundary line during the traversing of the lattice.

We start by building up the first row at the base CL of the lattice. We then build up the part of lattice above the cut from row CL to row EL’. Next the boundary line expands along the line-piece ES until it reach the position ESL” and the last site (at L”) is flipped to the other site of the top-most triangle (after this the boundary line is in the position marked by the open circles). Then we work our way down the right side of the lattice past R to position ESB. Finally the boundary line is moved down along the line-piece SEC after which the whole lattice has been build up. This process is then repeated for each configuration of the cut. Since the calculations for different cut-configurations are independent of each other this algorithm is perfectly suited to take full advantage of massively parallel computers.

Using this algorithm we calculated $P_N(q)$ for $N \leq 23$ for the bond and site-bond problems. The integer coefficients of $P_N(q)$ become very large so the calculation was performed using modular arithmetic (see, for example, Knuth 1969). Each run with $N = 23$, using a different moduli, took approximately 70 hours for the bond problem and 55 hours for the site-bond using 50 nodes on an Intel Paragon. For the site problem the weights only depend on whether or not there are any +’s among the neighbours of the top-most site. As was the case for the square site problem this may be used to sum over many configurations of the cut simultaneously (see Jensen and Guttmann (1995) for further details). This allowed us to calculate $P_N(q)$ for $N \leq 25$. Each run for $N = 25$ took about 85 hours using 50 nodes.

3 Extrapolation of the series

As mentioned, the coefficients of the polynomials $P_N(q) = \sum_{m\geq 0} a_{N,m}q^m$ will generally agree with those of the series for $P(q) = \sum_{m\geq 0} a_m q^m$ up to some order, $\tilde{N}$, determined by N, but depending on the specific problem. In the case of directed bond percolation on the square lattice Baxter and Guttmann (1988) found that the series for $P(q)$ could be extended significantly by determining correction terms to $P_N(q)$. Let us look at

$$P_N - P_{N+1} = q^{\tilde{N}} \sum_{r\geq 0} q^r d_{N,r} \tag{3}$$

then we call $d_{N,r} = a_{N,\tilde{N}+r} - a_{N+1,\tilde{N}+r}$ the rth correction term. If formulas can be found for $d_{N,r}$ for all $r \leq K$ then, using the series coefficients of $P_N(q)$, one can extend the series for $P(q)$ to order $\tilde{N} + K$ since

$$a_{\tilde{N}+r} = a_{N,\tilde{N}+r} - \sum_{m=1}^{r} d_{N+r-m,m} \tag{4}$$

for all $r \leq K$. That this method can be very efficient was demonstrated by Baxter and Guttmann, who identified the first twelve correction terms for the square bond problem, and used $P_{29}(q)$ to extend the series for $P(q)$ to order 41. To really appreciate this advance one should bear in mind that the time it takes to calculate $P_N(q)$ grows exponentially with N, so a direct calculation correct to the same order would have taken years rather than days. In the following we will give details of the correction terms for the various directed percolation problems on the triangular lattice.

3.1 The site problem

For the site problem the coefficients of $P_N(q)$ agree with those of $P(q)$ to order N. In this case the first correction term is very simple as $d_{N,0} = 2$ for $N \geq 2$, i.e., the first correction term is simply a constant. For the second correction term $d_{N,1}$ we find the following sequence

$$0, 0, 3, 18, 32, 50, 72, 98, \ldots$$

It is thus immediately clear that

$$d_{N,1} = 2N^2 \quad \text{for} \quad N \geq 3. \tag{5}$$

Note that for convenience we assume that the sequence starts from $N = 0$. And indeed we find that for $N \geq r+1$, $d_{N,r}$ can be expressed as a polynomial in N of order $2r$. We have been able to calculate these polynomials for the the first 10 correction terms. It

					c_r^k				
k/r	1	2	3	4	5	6	7	8	9
0	0	24	0	5760	-345600	-65318400	-15850598400	-2984789606400	-539895767040000
1	0	-24	-48	-6720	662400	86728320	15417077760	3039204188160	681914690150400
2	4	4	160	-2256	-299136	-54616320	-10042993152	-2801552624640	-758646639912960
3		-12	-456	-5592	-155040	29156640	6930400512	1683396497664	492391103938560
4		8	112	6968	262400	3721088	-1895857152	-641242189440	-236796916234752
5			-72	-4680	-211440	-13781520	-275292864	183056948928	80349078951936
6			16	1016	117072	9766720	775939360	32888441824	-13942053553664
7				-288	-35760	-3900960	-484442784	-52810790592	-3002221192320
8				32	6000	1183584	180360704	27746932192	4062978111936
9					-960	-222000	-46002432	-9468263616	-1860005271168
10					64	28000	8946336	2268003232	567526218432
11						-2880	-1175328	-405615168	-128527251840
12						128	112448	55739936	21947992384
13							-8064	-5494272	-2918143872
14							256	406784	301743168
15								-21504	-23270400
16								512	1362432
17									-55296
18									1024

Table I: The coefficients c_r^k in the extrapolation formulas Eq. (6) for the site problem.

turns out that it is useful to pull out a factor $1/(r!(r+1)!)$ and express the correction terms as

$$d_{N,r} = \frac{1}{r!(r+1)!}\sum_{k=0}^{2r} c_r^k N^k. \tag{6}$$

This ensures that the coefficients c_r^k in the extrapolation formulas are integers. We have listed these coefficients in Table I.

Obviously since these formulas are correct for $N \geq r+1$ and we have calculated $P_N(q)$ for $N \leq 25$ we did not have enough terms in the correction sequences to calculate all the coefficients in these polynomials for the largest values of r. However, from the table of coefficients, it is immediately clear that $c_r^{2r} = 2^{r+1}$. And in general we found that $c_r^{2r-m}/2^{r+1}$ is a polynomial in r of order $2m$

$$c_r^{2r-m} = \frac{2^{r+1}}{(-4)^m m!}\sum_{k=0}^{2m} b_m^k r^k, \tag{7}$$

where the prefactor has been chosen so as to make the leading coefficients particularly simple. In Table II we have listed the coefficients b_m^k for the first six polynomials.

This time we note that $b_m^{2m} = 3^m$. And indeed as before we find that $b_m^{2m-j}/3^m$ is a polynomial in m of order $2j$. In particular we have,

				b_m^k		
k/m	1	2	3	4	5	6
1	3	$-3\frac{1}{3}$	192	$4662\frac{2}{5}$	-76800	$2752914\frac{2}{7}$
2	3	19	-126	$-20702\frac{2}{3}$	-969328	-61888160
3		$24\frac{2}{3}$	-411	7092	1554956	$1312798444\frac{4}{9}$
4		9	459	$21958\frac{1}{3}$	196840	-55417284
5			141	$-17022\frac{2}{5}$	-1359655	$-81930639\frac{1}{3}$
6			27	$4615\frac{1}{3}$	860155	105874935
7				684	-236446	$-52835386\frac{20}{21}$
8				81	33050	14159255
9					3015	$-2180338\frac{4}{9}$
10					243	196605
11						12474
12						729

Table II: The coefficients b_m^k in the extrapolation formulas Eq. (7) for the site problem.

$$b_m^{2m-1} = 3^m m(17/27 + 10/27m),$$

and

$$b_m^{2m-2} = 3^m m(1015/486 - 5137/1458m + 332/243m^2 + 50/729m^3).$$

So when calculating the extrapolation formulas Eq. (6) we first used the sequences for the correction terms to predict as many polynomials as possible. When we ran out of terms we then predicted as many of the leading coefficients from Eq. (7) as possible. This in turn allowed us to find more extrapolation formulas, which we could use (together with the formulas for b_m^{2m-j}) to find more of the formulas for c_k^{2r-m}. And so on until the process stopped with the 10 extrapolation formulas we listed above.

Using the 10 extrapolation formulas and $P_{25}(q)$ we extended the series for $P(q)$ through order 35. The resulting series is listed in Table III.

3.2 The site-bond problem

For the site-bond problem the coefficients of $P_N(q)$ agree with those of $P(q)$ to order N. In this case the correction terms are very similar to those of the site problem. In particular we find that $d_{N,0} = 12$ and in general $d_{N,r}$ is a polynomial in N of order $2r$,

$$d_{N,r} = \frac{2^r}{r!(r+1)!} \sum_{k=0}^{2r} c_r^k N^k. \tag{8}$$

n	a_n	n	a_n
0	1	18	-111307
1	0	19	-255236
2	0	20	-590543
3	-1	21	-1362919
4	-2	22	-3182137
5	-5	23	-7362611
6	-10	24	-17377129
7	-20	25	-40125851
8	-41	26	-96106251
9	-86	27	-219681825
10	-182	28	-539266908
11	-393	29	-1200140540
12	-853	30	-3087966932
13	-1887	31	-6454135923
14	-4208	32	-18281313306
15	-9445	33	-33072764132
16	-21350	34	-114854030873
17	-48612	35	-145978838818

Table III: The coefficients a_n in the series expansion of the percolation probability $P(q)$ for directed site percolation on the triangular lattice.

We have identified the first 9 correction terms for the site-bond problem and have listed the coefficients c_r^k in the extrapolation formulas in Table IV.

From this table it is immediately clear that the coefficient of the leading order $c_r^{2r} = 3 \times 4^r$. As in the site case we find that $c_r^{2r-m}/4^{r+1}$ is a polynomial in r of order $2m$.

$$c_r^{2r-m} = \frac{4^{r+1}}{(-4)^m m!} \sum_{k=0}^{2m} b_m^k r^k, \tag{9}$$

where the prefactor has been chosen so as to make the leading coefficients particularly simple. In Table V we have listed the coefficients b_m^k for the first six polynomials.

In this case $b_m^{2m} = 3^{m+1}$ and $b_m^{2m-1} = 3^{m+1} m(10/27 - 16/27m)$, which, using the same procedure as before allowed us to find the first 9 extrapolation formulas. From $P_{23}(q)$ we were thus able to extend the series for $P(q)$ through order 32. The resulting series is listed in Table VI.

3.3 The bond problem

For the bond problem the coefficients of $P_N(q)$ agree with those of $P(q)$ to order $2N$. In this case the first correction term is more complicated. For the first correction term

					c_r^k			
k/r	1	2	3	4	5	6	7	8
0	-22	372	-6948	228960	-15136200	1002796200	-148319942400	16196987318400
1	-28	-88	-3570	26052	532350	202151160	54036574200	7153213667040
2	48	66	12222	-66190	16300863	-1072631628	61870142088	-28771509693672
3		-512	-6804	-464344	-9400240	-1026322032	27946386678	5012953659000
4		192	7512	428618	21649545	1760115147	84256658654	6746690054058
5			-4800	-249952	-23384790	-1734224880	-194249017018	-15249026722216
6			768	128960	12678024	1443885081	172767873502	22487197814172
7				-34816	-5084160	-762064416	-111221029556	-18388293899920
8				3072	1447680	274270176	53077387932	10265902430946
9					-220160	-72890880	-18083074464	-4339851543328
10					12288	13020672	4539617152	1389887209152
11						-1277952	-833487872	-335678443520
12						49152	101771264	61228145664
13							-6995968	-8139063296
14							196608	721256448
15								-36700160
16								786432

Table IV: The coefficients c_r^k in the extrapolation formulas Eq. (8) for the site-bond problem.

			b_m^k		
k/m	1	2	3	4	5
1	-2	$-30\frac{1}{2}$	-177	$-3187\frac{4}{5}$	-179760
2	9	$3\frac{1}{2}$	$198\frac{1}{2}$	$-3178\frac{1}{2}$	-101540
3		-44	-252	3962	$563989\frac{2}{3}$
4		27	$491\frac{1}{2}$	$8568\frac{1}{2}$	$-153182\frac{1}{2}$
5			-342	$-11196\frac{1}{5}$	$-381038\frac{1}{3}$
6			81	6733	$401698\frac{1}{2}$
7				-1944	$-199151\frac{1}{3}$
8				243	57705
9					-9450
10					729

Table V: The coefficients b_m^k in the extrapolation formulas Eq. (9) for the site-bond problem.

n	a_n	n	a_n
0	1	17	-86564874
1	0	18	-134834422
2	0	19	-1031059888
3	-8	20	-1842094489
4	-4	21	-12140138712
5	-70	22	-27303542028
6	-23	23	-133912895295
7	-640	24	-447687526744
8	-205	25	-1274069580864
9	-6272	26	-7565668332198
10	-2941	27	-10362711920204
11	-64028	28	-113855530577726
12	-47391	29	-131148651484930
13	-678361	30	-1188175707628214
14	-714246	31	-4485228802915811
15	-7495405	32	1963925987626925
16	-10059661		

Table VI: The coefficients a_n in the series expansion of the percolation probability $P(q)$ for directed site-bond percolation on the triangular lattice.

$d_{N,0}$ we find the following sequence

$$1, 3, 9, 27, 83, 263, 857, \ldots$$

which we have identified as

$$d_{N,0} = 2C_N - 1. \tag{10}$$

where $C_N = (2N)!/((N+1)!N!)$ are the Catalan numbers, which also occur in the correction terms for the square bond problem. In general we find that for $r \leq 4$ the correction terms are given, for $N \geq r-2$, by the formulas

$$d_{N,r} = \sum_{k=1}^{r+1} a_r^k C_{N+k-1} + \sum_{k=1}^{r} b_r^k \left(\begin{array}{c} N \\ k \end{array} \right) c_N + \frac{1}{r!r!} \sum_{k=0}^{2r} c_r^k N^k. \tag{11}$$

We have listed the coefficients a_r^k, b_r^k and c_r^k of these extrapolation formulas in Table VII. We note that as in the previous problems the leading coefficients are quite simple, $a_r^{r+1} = (-1)^r 2C_{r+1}$, $b_r^r = 2$, and $c_r^{2r} = -C_r$.

These 5 extrapolation formulas and $P_{23}(q)$ allowed us to extend the series for $P(q)$ through order 51. The resulting series is listed in Table VIII.

	a_r^k				b_r^k				c_r^k			
k/r	1	2	3	4	1	2	3	4	1	2	3	4
0									-1	-8	0	-2304
1	6	0	52	-418	2	-12	90	-748	2	12	108	1152
2	-4	-18	-56	88		2	-14	102	-1	-18	-176	-1112
3		10	72	288			2	-16		8	234	2392
4			-28	-284				2		-2	-125	-3526
5				84							36	2344
6											-5	-820
7												160
8												-14

Table VII: The coefficients a_r^k, b_r^k and c_r^k in the extrapolation formulas Eq. (11) for the bond problem.

n	a_n	n	a_n
0	1	26	1587391
1	0	27	-3535398
2	0	28	6108103
3	-1	29	-13373929
4	0	30	23438144
5	-3	31	-50592067
6	1	32	89703467
7	-9	33	-191306745
8	6	34	342473589
9	-29	35	-722890515
10	27	36	1304446379
11	-99	37	-2729084244
12	112	38	4957423139
13	-351	39	-10292036449
14	450	40	18800279417
15	-1275	41	-38769381587
16	1782	42	71154482443
17	-4704	43	-145869275322
18	6998	44	268798182822
19	-17531	45	-548189750051
20	27324	46	1013680069047
21	-65758	47	-2057857140279
22	106211	48	3816820768061
23	-247669	49	-7717195669953
24	411291	50	14352037073232
25	-935107	51	-28915083150931

Table VIII: The coefficients a_n in the series expansion of the percolation probability $P(q)$ for directed bond percolation on the triangular lattice.

4 Analysis of the series

We expect that the series for the percolation probability behaves like

$$P(q) \sim A(1 - q/q_c)^{\beta}[1 + a_{\Delta}(1 - q/q_c)^{\Delta} + \ldots], \tag{12}$$

where A is the critical amplitude, Δ the leading confluent exponent and the $\ldots$ represents higher order correction terms. In the following sections we present the results of our analysis of the series which include accurate estimates for the critical parameters q_c, β, A and Δ. For the most part the best results are obtained using Dlog Padé (or in some cases just ordinary Padé) approximants. A comprehensive review of these and other techniques for series analysis may be found in Guttmann (1989).

4.1 q_c and β

In Table IX we list various Dlog Padé approximants to the percolation probability series for directed site percolation on the triangular lattice. The defective approximants, those for which there is a spurious singularity on the positive real axis closer to the origin than the physical critical point, are marked with an asterisk. Most higher-order approximants yield estimates around the values $q_c = 0.4043528$ and $\beta = 0.27645$, with very little spread among the approximants. Opting for a conservative error estimate, it seems appropriate to estimate that the critical parameters lie in the ranges, $q_c = 0.4043528(10)$ and $\beta = 0.27645(10)$, where the figures in parenthesis indicate the estimated error on the last digits.

The results of the analysis of the series for the bond problem are listed in Table X. In this case the spread among the various approximants is quite substantial, there appears to be a marked downward drift in the estimates for both q_c and β, and the estimates do not settle down to definite values. It does however seem likely that the true critical parameters lie within the estimates: $q_c = 0.52198(1)$ and $\beta = 0.2769(10)$.

The analysis of the series for the site-bond problem yields the results in Table XI. Again we see a downward drift in the estimates for both q_c and β though the estimates are somewhat more stable than in the previous case. We estimate that the true critical parameters lie within the ranges: $q_c = 0.264173(3)$ and $\beta = 0.2766(3)$

4.2 The critical amplitudes

We can estimate the critical amplitude A by evaluating Padé approximants to $G(q) = (q_c - q)P^{-1/\beta}$ at q_c, since it follows from the leading critical behaviour in Eq. (12) that $G(q_c) \sim A^{-1/\beta}q_c$. This prodecure works well but requires knowledge of both q_c and β. As we have just shown, we know both q_c and β very accurately for the triangular site series. We estimated A using values of q_c between 0.4043524 and 0.4043534 and

N	[N-1,N]		[N,N]		[N+1,N]	
	q_c	β	q_c	β	q_c	β
5	0.4040928	0.27451	0.4034610	0.27045	0.4045236	0.27822
6	0.4038500	0.27301	0.4074251	0.31368	0.4048775	0.28115
7	0.4043787	0.27671	0.4043331	0.27633	0.4043677	0.27664
8	0.4043535	0.27651	0.4043803	0.27676	0.4043698	0.27666
9	0.4043615	0.27658	0.4043636	0.27660	0.4043555	0.27650
10	0.4043623	0.27658	0.4043582	0.27654	0.4043574	0.27653
11	0.4043567	0.27652	0.4043567	0.27652	0.4043576*	0.27653*
12	0.4043567*	0.27652*	0.4043610*	0.27656*	0.4043553	0.27650
13	0.4043525	0.27644	0.4043538	0.27647	0.4043580*	0.27653*
14	0.4043529	0.27645	0.4043526	0.27645	0.4043528	0.27645
15	0.4043527	0.27645	0.4043529	0.27645	0.4043528	0.27645
16	0.4043528	0.27645	0.4043528	0.27645	0.4043528	0.27645
17	0.4043528	0.27645				

Table IX: Dlog Padé approximants to the percolation series for directed site percolation on the triangular lattice.

N	[N-1,N]		[N,N]		[N+1,N]	
	q_c	β	q_c	β	q_c	β
10	0.5222235*	0.28059*	0.5241918*	0.25876*	0.5220853	0.27898
11	0.5221835	0.28019	0.5221078	0.27927	0.5220958	0.27912
12	0.5220691	0.27873	0.5220388	0.27823	0.5218366	0.27295
13	0.5221336*	0.27948*	0.5219680	0.27678	0.5222844*	0.28038*
14	0.5220278	0.27805	0.5220029	0.27755	0.5220086	0.27768
15	0.5220076	0.27765	0.5220064	0.27763	0.5219973	0.27741
16	0.5220101*	0.27770*	0.5219613	0.27616	0.5219942	0.27733
17	0.5220046	0.27759	0.5219895	0.27720	0.5219959*	0.27738*
18	0.5220774*	0.27768*	0.5218335	0.26612	0.5219770	0.27679
19	0.5220382*	0.27801*	0.5219944	0.27735	0.5219876	0.27715
20	0.5219795	0.27687	0.5219848	0.27706	0.5219846	0.27705
21	0.5219846	0.27705	0.5219848	0.27705	0.5219847	0.27705
22	0.5219847	0.27705	0.5219848*	0.27705*	0.5219780	0.27678
23	0.5219837	0.27702	0.5219820	0.27696	0.5219811	0.27692
24	0.5219767	0.27671	0.5219804	0.27689	0.5219830*	0.27699*
25	0.5219796	0.27686	0.5219827*	0.27698*		

Table X: Dlog Padé approximants to the percolation series for directed bond percolation on the triangular lattice.

N	[N-1,N]		[N,N]		[N+1,N]	
	q_c	β	q_c	β	q_c	β
5	0.2639552	0.27456	0.2639775	0.27475	0.2645066	0.28077
6	0.2647846	0.28559	0.2640753	0.27556	0.2641622	0.27647
7	0.2641695	0.27656	0.2641494	0.27632	0.2641560	0.27640
8	0.2641576	0.27642	0.2642476	0.27835	0.2641667	0.27654
9	0.2641679	0.27655	0.2641739	0.27665	0.2641747	0.27666
10	0.2641747	0.27666	0.2641734*	0.27664*	0.2641757	0.27668
11	0.2641758	0.27668	0.2641753	0.27667	0.2641754	0.27667
12	0.2641754	0.27667	0.2641753	0.27667	0.2641755*	0.27668*
13	0.2641755*	0.27668*	0.2641754*	0.27668*	0.2641755*	0.27668*
14	0.2641755*	0.27668*	0.2641750	0.27667	0.2641716	0.27654
15	0.2641724	0.27658	0.2641735	0.27663	0.2641726	0.27659
16	0.2641729	0.27660				

Table XI: Dlog Padé approximants to the percolation series for directed site-bond percolation on the triangular lattice.

values of β ranging from 0.2764 to 0.2765. For each (q_c, β) pair we calculate A as the average over all $[N+K, N]$ Padé approximants with $K = 0, \pm 1$ and $2N + K \geq 25$. The spread among the approximants is minimal for $q_c = 0.4043527$, $\beta = 0.27645$ where $A = 1.581883(5)$. Allowing for values of q_c and β within the full range we get $A = 1.5819(4)$.

For the bond problem we used values of q_c from 0.52196 to 0.52121 and β from 0.2763 to 0.2773 averaging over Padé approximants with $2N+K \geq 40$. In this case the spread is minimal for $q_c = 0.521985$, $\beta = 0.2767$ where $A = 1.48584(2)$. Again allowing for a wider choice of critical parameters we estimate that $A = 1.486(6)$.

For the site-bond series we restricted q_c to lie between 0.264170 and 0.264176 and β between 0.2763 to 0.2768 using all approximants with $2N + K \geq 25$. The minimal spread occurs at $q_c = 0.264173$, $\beta = 0.2766$ where $A = 1.477393(4)$. A wider choice for q_c and β leads to the estimate $A = 1.477(1)$.

4.3 The confluent exponent

We studied the series using two different methods in order to estimate the value of the confluent exponent. In the first method, due to Baker and Hunter (1973), one transforms the function P,

$$P(q) = \sum_{i=1}^{n} A_i (1 - q/q_c)^{-\lambda_i} = \sum_{n=0}^{\infty} a_n q^n \tag{13}$$

into an auxiliary function with simple poles at $1/\lambda_i$. We first make the change of variable $q = q_c(1 - e^{-\zeta})$ and find, after multiplying the coefficient of ζ^k by $k!$, the auxiliary function

$$\mathcal{F}(\zeta) = \sum_{i=1}^{N} \sum_{k=0}^{\infty} A_i(\lambda_i \zeta)^k = \sum_{i=1}^{N} \frac{A_i}{1 - \lambda_i \zeta}, \tag{14}$$

which has poles at $\zeta = 1/\lambda_i$ with residue $-A_i/\lambda_i$. The great advantage of this method is that one obtains simultaneous estimates for many critical parameters, namely, β (the dominant singularity), Δ (the sub-dominant singularity), and the critical amplitudes (the residues at the singularities), while there is only one parameter q_c in the transformation. Unfortunately this method does not appear to work well for this problem. For the site problem we find that the transformed series generally yields poor estimates for β and no estimates for the confluent exponent. For the bond and site-bond problem the situation is somewhat better. In Table XII we have listed estimates for the critical parameters obtained from various Padé approximants to the Baker-Hunter transformed series, using the values $q_c = 0.52198$ for the bond series and $q_c = 0.264173$ for the site-bond series.

Bond problem					
N	M	β	A	Δ	$A \times a_\Delta$
18	19	0.27662	1.48469	1.03897	2.21646
19	20	0.27705	1.48845	0.97124	1.81301
20	21	0.27678	1.48604	1.01327	2.04400
21	21	0.28038	1.49843	0.91120	1.68564
21	22	0.27677	1.48594	1.01530	2.05671
22	22	0.27673	1.48582	1.01656	2.06289
22	23	0.27677	1.48594	1.01530	2.05672
23	23	0.27559	1.48208	1.06473	2.34714
23	24	0.27676	1.48587	1.01657	2.06477
24	25	0.27680	1.48619	1.01064	2.02788
25	26	0.27679	1.48615	1.01133	2.03211
Site–Bond problem					
11	12	0.27788	1.48749	0.89858	1.62193
12	13	0.27651	1.47668	1.01068	2.16827
13	13	0.27342	1.46940	1.11395	3.15155
13	14	0.27651	1.47666	1.01091	2.16997
14	15	0.27661	1.47745	0.99950	2.08954
15	15	0.27828	1.48182	0.96056	1.91013
15	16	0.27659	1.47728	1.00194	2.10641

Table XII: The critical exponent β, confluent exponent Δ and critical amplitudes A and a_Δ obtained from $[N, M]$ Padé approximants to the Baker-Hunter transformed series for the bond and site-bond problems.

It should be noted that, obviously, all approximants yield estimates for the critical parameters. However, we have discarded many approximants from the table because we believe the results to be spurious. For all the discarded approximants we found that the amplitude of the confluent term was of order zero and generally the estimate for β was very far from the expected value. Among the remaining approximants we clearly see that the favoured value of the confluent exponent is $\Delta = 1$. We also note that the amplitude estimates are in full agreement with those of the previous section.

In the second method, due to Adler *et al.* (1981), one studies Dlog Padé approximants to the function $F(q)$, where

$$F(q) = \beta P(q) + (q_c - q)\mathrm{d}P(q)/\mathrm{d}q.$$

The logarithmic derivative of $F(q)$ has a pole at q_c with residue $\beta + \Delta$. We evaluate the Dlog Padé approximants for a range of values of q_c and β. In Table XIII we have listed the estimates for Δ obtained by averaging over all $[N, N+K]$ approximants for a few values of β with q_c fixed at the central value of our estimate range. For the site and site-bond problem we used all approximants with $2N + K \geq 25$ and for the bond problem all approximants with $2N + K \geq 40$. This analysis clearly indicates that $\Delta \simeq 1$ and thus that there is no sign of any non-analytic corrections to scaling.

Site problem		Site-bond problem		Bond problem	
β	Δ	β	Δ	β	Δ
0.27640	0.98587	0.27630	0.97076	0.27660	1.03471
0.27641	0.99003	0.27635	0.98220	0.27665	1.03079
0.27642	0.99378	0.27640	0.99136	0.27670	1.02537
0.27643	0.99683	0.27645	0.99796	0.27675	1.01846
0.27644	0.99890	0.27650	1.00176	0.27680	1.01013
0.27645	0.99979	0.27655	1.00262	0.27685	1.00042
0.27646	0.99942	0.27660	1.00047	0.27690	0.98941
0.27647	0.99782	0.27665	0.99533	0.27695	0.97716
0.27648	0.99514	0.27670	0.98732	0.27700	0.96377
0.27649	0.99164	0.27675	0.97663	0.27705	0.94934
0.27650	0.98755	0.27680	0.96352	0.27710	0.93394

Table XIII: Estimates for the confluent exponent Δ from the transformation due to Adler *et al.* (1981) for various values of β at the critical point q_c.

Problem	Unbiased estimates			Biased estimates		
	q_c	β	A	q_c	A	N_{min}
T bond	0.52198(1)	0.2769(10)	1.486(6)	0.521971(5)	1.4841(2)	45
T site	0.4043528(10)	0.27645(10)	1.5819(5)	0.4043523(3)	1.58183(2)	30
T site-bond	0.264173(3)	0.2766(3)	1.477(1)	0.264170(4)	1.4765(3)	25
S bond	0.3552994(10)	0.27643(10)	1.3292(5)	0.35529955(15)	1.32920(1)	45
S site	0.294515(5)	0.2763(3)	1.425(1)	0.294518(3)	1.42588(4)	30
H bond	0.177143(2)	0.2763(2)	1.106(1)	0.177144(2)	1.1064(3)	30
H site	0.160067(5)	0.2763(4)	1.167(1)	0.160069(2)	1.1680(3)	30

Table XIV: Estimates of critical parameters for the three problems on the triangular (T) lattice studied in this paper and for the site and bond problems on the directed square (S) and honeycomb (H) lattices. See the text for explanation of the biased estimates.

5 Conclusion

In this paper we have presented extended series for the percolation probability for site, bond and site-bond percolation on the directed triangular lattice. The analysis of the series leads to improved estimates for the percolation threshold and the order parameter exponent β. In Table XIV we summarise the critical parameter estimates for the percolation probability for the three problems on the triangular lattice studied here and the problems studied in our earlier paper. The estimates for $q_c = 1 - p_c$ for the triangular bond and site problems are in excellent agreement with those obtained by Essam *et al.* (1986, 1988), $q_c = 0.40437(7)$ and $q_c = 0.521975(7)$, respectively. The estimates for β clearly show, as one would expect, that all the models studied in this and our earlier paper belong to the same universality class. The unbiased estimates for β, derived in the manner described in the previous section, for the triangular site and square bond cases are in excellent agreement and have small error bars (we emphasize once more that our error estimates are conservative). This leads us to belive that an improved estimate $\beta = 0.27644(3)$ is reasonable. We used this highly accurate estimate to obtain the *biased* estimates in Table XIV as follows. First we formed the series for $P(q)^{-1/\beta}$ using $\beta = 0.27644$. This series has a simple pole at q_c which can be estimated from ordinary Padé approximants. By averaging over all $[N, N+K]$ approximants with $K = 0, \pm 1$ and $2N + K \geq N_{min}$ we obtained the biased estimates for q_c the error-bars are basicly twice the spread among the approximants. We then used the biased estimate for q_c (with β as before) to obtain the biased estimates for the amplitudes using the procedure decribed in the previous section. As previously noted (Jensen and Guttmann 1995), there is no simple rational fraction whose decimal expansion agrees with our estimate of β. Given that this model is not conformally invariant, and that the expectation of exponent rationality is a consequence of conformal invariance, it is perhaps naive to expect otherwise. It is nevertheless true that there is a widely held - if imprecisely expressed - view that two dimensional systems should have rational exponents. More precise numerical work such as the recent estimation of the longitudinal size exponent $\nu_{||}$ (Conway and Guttmann 1994) of directed animals and the present calculation, supports the conclusion that the critical exponents for these models should not be expected to be simple rational fractions. Finally note that none of the series show any evidence of non-analytic confluent correction terms. This provides a hint that the models might be exactly solvable.

Acknowledgements

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Appendix: The first extrapolation formulas

In this appendix we shall calculate the first correction term(s) $d_{N,r}$ for the various problems we have studied in this paper. In the following we rely heavily on the work of Bousquet-Mélou (1995) and we shall represent the directed percolation models in terms of directed animals. By a directed site (bond) animal A we simply understand any finite set of connected sites (bonds) starting at the origin O in Figure 1. The *area* (or size) $|A|$ of an animal is the number of sites in the animal and the *perimeter* $p(A)$ is the number of unoccupied sites (bonds) with a nearest neighbour in A. The *height* h of an animal is the last row to which the animal extends, i.e., there is at least one occupied site in row h belonging to A but none in row $h+1$. The percolation probability, for the site and site-bond cases, is

$$P(q) = 1 - \sum_{A \in \mathcal{A}} q^{p(A)} (1-q)^{|A|-1} \tag{A.1}$$

where $\mathcal{A}$ denotes the set of animals on the lattice. For bond percolation the power of $(1-q)$ in the above equation is $|A|$. The difference stems from the assumption that for site percolation the origin is occupied with probability 1. In analogy with the finite-lattice formulation we define subsets $\mathcal{A}_N$ of $\mathcal{A}$ as the set of animals of height less than N. It follows that

$$P_N(q) = 1 - \sum_{A \in \mathcal{A}_N} q^{p(A)} (1-q)^{|A|-1}, \tag{A.2}$$

and

$$P_N(q) - P_{N+1}(q) = 1 - \sum_{A \in \mathcal{A}_{N+1} \backslash \mathcal{A}_N} q^{p(A)} (1-q)^{|A|-1}. \tag{A.3}$$

It should be noted that in the site and site-bond cases the polynomials $P_N(q)$ defined above are identical to the polynomials $P_{N+1}(q)$ from Section 2. From Eq. (A.3) it is immediately clear that P_N and P_{N+1} agree up to an order $\tilde{N}$ determined by the animals in $\mathcal{A}_{N+1} \backslash \mathcal{A}_N$ with the smallest perimeter. In our cases $\tilde{N}$ is simply proportional to N and the polynomials $P_N(q)$ therefore have a formal limit $P_\infty(q)$ which we identify as the percolation probability $P(q)$. By expanding Eq. (A.3) one gets a very useful expression for the correction terms

$$P_N(q) - P_{N+1}(q) = q^{\tilde{N}} \sum_{r \geq 0} q^r d_{N,r} = q^{\tilde{N}} \sum_{r \geq 0} q^r \sum_{k=0}^{r} \sum_{A \in \mathcal{A}_{N,k}} (-1)^{r-k} \binom{|A|-1}{r-k}, \tag{A.4}$$

where $\mathcal{A}_{N,k} = \{A \in \mathcal{A}_{N+1} \backslash \mathcal{A}_N, \;\; p(A) = \tilde{N} + k\}$.

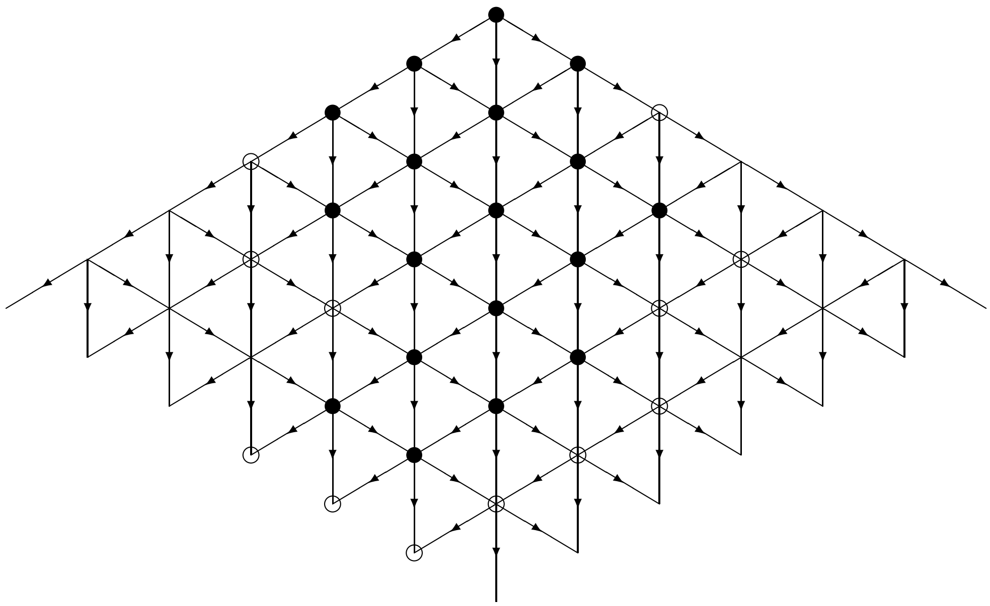

Figure 3: A compact directed site animal (filled circles) on the triangular lattice with perimeter sites marked by open circles.

The site case

An animal is *compact* if the occupied sites in any given row are consecutive, i.e., there are no holes in the animal (see Figure 3). Obviously, removing interior sites from a compact animal can never reduce the perimeter. Therefore, the animals in $\mathcal{A}_{N+1}\backslash\mathcal{A}_N$ with minimal perimeter are compact. The minimal perimeter of a compact animal of height N is $N+2$. This is proved by induction on N. It is obviously true for $N=1$ and one can easily see that by adding sites in row $N+1$ to a compact animal of height N at least one more perimeter-site is added. We also note that there are at least two animals of height N with perimeter $N+2$, namely a string of sites (one per row) running down either the left or right hand side of the lattice. This shows that $\tilde{N}=N+2$. It is also clear that these two animals must be the ones that give rise to the first correction term $d_{N,0}=2$. What remains is to prove that there can be no more animals in $\mathcal{A}_{N+1}\backslash\mathcal{A}_N$ with perimeter $N+2$. In order to do this we need a unique way of characterising the perimeter of compact animals of height N. Introduce lines R_k (L_k) parallel to the right-hand (left-hand) egde starting from row k. Since the animal is compact all sites in A intersecting R_k and L_k are consecutive. The number of perimeter sites on the left-hand side of the animal is $w_l = \max\{k, L_k \cap A \neq \emptyset\}$ because the last occupied site in line L_k has an unoccupied neighbour on L_k. Similar arguments apply for the number of perimeter sites w_r on the right side of A. Finally we note that the only perimeter site not accounted for is the one lying vertically below the last site in L_N and/or R_N. So the perimeter is $p(A)=w_l+w_r+1$. Furthermore if $A\in\mathcal{A}_{N+1}\backslash\mathcal{A}_N$ then either w_l or w_r (possibly both) has to equal N. The animals

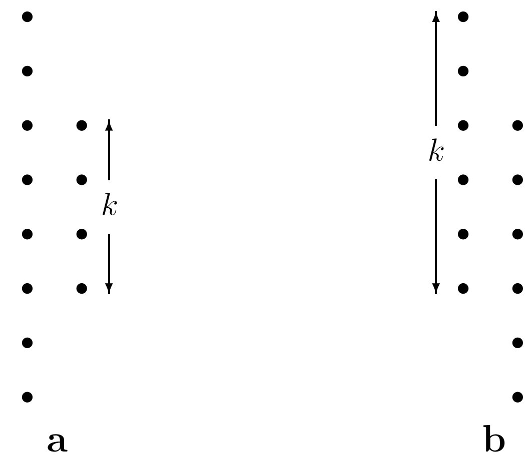

Figure 4: The two types of compact directed site animals with $w_l = 2$ which contribute to the second correction term.

with minimal perimeter $N+2$ are those with $w_l = 1$ or $w_r = 1$, obviously there can be only two such animals, which completes our proof that $d_{N,0} = 2$.

From Eq. (A.4) we get the second correction term

$$d_{N,1} = |\mathcal{A}_{N,1}| - \sum_{A \in \mathcal{A}_{N,0}} (|A| - 1), \tag{A.5}$$

where $|\mathcal{A}_{N,1}|$ is the number of animals of height N with a perimeter of length $N+3$. From the characterisation of compact animals derived above it follows that the animals in $\mathcal{A}_{N,1}$ are those with $w_l = 2$ or $w_r = 2$. Obviously there is the same number of animals in each case so we can restrict ourselves to the case $w_l = 2$, $w_r = N$. We are thus looking at animals restricted to the left-most two lines L_1 and L_2 of the lattice and either $L_1 \cap R_N$ or $L_2 \cap R_N$ has to be non-empty. The two types of animals are illustrated in Figure 4. If $L_1 \cap R_N \neq \emptyset$ (Figure 4a) then the first N sites of L_1 are occupied and $1 \leq k \leq N$ consecutive sites along L_2 are occupied; these k sites can be placed in $N - k + 1$ positions. If $L_1 \cap R_N = \emptyset$ (Figure 4b) and the first k sites ($1 \leq k \leq N-1$) of L_1 are occupied then the first j consecutive sites $0 \leq j \leq k$ of L_2 may be empty. Combining these two contributions with those from $w_r = 2$ we find

$$|\mathcal{A}_{N,1}| = 2\left(\sum_{k=1}^{N}(N-k+1) + \sum_{k=1}^{N-1}(k+1)\right) = 2N^2 + 2N - 2.$$

Since the number of sites in each of the two animals in $\mathcal{A}_{N,0}$ is N, Eq. (A.5) yields

$$d_{N,1} = 2N^2$$

thus proving the empirical formula derived previously.

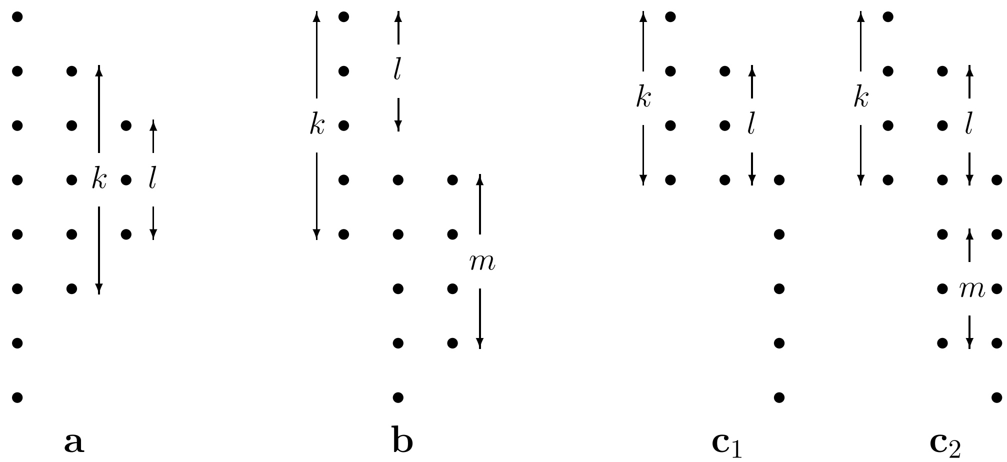

Figure 5: The types of compact directed site animals with $w_l = 3$ which contributes to the third correction term.

Next we prove the formula for $d_{N,2}$. From Eq. (A.4) we see that the third correction term is given by

$$d_{N,2} = |\mathcal{A}_{N,2}| - \sum_{A \in \mathcal{A}_{N,1}} (|A| - 1) + \sum_{A \in \mathcal{A}_{N,0}} \binom{|A|-1}{2}. \tag{A.6}$$

In this case there are two distinctly different sets of animals in $\mathcal{A}_{N,2}$, namely, compact animals with $w_l = 3$ as pictured in Figure 5, and animals formed from the compact animals of Figure 4 by removing consecutive sites from the second line of occupied sites leaving at least the first and last sites untouched. One easily sees that cutting such a 'hole' in these animals is the only way of increasing their perimeter by one site. From the animals in Figure 5 we get the following contributions

$$\begin{aligned}
\mathbf{a}: \quad & 2\sum_{k=1}^{N}\sum_{l=1}^{k}(N-k+1)(k-l+1) = && \frac{1}{12}N^4 + \frac{1}{2}N^3 + \frac{11}{12}N^2 + \frac{1}{2}N, \\
\mathbf{b}: \quad & 2\sum_{k=1}^{N-1}\sum_{l=1}^{k}\sum_{m=1}^{N-l}(N-l-m+1) = && \frac{1}{4}N^4 + \frac{5}{6}N^3 - \frac{1}{4}N^2 - \frac{5}{6}N, \\
\mathbf{c}_1: \quad & 2\sum_{k=1}^{N-1}\sum_{l=1}^{k}(l+1) = && \frac{1}{3}N^3 + N^2 - \frac{4}{3}N, \\
\mathbf{c}_2: \quad & 2\sum_{k=1}^{N-2}\sum_{l=0}^{k}\sum_{m=1}^{N-k-1}(m+l+1) = && \frac{1}{6}N^4 + \frac{1}{3}N^3 - \frac{7}{6}N^2 - \frac{4}{3}N + 2.
\end{aligned} \tag{A.7}$$

The animals in Figure 5a account for animals with $L_1 \cap R_N \neq \emptyset$, those of 5b for animals with $L_1 \cap R_N = \emptyset$ and $L_2 \cap R_N \neq \emptyset$, and lastly those of 5c for animals where $L_1 \cap R_N = \emptyset$ and $L_2 \cap R_N = \emptyset$. The contribution in each case is simply all the possible configurations which leads to an animal of the specified kind. The sums in Eq. (A.7) should be self-evident.

The animals in Figure 4 with a cut as described above yield the contributions

$$\begin{aligned}
\mathbf{a}: &\quad 2\sum_{k=3}^{N}\sum_{l=1}^{k-2}(N-k+1)(k-l-1) = \frac{1}{12}N^4 - \frac{1}{6}N^3 - \frac{1}{12}N^2 + \frac{1}{6}N, \\
\mathbf{b}: &\quad 2\sum_{k=2}^{N-1}\sum_{l=0}^{k-2}\sum_{m=1}^{k-l-1}(k-l-m) = \frac{1}{12}N^4 - \frac{1}{6}N^3 - \frac{1}{12}N^2 + \frac{1}{6}N. \qquad \text{(A.8)}
\end{aligned}$$

In case (a) the piece in the second line has to have at least three sites ($k \geq 3$) otherwise one could not cut out a hole of size $l \leq k-2$. The k sites can be placed in $(N-k+1)$ positions and the hole can be cut in $k-2-l+1 = k-l-1$ places, which leads to the first sum. In case (b) there can be from 2 to $N-1$ sites in the first line (the sum over k) with an overlap of $0 \leq m \leq k-2$ sites between the first line and the consecutive sites in the second line extending to the Nth row. Among the remaining $k-m$ sites in the second line $1 \leq l \leq k-m-1$ are occupied and they can be placed in $k-m-l$ positions, thus giving us the second sum.

The second term in Eq. (A.6) is the sum over $|A|-1$ of the compact animals in Figure 4 and we find the two contributions:

$$\begin{aligned}
\mathbf{a}: &\quad 2\sum_{k=1}^{N}(N-k+1)(N+k-1) = \frac{4}{3}N^3 + N^2 - \frac{1}{3}N, \\
\mathbf{b}: &\quad 2\sum_{k=1}^{N-1}\sum_{l=0}^{k}(N+k-l-1) = \frac{4}{3}N^3 - \frac{10}{3}N + 2. \qquad \text{(A.9)}
\end{aligned}$$

Finally the last term in Eq. (A.6) simply stems from the two animals in $\mathcal{A}_{N,0}$ and their contribution is

$$2\begin{pmatrix} N-1 \\ 2 \end{pmatrix} = N^2 - 3N + 2. \qquad \text{(A.10)}$$

By adding the contributions of Eqs. (A.7), (A.8) and (A.10) while subtracting those of Eq. (A.9) we get

$$d_{N,2} = \frac{2}{3}N^4 - N^3 + \frac{1}{3}N^2 - 2N + 2 = \frac{1}{12}(8N^4 - 12N^3 + 4N^2 - 24N + 24) \qquad \text{(A.11)}$$

in full agreement with the extrapolation formula listed in Table I, thus concluding the proof for $d_{N,2}$.

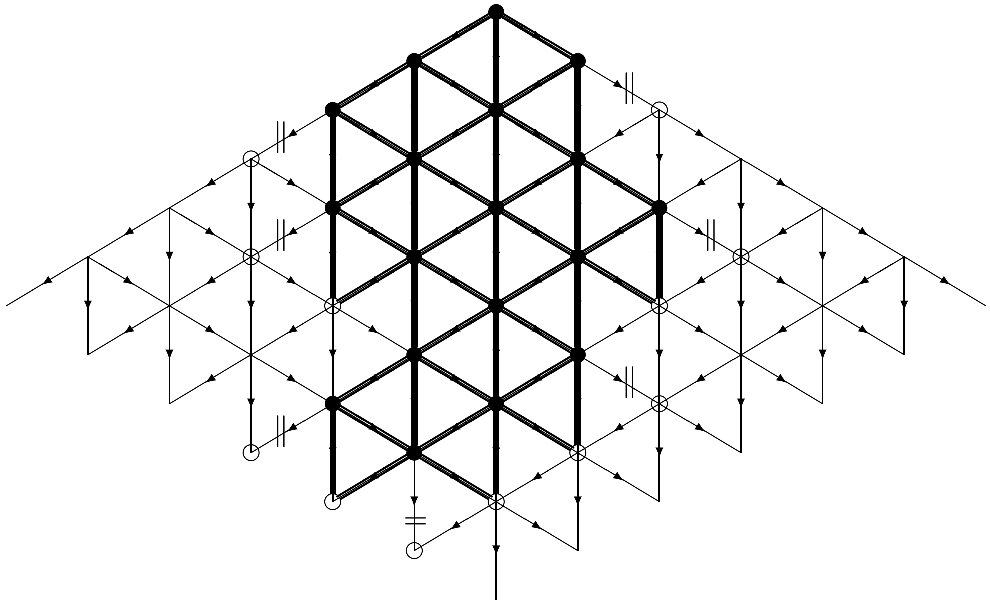

Figure 6: A site-compact directed site-bond animal (filled circles and thick bonds) on the triangular lattice with possible perimeter sites marked by open circles. Some of the perimeter sites have only one possible incident bond (marked by double lines) and in those cases the bond can be present (the site is part of the perimeter) or absent (the edge is part of the perimeter).

The site-bond case

From the emperical extrapolation formulas it it clear that the site-bond case is very similar to the site case and only a few generalisations are necessary. Again we look at compact animals and the ones we shall call *site-compact* have the minimal perimeter. A site-compact animal is one in which, as before, all occupied sites and bonds in a row are consecutive and in addition *all possible bonds to sites with more than one incident edge are present*. Figure 6 shows such an animal. Clearly the perimeter of such an animal is equal to the perimeter of the identical *site* animal. Thus the animals with minimal perimeter have $w_l = 1$ (or $w_r = 1$). Such animals consist of consecutive occupied sites down the left-hand side with most of the bonds emanating from these sites present. A few of the bonds can be either present or absent, namely, the bond from the top site pointing South-East and the bonds from the last site pointing South-West or South, though in this latter case at least one of the bonds has to be present. So all in all there are three possible bond configurations from the last site and two from the top site for a total of six possibilities. Taking into account the animals with $w_r = 1$ we have proved

$$d_{N,0} = 12$$

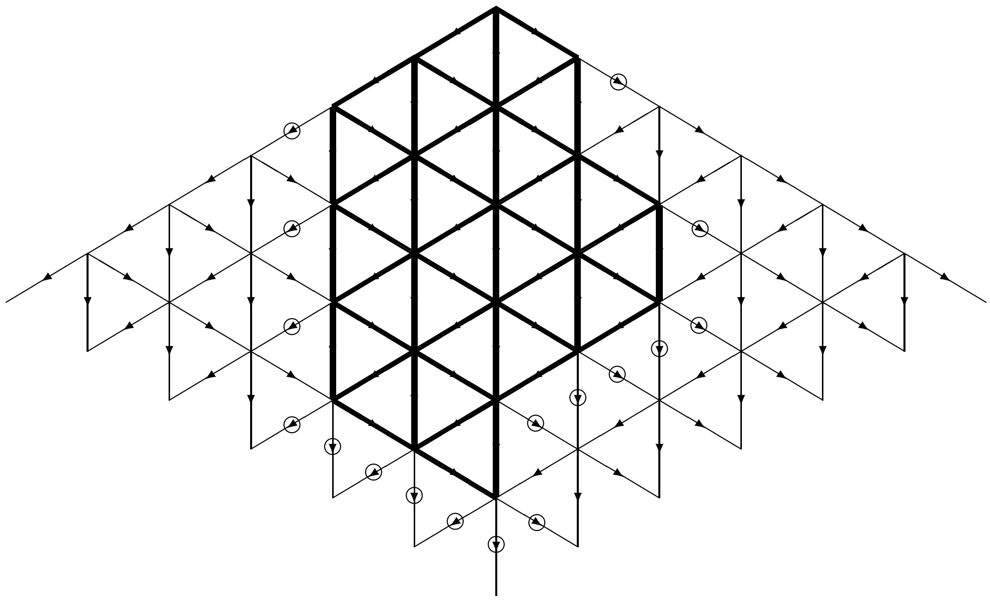

Figure 7: A compact directed bond animal (thick bonds) on the triangular lattice with perimeter bonds marked by open circles.

The bond case

The first correction term for the bond case, $d_{N,0} = 2C_N - 1$, involve the Catalan numbers C_N which equal the first correction term for the square bond problem (Baxter and Guttmann 1988). Bousquet-Mélou (1995) proved this result by noting that the square bond correction term arise from compact bond animals of directed height N. The first correction term for the triangular bond problem can be found by generalising the arguments from the square bond case. The first correction arise from compact animals constructed as follows. Choose two paths ω_1 and ω_2 consisting of bonds pointing only South and South-West starting from the origin and terminating at the same point on level N. The animal obtained by filling in all bonds between ω_1 and ω_2 has height N and perimeter $2N + 1$. These animals are just the *staircase animals* which are enumerated by the Catalan numbers and give rise to the first square bond correction term. Obviously the set of animals bounded by paths consisting of South and South-East bonds also contribute to the first correction term. The animal consisting entirely of south bonds (a line of bonds down the center of the lattice) is the only animal included in both sets. The first correction term is exactly due to these $2C_N - 1$ 'staircase animals' on the triangular lattice.