

Conjectured Enumeration of irreducible Multiple Zeta Values, from Knots and Feynman Diagrams

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Abstract Multiple zeta values (MZVs) are under intense investigation in three arenas – knot theory, number theory, and quantum field theory – which unite in Kreimer’s proposal that field theory assigns MZVs to positive knots, via Feynman diagrams whose momentum flow is encoded by link diagrams. Two challenging problems are posed by this nexus of knot/number/field theory: enumeration of positive knots, and enumeration of irreducible MZVs. Both were recently tackled by Broadhurst and Kreimer (BK). Here we report large-scale analytical and numerical computations that test, with considerable severity, the BK conjecture that the number, $D_{n,k}$, of irreducible MZVs of weight n and depth k , is generated by $\prod_{n \geq 3} \prod_{k \geq 1} (1 - x^n y^k)^{D_{n,k}} = 1 - \frac{x^3 y}{1-x^2} + \frac{x^{12} y^2 (1-y^2)}{(1-x^4)(1-x^6)}$, which is here shown to be consistent with all shuffle identities for the corresponding iterated integrals, up to weights $n = 44, 37, 42, 27$, at depths $k = 2, 3, 4, 5$, respectively, entailing computation at the petashuffle level. We recount the field-theoretic discoveries of MZVs, in counterterms, and of Euler sums, from more general Feynman diagrams, that led to this success.

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1. Introduction

In a recent Physics Letter [1] (hereafter BK), Dirk Kreimer and the author made a guess, informed by field theory, at the number, $D_{n,k}$, of irreducible multiple zeta values (MZVs) [2] of weight n and depth k that enter a minimal \mathbf{Q} -basis, to which all other MZVs may be reduced, as rational combinations of elements of the basis set, and their products. MZVs, and their extension [3] to alternating Euler sums, concerned BK, as practitioners of perturbative quantum field theory (pQFT), because of Kreimer's connection [4, 5, 6, 7] of MZVs with positive knots [8], via the counterterms of pQFT. This connection is strongly supported by BK's joint [1, 9, 10, 11, 12] and separate [13, 14, 15, 16, 17] calculations of multi-loop Feynman diagrams, and by those of others workers [18, 19, 20, 21].

At first parallel [22, 23, 24] to, and recently intertwining [25, 26, 27, 28] with, this flurry of field-theoretic activity, is an equally dramatic increase of understanding of MZVs [2] and of alternating [3] Euler sums, as richly structured number-theoretic objects, in their own right, irrespective of any field or knot theory. This is a rapidly progressing subject, where little is yet rigorously proven, but much is conjectured, on the basis of extensive [3, 26] analytical and numerical computations. Several conjectures command wide [2, 3, 24, 25, 26, 27] support. Yet the answer to a very obvious question – how many MZVs are left undetermined by the relations between MZVs? – appeared to be difficult, even to guess.

As in [1], we approach the question in three stages. First, we consider Euler sums, of weight $\sum_j s_j$ and depth k , which are k -fold nested sums of the form [1, 26]

$$\zeta(s_1, \dots, s_k; \sigma_1, \dots, \sigma_k) = \sum_{n_j > n_{j+1} > 0} \prod_{j=1}^k \frac{\sigma_j^{n_j}}{n_j^{s_j}},$$

with signs $\sigma_j = \pm 1$, positive integer exponents s_j , and $\sigma_1 s_1 \neq 1$, to prevent divergence of the outermost sum. As in [26], we combine the strings of exponents and signs into a single depth-length argument string, with s_j in the j th position when $\sigma_j = +1$, and \bar{s}_j in the j th position when $\sigma_j = -1$. The first question is then:

Q1 What is the number, $E_{n,k}$, of Euler sums of weight n and depth k , in a minimal \mathbf{Q} -basis for reducing all Euler sums to basic Euler sums?

Next, we define MZVs [2] as non-alternating Euler sums, with $\sigma_j = 1$, and ask:

Q2 What is the number, $M_{n,k}$, of Euler sums of weight n and depth k , in a minimal \mathbf{Q} -basis for reducing all MZVs to basic Euler sums?

Finally comes the most natural, yet most difficult question:

Q3 What is the number, $D_{n,k}$, of MZVs of weight n and depth k , in a minimal \mathbf{Q} -basis for reducing all MZVs to basic MZVs?

Section 2 gives the conjectured answers and focuses attention on the most difficult question, **Q3**, whose answer followed discoveries [1, 3, 11] of how Feynman diagrams assign MZVs to knots. Section 3 characterizes the two types of identity that are believed to achieve the conjectured reductions: depth-length shuffles, and weight-length shuffles. Section 4 presents the results of (very) large-scale computational tests, both analytical and numerical. Section 5 presents a brief bestiary of MZVs and Euler sums, obtained from Feynman diagrams that played vital roles in the inference of the conjectured enumeration. A conclusion appears, with several open questions, in Section 6.

2. Conjectured enumerations

It is conjectured that the answers to the three questions, above, are generated by

$$\prod_{n \geq 3} \prod_{k \geq 1} (1 - x^n y^k)^{E_{n,k}} \stackrel{?}{=} 1 - \frac{x^3 y}{1 - x^2} \frac{1}{1 - xy}, \quad (1)$$

$$\prod_{n \geq 3} \prod_{k \geq 1} (1 - x^n y^k)^{M_{n,k}} \stackrel{?}{=} 1 - \frac{x^3 y}{1 - x^2}, \quad (2)$$

$$\prod_{n \geq 3} \prod_{k \geq 1} (1 - x^n y^k)^{D_{n,k}} \stackrel{?}{=} 1 - \frac{x^3 y}{1 - x^2} + \frac{x^{12} y^2 (1 - y^2)}{(1 - x^4)(1 - x^6)}. \quad (3)$$

The solution to (1) is that given by the author in [3], in terms of what was there dubbed ‘‘Euler’s triangle’’, with the symmetric entries:

$$T(a, b) = \frac{1}{a + b} \sum_{d|a, b} \mu(d) P(a/d, b/d), \quad (4)$$

where $P(a, b) = (a + b)! / (a! b!)$ are the entries in Pascal’s triangle, the sum is over all positive integers d that divide both a and b , and a Möbius transformation is effected by

$$\mu(d) = \begin{cases} 1 & \text{when } d = 1 \\ 0 & \text{when } d \text{ is divisible by the square of a prime} \\ (-1)^k & \text{when } d \text{ is the product of } k \text{ distinct primes} \end{cases} \quad (5)$$

which is the Möbius function. When n and k have the same parity, and $n > k$, one obtains [3]

$$E_{n,k} = T\left(\frac{n-k}{2}, k\right). \quad (6)$$

With the exception of $\ln 2$ and π^2 (which act as seeds) all elements of the basis are thereby conjecturally enumerated.

The solution to (2) is that given by BK. It too involves Euler’s triangle. When n and k have the same parity, and $n > 3k$, one obtains [1]

$$M_{n,k} = T\left(\frac{n-3k}{2}, k\right). \quad (7)$$

With the exception of π^2 and $\zeta(3)$ (which act as seeds) all elements of the basis are thereby conjecturally enumerated.

Conjecture (3) appears in the final [1] version of BK, though no solution to it was given there. Here we motivate the Ansatz, and develop the generators for irreducibles of specific depths, up to $k = 6$. Observe, first, that the order y terms in (1) merely assert the irreducibility of the 2-braid torus knot-numbers [4], $\zeta(2n + 1)$ for $n > 0$. The claim of [3], there supported by a year of effort and 10^3 CPUhours of testing, is that all else is generated by including the factor of $1/(1 - xy)$ on the RHS, which gives (6), by Möbius inversion. It then took little time for BK to discover a RHS in (2) that was consistent with all known data. To celebrate the absence of $\ln 2$ from the reduction of both MZVs and counterterms, we omitted the factor of $1/(1 - xy)$, obtained (2), and performed CPU-intensive tests on the resulting prediction (7), which emerged unscathed. It is important

to remember the question thus answered: how many Euler sums in the basis for MZVs? It is ironical that the most complex of the three questions admits of the simplest answer. Consider weight-12 depth-2 irreducibles, with $M_{12,2} = 2$. The two basis elements may be taken as $\zeta(9, 3)$ and $\zeta(\overline{9}, \overline{3})$. The latter is not an MZV, but should be included, since it was shown in [3] that $\zeta(4, 4, 2, 2)$ reduces to a combination of $\zeta(\overline{9}, \overline{3})$ with MZVs of depths $k < 4$. Moreover, at weight 15 and depth 5, we found that $\zeta(6, 3, 2, 2, 2)$ reduces to a combination of $14\zeta(9, \overline{3}, \overline{3}) - 3\zeta(7, \overline{5}, \overline{3})$ with MZVs of depths $k < 5$. We call this phenomenon ‘‘pushdown’’. The simplicity of (2) relies on its existence and regularity.

The final step, to the most complex of the three Ansätze, in (3), was taken more falteringly¹. The question - how many MZVs in the basis for MZVs? - is the most natural to ask, in knot/number/field theory [7]. One now suspects that its answer will be the most complex of the three, since one is denied the use of non-MZV terms, like $\zeta(\overline{9}, \overline{3})$, in the reduction of MZVs. If pushdown makes (2) so simple, then its exclusion is expected to complicate (3). The final recipe of BK was the simplest possible, in these curious circumstances: to add a term that encapsulates what was already known about pushdown from depth 4 to depth 2, starting at weight 12. The additional term was constructed, with frank naivety, as follows. First, BK knew that the terms of order y^2 were given by $\frac{x^{12}y^2}{(1-x^4)(1-x^6)}$, merely by supposing the correctness of the conjectured [2, 24] enumeration of irreducible depth-2 MZVs. The factor of $(1 - y^2)$ was then inserted on the grounds that pushdown to depth $k = 2$ originated from depth $k = 4$. This pushdown demonstrably [3, 11] occurs at weight 12, where the ‘rule of three’, $D_{2n,2} = \lfloor \frac{n-1}{3} \rfloor$, for irreducible depth-2 MZVs, in [24], first differs from the ‘rule of two’, $M_{2n,2} = \lfloor \frac{n-2}{2} \rfloor$, for 3-braid knots, in [11]. It is supposed that pushdown to depth $k = 2$ occurs for greater even weights, at a rate that is metered [11] by comparing [3] with [24].

It should be stressed that recipe (3) is the simplest² one that BK could concoct, on the basis of depth-2 results. Its genesis does not guarantee any success at depths $k > 2$. It is hoped that the reader will share some the author’s amazement at the formidable success, below, of the highly specific predictions at depths $k = 3, 4, 5$. To extract these predictions: take the logarithm of each side of (3); set $a = x^2$ and $b = x^3y$; expand in powers of b ; determine the generator for $D_{3k+2n,k}$ by Möbius inversion. The result is

$$D_k(a) \equiv \sum_{n \geq 0} D_{3k+2n,k} a^n = \sum_{d|k} \frac{\mu(d)}{d} L_{k/d}(a^d), \quad (8)$$

where

$$\sum_{k \geq 1} L_k(a) b^k = \sum_{N \geq 1} \frac{1}{N} \left(\frac{b}{1-a} + \frac{b^2(b^2 - a^3)}{(1-a^2)(1-a^3)} \right)^N. \quad (9)$$

It follows that $D_k(a)$ is a ratio of polynomials, and that its singularities occur exclusively at roots of unity. In particular:

$$D_1(a) = \frac{1}{1-a}, \quad (10)$$

$$D_2(a) = \frac{a}{(1-a)(1-a^3)}, \quad (11)$$

¹A preliminary version of BK got it wrong, by failing to model (3) on the product forms (1,2).

²The aforementioned false start was more complicated, as it was not based on a product generator.

$$D_3(a) = \frac{a(1+a-a^2)}{(1-a)(1-a^2)(1-a^3)}, \quad (12)$$

$$D_4(a) = \frac{1+2a^2+a^3+a^4+2a^5+a^7-a^8}{(1-a)(1-a^3)(1-a^4)(1-a^6)}, \quad (13)$$

$$D_5(a) = \frac{1+2a+3a^2+3a^3+2a^4}{(1-a^2)^2(1-a^3)^2(1-a^5)}. \quad (14)$$

The complexity of the rational generators rapidly increases. For example,

$$D_6(a) = \frac{N_6(a)}{(1-a)(1-a^2)(1-a^3)(1-a^4)(1-a^6)(1-a^9)} \quad (15)$$

has a numerator

$$N_6(a) = 1+2a+3a^2+4a^3+6a^4+6a^5+6a^6+7a^7+4a^8+5a^9+4a^{10}+2a^{11}+2a^{12}-a^{16}+a^{17}. \quad (16)$$

As $n \rightarrow \infty$, a splendid behaviour of $D_{3k+2n,k}$ follows directly from (3):

$$k! \lim_{n \rightarrow \infty} n^{1-k} D_{3k+2n,k} = k \lim_{a \rightarrow 1} (1-a)^k D_k(a) = (3-\sqrt{3})^{-k} + (3+\sqrt{3})^{-k}, \quad (17)$$

requiring in (15) that $N_6(1) = 52$, which mental arithmetic shows to agree with (16).

Already a spectacular success emerges: the generator (12) is precisely that conjectured for depth-3 MZVs in [2, 25], though *no* term of order y^3 occurs in the Ansatz (3). Thus the simple-minded input of (10,11) predicts (12). The obvious question remains: how does the conjecture fare at greater depths? For $k > 3$, data prior to BK was extremely scanty. Personal communication from Don Zagier, reporting computations performed by him and by Dror Bar-Natan, were consistent with the first 5 terms predicted by (13) at depth 4. Warmed by this success, we here extend it, by *very* large-scale computation, to the first 16 terms. The method involves shuffle identities [2, 22, 23, 26], outlined in the next section, which provide rigorous upper bounds on irreducibles at specific depths and weights. Confirming that $D_{42,4} \leq 111$ involved 1.7 petashuffles. Having thus tuned and tested the code, we pushed on, to even larger systems of identities, at depth 5, where the first 7 terms of (14) have now been confirmed. The challenge of testing a significant number of terms in (16) was judged imprudent to attempt, unaided. Fortunately help is at hand, from Jon Borwein, David Bradley and Roland Girgensohn, in a collaboration [28] hosted by the Center for Experimental and Constructive Mathematics (CECM) at Simon Fraser University, which is a node of the Canadian High-Performance Computing Network.

3. Depth-length and weight-length shuffles

Depth-length shuffles were used in [3, 24, 25, 26]. The idea is simple, in the extreme, though less trivial to notate. For simplicity, consider the product of a depth-1 Euler sum, whose argument string is merely $\{s_1\}$, and an Euler sum of depth $k-1$, with argument string $\{s_2, \dots, s_k\}$. In the case of alternating Euler sums, signs may be included, as bars [1, 26] or minus signs [3], in these strings. Then the product clearly entails the k Euler sums [3] in which s_1 is inserted, in all possible ways, in the other string. The depth-length strings

$$\{s_1, s_2, \dots, s_k\}, \quad \{s_2, s_1, \dots, s_k\}, \quad \dots, \quad \{s_2, \dots, s_1, s_k\}, \quad \{s_2, \dots, s_k, s_1\},$$

result, each with unit coefficient, together with sums of lesser depth, which do not concern the present analysis of reducibility. Generalization [28] to the product of a string of depth r with one of depth $k - r$ is as might be expected: take all $\binom{k}{r}$ shuffles of the two depth-length strings that preserve the order of each. Combined with the reducibility [26] of

$$\zeta(\text{argument string of length } k) + (-1)^k \zeta(\text{reversed string})$$

such depth-length shuffles appear to exhaust that which can be concluded merely from the existence of a nested k -fold sum; all else must take account of the form of the summand.

In addition to such depth-length shuffle identities (called permutation identities in [3, 24, 25]) there are weight-length shuffle identities (corresponding to the partial-fraction identities of [3, 24, 25]). These follow from the existence of iterated-integral [22] representations for Euler sums. In the case of an MZV of weight n , a representation has been given [2, 22, 23] in terms of an n -fold iterated integral of the one-forms dx/x and $dx/(1-x)$. The extension to alternating sums was given in [26]: one has merely to include the one-form $dx/(1+x)$, which adds a third character to the weight-length alphabet.

Weight-length shuffles are then expressions of the ring structure of alternating [26] Euler sums, and their restriction [2] to MZVs:

$$\zeta(\text{string}_1)\zeta(\text{string}_2) = \sum_{\text{shuffles}} \zeta(\text{weight-length shuffled string}), \quad (18)$$

where each shuffled string preserves the order of each of the two constituent weight-length strings, in its iterated-integral representation. The product of strings of weights n_1 and n_2 entails $\binom{n_1+n_2}{n_1}$ weight-length shuffles, each of which has a weight and a depth that is the sum of those in the product, and many of which may occur many times.

4. Computational testing

For double sums, (11) was the input to (3). However the correctness of that input is still, according to the author's understanding of [2, 24], a conjecture. Thus it was tested, analytically, using REDUCE [29], to weight 44, which confirmed that the conjectured enumeration satisfied the identities of [24], and required no other.

For triple sums, (12) was a spectacular output of (3). However the correctness of that output is still, according to the author's understanding of [2, 25], a conjecture. Thus it was tested, analytically, also using REDUCE, to weight 37, which confirmed that the conjectured enumeration satisfied the identities of [25], and required no other.

The analytical results so obtained were consistent with the completeness and minimality of the concrete bases proposed in [26]. Minimality is more or less unprovable, since nothing, in principle, forbids further reductions, beyond those entailed by shuffles. However, the lattice algorithm PSLQ [30] resolutely failed to find anything new, wherever we had the CPUtime and patience to probe. More such tests may now be made with PSLQ, since its author, David Bailey, has very recently implemented Richard Crandall's fast algorithm [31] for computing Euler sums, including alternations of sign, and achieving a precision of 3200 digits at depth 5, and beyond. Confirmation has been obtained of instances of conjectures made in [26], and in the course of the present work. Results may

shortly appear in the high-performance computing literature. To date, none of the many numerical discoveries, made with PSLQ, erodes any conjecture in [1, 3, 26].

The strategy adopted to test (3) at depths 4 and 5, with high computational efficiency, was to pre-process all depth-length shuffles algebraically, using REDUCE, whose output was then converted to FORTRAN code. This was translated, by David Bailey's TRANSMP [30] code, into multiple precision calls of MPFUN [30] routines, which solved all the weight-length shuffles, modulo terms of lesser depth, enabling reducibility analysis at a numerical precision which provided overwhelming evidence that all known relations, and no others, were satisfied. The reader is (mercifully) spared programming details, which merely translate the transparent idea, of exhausting all shuffle identities, into computational practice. S/he is assured that the probability of misidentification of numerical solutions of the identities was less – often many orders of magnitude less – than 10^{-10} . The results that follow are hence not rigorously proven, though we discount the slender possibility that they fail to deliver the reductions that would have been achieved by computer algebra, with integer arithmetic, had one many gigabytes of core memory, and CPUyears of processing time, to expend on the huge integers that would be generated.

A measure of what was involved is provided by the following statistics. At depth $k = 4$ and weight $n = 42$, the ring structure (18) generates 1720620718074180 shuffles (1.7 petashuffles) each of which contributes unity to an element of a 16000×10660 matrix of constraints, whose rank deficiency was found to be 111. At depth $k = 5$ and weight $n = 27$, the rank deficiency of a 29900×14950 matrix was found to be 36.

In total, we confirmed the emboldened rank deficiencies in the integer sequence

$$\mathbf{1, 1, 3, 5, 7, 11, 16, 20, 27, 35, 43, 54, 66, 78, 94, 111}, 128, 150, 173, 196, 224, 254, 284, \dots \quad (19)$$

generated by (13), and those in the sequence

$$\mathbf{1, 2, 5, 9, 15, 23, 36}, 50, 71, 96, 127, 165, 213, 266, 333, 409, 498, 600, 720, 851, 1005, \dots \quad (20)$$

generated by (14). Neil Sloane has designated (19,20) as A019449 and A019450 in the on-line [32] encyclopedia of integer sequences.

For reasons that may be appreciated, the CPUtime required for the above was considerable: about 20 CPUdays on a 256MB AlphaStation 600 5/333. For that reason, testing of (3) at depth 6 awaits input from colleagues at CECM [28]. Nor is this an idle pursuit, since at depth 6 there appear to be new [28] features to solving the depth- and weight-length shuffle identities. If (3) is not correct, then depth 6 is the place to expect a failure. If a non-trivial number of terms in the amazing numerator (16) is, as hoped, confirmed, then the chances of failure at depths $k \geq 7$ appear remote, given the essentially depth-2 nature of the input.

5. Euler sums: a field theorist's sample bestiary

Lest Euler sums and MZVs appear a diversion from serious-minded field theory, we briefly recall some of their appearances in pQFT [1, 3, 5, 9, 11, 13, 14, 15, 17, 33, 34], often in advance of their systematic investigation by the maths community. To this traffic, from

field theory to number theory, Dirk Kreimer has added an equally noble trade route: from field theory to knot theory [7].

Depth-1 Every pQFT practitioner knows that the odd zetas, $\zeta(2n + 1)$, are everywhere dense in multiloop QCD and QED. In counterterms, they correspond to 2-braid torus knots [4, 5, 16]. Here one beast is gloriously absent [1] from pQFT: no subdivergence-free four-dimensional counterterm can spawn π^2 , for which there is no knot.

Depth-2 A decade ago [14], before MZVs attracted apparent interest in the maths community, the author failed to reduce a depth-2 weight-8 Euler sum, in the ε -expansion of dressed two-loop diagrams. The persistence [15] of this state of affairs led to the prediction that it would surface in 6-loop counterterms. It does [9]. It is the first depth-2 irreducible MZV, enumerated by $D_{8,2} = 1$. Even earlier [35], the St Petersburg group were unable to reduce ε -expansions of critical exponents to depth-1. It is now known why: an infinite series [11] of depth-2 irreducibles occurs, starting at weight 8. The corresponding knots start at 8 crossings, with the 3-braid [8] torus knot $8_{19} = (4, 3)$ [9]. Very recently [17], Anatoly Kotikov and the author have shown that *alternating* Euler sums result from the ε -expansion of critical exponents, in $D = 3 - 2\varepsilon$ dimensions. The first such beast is $\zeta(\overline{3}, \overline{1})$, which occurs with a multiple of $\zeta(4)$ that is process dependent. Thus odd-dimensional counterterms mimic even-dimensional massive diagrams, for it was observed in [3] that two of the most important 3-loop results in pQFT - for the electron anomaly [33] and the ρ -parameter [34] - have precisely that structure in their weight-4 terms.

Depth-3 Likewise, there was an obstacle to the reduction of triple sums of weight 11, obtained by cutting vacuum diagrams [13]. The irreducible occurs in the 7-loop beta-function of ϕ^4 -theory [9]. As at depth 2 [24], mathematicians later encountered [2, 25] this phenomenon. The associated knot is the sole positive 11-crossing 4-braid [9]. Its uniqueness is required by $D_{11,3} = 1$, and is confirmed by the knot enumeration of BK.

Depth-4 Perhaps the most far-reaching gift from pQFT to number theory occurred at depth 4 and weight 12 [3, 11]. Despite untiring effort, BK kept encountering an apparent mismatch between field, knot and (the then current state of) number theory. The (indubitably correct) result of [24], that $D_{12,2} \leq 1$, jarred with the existence of a *pair* [3] of 12-crossing 3-braid knots in the counterterms [11] of pQFT. The resolution was dramatic: the second knot corresponds to $\zeta(4, 4, 2, 2)$, which is pushed down [3] to $\zeta(\overline{9}, \overline{3})$, in the simpler enumeration (2), giving $M_{12,2} = D_{12,2} + D_{12,4} = 1 + 1 = 2$. Without this stimulus from pQFT, the author would not have discovered [3] such pushdowns, and the simplicity of (2) might have remained hidden. Instead of seeking a simple answer to the (now seen to be) difficult question **Q3**, BK backtracked to **Q2**, which has the simplest answer of the three. Then it was possible to return to **Q3**, input a minimal amount of information, interpreted in the light of Richard Feynman [1, 11] and Vaughan Jones [8], observe the success [2, 25] of (3) in giving (12), and arrive, via (13,14), at the predicted sequences of deficiencies in (19,20), tested here, non-trivially, at a cost of several petashuffles.

Such are the fruits of computational field theory.

5. Conclusion and questions

The conclusion is easy to state: the conjectured enumeration (3) emerged in the course of calculations in field theory [1, 3, 11], is illuminated by knot theory [4, 5, 7], and is shown by [24, 25, 28], and by the emboldened results in (19,20), to be in great shape, at all depths $k < 6$, however striking that may appear from the point of view of number theory [2, 26, 27].

Further questions are easier to pose than to answer:

Q4 Why?

Q5 Is (16) correct, at depth 6?

Q6 Is there an alternative to the CPU-intensive testing procedures, adopted here?

These, and related issues, are receiving due attention [6, 12, 17].

Acknowledgements

Of the many colleagues whose generosity of spirit has aided this work, I single out for especial thanks: David Bailey, of NASA-Ames, whose PSLQ [30] algorithm made computational number theory a joy; Jon Borwein, of CECM, for welcoming me to experimental mathematics; Bob Delbourgo, for a safe physics haven in Hobart, where this work began; Tony Hearn, of RAND, whose REDUCE [29] made computational knot theory much easier than I had supposed; Chris Stoddart, at the Open University, for management of a petashuffle-proof AlphaCluster; Don Zagier, for guiding my rash intrusion into his field with true grace, and for suggesting that any guess be cast in product form. Combined with these, David Bradley, Richard Crandall, Roland Girgensohn, John Gracey, Tolya Kotikov and Neil Sloane formed a valued network of advisors. At heart, I relied on the intuition, patience and tenacity of Dirk Kreimer, which provided daily sustenance.

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