

Asymptotic behavior of two-terminal series-parallel networks

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Abstract

This paper discusses the enumeration of two-terminal series-parallel networks, i.e. the number of electrical networks built with n identical elements connected in series or parallel with two-terminal nodes. They frequently occur in applied probability theory as a model for real networks. The number of networks grows asymptotically like R^n/n^α , as for some models of statistical physics like self-avoiding walks, lattice animals, meanders, etc. By using an exact recurrence relation, the entropy is numerically estimated at $R = 3.560\ 839\ 309\ 538\ 943\ 3\ (1)$, and we show that the sub-leading “universal” exponent α is $3/2$.

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I. INTRODUCTION

The series-parallel networks are one of the simplest models of real networks. Then even if it is at last one century old (Mac Mahon 1892), it still occurs frequently in both theoretical or applied studies. Some questions like the reliability (Feo and Johnson 1990), the survival probability after failures, the routing problem and the queuing delay (Tamir 1993), etc. are now an important branch of probability theory applied to communication, computer or power networks.

As usual in this kind of model, the elementary components (resistances or switches) are chosen all identical. Then the interest is in the behavior of the structure made of series or parallel connections.

Some authors (Riordan 1958, Lomnicki 1972) investigated this problem with a combinatorial point of view, and, in particular, give recurrences to compute the number of different structures built with n elements. Then, they consider the problem as formally “solved”.

In this article, we study this model by using analogies with statistical physics models like self-avoiding walks (Des Cloizeaux and Jannink 1987), lattice animals and percolation (Derida and De Seze 1982), folding and meanders model (Di Francesco et al. 1996), etc. In all these models, the number of configurations for n elements (steps, links, folds) grows asymptotically like R^n/n^α where R depends on the details of the model and α is an “universal” exponent, common to a broad set of models (“universality” class).

In Section 2, we define the problem. In Section 3, we explain how to enumerate the series-parallel networks. Finally, in Section 4, we give by numerical extrapolation a value for R and we show that $\alpha = 3/2$.

II. DEFINITIONS AND NOTATIONS

The two-terminal series-parallel networks are electric network considered geometrically. They are built with *elements* X_i which are conductor ($X_i = 1$) or insulator ($X_i = 0$). When two structures (with conductivities X_a and X_b) are in parallel, their (global) conductivity is $X_a + X_b$ where $+$ is the boolean operator *Or*. When they are in series, it is $X_1 X_2$ where the product is the boolean operator *And*.

A network with n elements, which connects two-terminal nodes, has a boolean representation $F = F(X_1, \dots, X_n)$. We consider that two networks are *equivalent* if their boolean representations differ only by a permutation of variables $\{X_i\}$. For example, $X_1 X_2 (X_3 + X_4)$, $X_1 (X_2 + X_3) X_4$ and $(X_1 + X_2) X_3 X_4$ are equivalent. We note that this definition is less restrictive than topological definition. In other words, equivalence consists of interchange inside a group of structures in series, or in parallel. The networks for $n = 1, 2, 3, 4$ are given in Table I.

Let us call a_n the number of different networks with n elements. It is also the number of non-equivalent boolean functions of n variables, built only with *And* and *Or*.

Except for the trivial case $n = 1$ with $a_1 = 1$, we see in Table I two classes of networks : essentially series (σ -networks) and essentially parallel (π -networks). There is a *duality* : by changing the boolean *And* into *Or* and vice versa, each σ -network is dual of a π -network. Let us call b_n the number of σ -networks (or of π -networks). We have

TABLE I. Two-terminal series-parallel networks for $n = 1, 2, 3, 4$. On each row, the two networks are dual. The a_n 's (number of networks) are 1, 2, 4, 10, ... The b_n 's (number of essentially series or, by duality, essentially parallel networks) are 1, 1, 2, 5, ...

number of elements	essentially series		essentially parallel		number of networks
1		X_1		X_1	1
2		X_1X_2		$X_1 + X_2$	2
3		$X_1X_2X_3$		$X_1 + X_2 + X_3$	4
		$X_1(X_2 + X_3)$		$X_1 + X_2X_3$	
4		$X_1X_2X_3X_4$		$X_1 + X_2 + X_3 + X_4$	10
		$X_1X_2(X_3 + X_4)$		$X_1 + X_2 + X_3X_4$	
		$X_1(X_2 + X_3 + X_4)$		$X_1 + X_2X_3X_4$	
		$X_1(X_2 + X_3X_4)$		$X_1 + X_2(X_3 + X_4)$	
		$(X_1 + X_2)(X_3 + X_4)$		$X_1X_2 + X_3X_4$	

$$a_n = 2b_n - \delta_{n,1}. \quad (1)$$

III. ENUMERATION AND RECURRENCE GENERATING FUNCTION

In this section, we explain how to enumerate the different networks, i.e. compute the a_n or b_n . By use of the duality, we consider only a σ -network S_n with $n \geq 2$ elements. It is composed with k π -structures of w_1, w_2, \dots, w_k elements, with $k \geq 2$ and $\sum_i w_i = n$. Its boolean representation is $S_n = P_{w_1}P_{w_2} \dots P_{w_k}$. By grouping together the equal w_i , we can use a symbolic exponential notation $S_n = P_1^{r_1}P_2^{r_2} \dots P_q^{r_q}$.

The $\{r_j\}$ are non-negative ($r_j \geq 0$) integers ; they form a *partition* of n with at least two parts : $n = \sum_j j r_j$ is partitioned in $k = \sum_j r_j$ parts. We note a partition as $1^{r_1}2^{r_2} \dots q^{r_q}$. For example, the partitions of 5 are $\{1^5, 1^32, 1^23, 12^2, 14, 23\}$.

Then, to obtain b_n (the number of σ -network with n elements), we first must compute the contribution of a given partition, and finally, we will sum over all the partitions. Thanks to the above decomposition, the number of S_n with a given partition $\{r_j\}$ is a product of as many factors as j , dependent only on b_j and r_j .

For example, it is obvious that, for b_5 , the contribution of the partition (23) is b_2b_3 , because b_2 is the number of choices for the first π -structure of two elements, and b_3 for the second of three elements.

When two π -structures have the same number of elements, i.e. $r_j = 2$, their contribution is not b_j^2 but, to remove duplications obtained by permutations, $b_j(b_j + 1)/2$. More generally, for arbitrary $r_j \geq 0$, the factor is

$$\binom{b_j + r_j - 1}{r_j} = \sum_{1 \leq u_1 \leq u_2 \leq \dots \leq u_{r_j} \leq b_j} 1. \quad (2)$$

So, the b_n (or the a_n) can be computed by recurrence

$$b_n = \sum_{\{r_j\}} \prod_j \binom{b_j + r_j - 1}{r_j} \quad (3)$$

where the sum runs over all the partition $\{r_j\}$ of n with at least two parts, as explained above. For example, $b_5 = 1 + b_2 + b_3 + b_2(b_2 + 1)/2 + b_4 + b_2b_3 = 12$.

With a computer, the b_n can be calculated by recurrence. Unfortunately, the number of partitions of n grows like $\exp(\pi\sqrt{2n/3})/n$ (Hardy-Ramanujan formula). We will now describe how to obtain a more practical recurrence formula.

We will use the generating functions

$$b(t) = \sum_{i \geq 1} b_i t^i = t + t^2 + 2t^3 + 5t^4 + \dots, \quad (4)$$

$$a(t) = \sum_{i \geq 1} a_i t^i = t + 2t^2 + 4t^3 + 10t^4 + \dots = 2b(t) - t. \quad (5)$$

Following an argument similar from that used by Euler for the generating function for the partition of n , by using Eq. (3)

$$\begin{aligned} & (1 + \binom{b_1}{1} t + \binom{b_1 + 1}{2} t^2 + \binom{b_1 + 2}{3} t^3 + \dots) \\ & \times (1 + \binom{b_2}{1} t^2 + \binom{b_2 + 1}{2} t^4 + \binom{b_2 + 2}{3} t^6 + \dots) \\ & \times \dots \\ & \times (1 + \binom{b_j}{1} t^j + \binom{b_j + 1}{2} t^{2j} + \dots + \binom{b_j + r - 1}{r} t^{rj} + \dots) \\ & \times \dots \\ & = 1 + \sum_{n \geq 1} \binom{b_n}{1} t^n + \sum_{n \geq 2} b_n t^n = 1 + a(t). \end{aligned} \quad (6)$$

As a generic line of the l.h.s. of Eq. (6) is equal to $(1 - t^j)^{-b_j}$, we obtain a formula attributed to Mac Mahon (1892) :

$$1 + a(t) = \prod_{j \geq 1} (1 - t^j)^{-b_j} \quad (8)$$

Eq (8) can be rewritten as a non-local functional relation

$$\ln[1 + a(t)] = \sum_{k \geq 1} \sum_{j \geq 1} \frac{1}{k} b_j t^{kj} = \sum_{k \geq 1} \frac{1}{k} b(t^k). \quad (9)$$

We obtain a useful recurrence taking the derivative of Eq. (9)

TABLE II. The a_n for $n \leq 45$

n	a_n	n	a_n	n	a_n
1	1	16	4 507 352	31	306 481 637 180 676
2	2	17	14 611 576	32	1 039 825 579 149 516
3	4	18	47 633 486	33	3 533 238 105 431 144
4	10	19	156 047 204	34	12 022 695 376 246 364
5	24	20	513 477 502	35	40 964 814 174 562 504
6	66	21	1 696 305 728	36	139 754 932 965 689 486
7	180	22	5 623 993 944	37	477 353 256 699 920 544
8	522	23	18 706 733 128	38	1 632 304 839 666 178 522
9	1 532	24	62 408 176 762	39	5 587 602 984 719 422 092
10	4 624	25	208 769 240 140	40	19 146 480 336 548 243 722
11	14 136	26	700 129 713 630	41	65 670 379 223 051 730 824
12	43 930	27	2 353 386 723 912	42	225 448 266 880 936 115 440
13	137 908	28	7 927 504 004 640	43	774 644 024 342 635 183 112
14	437 502	29	26 757 247 573 360	44	2 663 899 039 964 678 479 766
15	1 399 068	30	90 479 177 302 242	45	9 168 056 380 423 173 365 752

$$t \frac{a'(t)}{1+a(t)} = \sum_{k \geq 1} \sum_{j \geq 1} j b_j t^{kj} \quad (10)$$

Defining $e_i = \sum_{j|i} j a_j$, where the sum runs over the divisors of i including $j = 1$ and $j = i$,

$$2 \sum_{n \geq 1} n a_n t^n = (1 + \sum_{j \geq 1} a_j t^j) \sum_{i \geq 1} t^i (1 + e_i) \quad (11)$$

Defining $e'_i = \sum_{j|i, j \neq i} j a_j = e_i - i a_i$, we can compute the a_n with the recurrence formula (Lomnicki 1972)

$$n a_n = 1 + e'_n + \sum_{i=1}^{n-1} a_{n-i} (1 + e_i). \quad (12)$$

for which the computational effort grows like n^2 . By programming this recurrence on a computer with the use of *real*16* (i.e. real numbers described by 128 bits), it is possible to obtain several thousands of terms with 30 decimal digits of precision. It is more than enough to determine the asymptotic behavior.

This integer sequence belongs to the encyclopedia of Sloane (1996). The beginning is given in Table II.

IV. ASYMPTOTIC BEHAVIOR

A brief investigation of Table II shows that $a_{n+1} \approx 3.5 a_n$. Then, we can define a non-vanishing entropy per element $\ln a_n/n \sim \ln R$. More generally, it is expected that the exponential dominant behavior is corrected by sub-leading power-law for large n

$$a_n \sim C \frac{R^n}{n^\alpha} \quad (13)$$

where C is a constant.

In fact, in Eq. (12), for large n , $e_n \sim na_n$, then $a_n \sim \sum_{i=1}^{n-1} a_i a_{n-i}$. This last relation is identical apart from a shift ($n \rightarrow n+2$) to the definition $c_{n+1} \equiv \sum_{j=0}^n c_j c_{n-j}$ of the Catalan numbers $c_n = (2n)!/[n!(n+1)!]$. Using the Stirling formula, we see that c_n have the asymptotic behavior of Eq. (13) with $R = 4$ and $\alpha = 3/2$. The fact that the a_n and c_n have the same asymptotic relation do not imply that they have the same R . As $\alpha > 0$, in the sum, the extremal terms are dominant : $c_0 c_n > c_1 c_{n-1} > c_2 c_{n-2} > \dots$. Then the value of R strongly depends of the value of the first c_n (n small) and the exponential increase of a_n and c_n are not described by the same R . On the other hand, we expected the same α as we see below.

This behavior is common in statistical physics. The enumeration of random walks, self-avoiding walk, animals, etc. of n step (or links) drawn on a regular lattice always exhibits a exponential increase R^n where R depend on the connectivity of the lattice, and a sub-leading $n^{-\alpha}$, where α is ‘‘universal’’, i.e. is dependent on the physical nature of the problem (dimension, phases, ...) but independent on the details of the lattice.

Let us take a big two-terminal series-parallel network. At each scale, it is ‘‘decorated’’ by small parts b_n ($n = 1, 2, \dots$). Then, the contribution of these small b_n is a large fraction of the total entropy $\ln R$; they are similar to the connectivity of the lattice and govern the dominant behavior R^n . Nevertheless, the sub-leading $n^{-\alpha}$ depends on more general properties and α can be see as an ‘‘universal’’ exponent. It described the singularity of the generating function.

We suppose that the a_n have the following large n expansion

$$a_n \sim C \frac{R^n}{n^\alpha} \left(1 + \frac{u_1}{n} + \frac{u_2}{n^2} + \dots\right) \quad (14)$$

and we want estimate the values of R and α by numerical extrapolation. It is more convenient to study V_n and W_n

$$V_n \equiv \ln \frac{a_n}{a_{n-1}} = \ln R - \frac{\alpha}{n} + \frac{v_2}{n^2} + \frac{v_3}{n^3} + \dots \quad (15)$$

$$W_n \equiv n^2 \ln \frac{a_n a_{n-2}}{a_{n-1}^2} = \alpha + \frac{w_1}{n} + \frac{w_2}{n^2} + \dots \quad (16)$$

By plotting V_n (resp. W_n) versus $1/n$, the hypothesis (14) is consistent if the data draw a straight line ; its intersection with the y-axis gives $\ln R$ (resp. α).

A more systematic scheme of extrapolation is the Richardson polynomial extrapolation process (Brezinski and Redivo Zaglia 1991). Supposing Eq. (14),

$$\begin{aligned} \frac{1}{p!} \Delta^p (n^p V_n) &= \ln R + O(1/n^{p+1}) \\ \frac{1}{p!} \Delta^p (n^p W_n) &= \alpha + O(1/n^{p+1}) \end{aligned} \quad (17)$$

where Δ is the difference operator $\Delta f_n \equiv f_n - f_{n-1}$. This method gives the limits (R and α) and show whether the hypothesis (14) is consistent or not. When p is large, the acceleration

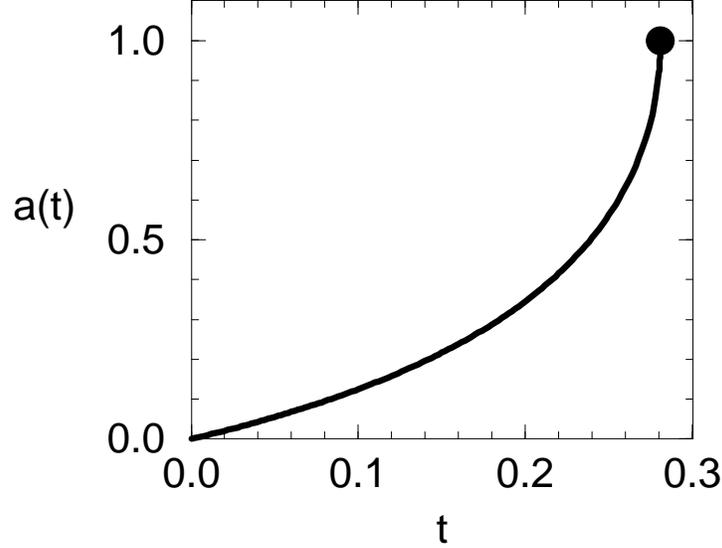


FIG. 1. The generating function $a(t)$. Its singularity occurs at $t^* = 1/R \approx 0.28$. At this point, $a(t^*) = 1$ and the function has a square root behavior because $\alpha = 3/2$.

of convergence is faster ; unfortunately, the numerical instabilities are amplified and after a given n , extrapolated values (Eq. 17) become chaotic. In this work, better extrapolations, obtained with p between 3 and 5, are

$$R = 3.560\ 839\ 309\ 538\ 943\ 3 \pm 10^{-16}, \quad (18)$$

$$\alpha = 1.5 \pm 10^{-13}. \quad (19)$$

We will now prove that $\alpha = 3/2$, by examining the behavior of the generating function, which is numerically plotted on Fig. 1. Clearly, the radius of convergence of the series $a(t) = \sum_{i \geq 1} a_i t^i$ (i.e. the location of its singularity) is $t^* = 1/R$. At $t = t^*$, $a(t^*) \sim \sum_i 1/i^\alpha$ converges if $\alpha > 1$, and its first derivative $a'(t^*) \sim \sum_i 1/i^{\alpha-1}$ diverges if $\alpha < 2$. Then, Fig. 1 indicates that $1 < \alpha < 2$.

More generally, the behavior of $a(t)$ around t^* is given by α . Let us consider the singular part $y(\epsilon) = a(t^*) - a(t)$ with $t = t^* e^{-\epsilon}$. For small ϵ , $y(\epsilon) \sim \sum_i [1 - e^{-\epsilon i}]/i^\alpha$. For $\alpha > 2$, the sum is dominated by small i and $y(\epsilon) = O(\epsilon)$, i.e. the first derivative $a'(t^*)$ is finite. For $1 < \alpha < 2$, the sum is dominated by $i = O(1/\epsilon)$ and $y(\epsilon) = O(\epsilon^{\alpha-1})$. The case $\alpha = 3/2$ corresponds to $y(\epsilon) = O(\sqrt{\epsilon})$.

Rewriting Eq. (9)

$$\ln[1 + a(t)] = \frac{1}{2}[a(t) + t] + r(t) \quad \text{with} \quad r(t) = \sum_{k \geq 2} \frac{1}{2k}[a(t^k) + t^k], \quad (20)$$

for $t \leq t^*$, $r(t)$ is a regular and increasing function. By defining $g(x) = 2 \ln(1 + x) - x$, $t + 2r(t) = g(a(t))$. The function $g(x)$ has a maximum at $x^* = 1$ with $g(x^*) = 2 \ln 2 - 1$ and a parabolic shape (i.e. a finite and negative second derivative). As $a(t) = g^{-1}(t + 2r(t))$, this gives a singularity for

$$t^* + 2r(t^*) = 2 \ln 2 - 1 \quad \text{and} \quad a(t^*) = 1 \quad (21)$$

with a square root shape $a(t^*) - a(t) = O(\sqrt{t^* - t})$. This proves that $\alpha = 3/2$. Unfortunately, with this argument, we can not compute the value of $t^* = 1/R$, because of the complexity of $r(t)$. Then R is sensitive to the details of the regular part of the recurrence : R is not “universal”. But α depends only of the kind of singularity : it is an “universal” exponent. As we have seen above, it is not surprising that it is equal to the α of the Catalan numbers.

V. CONCLUSION

We have seen that some concepts of statistical physics can be transposed to a problem of graph theory. The partition function of two-terminal series-parallel networks have an asymptotic behavior similar to those of self-avoiding walks or lattice animals. Its exponential growth depends on details on the model. On the other hand, a power-law correction, with an “universal” exponent can be computed exactly and have a fractional value $-3/2$. For the self-avoiding walks, this exponent is sensitive to the dimension and the kind of boundaries conditions and others independent exponents are related to geometrical properties. In our problem, it will be interesting to have a similar interpretation. For example, the number of terminal nodes (two in this article) could play the role of boundary conditions. By studying quantities like the average depth of elements (i.e. the number of decomposition needed to isolate this element), it is probable that new exponents would appear.

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