The Ramanujan Journal, , 1–15 ()

A Class of Series Acceleration Formulae for Catalan's Constant

DAVID M. BRADLEY dbradley@cecm.sfu.ca
Department of Mathematics, Dalhousie University, Halifax, N.S., Canada B3H 3J5

Editor: Jonathan M. Borwein

Abstract. In this note, we develop transformation formulae and expansions for the log tangent integral, which are then used to derive series acceleration formulae for certain values of Dirichlet L-functions, such as Catalan's constant. The formulae are characterized by the presence of an infinite series whose general term consists of a linear recurrence damped by the central binomial coefficient and a certain quadratic polynomial. Typically, the series can be expressed in closed form as a rational linear combination of Catalan's constant and π times the logarithm of an algebraic unit.

Keywords: log tangent integral, central binomial coefficient, algebraic unit, Catalan's constant.

1. Introduction

Catalan's constant may be defined by means of [1]

$$G := \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} = L(2, \chi_4), \tag{1}$$

where χ_4 is the non-principal Dirichlet character modulo 4. It is currently unknown whether or not G is rational.

The purpose of this note is to develop and classify acceleration formulae for slowly convergent series such as (1), based on transformations of the log tangent integral. The simplest acceleration formula of its type that we wish to consider is

$$G = \frac{\pi}{8}\log(2+\sqrt{3}) + \frac{3}{8}\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2\binom{2k}{k}}, \tag{2}$$

due to Ramanujan [4, 14]. We shall see that Ramanujan's formula (2) is the first of an infinite family of series acceleration formulae for G, each of which is characterized by the presence of an infinite series whose general term consists of a linear recurrence damped by the summand in (2). In each case, the series evaluates to a rational linear combination of G and π times the logarithm of an algebraic unit (i.e. an invertible algebraic integer). Perhaps the most striking example of this phenomenon is

$$G = \frac{\pi}{8}\log\left(\frac{10+\sqrt{50-22\sqrt{5}}}{10-\sqrt{50-22\sqrt{5}}}\right) + \frac{5}{8}\sum_{k=0}^{\infty} \frac{L(2k+1)}{(2k+1)^2\binom{2k}{k}}, \tag{3}$$

where $L(1)=1$, $L(2)=3$, and $L(n)=L(n-1)+L(n-2)$ for $n>2$ are the Lucas numbers [11], (M0155 in [15]).

We shall see that series acceleration results such as (2) and (3) have natural explanations when viewed as consequences of transformation formulae for the log tangent integral, although we should remark that Ramanujan apparently derived his result (2) by quite different methods. The connection with log tangent integrals is best explained by the equation

$$G=-\int_0^{\pi/4}\log(\tan\theta)\,d\theta, \tag{4}$$

obtained by expanding the integrand into its Fourier cosine series and integrating term by term. It will be shown that Ramanujan's result (2) arises from the transformation

$$2\int_0^{\pi/4}\log(\tan\theta)\,d\theta=3\int_0^{\pi/12}\log(\tan\theta)\,d\theta. \tag{5}$$

The roccoco formula (3) arises in a similar manner from the the transformation

$$2\int_0^{\pi/4}\log(\tan\theta)\,d\theta=5\int_0^{3\pi/20}\log(\tan\theta)\,d\theta-5\int_0^{\pi/20}\log(\tan\theta)\,d\theta. \tag{6}$$

Heuristically, one expects such transformations to succeed because the reduced range of integration on the right hand side, when re-expanded into a series, provides a continuous analog of bunching together many terms of the original series.

2. The Log Tangent Integral

There is a limitless supply of transformation formulae for the log tangent integral. In subsection 2.2, an infinite family of linear relations, of which both (5) and (6) are members, will be derived. These relations will be used in conjunction with the series expansions given in subsection 2.1 to develop a corresponding infinite family of series acceleration formulae which includes both (2) and (3) as special cases.

2.1. Series Expansions

We shall be concerned with only two series expansions for the log tangent integral. These are given in Theorems 1 and 2 below.

Theorem 1. *For* $0\le x\le \frac{1}{2}\pi$,

$$\int_0^x\log(\tan\theta)\,d\theta=-\sum_{k=0}^{\infty}\frac{\sin((4k+2)x)}{(2k+1)^2}.$$

Proof: Expand the integrand into its Fourier cosine series. Integrating term by term is justified by the fact that the Fourier series is boundedly convergent on compact subintervals of $(0, \frac{1}{2}\pi]$. ■

For us, the significance of Theorem 1 derives mostly from the specialization $x = \frac{1}{4}$, which yields the relationship (4) between Catalan's constant and the log tangent integral. On the other hand, the expansion in powers of sines provided by Theorem 2 below is more widely applicable.

THEOREM 2. *For* $0 \le x \le \frac{1}{4}\pi$,

$$\int_0^x \log(\tan\theta)\, d\theta = x \log(\tan x) - \frac{1}{4} \sum_{k=0}^{\infty} \frac{(2\sin 2x)^{2k+1}}{(2k+1)^2 \binom{2k}{k}}.$$

Proof: First integrate by parts, rescale, and use the double angle formula for sine:

$$\begin{aligned}
\int_0^x \log(\tan\theta)\, d\theta - x\log(\tan x) &= -\int_0^x \frac{\theta \sec^2\theta}{\tan\theta}\, d\theta \\
&= -\int_0^{2x} \frac{\theta\, d\theta}{4\tan(\frac{1}{2}\theta)\cos^2(\frac{1}{2}\theta)} \\
&= -\int_0^{2x} \frac{\theta\, d\theta}{2\sin\theta} \\
&= -\int_0^{\sin(2x)} \frac{2t\sin^{-1} t}{\sqrt{1-t^2}} \cdot \frac{dt}{4t^2}.
\end{aligned}$$

Now employ the power series expansion [6]

$$\frac{2t\sin^{-1} t}{\sqrt{1-t^2}} = \sum_{k=1}^{\infty} \frac{(2t)^{2k}}{k\binom{2k}{k}}, \qquad |t| < 1,$$

and integrate term by term. The result follows. ■

In addition to Theorem 1, the following representations were also more or less known to Ramanujan, and can be easily verified by differentiation:

$$\begin{aligned}
\int_0^x \log(\tan\theta)\, d\theta &= x\log(\tan x) + \sum_{k=0}^{\infty} \frac{(-1)^{k+1}(\tan x)^{2k+1}}{(2k+1)^2}, \quad 0 \le x \le \tfrac{1}{4}\pi, \\
\int_0^x \log(\tan\theta)\, d\theta &= (\tfrac{1}{2}\pi - x)\log(\cos x) - \sum_{k=1}^{\infty} \frac{(\cos x)^k (\sin kx)}{k^2}, \quad 0 \le x \le \tfrac{1}{2}\pi, \\
\int_0^x \log(\tan\theta)\, d\theta &= x\log(\tan x) + \tfrac{1}{2}\pi\log(2\cos x) \\
&\quad - \sum_{k=0}^{\infty} \binom{2k}{k} \frac{(\cos x)^{2k+1} + (\sin x)^{2k+1}}{4^k (2k+1)^2}, \quad 0 \le x \le \tfrac{1}{2}\pi.
\end{aligned}$$

2.2. *Transformation Formulae*

It will be convenient to define

$$T(r) := \int_0^{r\pi} \log(\tan\theta)\,d\theta, \qquad 0 \le r \le \tfrac{1}{2}. \tag{7}$$

Our development will provide two distinct transformation formulae for the T-function: the multiplication formula, which expresses T at odd multiples of the argument in terms of a multitude of other T-values; and the reflection formula, which makes it possible to restrict the domain to the interval $0 \le r \le \frac{1}{4}$, and which will effect a number of simplifications in our intermediate calculations, as we shall see.

THEOREM 3. *For all $0 \le r \le \frac{1}{2}$, the reflection formula*

$$T(r) = T(\tfrac{1}{2} - r)$$

holds.

Proof: First, note that $T(\frac{1}{2}) = 0$. This can be seen either by putting $x = \frac{1}{2}\pi$ in Theorem 1, or by observing that

$$T(\tfrac{1}{2}) = \int_0^{\pi/2} \log(\tan\theta)\,d\theta = \int_0^{\pi/2} \log(\sin\theta)\,d\theta - \int_0^{\pi/2} \log\sin(\tfrac{1}{2}\pi - \theta)\,d\theta = 0.$$

It follows that

$$\begin{aligned}
T(r) &= \int_0^{r\pi} \log(\tan\theta)\,d\theta = \int_0^{\pi/2} \log(\tan\theta)\,d\theta - \int_{r\pi}^{\pi/2} \log(\tan\theta)\,d\theta \\
&= \int_{\pi/2-r\pi}^{0} \log(\tan(\tfrac{1}{2}\pi - \theta))\,d\theta \\
&= \int_{\pi/2-r\pi}^{0} \log(\cot\theta)\,d\theta \\
&= \int_0^{(1/2-r)\pi} \log(\tan\theta)\,d\theta \\
&= T(\tfrac{1}{2} - r),
\end{aligned}$$

as stated. ■

To prove the multiplication formula, we require the following product expansion for the tangent function.

LEMMA 1. *Let $m = 2n+1$ be an odd positive integer, and let $x \in \mathbf{R}$. Then*

$$\frac{\tan(mx)}{\tan(x)} = \prod_{j=1}^{n} \tan\left(\frac{j\pi}{m} + x\right) \tan\left(\frac{j\pi}{m} - x\right).$$

Proof: Let $w = e^{ix}$. Then

$$\begin{aligned}
\frac{\tan(mx)}{\tan(x)} &= \left(\frac{w^{2m}-1}{w^{2m}+1}\right)\left(\frac{w^2+1}{w^2-1}\right) \\
&= \prod_{k=1}^{n}\left(\frac{w^2-e^{2k\pi i/m}}{w^2-e^{(2k-1)\pi i/m}}\right)\left(\frac{w^2-e^{-2k\pi i/m}}{w^2-e^{-(2k-1)\pi i/m}}\right) \\
&= \prod_{k=1}^{n}\left(\frac{we^{-k\pi i/m}-w^{-1}e^{k\pi i/m}}{we^{-(2k-1)\pi i/2m}-w^{-1}e^{(2k-1)\pi i/2m}}\right) \\
&\qquad \times\left(\frac{we^{k\pi i/m}-w^{-1}e^{-k\pi i/m}}{we^{(2k-1)\pi i/2m}-w^{-1}e^{-(2k-1)\pi i/2m}}\right) \\
&= \prod_{k=1}^{n}\frac{\sin(k\pi/m-x)\sin(k\pi/m+x)}{\sin((2k-1)\pi/2m-x)\sin((2k-1)\pi/2m+x)}.
\end{aligned}$$

After expressing the the sines in the denominator in terms of cosines and letting $j = n-k+1$, we have

$$\begin{aligned}
\frac{\tan(mx)}{\tan(x)} &= \prod_{j=1}^{n}\frac{\sin(j\pi/m-x)\sin(j\pi/m+x)}{\cos(j\pi/m-x)\cos(j\pi/m+x)} \\
&= \prod_{j=1}^{n}\tan\left(\frac{j\pi}{m}+x\right)\tan\left(\frac{j\pi}{m}-x\right)
\end{aligned}$$

as required. ∎

Theorem 4. *Let $m = 2n+1$ be an odd positive integer, and let $0 \le r \le 1/(2m)$. Then the multiplication formula*

$$T(mr) = m\sum_{j=0}^{n} T\left(\frac{j}{m}+r\right) - m\sum_{j=1}^{n} T\left(\frac{j}{m}-r\right)$$

holds.

Proof: By Lemma 1,

$$\begin{aligned}
T(mr) &= \int_0^{mr\pi}\log(\tan\theta)\,d\theta = m\int_0^{r\pi}\log(\tan(mx))\,dx \\
&= m\int_0^{r\pi}\log(\tan x)\,dx + m\sum_{j=1}^{n}\int_0^{r\pi}\log\tan\left(\frac{j\pi}{m}+x\right)\,dx \\
&\quad + m\sum_{j=1}^{n}\int_0^{r\pi}\log\tan\left(\frac{j\pi}{m}-x\right)\,dx \\
&= mT(r) + m\sum_{j=1}^{n}\left\{T\left(\frac{j}{m}+r\right) - T\left(\frac{j}{m}\right)\right\}
\end{aligned}$$

$$
\begin{aligned}
&-m\sum_{j=1}^{n}\left\{T\left(\frac{j}{m}-r\right)-T\left(\frac{j}{m}\right)\right\}\\
&= m\sum_{j=0}^{n}T\left(\frac{j}{m}+r\right)-m\sum_{j=1}^{n}T\left(\frac{j}{m}-r\right),
\end{aligned}
$$

as stated. ■

To obtain transformations such as (5) and (6), we apply the reflection formula (Theorem 3) and the multiplication formula (Theorem 4) with r chosen so as to express $T(\frac{1}{4})$ in terms of the T-function at values of the argument less than $\frac{1}{4}$. The resulting transformations are distinguished according to the parity of n in the multiplier $m = 2n + 1$.

Theorem 5. *Let n be an odd positive integer. Then*

$$G = -T\left(\frac{1}{4}\right) = \frac{2n+1}{n+1}\sum_{j=1}^{n}(-1)^{j}T\left(\frac{2j-1}{8n+4}\right).$$

Proof: Let p be a nonnegative integer. In the multiplication formula, let $n = 2p + 1$, so that $m = 4p + 3$, and put $r = 1/(4m)$. Then

$$
\begin{aligned}
T\left(\frac{1}{4}\right) &= m\sum_{j=0}^{p}\left\{T\left(\frac{4j+1}{4m}\right)+T\left(\frac{4(n-j)+1}{4m}\right)\right\}\\
&\quad -m\sum_{j=1}^{p}\left\{T\left(\frac{4j-1}{4m}\right)+T\left(\frac{4(n-j+1)-1}{4m}\right)\right\}\\
&\quad -mT\left(\frac{4(p+1)-1}{4m}\right).
\end{aligned}
$$

Applying the reflection formula (Theorem 3) to each term in the preceding sums yields the simplification

$$T\left(\frac{1}{4}\right) = 2m\sum_{j=0}^{p}T\left(\frac{4j+1}{4m}\right) - 2m\sum_{j=1}^{p}T\left(\frac{4j-1}{4m}\right) - mT\left(\frac{1}{4}\right).$$

The preceding expression can be simplified further by combining the two sums into a single alternating sum. Thus,

$$-T\left(\frac{1}{4}\right) = \frac{2m}{m+1}\sum_{j=1}^{2p+1}(-1)^{j}T\left(\frac{2j-1}{4m}\right).$$

Writing p and m in terms of n completes the proof. ■

Theorem 6 below addresses the alternative case in which the multiplier is congruent to 1 modulo 4.

THEOREM 6. *Let n be an even positive integer. Then*

$$G = -T\left(\frac{1}{4}\right) = \frac{2n+1}{n}\sum_{j=1}^{n}(-1)^{j+1}T\left(\frac{2j-1}{8n+4}\right).$$

Proof: Let p be a nonnegative integer. In the multiplication formula let $n = 2p$, so that $m = 4p+1$, and again put $r = 1/(4m)$. Then

$$\begin{aligned} T\left(\frac{1}{4}\right) &= m\sum_{j=0}^{p-1}\left\{T\left(\frac{4j+1}{4m}\right) + T\left(\frac{4(n-j)+1}{4m}\right)\right\} \\ &\quad -m\sum_{j=1}^{p}\left\{T\left(\frac{4j-1}{4m}\right) + T\left(\frac{4(n-j+1)-1}{4m}\right)\right\} \\ &\quad +mT\left(\frac{4p+1}{4m}\right). \end{aligned}$$

Applying the reflection formula (Theorem 3) to each term in the preceding sums yields the simplification

$$T\left(\frac{1}{4}\right) = 2m\sum_{j=0}^{p-1}T\left(\frac{4j+1}{4m}\right) - 2m\sum_{j=1}^{p}T\left(\frac{4j-1}{4m}\right) + mT\left(\frac{1}{4}\right).$$

The preceding expression can be simplified further by combining the two sums into a single alternating sum. Thus,

$$-T\left(\frac{1}{4}\right) = \frac{2m}{m-1}\sum_{j=1}^{2p}(-1)^{j+1}T\left(\frac{2j-1}{4m}\right),$$

Writing p and m in terms of n completes the proof. ■

EXAMPLE: Putting $n = 1$ in Theorem 5 yields the transformation $2T(\frac{1}{4}) = 3T(\frac{1}{12})$, which is a restatement of (5). Putting $n = 2$ in Theorem 6 yields the transformation $2T(\frac{1}{4}) = 5T(\frac{3}{20}) - 5T(\frac{1}{20})$, which is (6). □

3. Applications to Series Acceleration

3.1. Catalan's Constant

THEOREM 7. *Let n be an odd positive integer. For nonnegative integers k, define a sequence*

$$F_n(k) := \sum_{j=1}^{n}\left((-1)^{n-j+1}2\cos\left(\frac{j\pi}{2n+1}\right)\right)^k,$$

and let

$$u_n := \prod_{j=1}^{n} \left(\tan\left(\frac{2j-1}{8n+4}\right)\pi \right)^{(2j-1)(-1)^j}.$$

Then u_n is a unit algebraic integer, and Catalan's constant has the series acceleration formula

$$G = \left(\frac{\pi}{4n+4}\right)\log u_n - \left(\frac{2n+1}{4n+4}\right)\sum_{k=0}^{\infty}\frac{F_n(2k+1)}{(2k+1)^2\binom{2k}{k}}.$$

Proof: Apply Theorems 1 and 2 to the right hand side of Theorem 5. Thus,

$$\begin{aligned}
G &= \left(\frac{2n+1}{n+1}\right)\sum_{j=1}^{n}(-1)^j\left(\frac{2j-1}{8n+4}\right)\pi\log\left(\tan\left(\frac{2j-1}{8n+4}\right)\pi\right) \\
&\quad - \left(\frac{2n+1}{4n+4}\right)\sum_{j=1}^{n}(-1)^j\sum_{k=0}^{\infty}\frac{(2\sin((2j-1)\pi/(4n+2)))^{2k+1}}{(2k+1)^2\binom{2k}{k}} \\
&= \frac{\pi}{4}\sum_{j=1}^{n}(-1)^j\left(\frac{2j-1}{n+1}\right)\log\left(\tan\left(\frac{2j-1}{8n+4}\right)\pi\right) \\
&\quad - \left(\frac{2n+1}{4n+4}\right)\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2\binom{2k}{k}}\sum_{j=1}^{n}(-1)^j\left(2\sin\left(\frac{2j-1}{4n+2}\right)\pi\right)^{2k+1}. \qquad (8)
\end{aligned}$$

The inner sum in (8) simplifies somewhat if the sines are expressed in terms of cosines. Thus,

$$\begin{aligned}
&\sum_{j=1}^{n}(-1)^j\left(2\sin\left(\frac{2j-1}{4n+2}\right)\pi\right)^{2k+1} \\
&= \sum_{j=1}^{n}(-1)^j\left(2\cos\left(\frac{2n+1-(2j-1)}{4n+2}\right)\pi\right)^{2k+1} \\
&= \sum_{j=1}^{n}(-1)^j\left(2\cos\left(\frac{n-j+1}{2n+1}\right)\pi\right)^{2k+1} \\
&= \sum_{j=1}^{n}(-1)^{n-j+1}\left(2\cos\left(\frac{j\pi}{2n+1}\right)\right)^{2k+1}. \qquad (9)
\end{aligned}$$

Substituting (9) into (8) completes the derivation of the stated formula.

It now remains to show that u_n is indeed an algebraic unit (i.e. an invertible algebraic integer) as claimed. Let $x = (2j-1)\pi/(8n+4)$. Since the units in any ring form a multiplicative group, it suffices to show that the numbers $t := \tan x$ are all algebraic units, or equivalently, that the numbers t satisfy monic polynomials with integer coefficients and constant term ± 1.

From the addition formula for the tangent function, one sees that $\tan(kx)$ is a rational function of t for each nonnegative integer k. Indeed, if polynomials $p_k, q_k \in \mathbf{Z}[t]$ are defined by the recursion

$$\begin{pmatrix} p_{k+1} \\ q_{k+1} \end{pmatrix} = \begin{pmatrix} 1 & t \\ -t & 1 \end{pmatrix} \begin{pmatrix} p_k \\ q_k \end{pmatrix}, \quad \text{for} \quad k \geq 0, \qquad \begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{10}$$

then for all nonnegative integers k,

$$\tan((k+1)x) = \frac{\tan x + \tan(kx)}{1 - \tan(x)\tan(kx)} = \frac{tq_k + p_k}{q_k - tp_k} = \frac{p_{k+1}}{q_{k+1}}.$$

Since $\tan((2n+1)x) = \tan((2j-1)\pi/4) = (-1)^{j+1}$, it follows that $t = \tan((2j-1)\pi/(8n+4))$ satisfies the polynomial equation

$$p_{2n+1}(t) \pm q_{2n+1}(t) = 0.$$

It remains to show that $p_{2n+1} \pm q_{2n+1}$ has both highest degree coefficient and constant coefficient equal to ± 1.

Let k be an odd positive integer. From the recursion (10), it follows that

$$\begin{aligned} p_{k+2} + q_{k+2} &= (1 - 2t - t^2)p_k + (1 + 2t - t^2)q_k, \qquad (11) \\ p_{k+2} - q_{k+2} &= (1 + 2t - t^2)p_k - (1 - 2t - t^2)q_k. \qquad (12) \end{aligned}$$

An easy induction shows that the respective degrees of p_k and q_k are k and $k-1$, for all odd positive integers k. This fact, combined with a second induction, shows that the highest degree coefficient of $p_k \pm q_k$ is equal to ± 1 for all odd positive integers k. Finally, (11) and (12) show that

$$p_{k+2}(0) \pm q_{k+2}(0) = p_k(0) \pm q_k(0)$$

and so a final induction proves that $p_k \pm q_k$ has constant coefficient equal to ± 1 for all odd positive integers k. ∎

Remark. Suppose n is fixed, and we partition the algebraic numbers

$$(-1)^{n-j+1} 2\cos\left(\frac{j\pi}{2n+1}\right)$$

into disjoint sets of mutual conjugates. Then the product of the minimum polynomials for each set of conjugates is precisely the characteristic polynomial of the linear recurrence satisfied by the sequence $\{F_n(k)\}_{k=0}^{\infty}$.

Example: Putting $n = 1$ in Theorem 7 gives

$$G = -\frac{\pi}{8}\log\left(\tan\left(\frac{\pi}{12}\right)\right) + \frac{3}{8}\sum_{k=0}^{\infty} \frac{(2\cos(\pi/3))^{2k+1}}{(2k+1)^2\binom{2k}{k}},$$

which is Ramanujan's formula (2). □

Theorem 7 has its even counterpart in Theorem 8 below.

THEOREM 8. *Let n be an even positive integer. For nonnegative integers k, define a sequence*

$$F_n(k) := \sum_{j=1}^{n} \left((-1)^j 2\cos\left(\frac{j\pi}{2n+1}\right)\right)^k,$$

and let

$$u_n := \prod_{j=1}^{n} \left(\tan\left(\frac{2j-1}{8n+4}\right)\pi\right)^{(2j-1)(-1)^{j+1}}.$$

Then u_n is a unit algebraic integer, and Catalan's constant has the series acceleration formula

$$G = \left(\frac{\pi}{4n}\right)\log u_n + \left(\frac{2n+1}{4n}\right)\sum_{k=0}^{\infty} \frac{F_n(2k+1)}{(2k+1)^2\binom{2k}{k}}.$$

We omit the proof of Theorem 8, as it closely mimicks the proof of Theorem 7. Instead, we derive the formula (3) which relates Catalan's constant and the Lucas sequence.

COROLLARY. *Let $L(1) = 1$, $L(2) = 3$, and $L(n) = L(n-1) + L(n-2)$ for $n > 2$ be the Lucas numbers. Then Catalan's constant has the series acceleration formula*

$$G = \frac{\pi}{8}\log\left(\frac{10+\sqrt{50-22\sqrt{5}}}{10-\sqrt{50-22\sqrt{5}}}\right) + \frac{5}{8}\sum_{k=0}^{\infty} \frac{L(2k+1)}{(2k+1)^2\binom{2k}{k}}.$$

Proof: Put $n = 2$ in Theorem 8. Letting $\phi := 2\cos(2\pi/5) = \frac{1}{2}(\sqrt{5}-1)$ and $\tau := 2\cos(\pi/5) = \frac{1}{2}(\sqrt{5}+1)$, we have

$$G = \frac{\pi}{8}\log\left(\frac{\tan(\pi/20)}{\tan^3(3\pi/20)}\right) - \frac{5}{8}\sum_{k=0}^{\infty} \frac{\phi^{2k+1} - \tau^{2k+1}}{(2k+1)^2\binom{2k}{k}}.$$

Now recall [11] that

$$L(k) = \left(\frac{1+\sqrt{5}}{2}\right)^k + \left(\frac{1-\sqrt{5}}{2}\right)^k$$

for all nonnegative integers k. It follows that

$$G = \frac{\pi}{8}\log\left(\frac{\tan(\pi/20)}{\tan^3(3\pi/20)}\right) + \frac{5}{8}\sum_{k=0}^{\infty} \frac{L(2k+1)}{(2k+1)^2\binom{2k}{k}},$$

and so it remains only to verify the non-trivial denesting relationship

$$\frac{\tan(\pi/20)}{\tan^3(3\pi/20)} = \frac{10+\sqrt{50-22\sqrt{5}}}{10-\sqrt{50-22\sqrt{5}}}. \tag{13}$$

To express the tangent values in (13) in terms of radicals, we follow [13], p. 50. Let $t := \tan(\pi/20)$. Then

$$\tan\frac{3\pi}{20} = \frac{3t-t^3}{1-3t^2} = \tan\left(\frac{\pi}{4}-\frac{2\pi}{20}\right) = \frac{1-2t/(1-t^2)}{1+2t/(1-t^2)}.$$

Equating the previous rational expressions in t gives the quintic equation

$$(t-1)^5 = 20t^2(t-1), \quad \text{or} \quad (t-1)^2 = 2t\sqrt{5},$$

since $t \neq 1$. Putting $t = (1-\varepsilon)/(1+\varepsilon)$, it follows that $\varepsilon\sqrt{5+2\sqrt{5}} = \sqrt{5}$, and

$$\tan\frac{\pi}{20} = \frac{\sqrt{5+2\sqrt{5}}-\sqrt{5}}{\sqrt{5+2\sqrt{5}}+\sqrt{5}}, \quad \tan\frac{3\pi}{20} = \frac{\sqrt{5+2\sqrt{5}}-1}{\sqrt{5+2\sqrt{5}}+1}.$$

Therefore, we may write

$$\frac{\tan(\pi/20)}{\tan^3(3\pi/20)} = \left(\frac{\sqrt{5+2\sqrt{5}}+1}{\sqrt{5+2\sqrt{5}}-1}\right)^3 \frac{\sqrt{5+2\sqrt{5}}-\sqrt{5}}{\sqrt{5+2\sqrt{5}}+\sqrt{5}} = \frac{a+b}{a-b}, \tag{14}$$

where a and b are to be determined. Cross multiplying and expanding both sides, we have

$$5b(3+\sqrt{5}) = a(3-\sqrt{5})\sqrt{5+2\sqrt{5}}. \tag{15}$$

Since $(3-\sqrt{5})/(3+\sqrt{5}) = \frac{1}{2}(7-3\sqrt{5})$, we may write (15) in the form

$$10b = a\sqrt{(7-3\sqrt{5})^2(5+2\sqrt{5})} = a\sqrt{50-22\sqrt{5}}.$$

Therefore, if in (14), we take $a = 10$ and $b = \sqrt{50-22\sqrt{5}}$, then (13) holds, and the proof is complete. ■

Remark. It is unlikely that (13) will simplify any further. Zippel [16] gives two formulae (caution: there are misprints) for denesting expressions involving square roots. Borodin et. al. [5] show that these are the only two ways that such expressions can be denested over the rational number field. In particular, $\sqrt{50-22\sqrt{5}}$ cannot be denested, because $50^2 - 5\times 22^2 = 80$ and $22^2\times 5^2 - 50^2\times 5 = -400$ are not squares of rational numbers.

3.2. Some Additional Examples

One can derive additional acceleration formulae by specializing the value of x in Theorems 1 and 2 and equating the two results. In general, convergence improves as the value of x decreases. The following selections provide a representative sample of perhaps the most interesting results that can obtained using this approach.

EXAMPLE: Putting $x = \frac{1}{4}\pi$ gives (cf. (1))

$$G = L(2, \chi_4) = \frac{1}{2}\sum_{k=0}^{\infty} \frac{4^k}{(2k+1)^2\binom{2k}{k}}, \tag{16}$$

which Ramanujan [4] derived previously by other methods. We remark that (16) is actually a series *deceleration* result. The reason for the poor convergence is we have used the trivial transformation $T(\frac{1}{4}) = T(\frac{1}{4})$ which fails to exploit the reduced range of integration present in the other transformations. □

EXAMPLE: Putting $x = \frac{1}{6}\pi$ gives

$$L(2, \chi_6) = \frac{\pi\sqrt{3}}{18}\log 3 + \frac{1}{2}\sum_{k=0}^{\infty} \frac{3^k}{(2k+1)^2\binom{2k}{k}},$$

where χ_6 is the non-principal Dirichlet character modulo 6 (i.e. $\chi_6(5) = -1$). □

EXAMPLE: Putting $x = \frac{1}{8}\pi$ gives

$$L(2, \chi_8) = \frac{\pi\sqrt{2}}{8}\log(1+\sqrt{2}) + \frac{1}{2}\sum_{k=0}^{\infty} \frac{2^k}{(2k+1)^2\binom{2k}{k}},$$

where χ_8 is the Dirichlet character modulo 8 given by $\chi_8(1) = \chi_8(3) = 1$, and $\chi_8(5) = \chi_8(7) = -1$. □

Acknowledgments

I'm grateful to Chris Hill, Jonathan Borwein, Petr Lisonek, and John Zucker for their helpful observations.

Appendix

Here, we outline the role that inverse symbolic computation – in particular, Maple's integer relations algorithms – played in the discovery process.

A vector $\vec{v} = (v_1, v_2, \ldots, v_n)$ of real numbers is said to possess an integer relation if there exists a vector $\vec{a} = (a_1, a_2, \ldots, a_n)$ of integers not all zero such that the scalar product vanishes, i.e. $a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0$. In the past two decades, several algorithms which recover $\vec{a}$ given $\vec{v}$ have been discovered [2, 3, 9, 10, 12]. One of these, "LLL" [12], has been implemented in Maple V, and with its help, the authors of [7] and [8] discovered new formulae for values of the Riemann Zeta function. The obstacle which initially confounded efforts to extend the classical results

$$\zeta(2) = 3\sum_{k=1}^{\infty} \frac{1}{k^2\binom{2k}{k}}, \quad \zeta(3) = \frac{5}{2}\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^3\binom{2k}{k}}, \quad \zeta(4) = \frac{36}{17}\sum_{k=1}^{\infty} \frac{1}{k^4\binom{2k}{k}}$$

to higher zeta values was circumvented by the introduction of harmonic sums into the search space. Thus, for example, by searching for an identity of the form

$$\zeta(7) = r_1 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^7 \binom{2k}{k}} + r_2 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^5 \binom{2k}{k}} \sum_{j=1}^{k-1} \frac{1}{j^2} + r_3 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3 \binom{2k}{k}} \sum_{j=1}^{k-1} \frac{1}{j^4},$$

we [7] found

$$\zeta(7) = \frac{5}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^7 \binom{2k}{k}} + \frac{25}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3 \binom{2k}{k}} \sum_{j=1}^{k-1} \frac{1}{j^4},$$

and infinitely many more, as well as some lovely integral and hypergeometric series evaluations, besides.

We suspected that a similar reverse-engineered approach might work for certain Dirichlet L-series values, such as Catalan's constant, but searching for similar variations on Ramanujan's example (2) failed. In view of the ornate complexity of (3) and its relatives (Theorem 7 and Theorem 8), we can now understand the reason for this failure. For a direct attack, one would have had to introduce, among other things, logarithms of algebraic units into the model, so that in effect, one would have needed to know beforehand the formula one was searching for in order to find it. Models based on the inverse tangent integral [13, 14] suffer the same drawbacks. On the other hand, the model based on the log tangent integral is suited perfectly.

The author arrived at the log tangent integral model while attempting to give an alternative proof of Ramanujan's acceleration formula (2). It was found that the proof reduced to that of proving the integral transformation (5). Isolating the T-function of section 3 for study was then a natural choice. After directing Maple's integer relations finding algorithms to hunt for linear relations amongst various T-values, the following list was produced:

$$T(1/2) = 0, \tag{A.1}$$
$$T(1/3) = T(1/6), \tag{A.2}$$
$$T(1/8) = T(3/8), \tag{A.3}$$
$$3T(4/9) = T(1/3) + T(2/9) - 3T(1/9), \tag{A.4}$$
$$T(2/10) = T(3/10), \tag{A.5}$$
$$T(1/10) = T(2/5), \tag{A.6}$$
$$T(1/12) = T(5/12), \tag{A.7}$$
$$2T(1/4) = 3T(1/12), \tag{A.8}$$
$$T(3/14) = T(4/14), \tag{A.9}$$
$$T(5/14) = T(1/7), \tag{A.10}$$
$$T(1/14) = T(3/7), \tag{A.11}$$
$$3T(2/5) = -3T(1/15) + T(1/5) + 3T(4/15), \tag{A.12}$$
$$3T(7/15) = -3T(2/15) + 3T(1/5) + T(2/5), \tag{A.13}$$
$$15T(1/15) = 15T(2/15) - 5T(1/5) + 9T(1/3) - 10T(2/5), \tag{A.14}$$

$$
\begin{aligned}
T(3/16) &= T(5/16), && \text{(A.15)}\\
T(1/16) &= T(7/16), && \text{(A.16)}\\
T(2/9) &= T(5/18), && \text{(A.17)}\\
T(1/9) &= T(7/18), && \text{(A.18)}\\
T(1/18) &= T(4/9), && \text{(A.19)}\\
3T(1/18) &= 3T(5/18) + T(1/3) - 3T(7/18), && \text{(A.20)}\\
T(3/20) &= T(7/20), && \text{(A.21)}\\
T(1/20) &= T(9/20), && \text{(A.22)}\\
5T(3/20) &= 5T(1/20) + 2T(1/4) = 5T(7/20). && \text{(A.23)}
\end{aligned}
$$

Aside from trivial substitutions arising from the reflection formula (Theorem 3), the list evidently exhausts all linear relations amongst T-values with rational arguments having denominator no greater than 20. In fact, each list entry is a consequence of the reflection formula and the multiplication formula (Theorem 4). For example, (A.4) follows from the multiplication formula with $m = 3$ and $r = 1/9$. The slightly trickier (A.14) follows from three applications of the multiplication formula. One takes $m = 3$ with $r = 1/15$ and $r = 2/15$, and then one takes $m = 5$ with $r = 1/15$. This gives three equations. Multiplying the first through by $5/2$, the second through by $-5/2$, and the third through by $3/2$ and adding the three resulting equations gives (A.14).

From the list, it was easy to deduce and subsequently prove the reflection formula. At the same time, Chris Hill of the University of Illinois used the $m = 3$ case of Lemma 1 to prove (5) i.e. (A.8). This broke the dam, leading to the proof of Lemma 1, the multiplication formula (Theorem 4), and the remaining results of sections 2 and 3.

References

1. M. Abramowitz & I. Stegun, *Handbook of Mathematical Functions*, Dover, New York, 1972, p. 807.
2. D. H. Bailey & H. R. P. Ferguson, "Numerical Results on Relations Between Numerical Constants Using a New Algorithm," *Mathematics of Computation*, Vol. 53 (October 1989), pp. 649–656.
3. D. H. Bailey & H. R. P. Ferguson, "A Polynomial Time, Numerically Stable Integer Relation Algorithm," *RNR Technical Report*, RNR-91-032.
4. B. C. Berndt, *Ramanujan's Notebooks: Part I*, Springer-Verlag, 1985, p. 289.
5. A. Borodin, R. Fagin, J. E. Hopcroft, & M. Tompa, "Decreasing the nesting depth of expressions involving square roots," *J. Symbolic Comp.* **1** (1985), pp. 169–188.
6. J. M. Borwein & P. B. Borwein, *Pi and the AGM*, Wiley-Interscience, John Wiley & Sons, Toronto, 1987, p. 384.
7. J. M. Borwein & D. M. Bradley, "Empirically Determined Apéry-Like Formulae for $\zeta(4n+3)$," *Experimental Mathematics*, Vol. 6, Issue 3, October 1997, pp. 181–194.
8. J. M. Borwein & D. M. Bradley, "Searching Symbolically for Apéry-Like Formulae for Values of the Riemann Zeta Function," *SIGSAM Bulletin of Symbolic and Algebraic Manipulation*, Vol. 30, No. 2, Issue 116, (June 1996), pp. 2–7.
9. H. R. P. Ferguson & R. W. Forcade, "Generalization of the Euclidean Algorithm for Real Numbers to All Dimensions Higher Than Two," *Bulletin of the American Mathematical Society*, Vol. 1 (1979), pp. 912–914.

10. J. Hastad, B. Just, J. C. Lagarias, & C. P. Schnorr, "Polynomial Time Algorithms for Finding Integer Relations Among Real Numbers," *SIAM Jouranl on Computing*, Vol. 18 (1988), pp. 859–881.
11. G. H. Hardy & E. M. Wright, *An Introduction to the Theory of Numbers*, (5th ed.) Oxford University Press, New York, 1979, p. 148.
12. A. K. Lenstra, H. W. Lenstra, & L. Lovasz, "Factoring Polynomials with Rational Coefficients," *Math. Annalen*, Vol. 261 (1982), pp. 515–534.
13. L. Lewin, *Polylogarithms and Associated Functions*, Elsevier North Holland, New York, 1981.
14. S. Ramanujan, "On the integral $\int_0^x \frac{\tan^{-1} t}{t}\, dt$," *Journal of the Indian Mathematical Society*, VII (1915), pp. 93–96.
15. N. J. A. Sloane & S. Plouffe, *The Encyclopedia of Integer Sequences*, Academic Press, San Diego, 1995.
16. R. Zippel, "Simplifications of expressions involving radicals," *J. Symbolic Comp.* **1** (1985), pp. 189–210.